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AN EXTREMAL PROBLEM IN GEODETIC GRAPHS

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An upper bound for the number of lines in a geodetic block of diameter d on p points is obtained, using some new general properties of geodetic blocks which are also of independent interest.

1. Introduction

The concept of a geodetic graph is a natural generalization of that of a tree: In a tree there is a unique path between any two points; in a geodetic graph there is a unique shortest path (distance path) between any two points. In this paper we address ourselves to the problem of finding bounds on the number of lines in a geodetic graph. If the class of graphs is kept as general as all connected graphs an obvious lower bound for a p-point graph is $(p-1)$, since any tree is geodetic. A fairly easy upper bound is also obtained in this case, viz., $(d-1)+\binom{p+1-d}{2}$, (Theorem 1). The problem becomes interesting only when we restrict the class to geodetic blocks. The main result of this paper is an upper bound for the number of lines in this case (Theorem 2 and 3). In the course of deriving this bound several general properties of geodetic blocks have been discovered, which are also of independent interest (Propositions 1 to 7 and the corollaries).

2. Terminology and notation

The general terminology we follow is that of Harary [1]. We consider only ordinary graphs, without loops and multiple lines. For a graph G, the set of points is $V(G)=V$ and the set of lines is $E(G)=E$. The induced graph on the point set $U\subseteq V$ is denoted by $\langle U\rangle$. The induced graph on the line set $F\subseteq E$ is denoted by $\langle F\rangle$. For $v\in V$ we denote the set $\{u \mid d(v,u)=i\}$ by $N_i(v)$ and call it the ith neighbourhood of v. We set $E_i(v)=E\langle N_i(v)\rangle$, the set of lines in $\langle N_i(v)\rangle$. When it is clear from the context we may omit the reference to v. For $u\in N_i(v)$ a point $w\in N_{i-1}(v)$ such that $uw\in E(G)$ is called a *predecessor* of u and a point $t\in N_{i+1}(v)$

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such that $ut \in E(G)$ is called a *successor* of u. If there is a sequence of points $u_{j+s} \in N_{j+s}(v)$ $s = 0, 1, \ldots, i-j-1, i-j$ $(j \leq i)$, each a predecessor of the next, we say u_j is an *ancestor* ($(i-j)$th ancestor) of u_i. In such a situation we also refer to u_i as the $(i-j)$th *progeny* of u_j. We extend these notations to sets of points as follows. If $C \subseteq N_i(v)$,

$$P = \{u \in N_{i-1}(v) \mid \exists w \in C \text{ with } wu \in E(G)\}$$

is called the *predecessor set* of C and

$$S = \{u \in N_{i+1}(v) \mid \exists w \in C \text{ with } wu \in E(G)\}$$

is called the *successor set* of C. The $(i-j)$th *ancestral set* and *progeny set* are defined analogously. Note that all this terminology is relative to a chosen $v \in V$, which we call the *progenitor*.

It is well known that the centre (set of all points with minimum eccentricity) of a graph lies in a block. When G is a block, if every point of G is a central point, we refer to G as a *central block*. Every point of G is then, also, a peripheral point and $r(G) = d(G)$, where $r(G)$ and $d(G)$ denote the radius and the diameter of G respectively. The properties of geodetic graphs have also been studied by [2, 3, 6–9].

3. Some general results on geodetic blocks

The first general result is a simple characterization of geodetic graphs proved in [4].

Proposition 1 (The Unique Predecessor Theorem). *A graph G is geodetic iff for each $v \in V$, every $u \in N_i(v)$ has a unique predecessor, for $2 \leq i \leq k = e(v)$.*

Corollary 1. *Each point in a geodetic graph has a unique kth ancestor.*

The next is an important necessary condition for a graph to be geodetic.

Proposition 2. *Let G be a geodetic graph. Let $v \in V(G)$ and $xy \in E_i(v)$. Let a, $b \in N_j(v)$, $j \neq i$ such that $a \neq b$ and $d(x, a) = d(y, b) = |j - i|$. Then $ab \notin E(G)$.*

Proof. By hypothesis $d(v, x) = d(v, y) = i$. If possible let there be a $b \in N_j(v)$, $j \neq i$, $a \neq b$ with $d(x, a) = d(y, b) = |j - i|$ and $ab \in E(G)$, We may assume that $j > i$ (otherwise, we can interchange ab and xy). Now $d(x, b) = j - i$ or $j - i + 1$. In the former case, $d(v, b) = j$ and we get two j-distance paths between v and b, one through x and another through y. In the latter case, $d(x, b) = j - i + 1$ and we get two distance paths between x and b, one through y and the other through s. In either case geodetiacity of G is violated. This contradiction establishes the proposition.

The next two propositions give some insight into the structural properties of the neighbourhood sets of an arbitrary point in a geodetic block.

Proposition 3. *A block G is geodetic if the smallest cycle containing any pair of points is odd.*

Proof. Let G be not a geodetic block. Then there exists a pair of points x_1, x_2 such that x_1, x_2 have been them two distance paths. Among all such pairs consider a pair which have smallest distance between them. The smallest cycle containing that pair of points is of even length and hence the proposition.

Note 1. That the condition is not necessary is established by the following example, (Fig. 1). The smallest cycle containing x_1 and x_2 of the graph in Fig. 1 is 0 and yet the block is geodetic.

Definition 1. We say a graph G_1 is the extension of a graph G at a point $v \in V$, if G_1 is formed from G by subdividing each line incident with v by the insertion of a new point. We now describe the graphs of the type $K_n^{(i)}$ obtained from a given complete graph K_n $(n \geqslant 2)$ whose points are, in this context called *basic points*. A graph G_1 is of the type $K_n^{(i)}$ where $i \geqslant 0$ is an integer if either $i = 0$ and $G_1 = K_n$ or $i \geqslant 1$ and there is a graph G of the type $K_n^{(i-1)}$ and a basic point v of G such that G_1 is the extension of G at v. The graph G_1 has the same basic points as G. In general, a $K_n^{(i)}$ has n basic points and $i(n-1)$ non-basic points. Any $K_n^{(i)}$ and K_n are homeomorphic. We see that the number i does not determine a $K_n^{(i)}$ uniquely (Fig. 2).

Proposition 4. *Let G be a geodetic block of diameter $d \geqslant 2$. If $\langle N_i(v) \rangle$ is a clique for some $v \in V$, then $i = e(v) = d$.*

Proof. *Case* (i): $i < e(v)$. Suppose $\langle N_i(v) \rangle$ is a clique. By Proposotion 1 the $P(x_i)$ are disjoint. Since G is a block, there exists x_i, $y_i \in N_i(v)$ and $z \in P(x_i)$, $t \in P(y_i)$ such that $zt \in E(G)$. By Proposition 1, z and t belong to same $N_j(v)$ since $x_i y_i \in E(G)$, which contradicts Proposition 2.

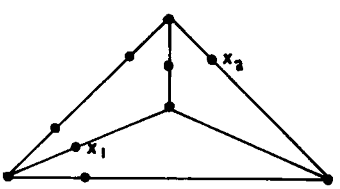

Fig. 1.

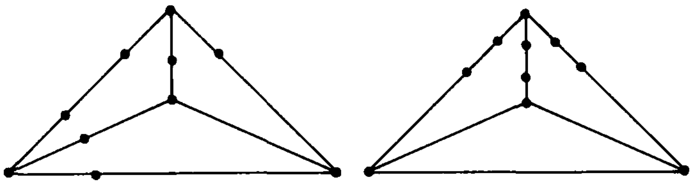

Fig. 2. Two non-isomorphic K_4.

Case (ii): $i=e(v)<d$. If $\langle N_i(v)\rangle$ is a clique then by Proposition 2, each $N_j(v)$, $1\leq j\leq i$ are independent sets and hence diameter of G is $i<d$—a contradiction. Hence $i=e(v)=d$.

Remark 1. If $N_d(v)$ is a clique, then by Proposition 2, each $\langle N_i(v)\rangle$ $1\leq i\leq d-1$ is an independent set and by Proposition 3 G is a geodetic block.

Remark 2. G is a particular type of $K_n^{(d-1)}$.

Proposition 5. *In any geodetic block G of diameter $d\geq 2$, for any $v\in V(G)$, $\langle N_1(v)\rangle$ is a disjoint union of at least two cliques.*

Proof. By Proposition 4, $N_1(v)$ is not a clique. If $\langle N_1(v)\rangle$ is connected, then there are non-adjacent points x, y in N_1 which are joined by a 2-path xzy in $\langle N_1(v)\rangle$. But then xvy is another 2-path between x and y which are at distance 2 from each other, contradicting geodeticity of G. Thus, $\langle N_1(v)\rangle$ is disconnected. Repeating the argument for each component of $\langle N_1(v)\rangle$, we see that each is a complete subgraph.

Proposition 6. *If G is a geodetic block of diameter $d\geq 2$, for any $v\in V$, every point of $N_1(v)$ is adjacent to at least one point of $N_2(v)$.*

Proof. Let $u\in N_1(v)$ be a point without a successor. Let C_1^1 be the component of $\langle N_1(v)\rangle$ containing u. By Proposition 5, C_1^1 is a complete subgraph and there is at least one more component C_2^1 of $\langle N_1(v)\rangle$ and it is also a complete subgraph. The progeny sets of C_1^1 and C_2^1 are disjoint by Proposition 1. Since G is a block, there exist C_1^i and C_2^i, the progeny sets of C_1^1 and C_2^1 in a certain $N_i(v)$, and $x_i\in C_1^i$ and $y_i\in C_2^i$ are connected in $\langle N_i(v)\rangle$ by a path x_i, t_1, $t_2,\ldots,t_k$, y_i $(t_1\neq x_i)$. Then $d(u,t_1)=i+1$ and there are two $(i+1)$-paths between u and t_1 on through x_i and another through v, contradicting geodeticity of G. This contradiction establishes the proposition.

Corollary 2. *In any geodetic block G of diameter $d\geq 2$, for $v\in V(G)$ $|N_1(v)|\leq |N_2(v)|$.*

Proposition 7. In a geodetic block of diameter d, which is neither a K_2, nor an odd cycle, there exists at least one pair of lines $xy \in E$, $uv \in E$ such that $d(x, u) = d(x, v) = d(y, u) = d(y, v) = d$.

Proof. The proposition is obvious if G is complete and $p \geq 4$. Let then G be a geodetic block of diameter $d \geq 2$, which is not an odd cycle. Then there exists a point $v \in V$, such that $|N_d(v)| \geq 2$. Clearly, there exist two points x^d, $y^d \in N_d(v)$ such that $x^d y^d \in E$. By Corollary 1, x^d and y^d have unique ancestors in $N_1(v)$.

Case (i). x^d and y^d have a common ancestor $x \in N_1(v)$. Since G is block degree $(v) \geq 2$ and there exists another point $y \in N_1(v)$ (Fig. 4). We claim that $d(y, x^d) = d(y, y^d) = d$. If not, suppose, for example $d(y, x^d) \neq d$. Then $d(y, x^d) = (d-1)$ and this will result in the existence of two d-paths between v and x^d, one through x and the other through y, contradicting the geodeticity of G. The lines vy, $x^d y^d$ of G establish the claim in the proposition.

Case (ii). x^d and y^d have distinct ancestors x^1, $y^1 \in N_1(v)$. Then by Corollary 1, x^d and y^d cannot have common ancestors in $N_i(v)$, $2 \leq i \leq (d-1)$. Let x^i, y^i be the distinct ancestors of x^d, y^d in $N_i(v)$, $1 \leq i \leq (d-1)$. By Proposition 1, Corollary 1 and Proposition 2 the points v, x^i, y^i with $1 \leq i \leq d$ constitute an induced odd cycle C (Fig. 3). We claim that pair of opposite points on C are at distance d. The claim is established sequentially as follows: $d(x^1, y^d) \neq d$ implies the existence of two d-paths between x and y^d, violating the geodicity of G. $d(x^1, y^{d-1}) \neq d$ implies the existence of two d-paths between x^1 and y^d. $d(x^2, y^{d-1}) \neq d$ implies the existence of two d-paths between x^1 and $y^{d-1}, \ldots$, etc.

Thus all points on C are peripheral. Since $G \not\equiv C$, there is at least one point, say x^i on C with degree $(x^i) > 2$. Let $w \notin C$ be adjacent to x^i. Then $d(w, y^{d-i}) = d(w, y^{d-i+1}) = d$ by arguments similar to the ones used above. The lines wx^i, y^{d-i} y^{d-i+1} of G establish the claim in the proposition.

Remark 3. Case (ii) of above proposition establishes that a geodetic block G of diameter d, which is neigther a K_2 nor an odd cycle, containing an induced C_{2d+1}, contains $(2d+2)$ peripheral points.

Problem 1. A geodetic block with diameter $d \geq 2$ contains an induced C_{2d+1}.

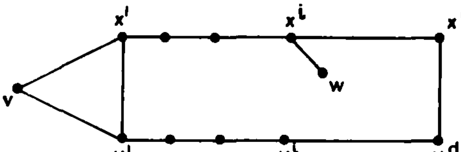

Fig. 3.

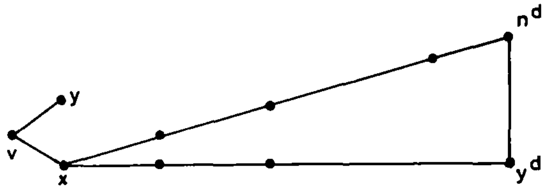

Fig. 4.

Corollary 3. *In a geodetic block of diameter d, which is neither a K_2 nor an odd cycle, there exist at least*

4 points with eccentricity d,
4 points with eccentricity $(d-1)$,
4 points with eccentricity $(d-2)$,
$\vdots$
4 points with eccentricity $\lfloor \frac{1}{2}(d+2) \rfloor$.

Proof. By Proposition 7 there exist 4 points x, y, u, v with eccentricity d, such that $d(x,u)=d(y,u)=d=d(v,x)=d(v,y)$ and xy, $uv \in E(G)$. Clearly u, $v \in N_d(x)$ and $y \in N_1(x)$. Let uu_{d-1}, $vv_{d-1} \in E(G)$, u_{d-1}, $v_{d-1} \in N_{d-1}(x)$. Now $e(u_{d-1})$, $e(v_{d-1})$ are at least $(d-1)$ because $d(u_{d-1},x)=d(v_{d-1},x)=d-1$. Again x, $y \in N_d(u)$, $xy \in E(G)$. Let xu_1, $yu_2 \in E(G)$, u_1, $u_2 \in N_{d-1}(u)$. Obviously $e(u_1)$ and $e(u_2)$ are at least $(d-1)$ because $d(u_1,u)=(d-1)=d(u_2,u)$. Note that $d(u,x)=d=d(u,y)$. Clearly u_{d-1}, v_{d-1}, u_1, u_2 are different points if $d \geqslant 3$. Hence there exist at least 4 points with eccentricity at least $(d-1)$. Similarly we can extend the above arguments to show that there exist sets of at least 4 points each with eccentricity at least d-2, d-3, ..., $\lfloor \frac{1}{2}(d+2) \rfloor$.

4. Bounds for the number of lines in a geodetic graph

Theorem 1. *If G is a connected geodetic graph on p points with q lines and diameter d, then* $p-1 \leqslant q \leqslant (d-1)+\binom{p+1-d}{2}$.

Proof. Obviously any tree on p points is a connected geodetic graph with diameter d, and with minimum number lines. Hence $q \geqslant (p-1)$.

Let G be a connected geodetic graph on p points with diameter d. It should have a diametral path P of length d. G has maximum number of lines only when the remaining $(p-d-1)$ points form a complete graph and each of these points are made adjacent to adjacent points of P. This G has $d+2(p-d-1)+\binom{p-d-1}{2}$ lines establishing the upper bound.

The graph realising the upper bound is a K_{p+1-d} to two points of which are attached paths P_{r_1}, P_{r_2} ($r_1, r_2 \geqslant 1$) such that $r_1+r_2=(d+1)$ (Fig. 5), and the graph realising the lower bound is a tree.

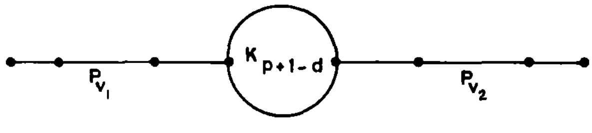

Fig. 5.

Proposition 8. *In a geodetic block G for every $v \in V(G)$ with eccentricity e,* degree$(v) \leqslant \frac{1}{2}(p-2e+3)$.

Proof. Since eccentricity $v = e$, $N_e(v) \neq \emptyset$ and $|N_i(v)| \geqslant 2$ for $1 \leqslant i \leqslant e(v)$. By Proposition 6, Corollary 2, $|N_1(v)| \leqslant |N_2(v)|$. But $|N_1(v)| + |N_2(v)| + \sum_{i=3}^{e(v)} |N_i(v)| = (p-1)$. Hence we get $2\,|N_1(v)| \leqslant (p-1) - 2(e-2) = (p-2e+3)$. Hence the proposition.

When G is a block, the following theorems give upper bounds for the number of lines. The result of Theorem 2 is sharper, since every point of a central block is a peripheral points.

Theorem 2. *If G is a central geodetic block on p points with diameter $d \geqslant 2$, then* $q \leqslant \frac{1}{4}p(p-2d+3)$.

Proof. Since G is central geodetic block with diameter $d \geqslant 2$, each $v \in V(G)$ has eccentricity d and hence degree $(v) = \frac{1}{2}(p-2d+3)$ by Proposition 8. This proves that $q \leqslant \frac{1}{4}p(p-2d+3)$.

Note 2. Stemple [8] has proved that geodetic blocks of diameter 2 are central blocks. In [5] we have extended this result to geodetic blocks of diameter 3. In [4] we have shown that for $d \geqslant 4$ there are geodetic non-selfcentered blocks.

Note 3. This bound is attained when G is an odd cycle of diameter $\frac{1}{2}(p-1)$.

Theorem 3. *If G is a non-selfcentered geodetic block of diameter d on p points with q lines, then*

$$q \leqslant \begin{cases} \frac{1}{4}[p^2 - p(d+2) - (d-3)(d-9) - 2k(d-8)] & \text{if } d \text{ is odd,} \\ \frac{1}{4}[p^2 - p(d+1) - (d-4)(d-6) - 2k(d-7)] & \text{if } d \text{ is even,} \end{cases}$$

where k is the maximum degree of a point with eccentricity d.

Proof. By Proposition 8, a point v with eccentricity $(d-i)$ has degree $(v) \leqslant \frac{1}{2}(p-2d+2i+3)$. Let k be the maximum degree of a point v with eccentricity d. Then

$$|N_1(v)| = k, \qquad |N_2(v)| \geqslant k, \qquad |N_i(v)| \geqslant 2 \quad \text{for } 3 \leqslant i \leqslant d.$$

We have by Corollary 3 at least 4 points with eccentricity d and with degree at most k; $(k+2)$ points with eccentricity at least $(d-1)$ and with degree at most $\frac{1}{2}(p-2d+5)$; $(k+2)$ points with eccentricity at least $(d-2)$ and with degree at most $\frac{1}{2}(p-2d+7)$; 4 points with eccentricity at least $(d-3)$ and with degree at most $\frac{1}{2}(p-2d+9); \ldots$, etc.

When d is odd then the remaining $(p-2d-2k+6)$ points have eccentricity $\frac{1}{2}(d+1)$ and each point has degree at most $\frac{1}{2}(p-d+2)$. When d is even then the remaining $(p-2d-2k+4)$ points have eccentricity $\frac{1}{2}d$ and each point has degree at most $\frac{1}{2}(p-d+3)$.

Case (i). When d is odd

$$\begin{aligned}2q \leqslant 4k &+ (k+2)\left(\frac{p-2d+5+p-2d+7}{2}\right)\\ &+4\left(\frac{p-2d+9}{2}+\frac{p-2d+11}{2}+\cdots+\frac{p-d}{2}\right)\\ &+(p-2d-2k+6)\left(\frac{p-d+2}{2}\right)\end{aligned}$$

which simplifies to

$$q \leqslant \tfrac{1}{4}[p^2-p(d+2)-(d-3)(d-9)-2k(d-8)].$$

Case (ii). When d is even

$$\begin{aligned}2q \leqslant 4k &+ (k+2)\left(\frac{p-2d+5+p-2d+7}{2}\right)\\ &+4\left(\frac{p-2d+9}{2}+\frac{p-2d+11}{2}+\cdots+\frac{p-d+1}{2}\right)\\ &+(p-2d-2k+4)\left(\frac{p-d+3}{2}\right)\end{aligned}$$

which simplifies to

$$q \leqslant \tfrac{1}{4}[p^2-p(d+1)-(d-4)(d-6)-2k(d-7)].$$

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