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Quasi-filiform Lie algebras of great length

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Abstract

We give a complete classification up to isomorphisms of complex graded quasi-filiform Lie algebras of dimension $n \geqslant 15$ with a finite number of subspaces greater than their nilindex $n-2$.

Keywords: Nilpotent and graded Lie algebras

1. Background and notation

The difficulties to obtain the classification of a class of nilpotent Lie algebras lead to study the algebras which can give useful information about such a class. In this way, the graded Lie algebras play an important role.

If $\mathfrak{g}$ is a nilpotent Lie algebra of dimension n and nilindex k (index of nilpotency), it is naturally filtered by the descending central sequence of $\mathfrak{g}$, $(C^i\mathfrak{g})_{0\leqslant i\leqslant k}$, $(C^0\mathfrak{g}=\mathfrak{g}, C^i\mathfrak{g}=[\mathfrak{g}, C^{i-1}\mathfrak{g}])$. We consider the filtration given by (S_{i+1}), where $S_{i+1}=\mathfrak{g}$, if $i\leqslant 0$; $S_{i+1}=C^i\mathfrak{g}$, if $1\leqslant i\leqslant k-1$, and $S_{i+1}=\{0\}$, if $i\geqslant k$. Associated to $\mathfrak{g}$ there exists a graded Lie algebra $\operatorname{gr}\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, where $\mathfrak{g}_i=S_i/S_{i+1}$. Thus, we have

$$\operatorname{gr}\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i,\qquad \mathfrak{g}_i=C^{i-1}\mathfrak{g}/C^i\mathfrak{g}.$$

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When $\operatorname{gr}\mathfrak{g}$ and $\mathfrak{g}$ are isomorphic, denoted by $\operatorname{gr}\mathfrak{g} = \mathfrak{g}$, we will say that the algebra is naturally graded.

From Vergne's work, if $\mathfrak{g}$ is an n-dimensional naturally graded filiform Lie algebra, then $\mathfrak{g}$ is isomorphic to L_n or Q_n [12]. This classification plays a fundamental role in the cohomological study of the variety of nilpotent Lie algebras laws.

Let us note that, for the filiform case, Goze and Khakimdjanov [8] give the complete classification of graded (not necessarily naturally graded) filiform Lie algebras and they give the geometric description of the characteristically nilpotent filiform Lie algebras by using those graded filiform Lie algebras. We note that, in this last classification, the case denoted $\mathcal{C}_r$ is not necessary because these algebras are isomorphic to Q_n (this remark was given by one of the authors). In this paper, we are interested by the quasi-filiform Lie algebras, that is nilpotent Lie algebras $\mathfrak{g}$ whose nilindex is equal to $\dim(\mathfrak{g}) - 2$. This class of algebras correspond to the class whose Goze' invariant is $(n-2, 1, 1)$. In a previous paper [3] we have given the classification of naturally graded quasi-filiform Lie algebras. Here we approach the general classification by considering a hypothesis called "length-condition": let $\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_{n_i}$ be a connected gradation, that is:

$$(1) \quad [\mathfrak{g}_{n_i}, \mathfrak{g}_{n_j}] \subset \mathfrak{g}_{n_i+n_j},$$

$$(2) \quad \mathfrak{g}_{n_i} \neq \{0\}.$$

Such a gradation is said of length p. The length $l(\mathfrak{g})$ of $\mathfrak{g}$ is the maximum of the lengths of connected gradations. It is clear that $l(\mathfrak{g}) \leqslant \dim(\mathfrak{g})$. In [5] we have given the classification when $l(\mathfrak{g}) = n = \dim(\mathfrak{g})$ and we have studied easily some of their cohomological properties by considering a gradation with n subspaces [7]. We remark that a graded Lie algebra with a gradation with a large number of subspaces facilitates the study of some cohomological properties for these algebras (see [1,2,9,10]).

The quasi-filiform Lie algebra $\mathfrak{g} = Q_{n-1} \oplus \mathbb{C}$ is graded and satisfies $l(\mathfrak{g}) = n-1$ while the natural gradation has only $n-2$ non-zero subspaces. The aim of this paper is to give the classification of graded quasi-filiform Lie algebras with $l(\mathfrak{g}) = n-1$.

This paper is structured in the following way. In Section 2, we define the quasi-filiform Lie algebras of maximum length. Then we introduce the quasi-filiform Lie algebras with a gradation of length $n-1$, and we state the main classification theorem. In Section 3 we study the existence of adequate homogeneous bases, which allows us to obtain the basis structure of quasi-filiform Lie algebras with a gradation of length $n-1$. Finally, in the last section we discuss how the proof of the classification theorem has been obtained. We also show how the programming language *Mathematica* [13] has been used as an assistant, with packages elaborated ad hoc in order to study similar problems.

2. Quasi-filiform Lie algebras of length bigger than $\dim\mathfrak{g} - 2$

We use the following definition of quasi-filiform Lie algebra which is a key concept in this paper.

Definition 2.1. An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be *quasi-filiform* if $C^{n-3}\mathfrak{g} \neq 0$ and $C^{n-2}\mathfrak{g} = 0$, where $C^0\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$.

Recently a new invariant has been introduced by Goze to study nilpotent Lie algebras. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let $\operatorname{gz}(X)$ be the ordered sequence of Jordan block's dimensions of nilpotent operator $\mathrm{ad}(X)$, where $X \in \mathfrak{g}$. In the set of these sequences, we consider the lexicographical order. Then the sequence

$$\mathrm{gz}(\mathfrak{g}) = \max_{X \in \mathfrak{g} \setminus [\mathfrak{g},\mathfrak{g}]} \{\mathrm{gz}(X)\}$$

is an invariant of $\mathfrak{g}$. An n-dimensional Lie algebra $\mathfrak{g}$ is quasi-filiform if and only if the invariant of Goze $\mathrm{gz}(\mathfrak{g})$ is $(n-2,1,1)$.

Thus, an n-dimensional quasi-filiform Lie algebra is characterized by its nilindex $k = n-2$, which determines all the algebras belonging to the quasi-filiform family. Hence, the number of subspaces for the natural gradation in the quasi-filiform case is $n-2$, but we can obtain other algebras with a gradation of length n.

2.1. *Quasi-filiform Lie algebras of length* $\dim \mathfrak{g}$

The algebras $\mathfrak{g} = \mathfrak{g}^i_{(n,1)}$, $i = 1,2,3$, defined in the basis $(X_0, X_1, \ldots, X_{n-2}, Y)$ as follows, can be graded $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with n one-dimensional subspaces $\mathfrak{g}_i$.
$\mathfrak{g}^1_{(n,1)}$ ($n \geqslant 5$, n odd):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} Y, & 1 \leqslant i \leqslant \dfrac{n-3}{2}. \end{cases}$$

$\mathfrak{g}^2_{(n,1)}$ ($n \geqslant 5$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, Y] = X_{i+2}, & 1 \leqslant i \leqslant n-4. \end{cases}$$

$\mathfrak{g}^3_{(n,1)}$ ($n \geqslant 7$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, Y] = X_{i+2}, & 1 \leqslant i \leqslant n-4, \\ [X_1, X_i] = X_{i+3}, & 2 \leqslant i \leqslant n-5. \end{cases}$$

These algebras can be graded with a number of subspaces equal to their dimension and for this reason they are said to be of maximum length. In [5] we prove the following theorem.

Theorem 2.2 *(Quasi-filiform Lie algebras of maximum length). Let $\mathfrak{g}$ be an n-dimensional non-split quasi-filiform Lie algebra of maximum length $l(\mathfrak{g}) = n$, $n \geqslant 13$. Then, the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^1_{(n,1)}$ (n odd), $\mathfrak{g}^2_{(n,1)}$ or $\mathfrak{g}^3_{(n,1)}$.*

2.2. *Quasi-filiform Lie algebras of length* $\dim \mathfrak{g} - 1$

We summarize now the classification of quasi-filiform Lie algebras with length $n-1$ in the following main theorem of this paper.

Theorem 2.3 *(Quasi-filiform Lie algebras of length $n-1$). Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra of length $n-1$, with $n \geqslant 15$. Then, the law of the algebra $\mathfrak{g}$ is isomorphic to one of the following laws.*

n even	*n odd*
$Q_{n-1} \oplus \mathbb{C}$	
$\mathfrak{g}^1_{(n,p)}$ $(3 \leqslant p \leqslant n-3)$	$\mathfrak{g}^1_{(n,p)}$ $(3 \leqslant p \leqslant n-3)$
$\mathfrak{g}^2_{(n,p)}$ $(3 \leqslant p \leqslant n-3)$	$\mathfrak{g}^2_{(n,p)}$ $(3 \leqslant p \leqslant n-3)$
$\mathfrak{g}^3_{(n,p)}$ $(3 \leqslant p \leqslant n-3,\ p\ odd;\ p = n-4)$	
$\mathfrak{g}^4_{(n,p)}$ $(n-5 \leqslant p \leqslant n-3)$	$\mathfrak{g}^4_{(n,p)}$ $(n-5 \leqslant p \leqslant n-3)$
$\mathfrak{g}^5_{(n,p)}$ $(n-5 \leqslant p \leqslant n-3)$	$\mathfrak{g}^5_{(n,p)}$ $(n-5 \leqslant p \leqslant n-3)$
$\mathfrak{g}^6_{(n,p)}$ $(5 \leqslant p \leqslant n-1,\ p\ odd)$	$\mathfrak{g}^6_{(n,p)}$ $(5 \leqslant p \leqslant n-2,\ p\ odd)$
$\mathfrak{g}^7_{(n,p)}$ $(5 \leqslant p \leqslant n-3,\ p\ odd)$	$\mathfrak{g}^7_{(n,p)}$ $(5 \leqslant p \leqslant n-2,\ p\ odd)$
$\mathfrak{g}^8_{(n,p)}$ $(5 \leqslant p \leqslant n-1,\ p\ odd)$	
$\mathfrak{g}^9_{(n,p)}$ $(p = 5,\ p = 7)$	$\mathfrak{g}^9_{(n,p)}$ $(p = 5,\ p = 7)$
$\mathfrak{g}^{10}_{(n,p)}$ $(p = 5,\ p = 7)$	$\mathfrak{g}^{10}_{(n,p)}$ $(p = 5,\ p = 7)$
$\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$	
	$\mathfrak{g}^{12}_{(n,n-4)}$
$\mathfrak{g}^{13}_{(n,n-5)}$	

This family of algebras has been divided into three subfamilies: algebras from extensions, principal algebras and extremal algebras, defined in a basis $(X_0, X_1, \dots, X_{n-2}, Y)$ as follows:

- **Algebras from extensions ($\dim C^1\mathfrak{g} = n-3$).**
  $Q_{n-1} \oplus \mathbb{C}$ ($n \geqslant 7$, n odd):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2}, & 1 \leqslant i \leqslant \dfrac{n-3}{2}. \end{cases}$$

$\mathfrak{g}^1_{(n,p)}$ ($n \geqslant 6$; $3 \leqslant p \leqslant n-3$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, Y] = X_{i+p}, & 1 \leqslant i \leqslant n-p-2. \end{cases}$$

$\mathfrak{g}^2_{(n,p)}$ ($n \geqslant 6$; $3 \leqslant p \leqslant n-3$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, Y] = X_{i+p}, & 1 \leqslant i \leqslant n-p-2, \\ [X_1, X_i] = X_{i+2}, & 2 \leqslant i \leqslant n-4. \end{cases}$$

$\mathfrak{g}^3_{(n,p)}$ ($n \geqslant 8$, n even; $3 \leqslant p \leqslant n-3$, p odd; $p = n-4$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, Y] = X_{i+p}, & 1 \leqslant i \leqslant n-p-2, \\ [X_i, X_{n-3-i}] = (-1)^{i-1} X_{n-2}, & 1 \leqslant i \leqslant \dfrac{n-4}{2}. \end{cases}$$

$\mathfrak{g}^4_{(n,p)}$ ($n \geqslant 13$; $n-5 \leqslant p \leqslant n-3$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{2\lfloor\frac{n-3}{2}\rfloor-1-i}] = (-1)^{i-1} X_{2\lfloor\frac{n-3}{2}\rfloor}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor, \\ [X_i, X_{2\lfloor\frac{n-3}{2}\rfloor-i}] = (-1)^{i-1} \left(\left\lfloor \frac{n-3}{2} \right\rfloor - i \right) X_{2\lfloor\frac{n-3}{2}\rfloor+1}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor, \\ [X_i, X_{n-3-i}] = (-1)^i \frac{(i-1)(n-4-i)}{2} \alpha X_{n-2}, & 2 \leqslant i \leqslant \frac{n-4}{2}, \\ [X_i, Y] = X_{i+p}, & 1 \leqslant i \leqslant n-p-2, \end{cases}$$

with $\alpha = 0$, if n is odd, and $\alpha = 1$, if n is even.
$\mathfrak{g}^5_{(n,p)}$ ($n \geqslant 13$; $n-5 \leqslant p \leqslant n-3$):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = \frac{6(i-1)!(j-1)!(j-i)}{(i+j)!} X_{i+j+1}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-4}{2} \right\rfloor,\ i < j \leqslant n-3-i, \\ [X_i, Y] = X_{i+p}, & 1 \leqslant i \leqslant n-p-2. \end{cases}$$

- **Principal algebras (dim $C^1\mathfrak{g} = n-2$).**

$\mathfrak{g}^6_{(n,p)}$ ($n \geqslant 6$; $5 \leqslant p \leqslant n-1$, p odd):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} Y, & 1 \leqslant i \leqslant \frac{p-3}{2}. \end{cases}$$

$\mathfrak{g}^7_{(n,p)}$ ($n \geqslant 9$; $5 \leqslant p \leqslant n-2$, p odd):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} \beta Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{2\lfloor\frac{n-3}{2}\rfloor-1-i}] = (-1)^{i-1} \big(X_{2\lfloor\frac{n-3}{2}\rfloor} + (1-\beta) Y \big), & 1 \leqslant i \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor, \\ [X_i, X_{2\lfloor\frac{n-3}{2}\rfloor-i}] = (-1)^{i-1} \left(\left\lfloor \frac{n-3}{2} \right\rfloor - i \right) X_{2\lfloor\frac{n-3}{2}\rfloor+1}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor, \\ [X_i, X_{n-3-i}] = (-1)^i \frac{(i-1)(n-4-i)}{2} \alpha X_{n-2}, & 2 \leqslant i \leqslant \frac{n-4}{2}, \end{cases}$$

where $\alpha = 0$, if n is odd, and $\alpha = 1$, if n is even; and where $\beta = 0$, if $p = 2\lfloor (n-1)/2 \rfloor - 1$, and $\beta = 1$, in other cases.
$\mathfrak{g}^8_{(n,p)}$ ($n \geqslant 8$, n even; $5 \leqslant p \leqslant n-1$, p odd):

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} \beta Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{n-3-i}] = (-1)^{i-1} \big(X_{n-2} + (1-\beta) Y \big), & 1 \leqslant i \leqslant \frac{n-4}{2}, \end{cases}$$

where $\beta = 0$ if $p = n-1$, and $\beta = 1$ in other cases.

- **Extremal algebras ($\dim C^1\mathfrak{g} = n-2$).**
$\mathfrak{g}^9_{(n,p)}$ ($n \geqslant 9$; $p=5$, $p=7$):

$$\begin{cases} [X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_1,X_j]=X_{j+2}+\delta_{p-3,j}Y, & 2\leqslant j\leqslant n-4,\\ [X_i,X_{p-2-i}]=(-1)^{i-1}Y, & 2\leqslant i\leqslant \dfrac{p-3}{2}. \end{cases}$$

$\mathfrak{g}^{10}_{(n,p)}$ ($n \geqslant 9$; $p=5$, $p=7$):

$$\begin{cases} [X_0,X_i]=X_{i+1}, \quad 1\leqslant i\leqslant n-3,\\ [X_i,X_j]=\dfrac{6(i-1)!(j-1)!(j-i)}{(i+j)!}X_{i+j+1}+\delta_{p-2-i,j}Y,\\ \quad 1\leqslant i\leqslant \left\lfloor \dfrac{n-4}{2}\right\rfloor,\ i<j\leqslant n-3-i. \end{cases}$$

$\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$ ($n \geqslant 12$, n even):

$$\begin{cases} [X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_i,X_{n-5-i}]=(-1)^{i-1}(\alpha X_{n-4}+Y), & 1\leqslant i\leqslant \dfrac{n-6}{2},\\ [X_i,X_{n-4-i}]=(-1)^{i-1}\left(\dfrac{n-4-2i}{2}\right)\alpha X_{n-3}, & 1\leqslant i\leqslant \dfrac{n-6}{2},\\ [X_i,X_{n-3-i}]=(-1)^{i}\left(\dfrac{(i-1)(n-4-i)}{2}\alpha+\dfrac{1}{\alpha}\right)X_{n-2}, & 1\leqslant i\leqslant \dfrac{n-4}{2},\\ [X_1,Y]=X_{n-2}, \end{cases}$$

where $\alpha = re^{i\theta}$, with $r \neq 0$, and $-\pi/2 \leqslant \theta < \pi/2$.
$\mathfrak{g}^{12}_{(n,n-4)}$ ($n \geqslant 13$, n odd):

$$\begin{cases} [X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_i,X_{n-6-i}]=(-1)^{i-1}(aX_{n-5}+Y), & 1\leqslant i\leqslant \dfrac{n-7}{2},\\ [X_i,X_{n-5-i}]=(-1)^{i-1}\left(\dfrac{n-5-2i}{2}\right)aX_{n-4}, & 1\leqslant i\leqslant \dfrac{n-7}{2},\\ [X_i,X_{n-4-i}]=(-1)^{i}\left(\dfrac{(i-1)(n-3-i)}{2}a+\dfrac{1}{a}\right)X_{n-3}, & 1\leqslant i\leqslant \dfrac{n-5}{2},\\ [X_i,X_{n-3-i}]=(-1)^{i-1}(i-1)\left(\dfrac{(i-2)(3n-15-2i)}{12}a+\dfrac{1}{a}\right)X_{n-2}, & 1\leqslant i\leqslant \dfrac{n-5}{2},\\ [X_i,Y]=X_{n-4+i}, & 1\leqslant i\leqslant 2, \end{cases}$$

where $a = \sqrt{-12/(n-6)(n-7)}$.

$\mathfrak{g}^{13}_{(n,n-5)}$ ($n \geqslant 13$, n even):

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-3, \\ [X_i, X_{n-7-i}] = (-1)^{i-1}(aX_{n-6} + Y), \quad 1 \leqslant i \leqslant \dfrac{n-8}{2}, \\ [X_i, X_{n-6-i}] = (-1)^{i-1}\left(\dfrac{n-6-2i}{2}\right)aX_{n-5}, \quad 1 \leqslant i \leqslant \dfrac{n-8}{2}, \\ [X_i, X_{n-5-i}] = (-1)^{i}\left(\dfrac{(i-1)(n-6-i)}{2}a + \dfrac{1}{a}\right)X_{n-4}, \quad 1 \leqslant i \leqslant \dfrac{n-6}{2}, \\ [X_i, X_{n-4-i}] = (-1)^{i-1}(i-1)\left(\dfrac{(i-2)(3n-18-2i)}{12}a + \dfrac{1}{a}\right)X_{n-3}, \\ \quad 1 \leqslant i \leqslant \dfrac{n-6}{2}, \\ [X_i, X_{n-3-i}] = (-1)^{i}(i-1)(i-2)\left(\dfrac{(i-3)(2n-12-i)}{24}a + \dfrac{1}{2a}\right)X_{n-2}, \\ \quad 1 \leqslant i \leqslant \dfrac{n-4}{2}, \\ [X_i, Y] = X_{n-5+i}, \quad 1 \leqslant i \leqslant 3, \end{cases}$$

where $a = \sqrt{-12/(n-7)(n-8)}$.

We will prove in the next sections that there are no n-dimensional quasi-filiform Lie algebra of length $n-1$ different from the above algebras.

3. Structure of quasi-filiform Lie algebras of length $\dim \mathfrak{g} - 1$

The initial problems in the study of graded Lie algebras are the existence of basis such that the expression of the brackets for the law of such algebras is reduced and to determine one of those bases. We will now restrict our attention to obtain an *adapted homogeneous* basis for any possible gradation of length $\dim \mathfrak{g} - 1$ on a quasi-filiform Lie algebra [5], i.e., a basis $(X_0, X_1, \dots, X_{n-2}, Y)$ formed by homogeneous vectors such that

$$[X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-3,$$
$$[X_0, X_{n-2}] = [X_0, Y] = 0.$$

We can note that the algebras $\mathfrak{g}^i_{(n,p)}$, $1 \leqslant i \leqslant 13$ of Theorem 2.3 are defined by considering such adapted and homogeneous basis.

3.1. Existence of adapted homogeneous basis

The n-dimensional quasi-filiform algebras with length $n-1$ admit a decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$. We will study the cases $n_1 \geqslant 0$, $n_1 < 0 < n_1 + n - 2$ and $n_1 + n - 2 \leqslant 0$; and we will show that there exists an adapted and homogeneous basis in each case. In the following lemma we show that if $n_1 \geqslant 0$ then $n_1 = 1$ or $n_1 = 0$.

Lemma 3.1. *Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra of length $l(\mathfrak{g}) = n-1$, that admits the decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$ with $n_1 \geqslant 0$, then $n_1 = 1$ or $n_1 = 0$ with* $\dim \mathfrak{g}_0 = 1$.

Proof. Indeed, if $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$ is a quasi-filiform Lie algebra with $l(\mathfrak{g}) = n-1$ and $n_1 \geqslant 2$, then $C^1\mathfrak{g} \subset \mathfrak{g}_{2n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$, $C^2\mathfrak{g} \subset \mathfrak{g}_{3n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}, \ldots, C^{n-3}\mathfrak{g} \subset \mathfrak{g}_{(n-2)n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$ thus $(n-2)n_1 \leqslant n_1 + n - 2$ so $(n-3)n_1 \leqslant n-2$. If $n_1 \geqslant 2$ then $2(n-3) \leqslant n-2$ thus $n \leqslant 4$ which is impossible.

If $n_1 = 0$ and $\dim \mathfrak{g}_0 = 2$, then $C^1\mathfrak{g} \subset \mathfrak{g}_3 \oplus \cdots \oplus \mathfrak{g}_{n-2}$ thus $C^2\mathfrak{g} \subset \mathfrak{g}_4 \oplus \cdots \oplus \mathfrak{g}_{n-2}, \ldots,$ $C^{n-3}\mathfrak{g} \subset \mathfrak{g}_{n-1} \oplus \cdots \oplus \mathfrak{g}_{n-2}$, which is impossible, thus $\dim \mathfrak{g}_0 = 1$ when $n_1 = 0$. □

We now prove that there exists an adapted homogeneous basis of $\mathfrak{g}$.

Theorem 3.2. *If $\mathfrak{g}$ is an n-dimensional quasi-filiform Lie algebra of length $l(\mathfrak{g}) = n-1$, that admits the decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$, then there exists an adapted homogeneous basis of $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra that admits the decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$. We can consider the cases $n_1 < 0 < n_1 + n - 2$ and $n_1 \geqslant 0$. In a similar way, we obtain an adapted homogeneous basis of $\mathfrak{g}$, in both cases.

For instance, when $n_1 \geqslant 0$ by Lemma 3.1 we have to consider $n_1 = 0$ and $n_1 = 1$.

$\diamond\ n_1 = 1$.

Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra that admits the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. Then there exists $m \in \mathbb{Z}$ with $1 \leqslant m \leqslant n-1$, such that $\dim \mathfrak{g}_i = 1$ for $i \neq m$ and $\dim \mathfrak{g}_m = 2$. As the nilindex of $\mathfrak{g}$ is $k = n-2$, there exists $p \geqslant 1$ such that $\dim C^i\mathfrak{g} = n-1-i$ for $1 \leqslant i \leqslant p-1$ (this condition being empty when $p = 1$), and $\dim C^i\mathfrak{g} = n-2-i$ for $p \leqslant i \leqslant n-2$. We will distinguish the case $p \neq 1$, i.e., $\dim C^1\mathfrak{g} = n-2$, from the case $p = 1$ where $\dim C^1\mathfrak{g} = n-3$. The treatment is similar in both cases. We consider a homogeneous basis of $\mathfrak{g}$ for the gradation and the generators of the basis belonging to a supplementary space of $C^1\mathfrak{g}$. Then, as $\dim C^{i-1}\mathfrak{g} - \dim C^i\mathfrak{g}$ is equal to 1 or 2, we can determine $\mathfrak{g}/C^i\mathfrak{g}$. Hence, we can obtain an appropriate reordering of the vectors which leads to the adapted homogeneous basis of $\mathfrak{g}$.

- Case $\dim C^1\mathfrak{g} = n-2$.

Let $(U_1, \ldots, U_m, U'_m, \ldots, U_{n-1})$ be a homogeneous basis of $\mathfrak{g}$ such that $\mathfrak{g}_i = \langle U_i \rangle$ for $i \neq m$ and $\mathfrak{g}_m = \langle U_m, U'_m \rangle$. If $m = 1$ then $C^1\mathfrak{g} = \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, $C^2\mathfrak{g} = \mathfrak{g}_3 \oplus \cdots \oplus \mathfrak{g}_{n-1}, \ldots, C^{n-2}\mathfrak{g} = \mathfrak{g}_{n-1}$, $C^{n-1}\mathfrak{g} = (0)$ which is impossible. If $m \neq 1$, then $\mathfrak{g}/C^1\mathfrak{g} = \langle U_1, U_2 \rangle$, and we have $\mathfrak{g}/C^2\mathfrak{g} = \langle U_1, U_2, U_3 \rangle$, where $[U_1, U_2] = a_{1,2}U_3$ with $a_{1,2} \neq 0$ replacing $a_{1,2}U_3$ by U_3 we can assume $[U_1, U_2] = U_3$; from the gradation we have $\mathfrak{g}/C^i\mathfrak{g} = \langle U_1, U_2, \ldots, U_{i+1} \rangle$ for $i \leqslant p-1$, where $[U_1, U_j] = U_{j+1}$ for $2 \leqslant j \leqslant i$, and $\mathfrak{g}/C^p\mathfrak{g} = \langle U_1, U_2, \ldots, U_p, U_{p+1}, [U_2, U_p] \rangle$. Moreover,

$$\mathfrak{g}/C^{p+1}\mathfrak{g} = \big\langle U_1, U_2, \ldots, U_p, U_{p+1}, [U_1, U_{p+1}], [U_2, U_p] \big\rangle,$$

and there is no other way to obtain $\mathfrak{g}/C^{p+1}\mathfrak{g}$ because if $C^p\mathfrak{g}/C^{p+1}\mathfrak{g} = \langle [U_1, [U_2, U_p]]\rangle$, the algebra $\mathfrak{g}$ has length equal to n. As $[U_2, U_p]$ and $[U_1, U_{p+1}]$ are independent vectors of $\mathfrak{g}_{p+2}$, we have $m = p + 2$. Thus,

$$\mathfrak{g}/C^m\mathfrak{g} = \big\langle U_1, U_2, \dots, [U_1, U_{p+1}], [U_2, U_p], \big[U_1, [U_1, U_{p+1}]\big]\big\rangle.$$

If $[U_1, [U_2, U_p]] \neq 0$, as $\langle [U_1, U_{p+1}], [U_2, U_p]\rangle = \langle U_m, U'_m\rangle$, taking the vectors $U_m = [U_1, U_{p+1}]$ and $U'_m = [U_2, U_p] - k[U_1, U_{p+1}]$, with k defined by $[U_1, [U_2, U_p]] = kU_{m+1}$ we can suppose $[U_1, U_m] = U_{m+1}$ and $[U_1, U'_m] = 0$. On the other hand, if $[U_1, [U_2, U_p]] = 0$, taking the vectors $U_m = [U_1, U_{p+1}]$ and $U'_m = [U_2, U_p]$, we have $[U_1, U_m] = U_{m+1}$ and $[U_1, U'_m] = 0$. Hence, $(U_1, \dots, U_{n-1}, U'_m)$ is an adapted basis of $\mathfrak{g}$, that is, $[U_1, U_i] = U_{i+1}$ for $2 \leqslant i \leqslant n - 1$, and $[U_1, U'_m] = 0$.

- Case $\dim C^1\mathfrak{g} = n - 3$.

When $\dim C^1\mathfrak{g} = n - 3$, the subspace $\mathfrak{g}/C^1\mathfrak{g}$ could be either $\langle U_1, U'_1, U_{n-1}\rangle$ or $\langle U_1, U_2, U'_j\rangle$. In both cases we can obtain an adapted homogeneous basis of $\mathfrak{g}$ in a similar way as above. Then, in the first case we have $(U_1, U'_1, U_2, \dots, U_{n-2}, U_{n-1})$ with $[U_1, U'_1] = U_2$, and $[U_1, U_i] = U_{i+1}$ for $2 \leqslant i \leqslant n - 3$. Hence, the algebra $\mathfrak{g}$ with this gradation is an algebraic extension of a graded filiform Lie algebra. In the second case, we have $(U_1, U_2, \dots, U_j, U'_j, \dots, U_{n-2}, U_{n-1})$, with $[U_1, U_i] = U_{i+1}$ for $2 \leqslant i \leqslant n - 2$.

$\diamond$ $n_1 = 0$.

Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra that admits the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-2}$, with $\dim \mathfrak{g}_0 = 1$, and let $(U_0, \dots, U_m, U'_m, \dots, U_{n-2})$ be a basis of $\mathfrak{g}$ with $\mathfrak{g}_m = \langle U_m, U'_m\rangle$, $m \neq 0$, and $\mathfrak{g}_j = \langle U_j\rangle$ for $j \neq m$, then, in a similar reasoning, the algebra is graded by

$$\mathfrak{g} = \langle U_0\rangle \oplus \big\langle U_1, U'_1\big\rangle \oplus \cdots \oplus \langle U_{n-3}\rangle \oplus \langle U_{n-2}\rangle.$$

$\diamond$ $n_1 < 0 < n_1 + n - 2$.

A similar analysis if $n_1 \geqslant 0$ shows that there exists an adapted basis of the algebra $\mathfrak{g}$. Thus, if $\dim C^1\mathfrak{g} = n - 3$, we obtain the basis $(U_i, U_j, \dots, U_{(n-3)i+j}, U'_k)$, where the algebra $\mathfrak{g}$ is graded by

$$\mathfrak{g} = \langle U_{n_1}\rangle \oplus \cdots \oplus \big\langle U_1, U'_1\big\rangle \oplus \cdots \oplus \langle U_{n_1+n-2}\rangle,$$

with $[U_1, U_0] = U'_1$, $[U_1, U'_1] = U_2$, $[U_1, U_i] = U_{i+1}$ for $n_1 \leqslant i \leqslant n_1 + n - 4$ ($U'_k = U_{n_1+n-2}$), or by

$$\mathfrak{g} = \langle U_{n_1}\rangle \oplus \cdots \oplus \big\langle U_1, U'_1\big\rangle \oplus \cdots \oplus \langle U_{n_1+n-2}\rangle,$$

with $[U_1, U_0] = U'_1$, $[U_1, U'_1] = U_2$, $[U_1, U_i] = U_{i+1}$ for $n_1 + 1 \leqslant i \leqslant n_1 + n - 3$ ($U'_k = U_{n_1}$), or by

$$\mathfrak{g} = \langle U_{3-n}\rangle \oplus \cdots \oplus \big\langle U_k, U'_k\big\rangle \oplus \cdots \oplus \langle U_1\rangle,$$

with $[U_1, U'_k] = 0$, $[U_1, U_i] = U_{i+1}$ for $3 - n \leqslant i \leqslant -1$, and $3 - n \leqslant k \leqslant 1$. If $\dim C^1\mathfrak{g} = n - 2$, there exists an adapted basis $(U_i, U_j, U_{i+j}, \dots, U_{(n-3)i+j}, U'_{pi+2j})$ of the algebra $\mathfrak{g}$ formed by homogeneous vectors, and the algebra $\mathfrak{g}$ admits the gradation

$$\mathfrak{g} = \langle U_{n_1}\rangle \oplus \langle U_{n_1+1}\rangle \oplus \cdots \oplus \langle U_1, U'_1\rangle \oplus \cdots \oplus \langle U_{n_1+n-2}\rangle,$$

where the length of the algebra implies that $pi + 2j = n_1$, with $[U_1, U_i] = U_{i+1}$ for $n_1 + 1 \leqslant i \leqslant n_1 + n - 3$, $[U_1, U'_1] = U_2$. □

Note that if an algebra $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n+n_1-2}$, then it admits the equivalent decomposition $\mathfrak{g} = \mathfrak{g}_{-n-n_1+2} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-n_1}$. Analogously, we will not distinguish for an algebra $\mathfrak{g}$ the decompositions $\mathfrak{g} = \mathfrak{g}_{2-n} \oplus \cdots \oplus \mathfrak{g}_0$, $\mathfrak{g} = \mathfrak{g}_{-n+1} \oplus \cdots \oplus \mathfrak{g}_{-1}$, from their respectively equivalent $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-2}$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. Thus, there exists an adapted homogeneous basis for each n-dimensional quasi-filiform Lie algebra of length $n - 1$, which admits the gradation $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n+n_1-2}$.

Corollary 3.3. *Let $\mathfrak{g}$ be an n-dimensional quasi-filiform Lie algebra of length $l(\mathfrak{g}) = n - 1$, that admits the decomposition $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$. Then there exists an adapted homogeneous basis $(X_0, X_1, \dots, X_{n-2}, Y)$ of $\mathfrak{g}$ such that*:

(a) *If the decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, then*

$$\mathfrak{g} = \langle X_0, X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle \oplus \langle Y\rangle$$

or

$$\mathfrak{g} = \langle X_0\rangle \oplus \langle X_1\rangle \oplus \cdots \oplus \langle X_{m-1}, Y\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle$$

with $\mathfrak{g}_m = \langle X_{m-1}, Y\rangle$ and $1 \leqslant m \leqslant n - 1$.

(b) *If the decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-2}$, then*

$$\mathfrak{g} = \langle Y\rangle \oplus \langle X_0, X_1\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle$$

or

$$\mathfrak{g} = \langle X_1\rangle \oplus \langle X_0, X_2\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle \oplus \langle Y\rangle.$$

(c) *If the decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \mathfrak{g}_{n_1+1} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n_1+n-3} \oplus \mathfrak{g}_{n_1+n-2}$ with $4 - n \leqslant n_1 \leqslant -1$, then*

$$\mathfrak{g} = \langle X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_{1-n_1}\rangle \oplus \langle X_{2-n_1}, X_0\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle \oplus \langle Y\rangle$$

or

$$\mathfrak{g} = \langle Y\rangle \oplus \langle X_1\rangle \oplus \cdots \oplus \langle X_{-n_1}\rangle \oplus \langle X_{1-n_1}, X_0\rangle \oplus \langle X_{n-3}\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle.$$

(d) *If the decomposition of* $\mathfrak{g}$ *is* $\mathfrak{g} = \mathfrak{g}_{3-n} \oplus \cdots \oplus \mathfrak{g}_{2-n+m} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, *then*

$$\mathfrak{g} = \langle Y \rangle \oplus \langle X_1 \rangle \oplus \cdots \oplus \langle X_0, X_{n-2} \rangle$$

or

$$\mathfrak{g} = \langle X_1 \rangle \oplus \cdots \oplus \langle X_m, Y \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle \oplus \langle X_0 \rangle$$

with either $\mathfrak{g}_{2-n+m} = \langle X_m, Y \rangle$ *and* $1 \leqslant m \leqslant n-2$, *or with* $\mathfrak{g}_1 = \langle X_0, Y \rangle$.

Proof. Obtaining the decomposition of $\mathfrak{g}$ by using an adapted homogeneous basis is a similar problem in each case. For instance, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, then Theorem 3.2 guarantees the existence of an adapted homogeneous basis such that $\mathfrak{g} = \langle U_1, U_1' \rangle \oplus \langle U_2 \rangle \oplus \cdots \oplus \langle U_{n-1} \rangle$ verifying $[U_1, U_1'] = U_2$, and $[U_1, U_i] = U_{i+1}$ for $2 \leqslant i \leqslant n-3$, or $\mathfrak{g} = \langle U_1 \rangle \oplus \cdots \oplus \langle U_m, U_m' \rangle \oplus \cdots \oplus \langle U_{n-1} \rangle$, with $[U_1, U_i] = U_{i+1}$, for $2 \leqslant i \leqslant n-2$. In the first case, by putting $X_0 = U_1$, $X_1 = U_1'$, $X_i = U_i$ for $2 \leqslant i \leqslant n-2$ and $Y = U_{n-1}$, we have the algebra $\mathfrak{g}$ with the decomposition

$$\mathfrak{g} = \langle X_0, X_1 \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle \oplus \langle Y \rangle$$

and in the second case by putting $X_i = U_{i+1}$ for $0 \leqslant i \leqslant n-2$, and $Y = U_m'$, we have that the algebra $\mathfrak{g}$ admits the decomposition

$$\mathfrak{g} = \langle X_0 \rangle \oplus \cdots \oplus \langle X_{m-1}, Y \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle.$$

We can obtain the other decompositions in a similar way. □

We remark that if $\mathfrak{g}$ is a quasi-filiform Lie algebra of length $l(\mathfrak{g}) = n-1$, we can determine its decomposition in an adapted homogeneous basis $(X_0, X_1, \ldots, X_{n-2}, Y)$ by knowing the index n_1 of the first subspace on the gradation, the index a of the subspace $\mathfrak{g}_a$ containing the vector Y, and the index b of the subspace verifying $\dim \mathfrak{g}_b = 2$. These data will be called the type of the algebra $\mathfrak{g}$.

Notation 3.4. Let $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$ be an n-dimensional quasi-filiform Lie algebra of length $n-1$, and let $(X_0, X_1, \ldots, X_{n-2}, Y)$ be an adapted homogeneous basis of $\mathfrak{g}$. We will say that $\mathfrak{g}$ is an algebra of *type* $\mathfrak{g} = \mathfrak{g}_{(n,n_1,a,b)}$ when $Y \in \mathfrak{g}_a$ and $\dim \mathfrak{g}_b = 2$.

This notation simplifies the computations during the classification of the quasi-filiform Lie algebras of length $n-1$. For instance, we can express easily the previous decompositions in Corollary 3.3 by using the above notation.

Corollary 3.5. *Let* $\mathfrak{g}$ *be an* n*-dimensional quasi-filiform Lie algebra with length* $n-1$. *Then, there exists an adapted homogeneous basis* $(X_0, X_1, \ldots, X_{n-2}, Y)$ *of the algebra* $\mathfrak{g} = \mathfrak{g}_{n_1} \oplus \cdots \oplus \mathfrak{g}_{n_1+n-2}$ *such that* $\mathfrak{g}$ *has one of the following types*:

(i) $\mathfrak{g} = \mathfrak{g}_{(n,3-n,p,p)}$ *with* $3-n \leqslant p \leqslant 1$,
(ii) $\mathfrak{g} = \mathfrak{g}_{(n,2-p,2-p,1)}$ *with* $2 \leqslant p \leqslant n-1$,
(iii) $\mathfrak{g} = \mathfrak{g}_{(n,3-p,n-p+1,1)}$ *with* $2 \leqslant p \leqslant n-1$,
(iv) $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ *with* $1 \leqslant p \leqslant n-1$.

Proof. We can see that these gradations are suitable with the decompositions in Corollary 3.3. For instance, when $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with $1 \leqslant p \leqslant n-1$ its gradation is obtained from a part of (a). Indeed, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, the algebra can be $\mathfrak{g} = \langle X_0\rangle \oplus \langle X_1\rangle \oplus \cdots \oplus \langle X_{m-1}, Y\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle$ with $\mathfrak{g}_m = \langle X_{m-1}, Y\rangle$ for $1 \leqslant m \leqslant n-1$, and then $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with $p = m$. More precisely let us split the cases (i)–(iv)(a)–(d) in subcases and let us compare them:

(i) $(n, 3-n, p, p), 3-n \leqslant p \leqslant 1$;
(ii)′ $(n, 2-p, 2-p, 1), 3 \leqslant p \leqslant n-2$, (ii)″ $(n, 0, 0, 1)$, (ii)‴ $(n, 3-n, 3-n, 1)$;
(iii)′ $(n, 3-p, n-p+1, 1), 4 \leqslant p \leqslant n-1$, (iii)″ $(n, 1, n-1, 1)$, (iii)‴ $(n, 0, n-2, 1)$;
(iv) $(n, 1, p, p), 1 \leqslant p \leqslant n-1$;
(a)′ $(n, 1, n-1, 1)$, (a)″ $(n, 1, m, m), 1 \leqslant m \leqslant n-1$;
(b)′ $(n, 0, 0, 1)$, (b)″ $(n, 0, n-2, 1)$;
(c)′ $(n, n_1, n+n_1-2, 1), 4-n \leqslant n_1 \leqslant -1$, (c)″ $(n, n_1, n_1, 1), 4-n \leqslant n_1 \leqslant -1$;
(d)′ $(n, 3-n, 3-n, 1)$, (d)″ $(n, 3-n, 2-n+m, 2-n+m), 1 \leqslant m \leqslant n-1$;

then (i) = (d)″ $(p = 2-n+m)$, (ii)′ = (c)″ $(p = 2-n_1)$, (ii)″ = (b)′, (ii)‴ = (d)′, (iii)′ = (c)′ $(p = 3-n_1)$, (iii)″ = (a)′, (iii)‴ = (b)″, (iv) = (a)″ $(p = m)$. □

In order to obtain the classification of the n-dimensional quasi-filiform Lie algebras of length $n-1$, we have to determine the structure for the family of those algebras. We will determine the structure of the algebras $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ and then we will give some relations about the structure constants to use later in order to obtain the general classification.

3.2. *Structure of algebras of type* $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$

We remark that the quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ admits the decomposition $\mathfrak{g} = \langle X_0\rangle \oplus \cdots \oplus \langle X_{p-1}, Y\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle$, with $X_i \in \mathfrak{g}_{i+1}$ for $0 \leqslant i \leqslant n-2$, and $Y \in \mathfrak{g}_p$, where $(X_0, X_1, \ldots, X_{n-2}, Y)$ is an adapted basis of algebra $\mathfrak{g}$. Next lemma indicates the structure of those families of algebras.

Lemma 3.6. *Let $(X_0, X_1, \ldots, X_{n-2}, Y)$ be an adapted basis of a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with $1 \leqslant p \leqslant n-1$. Then, the law of the algebra is*

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{i,j} X_{i+j+1} + \alpha \delta_{p-2-i,j} B_i Y, & 1 \leqslant i < j \leqslant n-3-i, \\ [X_i, Y] = A_i X_{i+p}, & 1 \leqslant i \leqslant n-2-p, \end{cases}$$

where $\alpha = 0$ if $p \leqslant 4$, $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \neq j$, and the structure constants $\{a_{i,j}, B_i, A_i\}$ verify Jacobi's relations.

Proof. Everything is obvious except maybe $\alpha = 0$ if $p \leqslant 4$. Indeed $[X_i, X_{p-2-i}] = a_{i,p-2-i} X_{p-1} + \alpha B_i Y$, and one must have $i < p-2-i$ that is $p > 2i+2 \geqslant 4$. □

We now can obtain the relations among the structure constants $a_{i,j}$ for a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$.

Proposition 3.7. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with the law of the algebra as in Lemma 3.6. Then*:

(a) $A_i = A_1$ *for* $1 \leqslant i \leqslant n-2-p$.

(b) $\begin{cases} B_i = (-1)^{i-1}B_1, & i \leqslant \left\lfloor \dfrac{p-3}{2} \right\rfloor, \quad \textit{if } p \textit{ is odd}; \\ B_i = 0, & \textit{if } p \textit{ is even}. \end{cases}$

(c) $a_{i,j} = \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} a_{1,j+k}$, *with* $2 \leqslant i < j \leqslant n-3-i$.

(d) $\sum_{k=0}^{i} (-1)^k \binom{i}{k} a_{1,i+1+k} = 0$, *for* $1 \leqslant i \leqslant \left\lfloor \dfrac{n-5}{2} \right\rfloor$.

Proof. Let $(X_0, X_1, \ldots, X_{n-2}, Y)$ be an adapted basis of a quasi-filiform Lie algebra $\mathfrak{g}$ of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$. Then $\mathfrak{g}$ belongs to the family given in Lemma 3.6. The proof is a consequence of some Jacobi identities $J(X,Y,Z) = 0$ for appropriate $X, Y, Z \in \mathfrak{g}$.

For instance, from Jacobi relations $J(X_0, X_i, Y) = 0$ for $1 \leqslant i \leqslant n-3-p$ and $J(X_0, X_i, X_{p-3-i}) = 0$ for $1 \leqslant i \leqslant \lfloor (p-5)/2 \rfloor$ (a) and (b) are obtained respectively.

We will consider Jacobi's relations $J(X_0, X_m, X_j) = 0$ for $1 \leqslant m < j \leqslant n-3-m$ and we will prove (c) by induction. Indeed, from the Jacobi relations $J(X_0, X_m, X_j) = 0$, $1 \leqslant m < j-1$, we have that $a_{m+1,j} = a_{m,j} - a_{m,j+1}$, and therefore the expression is true for the particular case $m = 1$, that is $i = 2$, and every j. We now suppose for $2 \leqslant i \leqslant s$ and every j that

$$a_{s,j} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} a_{1,j+k},$$

and we will obtain $a_{s+1,j}$ for every j. In fact, the Jacobi relation $J(X_0, X_s, X_j) = 0$ implies $a_{s+1,j} = a_{s,j} - a_{s,j+1}$, and we have

$$a_{s+1,j} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} a_{1,j+k} - \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} a_{1,j+1+k}.$$

But

$$\begin{aligned}
&(-1)^0 \binom{s-1}{0} a_{1,j} = (-1)^0 \binom{s}{0} a_{1,j}, \\
&(-1)^k \left(\binom{s-1}{k} + \binom{s-1}{k-1} \right) a_{1,j+k} = (-1)^k \binom{s}{k} a_{1,j+k}, \quad 1 \leqslant k \leqslant s-1, \\
&(-1)^s \binom{s-1}{s-1} a_{1,j+s} = (-1)^s \binom{s}{s} a_{1,j+s}.
\end{aligned}$$

Thus

$$a_{s+1,j} = \sum_{k=0}^{s} (-1)^k \binom{s}{k} a_{1,j+k}.$$

Finally, (d) is obtained in a similar way from $J(X_0, X_i, X_{i+1}) = 0$, $1 \leqslant i \leqslant \lfloor (n-5)/2 \rfloor$. $\square$

We remark that when $A_1 \neq 0$ or $B_1 \neq 0$, we can always consider $A_1 = A = 1$ and $B_1 = B = 1$ in Proposition 3.7, by shifting to the appropriate adapted homogeneous basis if it is necessary. Thus, we can observe that if $\dim C^1\mathfrak{g} = n-3$ then $B = 0$ and when $\dim C^1\mathfrak{g} = n-2$ then $B = 1$ and $p \geqslant 5$ odd.

4. Proof of classification theorem

In this section we consider in detail the proof of Theorem 2.3. First of all, the next proposition shows that the algebras defined in Section 2.2 are actually different.

Proposition 4.1. *The quasi-filiform Lie algebras* $Q_{n-1} \oplus \mathbb{C}$, $\mathfrak{g} = \mathfrak{g}^i_{(n,p)}$, $1 \leqslant i \leqslant 13$, *are pairwise non-isomorphic. Two laws* $\mathfrak{g}^{11}_n(\alpha_1)$, $\mathfrak{g}^{11}_n(\alpha_2)$ *with* $\alpha_1 \neq \alpha_2$ *of the same family* $\mathfrak{g}^{11}_{(n,p)}(\alpha)$, *where* $\alpha = re^{i\theta}$, $r > 0$, $-\pi/2 \leqslant \theta < \pi/2$, *are also non-isomorphic.*

Proof. Let $\mathfrak{g} = \mathfrak{g}^i_{(n,p)}$, $1 \leqslant i \leqslant 13$. If $\dim C^1\mathfrak{g} = n-2$ then $\dim C^i\mathfrak{g} = n-1-i$ for $1 \leqslant i \leqslant p-3$, and $\dim C^i\mathfrak{g} = n-2-i$ for $p-2 \leqslant i \leqslant n-2$. Consequently, $\mathfrak{g}_1 = \mathfrak{g}^s_{(n,r)}$ and $\mathfrak{g}_2 = \mathfrak{g}^k_{(n,q)}$ are non-isomorphic for $r > q \geqslant 6$. In a similar way the result is true for $\dim C^1\mathfrak{g} = n-3$. In the following table, for any pair of algebras with the same p there exists an invariant showing that they are non-isomorphic.

$\mathfrak{g} \setminus \dim$	$C^1\mathfrak{g}$	$\operatorname{cen}\mathfrak{g}$	$[C^1\mathfrak{g}, C^1\mathfrak{g}]$	$\operatorname{cen}_{\mathfrak{g}} C^i\mathfrak{g}$
$Q_{n-1} \oplus \mathbb{C}$	$n-3$	2	–	–
$\mathfrak{g}^1_{(n,p)}$	$n-3$	1	0	$n-2$ and $i=1$
$\mathfrak{g}^2_{(n,p)}$	$n-3$	1	0	$n-3$ and $i=1$
$\mathfrak{g}^3_{(n,p)}$	$n-3$	1	1 if n is even	–
$\mathfrak{g}^4_{(n,p)}$	$n-3$	1	2 if n is odd	–
			3 if n is even	
$\mathfrak{g}^5_{(n,p)}$	$n-3$	1	$n-7$	–
$\mathfrak{g}^6_{(n,p)}$	$n-2$	2	0 $p=5$	$n-1$ $p=5$ and $i=2$
			1 $p \geqslant 7$	$n-2$ $p \geqslant 7$ and $i=2$
$\mathfrak{g}^7_{(n,p)}$	$n-2$	2	3 if n is odd	–
			4 if n is even	
$\mathfrak{g}^8_{(n,p)}$	$n-2$	2	2	–
$\mathfrak{g}^9_{(n,p)}$	$n-2$	2	0 $p=5$	$n-2$ $p=5$ and $i=2$
			1 $p=7$	$n-3$ $p=7$ and $i=2$
$\mathfrak{g}^{10}_{(n,p)}$	$n-2$	2	$n-6$	–
$\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$	$n-2$	1	3	–
$\mathfrak{g}^{12}_{(n,n-4)}$	$n-2$	1	4	–
$\mathfrak{g}^{13}_{(n,n-5)}$	$n-2$	1	5	–

Now, observe that the change of basis $X'_0 = X_0$, $X'_i = -X_i$, $1 \leqslant i \leqslant n-2$, $Y' = Y$, proves that the algebras $\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$ and $\mathfrak{g}^{11}_{(n,n-3)}(-\alpha)$ are isomorphic. We will show that if the algebras $\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$ and $\mathfrak{g}^{11}_{(n,n-3)}(\alpha')$ are isomorphic then $\alpha = \pm\alpha'$. Indeed, as X_0 and X_1 are the unique generators, we only need to consider two changes of basis determined by $X'_0 = X_0$ and $X'_1 = \sum_{i=0}^{i=n-2} a_i X_i + a_{n-1}Y$, or $X'_0 = \sum_{i=0}^{i=n-2} b_i X_i + b_{n-1}Y$ and $X'_1 = X_1$. By combining these changes of bases one can obtain any change of basis.

For example, consider the change of basis $X'_0 = X_0$ and $X'_1 = \sum_{i=0}^{i=n-2} a_i X_i + a_{n-1}Y$. By $[X'_0, X'_i] = X'_{i+1}$ for $1 \leqslant i \leqslant n-3$ we have $X'_k = \sum_{i=1}^{n-k-1} a_i X_{i+k-1}$, for $2 \leqslant i \leqslant n-2$. Suppose $Y' = \sum_{i=0}^{n-2} a'_i X_i + a'_{n-1}Y$. From $[X'_0, Y'] = 0$ and $[X'_1, Y'] = X'_{n-2}$ we get $a'_i = 0$ for $0 \leqslant i \leqslant n-3$. The brackets $[X'_1, X'_{n-3}]$, $[X'_1, X'_{n-5}]$ and $[X'_1, X'_{n-4}]$ imply that $\alpha' = \pm\alpha$.

When we consider $X'_0 = \sum_{i=0}^{i=n-2} b_i X_i + b_{n-1}Y$ and $X'_1 = X_1$, from the brackets $[X'_0, X'_{i-1}]$ we obtain the vectors X'_i with $i > 1$. Besides, the law of the algebra must verify $[X'_0, Y'] = 0$ and $[X'_1, Y'] = X'_{p+1}$, thus $\alpha' = b_0\alpha$ and $\alpha = {b_0}^2\alpha'$. Finally, from the bracket $[X'_2, X'_{p-2}]$ we have ${b_0}^2 = 1$, hence $\alpha = \alpha'$. Then $\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$ and $\mathfrak{g}^{11}_{(n,n-3)}(\alpha')$ are non-isomorphic for $\alpha \neq \alpha'$ in this case. □

We now study the algebras family law of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$. When the derived algebra of $\mathfrak{g}$ has dimension $n-3$ we can obtain $\mathfrak{g}$ by an appropriate extension of a filiform Lie algebra $\mathfrak{g}'$. When $\dim C^1\mathfrak{g} = n-2$ (p is odd) we obtain the three finite locally families (depending on n) of non-split algebras $\mathfrak{g}^i_{(n,p)}$, $i = 6, 7, 8$, introduced as *principal* in Section 4. We must consider in a different way the *extremal* cases where either p is very small ($p = 5, 7$) or is close to the dimension of $\mathfrak{g}$ ($n-5 \leqslant p \leqslant n-1$). In the particular case $p = n-3$ we obtain the parametric family $\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$. Finally, we will see that the algebras of type $\mathfrak{g} \neq \mathfrak{g}_{(n,1,p,p)}$ has been already computed. In a certain way, if $p \neq n-3$, every quasi-filiform Lie algebra $\mathfrak{g}$ of length $l(\mathfrak{g}) = n-1$ is either obtained from an extension of a filiform algebra of maximum length or is a principal algebra.

4.1. The algebras from extensions

By Proposition 3.7, we can observe that if the dimension of the derived algebra is $n-3$ and $A = 0$ then the algebra is a split quasi-filiform algebra, i.e. $Q_{n-1} \oplus \mathbb{C}$ and if $A \neq 0$, we will prove that these algebras can be obtained from an extension by derivations of a filiform Lie algebra with maximum length.

Proposition 4.2. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with $p \geqslant 1$ and $\dim C^1\mathfrak{g} = n-3$. Then, there exists an $(n-1)$-dimensional filiform Lie algebra $\mathfrak{g}'$ of maximum length such that $\mathfrak{g}$ is an extension by derivations of the algebra $\mathfrak{g}'$.*

Proof. Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ and $\dim C^1\mathfrak{g} = n-3$, from Corollary 3.3 we have guaranteed the existence of an adapted homogeneous basis $(X_0, X_1, \dots, X_{n-2}, Y)$, such that the law of the algebra is given by:

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{i,j} X_{i+j+1}, & 1 \leqslant i < j \leqslant n-3-i, \\ [X_i, Y] = A X_{i+p}, & 1 \leqslant i \leqslant n-2-p. \end{cases}$$

Let V be the vectorial space generated by the vectors $(X_0, X_1, \dots, X_{n-2})$ of $\mathfrak{g}$, and let $\mathfrak{g}' = (V, [,]_{\mathfrak{g}})$ be the Lie algebra defined by the restriction of the law of $\mathfrak{g}$ on V. Then, the algebra

$\mathfrak{g}'$ is a filiform Lie algebra because of $[X_0, X_i] = X_{i+1}$, $1 \leqslant i \leqslant n-3$. In [4] we showed that a filiform Lie algebra of maximum length $\mathfrak{g}'$ belongs to the family characterized by

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{i,j} X_{i+j+1}, & 1 \leqslant i < j \leqslant n-3-i. \end{cases}$$

Then, we can connect the vector Y with the homogeneous derivation defined by $d_p(X_i) = X_{i+p}$, $1 \leqslant i \leqslant n-2-p$, and $d_p(X_i) = 0$ in other case. Thus $\mathfrak{g} = \mathfrak{g}' \oplus d_p$. □

Conversely, from every $(n-1)$-filiform Lie algebra of maximum length we can obtain in this way a quasi-filiform Lie algebra of type $\mathfrak{g}_{(n,1,p,p)}$.

Proposition 4.3. *Let $\mathfrak{g}'$ be an $(n-1)$-dimensional filiform Lie algebra of maximum length defined in an adapted homogeneous basis $(X_0, X_1, \ldots, X_{n-2})$, and let d_p be the homogeneous derivation given by $d_p(X_i) = X_{i+p}$, $1 \leqslant i \leqslant n-2-p$, and $d_p(X_i) = 0$ in other cases. Then, there exists an n-dimensional quasi-filiform Lie algebra $\mathfrak{g}$ of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$, $p \geqslant 1$, such that* $\dim C^1\mathfrak{g} = n-3$ *and $\mathfrak{g}$ is an extension by derivations of the algebra $\mathfrak{g}'$.*

Proof. Indeed, let $\mathfrak{g}'$ be a filiform algebra of maximum length, whose law is described in the proof of Proposition 4.2. We can define the algebra $\mathfrak{g}$ generated by the vectors $(X_0, X_1, \ldots, X_{n-2}, Y)$, where the vector Y is obtained from the homogeneous derivation $ad_{-Y} = d_p$, where d_p is defined by $d_p(X_i) = X_{i+p}$, $1 \leqslant i \leqslant n-2-p$, and $d_p(X_i) = 0$ in other cases. Then, $\dim C^1\mathfrak{g} = n-3$ and the algebra $\mathfrak{g}$ is a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$, $p \geqslant 1$. □

In [4] we show that if $n \geqslant 12$, the n-dimensional filiform Lie algebras of maximum length are the algebras L_n, R_n, W_n, Q'_n and K_n. Thus, we can determine those quasi-filiform Lie algebras $\mathfrak{g}$ of length $l(\mathfrak{g}) = \dim\mathfrak{g} - 1$ obtained from such filiform algebras.

Corollary 4.4. *Every n-dimensional quasi-filiform Lie algebra $\mathfrak{g}$ of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ such that* $\dim C^1\mathfrak{g} = n-3$, $n \geqslant 13$ *is isomorphic to $Q_{n-1} \oplus \mathbb{C}$ or to one of the algebras $\mathfrak{g}^i_{(n,p)}$*, $1 \leqslant i \leqslant 5$.

We remark that if $p \leqslant 2$, a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ is a direct sum $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}$, where $\mathfrak{g}'$ is a filiform Lie algebra. Then the algebra $\mathfrak{g}$ is a split algebra, so it must be $Q_{n-1} \oplus \mathbb{C}$.

4.2. Principal and extremal algebras

In the next proposition we will determine the quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ with $\dim C^1\mathfrak{g} = n-2$. To obtain the classification we have considered some restrictions for the n-dimensional graded quasi-filiform family of length $n-1$, which agree with an $(n-1)$-dimensional graded filiform family of maximum length. In [11] Reyes considers a filiform Lie algebra $\mathfrak{g}$ admitting the gradation $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with $\mathfrak{g}_i = \langle X_{i-1} \rangle$, and determines what conditions must verify the structure constants of the family of such graded algebras. Those conditions are collected in the following lemma.

Lemma 4.5. *(See [5].) Let* $\mathfrak{g}$ *be an* n*-dimensional filiform Lie algebra, with* $n \geqslant 12$*, that admits a gradation* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$*, with* $\mathfrak{g}_i = \langle X_{i-1}\rangle$*. Then the law of the algebra* $\mathfrak{g}$ *belongs to the family*

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2,\\ [X_i, X_j] = a_{i,j} X_{i+j+1}, & 1 \leqslant i < j \leqslant n-i-2, \end{cases}$$

where the structure constants $\{a_{i,j}\}$ *take one of the following set of values*:

1. $a_{i,j} = 0$, *for every* i, j.
2. $\begin{cases} a_{1,j} = \beta, & 2 \leqslant j \leqslant n-3,\\ a_{i,j} = 0, & \text{for } i \neq 1. \end{cases}$
3. $\begin{cases} a_{i,n-2-i} = (-1)^{i-1}\beta, & 1 \leqslant i \leqslant \frac{n-3}{2},\\ a_{i,j} = 0, & \text{in other case}, \end{cases}$
   where n *is odd.*
4. $\begin{cases} a_{i,2\lfloor\frac{n-2}{2}\rfloor-1-i} = (-1)^{i-1}\beta, & 1 \leqslant i \leqslant \left\lfloor\frac{n-4}{2}\right\rfloor,\\ a_{i,2\lfloor\frac{n-2}{2}\rfloor-i} = (-1)^{i-1}\left(\left\lfloor\frac{n-2}{2}\right\rfloor - i\right)\beta, & 1 \leqslant i \leqslant \left\lfloor\frac{n-4}{2}\right\rfloor,\\ a_{i,n-2-i} = (-1)^{i}\frac{(i-1)(n-3-i)}{2}\beta\alpha, & 2 \leqslant i \leqslant \frac{n-3}{2},\\ a_{i,j} = 0, & \text{in other case}, \end{cases}$
   where $\alpha = 0$*, if* n *even, and* $\alpha = 1$*, if* n *odd.*
5. $\left\{ a_{i,j} = \frac{6(i-1)!(j-1)!(j-i)}{(i+j)!}\beta, \quad 1 \leqslant i < j \leqslant n-2-i. \right.$

We now consider a quasi-filiform Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ such that $\dim C^1\mathfrak{g} = n-2$ which admits the decomposition $\mathfrak{g} = \langle X_0\rangle \oplus \langle X_1\rangle \oplus \cdots \oplus \langle X_{p-1}, Y\rangle \oplus \cdots \oplus \langle X_{n-2}\rangle$, where $(X_0, X_1, \ldots, X_{n-2}, Y)$ is an adapted homogeneous basis of $\mathfrak{g}$. Then, we have the following proposition.

Proposition 4.6. *Every* n*-dimensional quasi-filiform Lie algebra* $\mathfrak{g}$ *of type* $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$ *such that* $\dim C^1\mathfrak{g} = n-2$*,* $n \geqslant 15$*, is isomorphic to one of the algebras* $\mathfrak{g}^i_{(n,p)}$, $6 \leqslant i \leqslant 13$.

Proof. Since $\dim C^1\mathfrak{g} = n-2$, by Proposition 3.7, we have $p \geqslant 5$ odd, and the structure of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3,\\ [X_i, X_j] = a_{i,j}X_{i+j+1} + (-1)^{i-1}BY, & 1 \leqslant i < j \leqslant n-3-i,\\ [X_i, Y] = AX_{i+p}, & 1 \leqslant i \leqslant n-2-p, \end{cases}$$

where $B \neq 0$ if $i+j+2 = p$, $B = 0$ if $i+j+2 \neq p$, and the structure constants $\{a_{i,j}, A, B\}$ verify Jacobi's relations. First of all, we suppose that the vector Y belongs to the center of the algebra, and then the other possible case.

• Case $Y \in \operatorname{cen}\mathfrak{g}$.

If $Y \in \operatorname{cen}\mathfrak{g}$, then we have $A = 0$ since $[Y, \mathfrak{g}] = 0$, and we only have to consider the Jacobi relations $J(X_i, X_j, X_k) = 0$ with $0 \leqslant i < j < k \leqslant n-4-i-j$. Note that this is the only possible situation for $p > n-3$. Then, if $p \geqslant 9$, we can consider the Jacobi relations $(X_i, X_j, X_k) = 0$, with $i + j + k = p - 3$ to obtain the equations

$$(-1)^{i-1} a_{j,p-3-i-j} = (-1)^{j-1} a_{i,p-3-i-j} + (-1)^{i+j} a_{i,j}.$$

Now, we can use Lemma 4.5 to modify these equations. Thus we show that the structure constants verifying these equations are exactly those given for the principal algebras laws $\mathfrak{g}^6_{(n,p)}$, $\mathfrak{g} = \mathfrak{g}^7_{(n,p)}$ or $\mathfrak{g} = \mathfrak{g}^8_{(n,p)}$ for the appropriate $9 \leqslant p \leqslant n-1$ odd. On the other hand, if $p = 5$ or $p = 7$, we have to consider the equations

$$a_{i,j+k} a_{j,k} = a_{i,j} a_{i+j+1,k} + a_{i,k} a_{j,i+k+1}$$

obtained from the Jacobi relations $J(X_i, X_j, X_k) = 0$ with $i + j + k \leqslant n - 4$. By using Lemma 4.5 to modify these equations we obtain in addition to the principal algebras laws $\mathfrak{g}^6_{(n,p)}$, $\mathfrak{g}^7_{(n,p)}$, $\mathfrak{g}^8_{(n,p)}$, the extremal algebras laws $\mathfrak{g}^9_{(n,p)}$ or $\mathfrak{g} = \mathfrak{g}^{10}_{(n,p)}$ for $p = 5, 7$.

• Case $Y \notin \operatorname{cen}\mathfrak{g}$.

When $Y \notin \operatorname{cen}\mathfrak{g}$, then $p \leqslant n-3$ and $A \neq 0$.

Assume $p = n-3$. Then, if $(X_0, X_1, \ldots, X_{n-2}, Y)$ is an adapted homogeneous basis of the algebra $\mathfrak{g}$ determined by Corollary 3.5, we have the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$, with

$$\mathfrak{g} = \langle X_0 \rangle \oplus \langle X_1 \rangle \oplus \cdots \oplus \langle X_{p-1}, Y \rangle \oplus \langle X_p \rangle \oplus \langle X_{p+1} \rangle.$$

Therefore, since $C^{n-3}\mathfrak{g} = \langle X_{n-2} \rangle$, denoting each class by the corresponding element, we obtain an adapted homogeneous basis $(X_0, X_1, \ldots, X_{n-3}, Y)$ of the quotient algebra

$$\mathfrak{g}/C^{n-3}\mathfrak{g} = \langle X_0 \rangle \oplus \langle X_1 \rangle \oplus \cdots \oplus \langle X_{p-1}, Y \rangle \oplus \langle X_p \rangle.$$

Hence, we have that $\mathfrak{g}' = \mathfrak{g}/\langle X_{p+1} \rangle$ is an $(n-1)$-dimensional quasi-filiform Lie algebra of type $\mathfrak{g}' = \mathfrak{g}_{(n-1,1,n-3,n-3)}$ with $Y \notin \operatorname{cen}\mathfrak{g}'$ and we have by induction $\mathfrak{g}' = \mathfrak{g}^6_{(p+2,p)}$ or $\mathfrak{g}' = \mathfrak{g}^7_{(p+2,p)}$. If $\mathfrak{g}' = \mathfrak{g}^6_{(p+2,p)}$, the law of the algebra is given by

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant p, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-i}] = a_i X_{p+1}, & 1 \leqslant i \leqslant \frac{p-1}{2}, \\ [X_1, Y] = A X_{p+1}, \end{cases}$$

and the Jacobi relation $J(X_1, X_2, X_{p-4}) = 0$ implies $Y \in \operatorname{cen}\mathfrak{g}$, which contradicts our assumption. Thus $\mathfrak{g}' = \mathfrak{g}^7_{(p+2,p)}$,

$$\mathfrak{g}' = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant p-1, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} X_{p-1} + (-1)^{i-1} Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-1-i}] = (-1)^{i-1} \frac{p-1-2i}{2} X_p, & 1 \leqslant i \leqslant \frac{p-3}{2}, \end{cases}$$

therefore

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant p, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} X_{p-1} + (-1)^{i-1} Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-1-i}] = (-1)^{i-1} \frac{p-1-2i}{2} X_p, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-i}] = a_i' X_{p+1}, & 1 \leqslant i \leqslant \frac{p-1}{2}, \\ [X_1, Y] = A X_{p+1}, \end{cases}$$

with $A \neq 0$. Let $X_0' = X_0$, $X_i' = \frac{1}{\sqrt{A}} X_i$, for $1 \leqslant i \leqslant n-2$, $Y' = \frac{1}{A} Y$, then

$$\mathfrak{g} = \begin{cases} [X_0', X_i'] = X_{i+1}', & 1 \leqslant i \leqslant p, \\ [X_i', X_{p-2-i}'] = (-1)^{i-1} \sqrt{A} X_{p-1}' + (-1)^{i-1} Y', & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i', X_{p-1-i}'] = (-1)^{i-1} \frac{p-1-2i}{2} \sqrt{A} X_p', & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i', X_{p-i}'] = a_i' \sqrt{A} X_{p+1}', & 1 \leqslant i \leqslant \frac{p-1}{2}, \\ [X_1', Y'] = X_{p+1}'. \end{cases}$$

Finally, we let $\alpha = \sqrt{A}$, $a_i = a_i' \sqrt{A}$ and replace X_i', Y' by X_i, Y. The law of $\mathfrak{g}$ then becomes:

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant p, \\ [X_i, X_{p-2-i}] = (-1)^{i-1} \alpha X_{p-1} + (-1)^{i-1} Y, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-1-i}] = (-1)^{i-1} \frac{p-1-2i}{2} \alpha X_p, & 1 \leqslant i \leqslant \frac{p-3}{2}, \\ [X_i, X_{p-i}] = a_i X_{p+1}, & 1 \leqslant i \leqslant \frac{p-1}{2}, \\ [X_1, Y] = X_{p+1}. \end{cases}$$

We will use now induction to prove that

$$a_i = (-1)^i \left(\frac{(i-1)(p-1-i)}{2} \alpha - a_1 \right) \quad \text{for } 1 \leqslant i \leqslant \frac{p-1}{2}.$$

Indeed, from the Jacobi relations $J(X_0, X_i, X_{p-1-i}) = 0$, $1 \leqslant i \leqslant (p-3)/2$, we obtain

$$a_{i+1} = (-1)^{i-1} \left(\frac{p-1-2i}{2} \right) \alpha - a_i, \tag{1}$$

for $1 \leqslant i \leqslant (p-3)/2$. Thus, we can express

$$a_2 = \frac{(2-1)(p-2-1)}{2} \alpha - a_1$$

and the relation is true for $i = 1$. Assume that

$$a_j = (-1)^j \left(\frac{(j-1)(p-1-j)}{2} \alpha - a_1 \right)$$

for $2 \leqslant j \leqslant k$ Then, the expression (1) for $i = k+1$ is

$$a_{k+1} = (-1)^{k-1} \left(\frac{p-1-2k+(k-1)(p-1-k)}{2} \right) \alpha - (-1)^{k+1} a_1$$

which can be expressed as

$$a_{k+1}=(-1)^{k+1}\left(\frac{k(p-k-2)}{2}\alpha-a_1\right),$$

whence the conclusion. Finally, the Jacobi relation $J(X_1,X_2,X_{p-4})=0$ implies $a_1=-\alpha^{-1}$, and $\mathfrak{g}$ is the algebra law $\mathfrak{g}^{11}_{(n,n-3)}(\alpha)$.

Now, consider $p=n-4$. Then, we have the decomposition

$$\mathfrak{g}=\langle X_0\rangle\oplus\langle X_1\rangle\oplus\cdots\oplus\langle X_{p-1},Y\rangle\oplus\langle X_p\rangle\oplus\langle X_{p+1}\rangle\oplus\langle X_{p+2}\rangle.$$

Therefore, since $\mathcal{C}^{n-3}\mathfrak{g}=\langle X_{n-2}\rangle$ denoting as above each class by the corresponding element, we obtain an adapted homogeneous basis $(X_0,X_1,\dots,X_{n-3},Y)$ of the quotient algebra

$$\mathfrak{g}/C^{n-3}\mathfrak{g}=\langle X_0\rangle\oplus\langle X_1\rangle\oplus\cdots\oplus\langle X_{p-1},Y\rangle\oplus\langle X_p\rangle\oplus\langle X_{p+1}\rangle.$$

Hence, we have that $\mathfrak{g}'=\mathfrak{g}/\langle X_{p+2}\rangle$ is an $(n-1)$-dimensional quasi-filiform Lie algebra of type $\mathfrak{g}'=\mathfrak{g}_{(n-1,1,n-4,n-4)}$, and the law of the algebra $\mathfrak{g}$ is now given by

$$\mathfrak{g}=\begin{cases}[X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant p,\\ [X_i,X_{p-2-i}]=(-1)^{i-1}(\alpha X_{p-1}+Y), & 1\leqslant i\leqslant \frac{p-3}{2},\\ [X_i,X_{p-1-i}]=(-1)^{i-1}(\frac{p-1-2i}{2})\alpha X_p, & 1\leqslant i\leqslant \frac{p-3}{2},\\ [X_i,X_{p-i}]=(-1)^{i}(\frac{(i-1)(p-1-i)}{2}\alpha+\frac{1}{\alpha})X_{p+1}, & 1\leqslant i\leqslant \frac{p-1}{2},\\ [X_i,X_{p+1-i}]=a_iX_{p+2}, & 1\leqslant i\leqslant \frac{p-1}{2},\\ [X_i,Y]=X_{p+i}, & 1\leqslant i\leqslant 2.\end{cases}$$

The Jacobi relations $J(X_0,X_i,X_{p-i})=0$ for $1\leqslant i\leqslant (p-1)/2$ imply

$$a_{i+1}+a_i=(-1)^i\left(\frac{(i-1)(p-1-i)}{2}\alpha+\frac{1}{\alpha}\right),\quad i\neq\frac{p-1}{2}, \tag{2}$$

$$a_{\frac{p-1}{2}}=(-1)^{\frac{p-1}{2}}\left(\frac{(\frac{p-3}{2})(\frac{p-1}{2})}{2}\alpha+\frac{1}{\alpha}\right), \tag{3}$$

and the Jacobi relation $J(X_1,X_2,X_{p-3})=0$ implies $a_1=0$. Thus, we can use again induction to obtain

$$a_i=(-1)^{i-1}\left(\frac{(i-1)(i-2)(3p-2i-3)}{12}\alpha+\frac{i-1}{\alpha}\right).$$

Now, from Eq. (3), we have $\alpha=\sqrt{-12/(p-2)(p-3)}$. Hence, $\mathfrak{g}$ is the algebra law $\mathfrak{g}=\mathfrak{g}^{12}_{(n,n-4)}$. In a similar way for $p=n-5$ we obtain that the algebra is $\mathfrak{g}=\mathfrak{g}^{13}_{(n,n-5)}$. We will finish the proof by induction for the case $p\leqslant n-6$ and conclude that $Y\in\operatorname{cen}\mathfrak{g}$ when $Y\in C^1\mathfrak{g}$. Indeed, for

$p=n-6$, we have $\mathfrak{g}'=\mathfrak{g}/\langle X_{p+4}\rangle$ an algebra of type $\mathfrak{g}'=\mathfrak{g}_{(n-1,1,n-6,n-6)}$ with $Y\in C^1\mathfrak{g}'$ and $Y\notin \operatorname{cen}\mathfrak{g}'$; hence $\mathfrak{g}'=\mathfrak{g}^{12}_{(p+5,p)}$, and the law of the algebra is given by

$$\mathfrak{g}=\begin{cases}[X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant p+3,\\ [X_i,X_{p-2-i}]=(-1)^{i-1}(\alpha X_{p-1}+Y), & 1\leqslant i\leqslant \frac{p-3}{2},\\ [X_i,X_{p-1-i}]=(-1)^{i-1}(\frac{p-1-2i}{2})\alpha X_p, & 1\leqslant i\leqslant \frac{p-3}{2},\\ [X_i,X_{p-i}]=(-1)^{i}(\frac{(i-1)(p-1-i)}{2}\alpha+\frac{1}{\alpha})X_{p+1}, & 1\leqslant i\leqslant \frac{p-1}{2},\\ [X_i,X_{p+1-i}]=(-1)^{i-1}(\frac{(i-1)(i-2)(3p-2i-3)}{12}\alpha+\frac{i-1}{\alpha})X_{p+2}, & 1\leqslant i\leqslant \frac{p-1}{2},\\ [X_i,X_{p+2-i}]=(-1)^{i}(\frac{(i-1)(i-2)(i-3)(2p-i-2)}{24}\alpha+\frac{(i-1)(i-2)}{2\alpha})X_{p+3}, & 1\leqslant i\leqslant \frac{p+1}{2},\\ [X_i,X_{p+3-i}]=a_iX_{p+4}, & 1\leqslant i\leqslant \frac{p+1}{2},\\ [X_i,Y]=X_{p+i}, & 1\leqslant i\leqslant 4,\end{cases}$$

with $\alpha=\sqrt{-12/(p-2)(p-3)}$. Then, from the Jacobi relations $J(X_0,X_1,X_{p+1})=0$ and $J(X_1,X_2,X_{p-1})=0$ we have $a_1=a_2=0$. Now, consider the expression

$$\sum_{k=0}^{i}(-1)^k\binom{i}{k}a_{1,i+1+k}=0,$$

that is, the identity (d) in Proposition 3.7, to obtain for $i=(p+1)/2$ an equation without solution in $n\in\mathbb{Z}$, $n\geqslant 15$, which contradicts our assumption $Y\notin\operatorname{cen}\mathfrak{g}$. By using as above the quotient algebra and that there are no extremal algebras for $p=5,7$ with $Y\notin\operatorname{cen}\mathfrak{g}$ we can use an induction to conclude that for $p\leqslant n-6$ the only quasi-filiform Lie algebras with length $n-1$ are those obtained in the case $Y\in\operatorname{cen}\mathfrak{g}$. □

4.3. The other types

Now we consider the other types of gradations on a quasi-filiform Lie algebra $\mathfrak{g}$ of length $l(\mathfrak{g})=n-1$. Then, if $(X_0,X_1,\dots,X_{n-2},Y)$ is an adapted homogeneous basis of the algebra $\mathfrak{g}$, determined by Corollary 3.5 we have the appropriate decomposition $\mathfrak{g}=\mathfrak{g}_{n_1}\oplus\cdots\oplus\mathfrak{g}_{n_1+n-2}$. We study in this section each type to conclude that we can just obtain one of the algebras $Q_{n-1}\oplus\mathbb{C}$, $\mathfrak{g}^1_{(n,p)}$ or $\mathfrak{g}^6_{(n,p)}$, with the adequate p, so we can find a gradation for these algebras in order to consider them of type $\mathfrak{g}=\mathfrak{g}_{(n,1,p,p)}$.

Next proposition shows that the family of Lie algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,3-n,p,p)}$ with $3-n\leqslant p\leqslant 1$ does not actually have length $n-1$.

Proposition 4.7. *There are no quasi-filiform Lie algebras of type $\mathfrak{g}=\mathfrak{g}_{(n,3-n,p,p)}$ with $3-n\leqslant p\leqslant 1$.*

Proof. Let $(X_0,X_1,\dots,X_{n-2},Y)$ be an adapted basis guaranteed by Corollary 3.5 for a quasi-filiform Lie algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,3-n,p,p)}$.

If $p=1$, the law of algebra is

$$\begin{cases}[X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_i,X_j]=a_{i,j}X_{i+j-n+2}, & 1\leqslant i<j\leqslant n-2,\ n-1\leqslant i+j,\\ [X_i,Y]=A_iX_{i+1}, & 1\leqslant i\leqslant n-3.\end{cases}$$

The Jacobi relations $J(X_0, X_i, Y) = 0$ for $1 \leqslant i \leqslant n-4$ imply $A_i = A_1$ for $1 \leqslant i \leqslant n-3$. The change of basis $X'_i = X_i$, $0 \leqslant i \leqslant n-2$, $Y' = Y + A_1 X_0$, permits to suppose $[X_i, Y] = 0$ for $0 \leqslant i \leqslant n-2$. Then $\mathfrak{g} = \mathfrak{g}' \oplus \langle Y \rangle$ where $\mathfrak{g}'$ is a graded filiform Lie algebra for $\mathfrak{g}' = \mathfrak{g}_{3-n} \oplus \cdots \oplus \mathfrak{g}_1$ with $X_0 \in \mathfrak{g}_1$ and $X_1 \in \mathfrak{g}_{3-n}$. In [4] we prove that $\mathfrak{g}' = L_{n-1}$, therefore the algebra is $\mathfrak{g} = L_{n-1} \oplus \mathbb{C}$ which has maximum length.

When $3 - n \leqslant p < 1$, and denoting $\lfloor x \rfloor$ the floor function of x, the law of $\mathfrak{g}$ is given by

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{i,j} X_{i+j-n+2}, & n-1 \leqslant i+j \leqslant 2n-4, \\ & 1 \leqslant i < j \leqslant n-2, i+j \neq 2n+p-4, \\ [X_i, Y] = A_i X_{i+p}, & 1-p \leqslant i \leqslant n-3, \\ [X_{n-2}, Y] = A_{n+p-2} X_{n+p-2} + BY, & \\ [X_{n+p-3+i}, X_{n-1-i}] = a_i X_{n+p-2} + B_i Y, & 1 \leqslant i \leqslant \left\lfloor \dfrac{1-p}{2} \right\rfloor, \end{cases}$$

where the structure constants a_i, $a_{i,j}$, A_i, B and B_i must verify Jacobi's relations. The nilpotency of $\mathfrak{g}$ implies $A_i = a_{i,j} = a_i = B = 0$ for all i, j. From the Jacobi relations $J(X_0, X_{n+p-3+i}, X_{n-2-i}) = 0$ for $1 \leqslant i \leqslant \lfloor -p/2 \rfloor$ we obtain $B_i = (-1)^{i-1} B_1$. If p is even we have $B_1 = 0$, so $\mathfrak{g} = L_{n-1} \oplus \mathbb{C}$ with maximum length, but if p is odd $\mathfrak{g}$ is a filiform Lie algebra. Thus, there are no quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{g}_{(n,3-n,p,p)}$. □

Now, we will show that the only quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{g}_{(n,3-p,n-p+1,1)}$ are the algebras $\mathfrak{g}^1_{(n,p)}$ for some suitable p.

Proposition 4.8. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,3-p,n-p+1,1)}$, $3 \leqslant p \leqslant n-1$. Then the algebra $\mathfrak{g}$ is isomorphic to one of the algebras $\mathfrak{g}^1_{(n,n-p+1)}$, with $4 \leqslant p \leqslant n-2$.*

Proof. In the basis $(X_0, X_1, \ldots, X_{n-2}, Y)$ guaranteed by Corollary 3.5 the law of an algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,3-p,n-p+1,1)}$ is given by

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{i,j} X_{i+j-p+2}, & 1 \leqslant i < j \leqslant n-2,\ p-1 \leqslant i+j \leqslant n+p-4 \\ [X_i, X_{n+p-3-i}] = B_i Y, & p-1 \leqslant i \leqslant \left\lfloor \dfrac{n+p-4}{2} \right\rfloor, \\ [X_i, Y] = A_i X_{i+n-p+1}, & 1 \leqslant i \leqslant p-3. \end{cases}$$

Now, we determine the structure constants. We will show that $a_{i,j} = 0$ for all i, j. Indeed, the nilpotency of the algebra $\mathfrak{g}$ imply $a_{i,j} = 0$ for every i, j, with $1 \leqslant i \leqslant p-2$. Suppose $a_{i,j} = 0$ for $1 \leqslant i \leqslant k-1$ and $a_{k,j} \neq 0$, then the Jacobi relation $J(X_0, X_{k-1}, X_j) = 0$ implies $[X_k, X_j] = 0$, thus we have $a_{k,j} = 0$. From the Jacobi relations $J(X_0, X_i, X_{n+p-4-i}) = 0$, for $p-1 \leqslant i \leqslant \lfloor (n+p-5)/2 \rfloor$, we have $B_{i+1} = -B_i$; then, $B_{(p-2)+i'} = (-1)^{i'-1} B_{p-1}$, with $1 \leqslant i' \leqslant \lfloor (n-p)/2 \rfloor$. If $B_{p-1} \neq 0$ the change of basis $X'_0 = X_0 + X_{p-1}$, $X'_i = X_i$, $1 \leqslant i \leqslant n-2$, $X'_{n-1} = B_{p-1} Y$, implies that the algebra $\mathfrak{g}$ is a filiform Lie algebra. Now, when $p = 3$ and $p = n-1$, one can check easily that the algebras have maximum length. Finally, if $4 \leqslant p \leqslant n-2$, from the Jacobi relations $J(X_0, X_i, Y) = 0$ for $1 \leqslant i \leqslant p-4$ we obtain that $A_i = A_{i+1}$, and therefore $\mathfrak{g} = \mathfrak{g}^1_{(n,n-p+1)}$. □

When the algebra has type $\mathfrak{g}=\mathfrak{g}_{(n,2-p,2-p,1)}$, the key to obtain the classification is that $Y\in\operatorname{cen}\mathfrak{g}$.

Proposition 4.9. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,2-p,2-p,1)}$, $2\leqslant p\leqslant n-1$. Then the algebra $\mathfrak{g}$ is either isomorphic to the algebra $\mathfrak{g}=Q_{n-1}\oplus\mathbb{C}$ or to the algebra $\mathfrak{g}=\mathfrak{g}^6_{(n,p)}$, with $p\geqslant 5$ odd.*

Proof. Let $(X_0,X_1,\dots,X_{n-2},Y)$ be an adapted basis guaranteed by Corollary 3.5 for the algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,2-p,2-p,1)}$. Then the law of the algebra is given by

$$\begin{cases}[X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_i,X_j]=a_{i,j}X_{i+j-p+2}, & 1\leqslant i<j\leqslant n-2,\ p-1\leqslant i+j\leqslant n+p-4,\\ [X_i,X_{p-2-i}]=B_iY, & 1\leqslant i\leqslant \left\lfloor\dfrac{p-3}{2}\right\rfloor,\\ [X_i,Y]=A_iX_{i-p+2}, & p-1\leqslant i\leqslant n+p-4.\end{cases}$$

Because of the nilpotency of $\mathfrak{g}$ we obtain $[X_i,Y]=0$, so $Y\in\operatorname{cen}\mathfrak{g}$. Then, if $p=2$, the algebra is $\mathfrak{g}=\mathfrak{g}'\oplus\mathbb{C}$ with $\mathfrak{g}'=\langle X_0,X_1\rangle\oplus\cdots\oplus\langle X_{n-2}\rangle$, with $\mathfrak{g}'=Q_{n-1}$. On the other hand, if $p\neq 2$, the nilpotency of $\mathfrak{g}$ implies $a_{i,j}=0$ for $1\leqslant i\leqslant p-2$ as well. A reasoning similar to Proposition 4.8 proves that $a_{i,j}=0$ for $p-1\leqslant i$. Now, from the Jacobi relations $J(X_0,X_i,X_{p-3-i})=0$, $1\leqslant i\leqslant\lfloor(p-5)/2\rfloor$, we obtain $B_i=(-1)^{i-1}B_1$ for $1\leqslant i\leqslant\lfloor(p-3)/2\rfloor$. If p is even, the Jacobi relation $J(X_0,X_{(p-4)/2},X_{(p-2)/2})=0$ implies $B_1=0$ and $\mathfrak{g}$ has maximum length. Thus we have $p\geqslant 5$ odd, and $\mathfrak{g}=\mathfrak{g}^6_{(n,p)}$. □

Finally, we check that the only algebra admitting the gradation $\mathfrak{g}=\mathfrak{g}_{(n,1,n-1,1)}$ is the split algebra $\mathfrak{g}=Q_{n-1}\oplus\mathbb{C}$.

Proposition 4.10. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,3-p,n-p+1,1)}$, with $p=2$. Then the algebra is $\mathfrak{g}=Q_{n-1}\oplus\mathbb{K}$, with $n\geqslant 7$ odd.*

Proof. Let $(X_0,X_1,\dots,X_{n-2},Y)$ be an adapted basis guaranteed by Corollary 3.5 for an algebra of type $\mathfrak{g}=\mathfrak{g}_{(n,1,n-1,1)}$. Then, the algebra admits the decomposition

$$\mathfrak{g}=\langle X_0,X_1\rangle\oplus\cdots\oplus\langle X_{n-2}\rangle\oplus\langle Y\rangle$$

with $Y\in\operatorname{cen}\mathfrak{g}$. Thus the quotient algebra $\mathfrak{g}/C^{n-2}\mathfrak{g}=\mathfrak{g}'$ is a naturally graded filiform Lie algebra. If $\dim\mathfrak{g}'$ is odd, the algebra $\mathfrak{g}$ has maximum length $l(\mathfrak{g})=n$. Thus, $\dim\mathfrak{g}$ is odd, and denoting each class by the corresponding element, we obtain an adapted homogeneous basis $(X_0,X_1,\dots,X_{n-2})$ of the quotient algebra $\mathfrak{g}/\langle Y\rangle$, and the law of the algebra $\mathfrak{g}$ is given by

$$\begin{cases}[X_0,X_i]=X_{i+1}, & 1\leqslant i\leqslant n-3,\\ [X_i,X_{n-2-i}]=(-1)^{i-1}\alpha X_{n-2}, & 1\leqslant i\leqslant \dfrac{n-3}{2},\\ [X_i,X_{n-1-i}]=A_iY, & 1\leqslant i\leqslant\left\lfloor\dfrac{n-2}{2}\right\rfloor,\end{cases}$$

with $\alpha=0$ or $\alpha=1$. Now, $\alpha=0$ implies that the algebra $\mathfrak{g}$ has maximum length again. Assume $\alpha=1$, then the Jacobi relations $J(X_0,X_i,X_{n-2-i})=0$, $1\leqslant i\leqslant(n-5)/2$, imply

$A_i = (-1)^{i-1} A_1$ for $1 \leqslant i \leqslant (n-3)/2$. Finally, from $J(X_0, X_{(n-3)/2}, X_{(n-1)/2}) = 0$ we obtain $A_{(n-3)/2} = 0$. Thus, we conclude that $\mathfrak{g} = Q_{n-1} \oplus \mathbb{C}$. □

We remark that each quasi-filiform Lie algebra in this section could be obtained assuming a gradation of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$. We gather the observation in the following corollary, which concludes the proof of Theorem 2.3.

Corollary 4.11. *Every n-dimensional quasi-filiform Lie algebra $\mathfrak{g}$ of length $l(\mathfrak{g}) = n-1$, $n \geqslant 15$, is an algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$.*

Actually, the above result is true for $n < 15$, so the classification obtained in [6] for those algebras of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$, $n \leqslant 15$, complete the classification for the family in low-dimension, where the theorem of classification obtained in this paper could not be applied.

4.4. Symbolic calculus and low-dimensional

In low-dimensional ($n \leqslant 15$) we have obtained in [6] the classification by using symbolic calculus. All brackets of an n-dimensional quasi-filiform Lie algebra of length $n-1$ are determined by the *fundamental brackets* $[X_0, X_i]$, $1 \leqslant i \leqslant n-3$, $[X_1, X_2]$, $[X_1, X_4]$, …, $[X_1, X_{2\lfloor (n-4)/2 \rfloor}]$, $[X_1, Y]$, in an adapted homogeneous basis $(X_0, X_1, \ldots, X_{n-2}, Y)$. Thus, the algebra $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = FB\big([X_1, X_2], [X_1, X_4], \ldots, [X_1, X_{2\lfloor \frac{n-4}{2} \rfloor}], [X_1, Y]\big)$$

in [6], where we show how to use the computer as assistant for the study of the graded Lie algebras. For instance, we can automate among other computations:

(1) To generate the initial family of Lie algebras to consider in every dimension.
(2) To compute the Jacobi identities for the family.
(3) To simplify the restriction given by the structure constants.
(4) To substitute the simplified parameter to reduce the family of Lie algebras.

With the simplification obtained in items above we can study the resulting family to obtain the classification.

Theorem 4.12. *Every n-dimensional complex quasi-filiform Lie algebra law of length $l(\mathfrak{g}) = n-1$ and type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$, with $n \leqslant 15$, is isomorphic to a law $\mathfrak{g}_n^i$ of the List of Laws given in* [6].

We remark that the list is increasing with the dimension, and, for example, there are 51 algebras when $n = 14$, including a parametric family $\mathfrak{g}_{14}^{50}(\alpha)$, with $\alpha \in \mathbb{C} - 0$. On the other hand, for $n = 15$, the 41 algebras obtained in [6] are actually in other notation the algebras which have been obtained in this paper. We can see, for instance, that the algebras $\mathfrak{g}_{15}^i$, $i = 27, 31, 35, 37, 40$, are the principal algebras $\mathfrak{g}_{(15,p)}^6$ with $5 \leqslant p \leqslant 14$, stated in Theorem 2.3.

The following functions show how the law of the algebras was generated by using the software *Mathematica* [13]. First, we obtained the brackets with the vector X_0 of the adapted homogeneous basis.

```
For[i=1, i<=dim-3, i++, mu[X[0], X[i]] = X[i+1]  ]; mu[X[0],
X[dim-2]]=0;  mu[X[0], X[dim-1]]=0;
```

Then, we assumed the conditions for a graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{g}_{(n,1,p,p)}$, where the vector X_{n-1} is the vector Y of the adapted basis.

```
For[i=1, i<=dim-2, i++,
  If[i<=dim-grad-2,
    mu[X[i],X[dim-1]]=X[i+grad], mu[X[i],X[dim-1]]=0  ]  ];
```

Finally, computing the rest of the bracket with the vectors of the basis, by using Proposition 3.7. For instance, when the vector Y is not in the derived algebra we have:

```
For[i=1, i<=dim-3, i++, mu[X[i],X[dim-2]]=0 ];

For[j=2, j<=dim-3, j++,
  If[j<=dim-4,
    If[j+3 !=grad, mu[X[1],X[j]]= a[1,j] X[j+2],
      mu[X[1],X[j]]= a[1,j] X[grad-1] ],
    mu[X[1],X[j]]= 0  ] ];

For[i=2, i<=dim-4, i++,
  For[j=i+1, j<= dim-3, j++,
    If[i+j<= dim-3,
      If[i+j+1 != grad-1,
        mu[X[i],X[j]]=
          Sum[(-1)^k Binomial[i-1,k] a[1,j+k],{k,0,i-1}]
           X[i+j+1],
        mu[X[i],X[j]]=
          Sum[(-1)^k Binomial[i-1,k] a[1,j+k],{k,0,i-1}]
           X[grad-1] ],
      mu[X[i],X[j]]=0    ]  ]  ];
```

In this way, we obtained the goal of the classification in concrete dimensions. Moreover, we have used the package to check or refuse some conjectures about more larger dimensions, which lead us to “guess” how should be the general result for any dimension. Indeed, Symbolic Calculus can be useful in order to obtain the classification of a family of Lie algebras in low dimensions. In those dimensions usually there exist algebras which do not appear on the general case, so the computations needed to obtain a classification could be very complicated. But, actually this approach is also useful to conjecture the existence of certain *patterns* on a family of algebras. And these patterns could lead to obtain the classification of that family in any arbitrary dimension. Some of the results introduced in Section 1 were obtained in this way, which has been successful again to obtain the quasi-filiform algebras $\mathfrak{g}$ of length $l(\mathfrak{g}) = n - 1$ in arbitrary dimension. Thus, this approach can be used to study similar problems.

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