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Local minimizers with vortex pinning to Ginzburg–Landau functional in three dimensions ☆

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ABSTRACT

We consider the pinning effect for full Ginzburg–Landau functional. The existence of local minimizers with vortices locating in the pinning regions is obtained.

1. Introduction

In this paper, we develop a variational inequality method to the local minimizing problems arisen in geometry and physics. Particularly, we consider the local minimizers with vortex pinning effect for full Ginzburg–Landau functional in three dimensions (see Fig. 1). The existence of the local minimizers with vortices locating in the pinning regions is obtained. Andre, Bauman, Phillips [2] considered vortex pinning with bounded fields in 2-dimensional case. They proved the pinning supercurrent patterns near the zeros of $a(x)$. In [13], Montero, Sternberg and Ziemer constructed local minimizers to Ginzburg–Landau functional in certain 3-dimensional domains where $a(x) \equiv 1$ (no pinning). In this paper, we give the local minimizers with vortex pinning to Ginzburg–Landau functional in 3-dimensional case where $0 < a(x) \leqslant 1$.

When a superconducting material is placed in an applied magnetic field, the state it adopts depends on the Ginzburg–Landau parameter κ which equals to the ratio of the penetration length to the coherence length. The materials with $\kappa < 1/\sqrt{2}$ are known as type I superconductors. Type II superconductors are defined to be those materials for which $\kappa > 1/\sqrt{2}$.

For type I superconductors, if the magnetic field strength H is sufficiently small, then the field will be excluded from the interior of the material. If H exceeds certain critical value, then the field will penetrate the material fully. There exists a third state (mixed state) for type II superconductors, in which there exists a partial of the magnetic field penetrating into the superconducting materials. An important feature of mixed state is the presence of vortices. These are narrow elongated filaments that carry with them quantized amounts of magnetic flux. At the center of the vortex, the density of the superconducting electrons is zero. It is important to comprehend the behavior of the superconductor while it is in the mixed state, since most superconductors are in this state in applications. The motion of vortices is of particular interest, since the motion dissipates energy and generates an electric field, which in turn results in an effective resistivity for the material.

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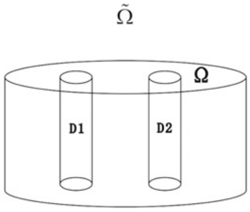

Fig. 1. Pining regions (D_1, D_2) and no pining region Ω.

The aim of this paper is to study the equilibrium distribution of vortices. Precisely, we research the distribution of vortices in the presence of pinning sites which are small regions in the superconductor where the material is tempered with impurities.

Ginzburg–Landau equation is widely accepted as a macroscopic model for superconductivity. It is the Euler–Lagrange equation of the following functional for the wave function Ψ of superconducting electrons, and the potential A of magnetic field:

$$\mathcal{H}_\lambda(\Psi, B) = \int_{\widetilde{\Omega}} \frac{1}{2}\left|(\nabla - iB)\Psi\right|^2 + \frac{\lambda}{4}\left(|\Psi|^2 - a^2(x)\right)^2 + \int_{\mathbb{R}^3} \frac{1}{2}|\mathrm{rot}\, B|^2, \tag{1.1}$$

here $\widetilde{\Omega} \subset \mathbb{R}^3$ is a bounded domain, Ψ is a $\mathbb{C}$-valued function in $\widetilde{\Omega}$, and A is an $\mathbb{R}^3$-valued function in $\mathbb{R}^3$, $\lambda > 0$ is a parameter. The function $a(x)$ describes the maximal density of the superconducting electrons. When the pinning effect is neglected, $a(x)$ is set to equal to 1 in full domain $\widetilde{\Omega}$.

There are many important papers on Ginzburg–Landau equation, recently (see [2–4,6,13–15,17,18], etc. and references therein). In [15], on two-dimensional domain with Dirichlet boundary condition, the pinning effect on vortex locations was considered by Rubinstein. In [11], we constructed local minimizers with vortices to Ginzburg–Landau functional where $a(x) \equiv 1$ (no pinning) by a domain perturbation method. Recently [2] considered vortex pinning with bounded fields in 2-dimensional case. They proved the pinning supercurrent patterns near the zeros of $a(x)$. In [13], Montero, Sternberg and Ziemer constructed local minimizers to Ginzburg–Landau functional in certain 3-dimensional domains where $a(x) \equiv 1$ (no pinning). The results of this paper give the local minimizers with vortex pinning to Ginzburg–Landau functional in 3-dimensional case where $0 < a(x) \leqslant 1$.

In this paper, the existence of local minimizers to (1.1) with vortices which are located in the pinning regions is proved where $a(x)$ may take non-zero value. For this purpose, we shall use a variational inequality method as well as the domain perturbation method. For a non-simply connected bounded domain Ω, it is well known that there is a harmonic map from Ω to S^1 in each homotopic class, and when λ is large enough, because the Ginzburg–Landau functional is approximated by the harmonic map functional, we can construct local minimizers $(\Phi_\lambda, A_\lambda) \in H^1(\Omega, \mathbb{C}) \times Z$ to the Ginzburg–Landau functional (see [7,9,10,20]). On the other hand, a simply connected domain $\widetilde{\Omega}$ can be regarded as a perturbation of a non-simply connected domain Ω by adding thin disks. So we can expect local minimizers $(\Psi_\lambda, B_\lambda)$ to the Ginzburg–Landau functional on the simply connected domain $\widetilde{\Omega}$ too. The domain perturbation method has been used in [8] and [11] successfully. In this paper we want to improve these results by using variational inequality to obtain the local minimizers $(\Psi_\lambda, B_\lambda)$ on the simply connected domain $\widetilde{\Omega}$ which is a perturbation of a non-simply connected domain Ω adding disks. We want to use the smallness of the value of pinning function $a(x)$ to replace the thinness of pinning regions. The key steps are to prove a non-degeneracy inequality (see Section 3), and to develop a special variational inequality argument (see Sections 4–5). We first obtain a minimizer of the Ginzburg–Landau functional with some constraints in Section 4. The constraints are removed in Section 5 by the non-degeneracy inequality and some other estimates. Thus, the minimizer of the variational inequality is a minimizer of the Ginzburg–Landau functional. Variational inequality methods are often used in geometric problems, such as minimizing surfaces, constant mean curvature surfaces, etc. (see [5,16]). Similar variational inequality method has been used in Ginzburg–Landau type equations [20].

2. Main Theorem

Let Ω be a non-simply connected bounded domain in $\mathbb{R}^3$ with C^3 boundary. From [10] (or see [19]), for large λ, there exists a local minimizer $(\Phi_\lambda, A_\lambda)$ to the Ginzburg–Landau functional $\mathcal{H}_\lambda^\Omega$ in $H^1(\Omega; \mathbb{C}) \times Z$:

$$\mathcal{H}_\lambda^\Omega(\Phi, A) = \int_\Omega \left(\frac{1}{2}\left|(\nabla - iA)\Phi\right|^2 + \frac{\lambda}{4}\left(1 - |\Phi|^2\right)^2\right) dx + \int_{\mathbb{R}^3} \frac{1}{2}|\mathrm{rot}\, A|^2\, dx, \tag{2.1}$$

where space Z is defined by

$$Z = \left\{B \in L^6\left(\mathbb{R}^3; \mathbb{R}^3\right) \;\middle|\; \nabla B \in L^2\left(\mathbb{R}^3; \mathbb{R}^{3\times 3}\right)\right\}. \tag{2.2}$$

Precisely, we have proved next theorem in [10] (or see [19]).

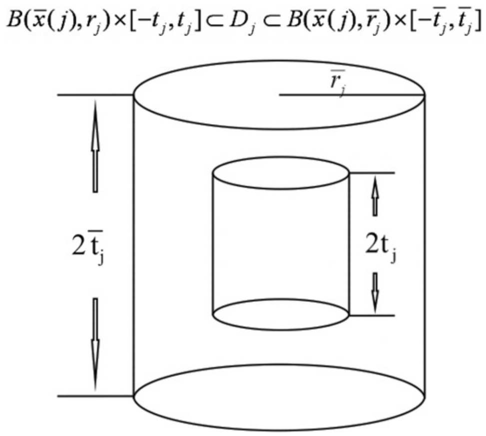

Fig. 2. Pining region D_j.

Theorem A. *Assume that Ω is non-simply connected and θ_0 is a continuous map from $\overline{\Omega}$ to S^1 which is not homotopic to a constant valued map. Then there exists $\lambda_0 > 0$ such that for $\lambda \geqq \lambda_0$, there exists a local minimizer $(\Phi_\lambda, A_\lambda) \in (H^1(\Omega) \times Z) \cap (C^{2+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\mathbb{R}^3))$ to (2.1). Moreover,*

$$\lim_{\lambda\to\infty} \sup_{x\in\Omega} \big| |\Phi_\lambda| - 1 \big| = 0$$

and $\Phi_\lambda/|\Phi_\lambda|$ is homotopic to θ_0.

Let $\widetilde{\Omega}$ be a simply connected domain in $\mathbb{R}^3$. We shall consider following Ginzburg–Landau model for pinning:

$$\mathcal{H}_\lambda(\Psi, B) = \int_{\widetilde{\Omega}} \frac{1}{2}\big|(\nabla - iB)\Psi\big|^2 + \frac{\lambda}{4}\big(\big(a(x)\big)^2 - |\Psi|^2\big)^2 + \int_{\mathbb{R}^3} \frac{1}{2}|\operatorname{rot} B|^2, \tag{2.3}$$

here $\widetilde{\Omega} = \Omega \cup (\bigcup_{j=1}^n D_j)$, Ω is a non-simply connected bounded domain in $\mathbb{R}^3$ with C^3 boundary (see Figs. 1 and 2),

$$B\big(\bar{x}(j), r_j\big) \times [-t_j, t_j] \subset D_j \subset B\big(\bar{x}(j), \bar{r}_j\big) \times [-\bar{t}_j, \bar{t}_j], \qquad B\big(\bar{x}(j), r_j\big) \subset B\big(\bar{x}(j), \bar{r}_j\big) \subset \mathbb{R}^2,$$

$0 < r_j < \bar{r}_j$, $0 < t_j \leqslant \bar{t}_j$, and $a(x)$ is a function satisfying

$$a(x) = \begin{cases} 1, & \text{for } x \in \Omega \setminus (\bigcup_{1\leqslant j\leqslant n} B(\bar{x}(j), \bar{r}_j) \times [-\bar{t}_j, \bar{t}_j]), \\ a_0 \in (0,1), & \text{for } x \in \bigcup_{1\leqslant j\leqslant n} B(\bar{x}(j), r_j) \times [-\bar{t}_j, \bar{t}_j]. \end{cases} \tag{2.4}$$

Main Theorem. *For fixed $\lambda \geqq \lambda_0$, there is $\delta(\lambda) > 0$, if we take $r_j \propto \bar{r}_j$, and $\bar{t}_j$, $\bar{r}_j$ and a_0 satisfying*

$$\sum_{1\leqslant j\leqslant n} \bar{t}_j \big\{ a_0^2 \ln[\bar{r}_j \sqrt{\lambda}] + a_0 + \bar{r}_j^2 \lambda \big\} \leqslant \delta,$$

then there exists a local minimizer $(\Psi_\lambda, B_\lambda) \in H^1(\widetilde{\Omega}) \times Z$ of (2.3) with vortices locating in D_j ($\forall j = 1, 2, \ldots, n$).

3. Non-degeneracy inequality

Let $(\Phi_\lambda, A_\lambda)$ be the minimizer obtained in Theorem A. Since the Ginzburg–Landau functional is invariant under the gauge transformation:

$$(\Phi_\lambda, A_\lambda) \mapsto \big(\Phi'_\lambda, A'_\lambda\big): \quad \Phi'_\lambda = e^{i\rho}\Phi_\lambda, \quad A'_\lambda = A_\lambda + \nabla\rho \quad \big(\rho : \mathbb{R}^3 \to \mathbb{R}\big),$$

varying ρ, we get a continuum of solutions from the solution $(\Phi_\lambda, A_\lambda)$. Let $T(\Phi'_\lambda, A'_\lambda)$ and $N(\Phi'_\lambda, A'_\lambda)$ denote the tangent space and the normal space of the continuum of solutions at $(\Phi'_\lambda, A'_\lambda)$, respectively. To study the stability of a solution, we only need to consider the variation of the solution in N-space.

Let $\Phi(x) = u(x) + iv(x)$. Then, we have

$$T(u, v, A) = \big\{(-v\xi, u\xi, \nabla\xi): \ \xi \in L^6_{\mathrm{loc}}\big(\mathbb{R}^3\big), \ \nabla\xi \in Z\big\}$$

by calculation. To obtain the expression of $N(u, v, A)$, we use the Helmholtz decomposition of $L^6(\mathbb{R}^3, \mathbb{R}^3)$ (cf. [12]):

$$L^6(\mathbb{R}^3,\mathbb{R}^3) = Y_1 \oplus Y_2:$$
$$Y_1 = \{\nabla\xi:\ \xi \in L^6_{\mathrm{loc}}(\mathbb{R}^3),\ \nabla\xi \in L^6(\mathbb{R}^3;\mathbb{R}^3)\},$$
$$Y_2 = \{B \in L^6(\mathbb{R}^3;\mathbb{R}^3):\ \operatorname{div} B = 0 \text{ in } \mathbb{R}^3\}.$$

We have (cf. [9,10] or see [19])

$$N(u,v,A) = \left\{(\phi,\psi,B) \in H^1(\Omega)^2 \times Z:\ \int_\Omega (v\phi - u\psi) = 0,\ B \in Y_2\right\}.$$

Next we prove an important non-degeneracy inequality.

Lemma 3.1. *Let $(\Psi, B) \in N(\Phi_\lambda, A_\lambda)$, where $\Psi = (\phi,\psi) \in C^0(\overline{\Omega})$ satisfies $\|\Psi\|_{C^0(\overline{\Omega})} \leqq M$ and $M \geqq 3$ is a constant. There exist $\delta_0 = \delta_0(\lambda) > 0$ and $c_0 > 0$ (c_0 is independent of λ) such that if*

$$\|\Psi\|_{L^2(\Omega)} \leqq \delta_0, \qquad \|\nabla B\|_{L^2(\mathbb{R}^3)} \leqq \delta_0,$$

then

$$\mathcal{H}^\Omega_\lambda(\Phi_\lambda + \Psi, A_\lambda + B) - \mathcal{H}^\Omega_\lambda(\Phi_\lambda, A_\lambda) \geqq c_0\left(\|\Psi\|^2_{H^1(\Omega)} + \|B\|^2_{L^2(\Omega)} + \|\nabla B\|^2_{L^2(\mathbb{R}^3)}\right)$$

provided that λ is large enough.

Proof. For any $\epsilon \in \mathbb{R}$,

$$\begin{aligned}\mathcal{H}^\Omega_\lambda(\Phi + \epsilon\Psi, A + \epsilon B) &= \int_\Omega \frac{1}{2}\left|\left(\nabla - i(A+\epsilon B)\right)(\Phi + \epsilon\Psi)\right|^2 + \int_\Omega \frac{\lambda}{4}\left(1 - |\Phi + \epsilon\Psi|^2\right)^2 + \int_{\mathbb{R}^3} \frac{1}{2}\left|\operatorname{rot}(A + \epsilon B)\right|^2 \\ &= \mathcal{H}^\Omega_\lambda(\Phi, A) + \epsilon^2 \mathcal{L}^\Omega_\lambda(\Phi, A; \Psi, B) + \epsilon^3 \int_\Omega \operatorname{Re}\left[\overline{(-iB)\Psi} \cdot \left(-iB\Phi + (\nabla - iA)\Psi\right) + \lambda(\Phi\overline{\Psi})|\Psi|^2\right] \\ &\quad + \epsilon^4 \int_\Omega \frac{1}{2}|B\Psi|^2 + \frac{\lambda}{4}|\Psi|^4,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}^\Omega_\lambda(\Phi, A; \Psi, B) &= \int_\Omega \frac{1}{2}\left|-iB\Phi + (\nabla - iA)\Psi\right|^2 - \operatorname{Re}\left[\overline{(iB\Psi)} \cdot (\nabla - iA)\Phi\right] \\ &\quad + \frac{\lambda}{4}\left(\left(2\operatorname{Re}(\Phi\overline{\Psi})\right)^2 - 2\left(1 - |\Phi|^2\right)|\Psi|^2\right) + \int_{\mathbb{R}^3} \frac{1}{2}|\operatorname{rot} B|^2\end{aligned}$$

is the second variation.

Fix $\epsilon = 1$. From [10] (the detailed proof for a similar inequality can been found in [9] or see [19]), we know that there exists $c_1 > 0$ such that for large λ,

$$\mathcal{L}^\Omega_\lambda(\Phi, A; \Psi, B) \geqq c_1\left(\|\Psi\|^2_{H^1(\Omega)} + \|B\|^2_{L^2(\Omega)} + \|\nabla B\|^2_{L^2(\mathbb{R}^3)}\right). \tag{3.1}$$

On the other hand, we have the following estimates:

$$\int_\Omega |B|^2|\Psi| \leqq C_1(\eta)\int_\Omega |B|^4 + \eta\int_\Omega |\Psi|^2,$$
$$\int_\Omega |B||\Psi||\nabla\Psi| \leqq C_2(\eta)\int_\Omega |B|^4 + \eta\int_\Omega |\Psi|^4 + \eta\int_\Omega |\nabla\Psi|^2,$$
$$\int_\Omega |B||\Psi|^2 \leqq C_3(\eta)\int_\Omega |\Psi|^4 + \eta\int_\Omega |B|^2,$$

and

$$\lambda\int_\Omega |\Psi|^2\left|\operatorname{Re}(\Phi\overline{\Psi})\right| \leqq \lambda M\int_\Omega |\Psi|^3,$$

with $\eta \in (0, c_1/4)$.

Let $\{v_j\}_{j=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ be the eigenfunctions and eigenvalues of Laplace operator $-\Delta$ in $H^1(\Omega,\mathbb{R})$:

$$\begin{cases} -\Delta v_j = \mu_j v_j & \text{in } \Omega, \\ \dfrac{\partial v_j}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $P : H^1(\Omega,\mathbb{R}) \to \operatorname{span}\{v_1, v_2, \ldots, v_{m-1}\}$ be a projector and $Q = I - P$, and

$$\Psi_p = (P\phi, P\psi) = \left(\sum_{j=1}^{m-1} \langle\phi, v_j\rangle v_j, \sum_{j=1}^{m-1} \langle\psi, v_j\rangle v_j\right),$$
$$\Psi_q = (\phi_q, \psi_q) = (Q\phi, Q\psi).$$

From

$$\int_\Omega \nabla\Psi_p \cdot \nabla\Psi_q = -\int_\Omega (\Delta\Psi_p)\Psi_q = 0,$$

we have

$$\begin{aligned} \|\nabla\Psi\|^2_{L^2(\Omega)} &= \|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)} \\ &= \|\nabla\phi_p\|^2_{L^2(\Omega)} + \|\nabla\phi_q\|^2_{L^2(\Omega)} + \|\nabla\psi_p\|^2_{L^2(\Omega)} + \|\nabla\psi_q\|^2_{L^2(\Omega)} \\ &\geqq \mu_2\big(\big\|\phi_p - \langle\phi, v_1\rangle v_1\big\|^2_{L^2(\Omega)} + \big\|\psi_p - \langle\psi, v_1\rangle v_1\big\|^2_{L^2(\Omega)}\big) + \mu_m\big(\|\phi_q\|^2_{L^2(\Omega)} + \|\psi_q\|^2_{L^2(\Omega)}\big). \end{aligned}$$

Then

$$\frac{c_1}{2}\|\Psi\|^2_{H^1(\Omega)} \geqq \frac{c_1}{2}\|\Psi\|^2_{L^2(\Omega)} + \frac{c_1}{2}\mu_m\|\Psi_q\|^2_{L^2(\Omega)}. \tag{3.2}$$

Next, we prove some estimates for Ψ,

$$\begin{aligned} \lambda\int_\Omega |\Psi|^3 &\leqq 2\lambda\int_\Omega |\Psi||\Psi_p|^2 + |\Psi||\Psi_q|^2 \\ &\leqq 2\lambda\left(\int_\Omega |\Psi|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\Psi_p|^4\right)^{\frac{1}{2}} + 2\lambda M\int_\Omega |\Psi_q|^2 \\ &\leqq 2\delta_0\lambda k_p^2\left(\int_\Omega |\Psi_p|^2\right) + 2\lambda M\int_\Omega |\Psi_q|^2, \end{aligned}$$

where we have used

$$\left(\int_\Omega |\Psi_p|^4\right)^{\frac{1}{4}} \leqq k_p\left(\int_\Omega |\Psi_p|^2\right)^{\frac{1}{2}}$$

for some constant k_p.

Similarly, we can get

$$\big(\eta + C_3(\eta)\big)\int_\Omega |\Psi|^4 \leqq 2M\big(\eta + C_3(\eta)\big)\delta_0 k_p^2\left(\int_\Omega |\Psi_p|^2\right) + 2M^2\big(\eta + C_3(\eta)\big)\int_\Omega |\Psi_q|^2.$$

If we take large m such that

$$2M^2\big(\eta + C_3(\eta)\big) + M\lambda \leqq \frac{c_1\mu_m}{2},$$

and take δ_0 such that

$$\big(2M\big(\eta + C_3(\eta)\big) + \lambda\big)\delta_0 k_p^2 \leqq \frac{c_1}{2},$$

then we get

$$\big(\eta + C_3(\eta)\big)\int_\Omega |\Psi|^4 + \lambda\int_\Omega |\Psi|^3 \leqq \frac{c_1}{2}\|\Psi\|^2_{H^1(\Omega)}.$$

On the other hand, it is easy to obtain the estimate

$$\int_{\Omega} |B|^4 \leqq |\Omega|^{\frac{1}{3}} \delta_0^2 \int_{\mathbb{R}^3} |\nabla B|^2.$$

We have completed the proof. □

4. Variational problem with constraint

Let

$$Z := \{B \in L^6(\mathbb{R}^3; \mathbb{R}^3) \mid \nabla B \in L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})\}. \tag{4.1}$$

Lemma 4.1. *Z is a Banach space with norm $\|B\|_{L^6(\mathbb{R}^3)} + \|\nabla B\|_{L^2(\mathbb{R}^3)}$, and is a Hilbert space with norm $\|\nabla B\|_{L^2(\mathbb{R}^3)}$.*

Proof. We only need to prove that Z is complete with norm $\|\nabla B\|_{L^2(\mathbb{R}^3)}$. Suppose $\{B_n\} \subset Z$ and

$$\lim_{n\to\infty} \left\| \frac{\partial}{\partial x_i} B_n - \overline{B} \right\|_{L^2(\mathbb{R}^3)} = 0.$$

Then, by Sobolev's inequality, $\{B_n\}$ converges to $\widetilde{B}$ in $L^6(\mathbb{R}^3)$. For any $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, by definition of weak derivative,

$$\int_{\mathbb{R}^3} \frac{\partial B_n}{\partial x_i} \cdot \varphi \, dx = -\int_{\mathbb{R}^3} B_n \cdot \frac{\partial \varphi}{\partial x_i} \, dx.$$

The left converges to

$$\int_{\mathbb{R}^3} \overline{B} \cdot \varphi \, dx$$

and the right converges to

$$\int_{\mathbb{R}^3} \widetilde{B} \cdot \frac{\partial \varphi}{\partial x_i} \, dx.$$

Thus $\frac{\partial}{\partial x_i} \widetilde{B} = \overline{B}$ in weak sense. That is, $\widetilde{B} \in Z$.

Then, Z is complete in norm $\|\nabla B\|_{L^2(\mathbb{R}^3)}$. □

Definition 4.2. For $\delta > 0$, we define

$$E_\delta = \{(\Psi, B) \in (H^1(\widetilde{\Omega}) \cap L^\infty(\Omega)) \times Z\colon\ (\Psi, B) \text{ satisfies (4.2)}\},$$

$$\begin{cases} \operatorname{div} B = 0 \quad \text{in } \mathbb{R}^3, \\ \|\nabla(A_\lambda - B)\|_{L^2(\mathbb{R}^3)} \leqq \delta, \\ \|\Psi\|_{L^\infty(\Omega)} \leqq M, \\ \inf_{\rho\in\mathbb{R}} \|\Psi - e^{i\rho}\Phi_\lambda\|_{L^2(\Omega)} \leqq \delta. \end{cases} \tag{4.2}$$

Lemma 4.3. *$\mathcal{H}_\lambda$ can take its minimum at $(\Psi_\delta, B_\delta) \in E_\delta$.*

Proof. Take $\{(\Psi_n, B_n)\} \in E_\delta$ such that

$$\lim_{n\to\infty} \mathcal{H}_\lambda(\Psi_n, B_n) = \min_{(\Psi, B)\in E_\delta} \mathcal{H}_\lambda(\Psi, B).$$

Since

$$\int_{\widetilde{\Omega}} |\nabla \Psi_n|^2 \leqq \int_{\widetilde{\Omega}} |(\nabla - iB_n)\Psi_n|^2 + |iB_n\Psi_n|^2 \leqq C + \int_{\widetilde{\Omega}} |B_n|^4 + |\Psi_n|^4$$

and

$$\int_{\widetilde{\Omega}} |B_n|^4 \leqq C_{\widetilde{\Omega}}\biggl(\int_{\widetilde{\Omega}} |B_n|^6\biggr)^{\frac{2}{3}} \leqq C_{\widetilde{\Omega}}\biggl(\int_{\mathbb{R}^3} |\operatorname{rot} B_n|^2\biggr)^2 \leqq C,$$

$$\int_{\widetilde{\Omega}} |\Psi_n|^4 \leqq C\biggl(\int_{\widetilde{\Omega}} \bigl(a^2 - |\Psi_n|^2\bigr)^2 + 1\biggr) \leqq C,$$

$$\int_{\widetilde{\Omega}} |\Psi_n|^2 \leqq C_{\widetilde{\Omega}}\biggl(\int_{\widetilde{\Omega}} |\Psi_n|^4\biggr)^{\frac{1}{2}} \leqq C,$$

$\{\Psi_n\}$ is bounded in $H^1(\widetilde{\Omega})$ and $\{B_n\}$ is bounded in Z. Thus there exists weak convergence subsequence (Ψ_{n_i}, B_{n_i}) which weakly converges to $(\Psi_\delta, B_\delta) \in H^1(\widetilde{\Omega}) \times Z$. Because

$$\Psi_n \to \Psi_\delta \quad \text{a.e. in } \Omega,$$
$$\|\Psi_n\|_{L^\infty(\Omega)} \leqq M,$$
$$\Psi_n \to \Psi_\delta \quad \text{strongly in } L^2(\widetilde{\Omega}),$$

and

$$(B_n - A_\lambda) \to (B_\delta - A_\lambda) \quad \text{weakly in } Z,$$

we get

$$\|\Psi_\delta\|_{L^\infty(\Omega)} \leqq M,$$
$$\inf_{\rho\in\mathbb{R}} \bigl\|\Psi_\delta - e^{i\rho}\Phi_\lambda\bigr\|_{L^2(\Omega)} \leqq \delta,$$

and

$$\bigl\|\nabla(B_\delta - A_\lambda)\bigr\|_{L^2(\mathbb{R}^3)} \leqq \liminf_{n\to\infty}\bigl\|\nabla(B_n - A_\lambda)\bigr\|_{L^2(\mathbb{R}^3)} \leqq \delta.$$

Thus, (Ψ_δ, B_δ) is a minimizer of $\mathcal{H}_\lambda$ on E_δ. □

5. Remove the constraints

Lemma 5.1. *For fixed $\lambda \geqq \lambda_0$, if we take $\bar{t}$, $\bar{r}$, r_1 and a_0 satisfying*

$$\bar{t}\biggl\{a_0\bigl(a_0 \ln[r_1\sqrt{\lambda}\,] + 1\bigr) + \bigl(\bar{r}^2 - r_1^2\bigr)\biggl(\frac{1}{r_1^2} + \frac{1}{(\bar{r} - r_1)^2} + \lambda\biggr)\biggr\} \leqslant Cc_0\delta,$$

then we have

$$\mathcal{H}_\lambda(\Psi_\delta, B_\delta) \leqq \mathcal{H}_\lambda^\Omega(\Phi_\lambda, A_\lambda) + \frac{c_0\delta}{2}.$$

Proof. For convenience, we assume that $n = 1$, $B(0, \bar{r})$ is a ball in $\mathbb{R}^2$ and

$$B(0, r_1) \times [-t_1, t_1] \subset D \subset B(0, \bar{r}) \times [-\bar{t}, \bar{t}].$$

We first rewrite Φ_λ on $\Omega \cap \partial(B(0, \bar{r}) \times [-\bar{t}, \bar{t}])$ as

$$\Phi_\lambda(\bar{r}\cos\theta, \bar{r}\sin\theta, t) = w_\lambda(\bar{r}\cos\theta, \bar{r}\sin\theta, t)e^{i\beta_\lambda(\bar{r},\theta,t)},$$

and define

$$V_\lambda(x) = \begin{cases} \Phi_\lambda(x), & x \in \Omega \setminus (B(0,\bar{r}) \times [-\bar{t}, \bar{t}]), \\ [(\frac{\bar{r}-r}{\bar{r}-r_1})a_0 + (\frac{r-r_1}{\bar{r}-r_1})w_\lambda(\bar{r}\cos\theta, \bar{r}\sin\theta, t)]e^{i\beta_\lambda(\bar{r},\theta,t)}, & r_1 \leqslant r \leqslant \bar{r},\ -\bar{t} \leqslant t \leqslant \bar{t},\ 0 \leqslant \theta < 2\pi, \\ \rho(\frac{r}{\epsilon})e^{i\beta_\lambda(\bar{r},\theta,t)}, & r \leqq r_1,\ -\bar{t} \leqslant t \leqslant \bar{t},\ 0 \leqslant \theta < 2\pi, \end{cases}$$

here $x = (r\cos\theta, r\sin\theta, t)$ for $x \in B(0, \bar{r}) \times [-\bar{t}, \bar{t}]$, $\epsilon = \frac{1}{\sqrt{\lambda}}$ and $\rho(s)$ for $s \in [0, 1]$ is the solution of ODE

$$\begin{cases} \rho'' + \frac{1}{s}\rho' - \frac{1}{s^2}\rho - \rho\bigl(a_0^2 - \rho^2\bigr) = 0, \\ \rho(0) = 0, \quad \rho(1) = a_0, \end{cases}$$

and

$$\rho(s) := a_0, \quad \forall s \in \left[1, \frac{r_1}{\epsilon}\right].$$

Note that $(V_\lambda, A_\lambda) \in E_\delta$ and

$$\mathcal{H}_\lambda(V_\lambda, A_\lambda) \leqslant \mathcal{H}_\lambda^\Omega(\Phi_\lambda, A_\lambda) + \int\limits_{(B(0,r_1)\times[-\bar{t},\bar{t}])} \frac{1}{2}\left|(\nabla - iA_\lambda)V_\lambda\right|^2 + \frac{\lambda}{4}\left(a^2 - |V_\lambda|^2\right)^2 + \int\limits_{(B(0,\bar{r})\setminus B(0,r_1))\times[-\bar{t},\bar{t}]} \frac{1}{2}\left|(\nabla - iA_\lambda)V_\lambda\right|^2 + \frac{\lambda}{4}\left(a^2 - |V_\lambda|^2\right)^2.$$

Step 1. We first estimate the integration on $(B(0,r_1) \times [-\bar{t},\bar{t}])$,

$$\int\limits_{(B(0,r_1)\times[-\bar{t},\bar{t}])} \frac{1}{2}\left|(\nabla - iA_\lambda)V_\lambda\right|^2 + \frac{\lambda}{4}\left(a^2 - |V_\lambda|^2\right)^2 \leqslant C_1\bar{t}\left(a_0^2 \ln[r_1\sqrt{\lambda}\,] + C_2\right).$$

Note that

$$\begin{aligned}
&\int\limits_{(B(0,r_1)\times[-\bar{t},\bar{t}])} \frac{1}{2}\left|(\nabla - iA_\lambda)V_\lambda\right|^2 + \frac{\lambda}{4}\left(a^2 - |V_\lambda|^2\right)^2 \\
&\leqq \int_{-\bar{t}}^{\bar{t}} \int\limits_{B(0,r_1)} \left\{\left|\nabla\rho\left(\frac{r}{\epsilon}\right)\right|^2 + \left|\rho\left(\frac{r}{\epsilon}\right)\nabla\beta_\lambda(\bar{r},\theta,t)\right|^2 + |A_\lambda|^2\left(\rho\left(\frac{r}{\epsilon}\right)\right)^2 + \frac{\lambda}{4}\left(a^2 - \left(\rho\left(\frac{r}{\epsilon}\right)\right)^2\right)^2\right\} r\,dr\,d\theta\,dt \\
&\leqq C_1\bar{t}\int_0^{r_1}\left\{\frac{1}{2\epsilon^2}(\rho')^2 + (\rho)^2 + \frac{1}{2r^2}(\rho)^2 + \frac{\lambda}{4}\left(a^2 - \rho^2\right)^2\right\} r\,dr,
\end{aligned}$$

with $C_1 = C_1(\max_D |A_\lambda|, \max_{\theta,t}(|\frac{\partial\beta}{\partial\theta}| + |\frac{\partial\beta}{\partial t}|))$. The last integration can be written as

$$\begin{aligned}
&\int_0^{\frac{r_1}{\epsilon}}\left\{\frac{1}{2}\left(\rho'(s)\right)^2 + \frac{1}{2s^2}\left(\rho(s)\right)^2 + \frac{1}{4}\left(a_0^2 - \rho(s)^2\right)^2\right\} s\,ds + \int_0^{r_1}\left(\rho\left(\frac{r}{\epsilon}\right)\right)^2 r\,dr \\
&= \int_0^1\left\{\frac{1}{2}\left(\rho'(s)\right)^2 + \frac{1}{2s^2}\left(\rho(s)\right)^2 + \frac{1}{4}\left(a_0^2 - \left(\rho(s)\right)^2\right)^2\right\} s\,ds + \int_1^{\frac{r_1}{\epsilon}}\left\{\frac{1}{2}\left(\rho'(s)\right)^2 + \frac{1}{2s^2}\left(\rho(s)\right)^2\right\} s\,ds + \int_0^{r_1}\left(\rho\left(\frac{r}{\epsilon}\right)\right)^2 r\,dr
\end{aligned}$$

which is smaller than $a_0^2 \ln\frac{\bar{r}}{\epsilon} + C_2 a_0$, because

$$\int_1^{\frac{r_1}{\epsilon}} \frac{1}{s}(a_0)^2\,ds = a_0^2 \ln\frac{r_1}{\epsilon},$$

$$\int_1^{\frac{r_1}{\epsilon}} (\rho'(s))^2 s\,ds = 0,$$

and

$$\begin{aligned}
\int_0^1\left\{\frac{1}{2}\left(\rho'(s)\right)^2 + \frac{1}{2s^2}\left(\rho(s)\right)^2 + \frac{1}{4}\left(a_0^2 - \rho^2(s)\right)^2\right\} s\,ds &= \frac{1}{2}[s\rho\rho']_0^1 - \frac{1}{2}\int_0^1\left\{\rho'' + \frac{1}{s}\rho' - \frac{1}{s^2}\rho\right\}\rho s\,ds + \frac{1}{4}\int_0^1\left(a_0^2 - \rho^2(s)\right)^2 s\,ds \\
&= \frac{1}{2}[s\rho\rho']_0^1 - \frac{1}{2}\int_0^1 \rho\left(a_0^2 - \rho^2\right)\rho s\,ds + \frac{1}{4}\int_0^1\left(a_0^2 - \rho^2(s)\right)^2 s\,ds \\
&\leqq Ca_0.
\end{aligned}$$

Here ρ' is estimated near $s=0$ by the equation

$$s^2\rho''(s)+s\rho'(s)-\rho=0,$$

and ρ is estimated by $0\leqslant\rho\leqslant a_0$ because

$$-s^2\rho''-s\rho'=f(s,\rho),\qquad f(s,0)\equiv 0,\qquad f(s,a_0)\leqslant 0,$$

from [1], there is a solution $\rho\in[0,a_0]$.

Step 2. Note that in $(B_{\bar r}(0)\setminus B_{r_1}(0))\times[-\bar t,\bar t]$,

$$|\nabla V_\lambda|^2\leqslant C\left(|\partial_t w_\lambda|^2+\frac{1}{r_1^2}|\partial_\theta w_\lambda|^2+\frac{1}{(\bar r-r_1)^2}+|\partial_t\beta_\lambda|^2+\frac{1}{r_1^2}|\partial_\theta\beta_\lambda|^2\right).$$

Then

$$\int\limits_{(B(0,\bar r)\setminus B(0,r_1))\times[-\bar t,\bar t]}\frac12\left|(\nabla-iA_\lambda)V_\lambda\right|^2+\frac{\lambda}{4}\left(a^2-|V_\lambda|^2\right)^2\leqslant C_3\bar t\left(\bar r^2-r_1^2\right)\left(1+\frac{1}{r_1^2}+\frac{1}{(\bar r-r_1)^2}+\lambda\right).$$

Step 3. To obtain the energy estimate, we only need

$$\bar t\left\{C_1a_0\left(a_0\ln[r_1\sqrt{\lambda}\,]+C_2\right)+C_3\left(\bar r^2-r_1^2\right)\left(1+\frac{1}{r_1^2}+\frac{1}{(\bar r-r_1)^2}+\lambda\right)\right\}\leqslant\frac{c_0\delta}{2}.\qquad\square$$

Lemma 5.2. *(Ψ_δ,B_δ) obtained in Lemma* 4.3 *is a minimizer of $\mathcal{H}_\lambda$ on*

$$\left(H^1(\widetilde\Omega)\cap\left\{\|\Psi\|_{L^\infty(\Omega)}\leqq M\right\}\right)\times Z.$$

Proof. Suppose that (Ψ_δ,B_δ) at least satisfies one of the following two equations:

$$\inf_{\rho\in\mathbb{R}}\left\|\Psi_\delta-e^{i\rho}\Phi_\lambda\right\|_{L^2(\Omega)}=\delta,$$
$$\left\|\nabla(A_\lambda-B_\delta)\right\|_{L^2(\mathbb{R}^3)}=\delta.\tag{5.1}$$

Since $\inf_{\rho\in\mathbb{R}}\|\Psi_\delta-e^{i\rho}\Phi_\lambda\|_{L^2(\Omega)}=\inf_{\rho\in[0,2\pi]}\|\Psi_\delta-e^{i\rho}\Phi_\lambda\|_{L^2(\Omega)}$, we can prove that there exists a constant ρ_0 such that

$$\left\|\Psi_\delta-e^{i\rho_0}\Phi_\lambda\right\|_{L^2(\Omega)}=\inf_{\rho\in\mathbb{R}}\left\|\Psi_\delta-e^{i\rho}\Phi_\lambda\right\|_{L^2(\Omega)}.$$

Note that

$$0=\frac{d}{d\rho}\bigg|_{\rho=\rho_0}\left\|\Psi_\delta-e^{i\rho}\Phi_\lambda\right\|_{L^2(\Omega)}=\left\langle\Psi_\delta-e^{i\rho_0}\Phi_\lambda,-ie^{i\rho_0}\Phi_\lambda\right\rangle_{L^2(\Omega)}=\left\langle e^{-i\rho_0}\Psi_\delta-\Phi_\lambda,-i\Phi_\lambda\right\rangle_{L^2(\Omega)},$$

and from the definition of $N(\Phi_\lambda,A_\lambda)$ (Section 3), we have

$$\left(\Phi_\lambda-e^{-i\rho_0}\Psi_\delta,A_\lambda-B_\delta\right)\in N(\Phi_\lambda,A_\lambda).$$

By Lemma 3.1 and (5.1), we get

$$\mathcal{H}_\lambda(\Psi_\delta,B_\delta)\geqq\mathcal{H}_\lambda^\Omega(\Psi_\delta,B_\delta)\geqq\mathcal{H}_\lambda^\Omega(\Phi_\lambda,A_\lambda)+c_0\delta,\tag{5.2}$$

provided $\delta\leqq\delta_0$. But (5.2) is contradicted with Lemma 5.1. $\square$

Lemma 5.3. $\|\Psi_\delta\|_{L^\infty(\widetilde\Omega)}\leqq 1$.

Proof. We first denote Ψ_δ by $w_\delta e^{i\theta_\delta}$, where $w_\delta\geqq 0$. Since

$$\mathcal{H}_\lambda(w_\delta,\theta_\delta,B_\delta)=\min\left\{\int\limits_{\mathbb{R}^3}\frac12|\operatorname{rot}B_\delta|^2+\int\limits_{\widetilde\Omega}\frac12|\nabla w_\delta|^2+\frac{\lambda}{4}\left(a-w_\delta^2\right)^2+\frac12\left|w_\delta(\nabla\theta_\delta-B_\delta)\right|^2\right\},$$

for $(w_\delta e^{i\theta_\delta},B_\delta)\in H^1(\widetilde\Omega)\times Z$ and $\|w_\delta\|_{L^\infty(\widetilde\Omega)}\leqq M$, we can take

$$v_t:=(1-t\varphi)w_\delta\quad\text{for }\varphi\in C^\infty\left(\overline{\widetilde\Omega};\mathbb{R}^+\right),$$

as test function provided that $t>0$ is small enough. From

$$\frac{d}{dt}\bigg|_{t=0^+} \mathcal{H}_\lambda(v_t, \theta_\delta, B_\delta) \geqq 0,$$

we get

$$0 \geqq \int_{\widetilde{\Omega}} \frac{1}{2}\nabla w_\delta^2 \cdot \nabla\varphi + \varphi|\nabla w_\delta|^2 - \lambda\big(a - w_\delta^2\big)w_\delta^2\varphi + |\nabla\theta_\delta - B_\delta|^2 w_\delta^2\varphi$$
$$\geqq \int_{\widetilde{\Omega}} \frac{1}{2}\nabla w_\delta^2 \cdot \nabla\varphi + \varphi|\nabla w_\delta|^2 - \lambda\big(1 - w_\delta^2\big)w_\delta^2\varphi + |\nabla\theta_\delta - B_\delta|^2 w_\delta^2\varphi$$

for any $\varphi \in C^\infty(\overline{\widetilde{\Omega}}; \mathbb{R}^+)$. Let

$$G = \big\{x \in \overline{\widetilde{\Omega}}:\ w_\delta(x) \geqq 1\big\}.$$

Then for any $\varphi \in C^\infty(\overline{\widetilde{\Omega}}; \mathbb{R}^+)$,

$$\int_G \nabla w_\delta^2 \cdot \nabla\varphi + \varphi\big(|\nabla w_\delta|^2 - \lambda\big(1 - w_\delta^2\big)w_\delta^2\big) \leqq 0.$$

By Maximum Principle, $w_\delta \equiv 1$ a.e. on G. □

Proof of Main Theorem. From Lemmas 5.2–5.3, we get that (Ψ_δ, B_δ) is a minimizer of $\mathcal{H}_\lambda$ in $H^1(\widetilde{\Omega}) \times Z$. Noting that Ψ_δ satisfies (4.2) and $\Phi_\lambda/|\Phi_\lambda|$ is homotopic to θ_0 (cf. Theorem A in Section 2) and by a compactness argument and the Schauder estimates for second-order elliptic boundary value problem,

$$\|\Psi_\delta - \Phi_\lambda\|_{C^2(\Omega)} \to 0, \quad \text{as } \bar{t}_j \to 0,\ \bar{r}_j \to 0,\ a_0 \to 0.$$

So there must be zeros of Ψ_δ locating in D. □

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