Multiplicatively range-preserving maps between Banach function algebras

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1. Introduction

Let $A$ and $B$ be two Banach function algebras on locally compact Hausdorff spaces $X$ and $Y$, respectively. Let $T$ be a multiplicatively range-preserving map from $A$ onto $B$ in the sense that $(TfTg)(Y) = (fg)(X)$ for all $f, g \in A$. We define equivalence relations on appropriate subsets $\hat{X}$ and $\hat{Y}$ of $X$ and $Y$, respectively, and show that $T$ induces a homeomorphism between the quotient spaces of $\hat{X}$ and $\hat{Y}$ by these equivalence relations. In particular, if all points in the Choquet boundaries of $A$ and $B$ are strong boundary points, then $X$ and $Y$ are equal to the Choquet boundaries of $A$ and $B$, respectively, and moreover, there exist a continuous function $h$ on the Choquet boundary of $B$ taking its values in $[-1, 1]$ and a homeomorphism $\psi$ from the Choquet boundary of $B$ onto the Choquet boundary of $A$ such that $Tf(y) = h(y)f(\psi(y))$ for all $f \in A$ and $y$ in the Choquet boundary of $B$. For certain Banach function algebras $A$ and $B$ on compact Hausdorff spaces $X$ and $Y$, respectively, we can weaken the surjectivity assumption and give a representation for maps belonging 2-locally to the family of all multiplicatively range-preserving maps from $A$ onto $B$.

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on their maximal ideal spaces. It is worth mentioning that the existence of the unit and so the compactness of the maximal ideal spaces of Banach algebras under consideration has an important role in their proofs and, as they remark at the end of [7], their proofs cannot be adopted directly for non-unital Banach algebras. We note that the main problem in dealing with multiplicatively range-preserving maps is the lack of linearity assumption. If we could prove that any multiplicatively range-preserving map between two Banach function algebras is linear, then extending such maps to the uniform closures of the Banach function algebras and using the general form of surjective linear isometries between uniformly closed subalgebras of continuous functions (see for example [17, Corollary 7.31] and [1, Theorem 4.1]), the Molnár’s results and their generalizations could be obtained immediately.

In this paper we first study surjective multiplicatively range-preserving maps between Banach function algebras $A$ and $B$ defined on locally compact Hausdorff spaces $X$ and $Y$, respectively, and show that such maps induce homeomorphisms between quotient spaces of appropriate subsets of $X$ and $Y$ by some equivalence relations (Theorem 3.1). If $A$ and $B$ are Banach function algebras whose Choquet boundaries are the same as the set of strong boundary points (in particular, if $A$ and $B$ are both completely regular), then these quotient spaces are equal to the Choquet boundaries. We then show that in this case for such preserving map $T : A \to B$ there exist a continuous function $h$ on the Choquet boundary of $B$ taking its values in $[-1, 1]$ and a homeomorphism $\varphi$ from the Choquet boundary of $B$ onto the Choquet boundary of $A$ such that $Tf(y) = h(y)f(\varphi(y))$ for all $f \in A$ and $y$ in the Choquet boundary of $B$ (Theorem 3.1).

Example 3.4 in [7] shows that surjectivity assumption is essential in their results. In this example, authors construct a non-linear and non-multiplicatively spectrum-preserving map (which is, in this case, multiplicatively range-preserving) on $C(\Gamma)$, for the Cantor ternary set $\Gamma$. We generalize a result proved by Kowalski and Slodkowski in [10] concerning $C$-linearity of an $\mathbb{R}$-linear map $\varphi : A \to C$, where $A$ is a Banach algebra, such that for every $x \in A$, $f(x)$ is contained in the spectrum of $x$. Using this extension we give a representation for the maps belonging 2-locally to the family of all surjective multiplicatively spectrum-preserving (respectively range-preserving) maps between certain Banach function algebras (Corollaries 4.3 and 4.4). Here, for two Banach function algebras $A$ and $B$ and a given family $\mathcal{G}$ of mappings from $A$ into $B$, an arbitrary map $T : A \to B$ is said to belong 2-locally to $\mathcal{G}$ if for every pairs $x, y \in A$ there exists an element $T_{x,y} \in \mathcal{G}$ with $T_{x,y}(x) = T(x)$ and $T_{x,y}(y) = T(y)$. We note that mappings belonging 2-locally to the family of all surjective multiplicatively range-preserving maps between two Banach function algebras are not necessarily surjective. Indeed in [5] Győry showed that for an uncountable discrete space $L$ there exists a non-surjective map belonging 2-locally to the family of all automorphisms (and hence to the family of all multiplicatively range-preserving maps) on $C_0(L)$.

2. Preliminaries

Let $X$ be a locally compact Hausdorff space and $X_\infty$ be its one point compactification. Let $\| \cdot \|_X$ denote the sup-norm on $C_0(X)$, the algebra of all continuous complex-valued functions on $X$ vanishing at infinity. A subalgebra $A$ of $C_0(X)$ is a function algebra on $X$ if $A$ separates the points of $X$, i.e. for each $x, z \in X$ with $x \neq z$, there exists $f \in A$ with $f(x) \neq f(z)$ and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. A function algebra $A$ on $X$ is a Banach function algebra on $X$ if $A$ is a Banach algebra with a norm. A uniform algebra on $X$ is a function algebra on $X$ which is a closed subalgebra of $(C_0(X), \| \cdot \|_X)$.

When $X$ is compact, all function algebras on $X$ are assumed to contain the constant functions.

Let $A$ be a function algebra on a locally compact Hausdorff space $X$. We denote the closure of $A$ in $(C_0(X), \| \cdot \|_X)$ by $\overline{A}$. A subset $E$ of $X$ is called a boundary for $A$ if every $f \in A$ assumes its maximum modulus at some point of $E$. The unique minimal closed boundary for $A$, which exists by [17, Theorem 7.4], is called the Šilov boundary for $A$ and is denoted by $\partial A$. The Choquet boundary $c(A)$ of $A$ is the set of all $x \in X$ for which $\delta_x$, the evaluation homomorphism at $x$, is an extreme point of the unit ball of the dual space of $(A, \| \cdot \|_X)$. Hence clearly $c(A) = c(\overline{A})$. Moreover, for a function algebra $A$, $\partial A$ is the closure of $c(A)$ [2, Theorem 1]. A subset $F$ of $X$ is a peak set for $A$ if there exists $f \in A$ such that $|f|_F = 1$ and $|f| < 1$ on $X \setminus F$. A function $f$, which peaks on $F$, i.e. $|f|_F = 1$ and $|f| < 1$ on $X \setminus F$, is called a peaking function for $F$. For $f \in A$ we take $M_f = \{ t \in X : |f(t)| = \|f\|_X \}$ and for $x \in X$ we take $F_x = \{ f \in A : |f(x)| = \|f\|_X \}$. A point $x \in X$ is a strong boundary point for $A$ if for every neighborhood $V$ of $x$, there exists a function $f \in A$ such that $\|f\|_X = |f(x)| = 1$ and $|f| < 1$ on $X \setminus V$. When $A$ is a Banach function algebra on $X$, by considering the function $\overline{M_f}$ instead of $f$, we can assume that the function $f$ in the latter definition is a peaking function. Following [9] we call the function algebra $A$ completely regular if all points in $X$ are strong boundary points for $A$.

If $A$ is a uniform algebra on a locally compact Hausdorff space $X$, then for an element $x_0 \in X$ the statements (i) and (iii) below are equivalent (see for instance [17, Theorem 7.30] for compact case and [16, Theorem 2.1] for the general case). Moreover, adapting the proof of [11, Theorem 4.7.22] one can see that (ii) and (iii) are equivalent too:

(i) $x_0 \in c(A)$.
(ii) For every neighborhood $V$ of $x_0$, there is a function $f \in A$ such that $\|f\|_X \leq 1$, $|f(x_0)| > \frac{3}{4}$, and $|f| < \frac{1}{4}$ on $X \setminus V$.
(iii) $x_0$ is a strong boundary point for $A$.

The example given in Section 3 of [3] shows that there exists a Banach function algebra $A$ on a compact metric space $X$ with a peak set which contains no strong boundary point. Since each peak set for $A$ intersects $c(A)$ it follows that condi-


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tions (i) and (iii) are not equivalent, in general, for Banach function algebras. The following proposition may be known but the authors do not know any reference.

**Proposition 2.1.** For every Banach function algebra $A$ on a locally compact Hausdorff space $X$ the conditions (i) and (ii) above are equivalent.

**Proof.** Suppose that $x_0 \in c(A) = c(\overline{A})$. Consider the subalgebra $A_1 = A + C1$ of $C(X_{\infty})$ and let $\overline{A}_1$ be the closure of $A_1$ in $(C(X_{\infty}), \|\cdot\|_{X_{\infty}})$. Then the proof of implication $a \Rightarrow b$ in [16, Theorem 2.1] can be applied to show that $x_0 \in c(\overline{A}_1) = c(A_1)$. Let $V$ be a neighborhood of $x_0$ in $X$ and let $u \in C_0(X)$ be such that $0 \leq u \leq 1$ on $X$, $u(x_0) = 1$ and $u = 0$ on $X \setminus V$. Set $\alpha = \sup\{\text{Re}h(x_0) : h \in \overline{A}_1, \text{Re}h \leq u\} = \sup\{\text{Re}h(x_0) : h \in A_1, \text{Re}h \leq u\}$. Since $\overline{A}_1$ is a uniform algebra on $X_{\infty}$, we have $\alpha = 1$ [17, Lemma 7.19]. So we can take an element $g \in A_1$ with $g \leq u$ on $X_{\infty}$ and $\text{Re}g(x_0) > 1/2\alpha$. Then for $c = 1 + \text{ln}16$, $f = c^{(c^{-1})} \in A_1$ with $\|f\|_{X_{\infty}} \leq 1$, $\|f(x_0)\| > 1/\alpha$ and $\|f\| \leq 1/\alpha$ on $X_{\infty} \setminus V$. Taking $\lambda = f(\infty)$ we see that $|\lambda| \leq 1/\alpha$ and $h = \frac{1-\lambda(\frac{\lambda}{2})}{16}A \in A$ such that $|h(x_0)| > \frac{3}{4}$ and $|h| < \frac{1}{4}$ on $X \setminus V$ as desired. Hence (i) implies (ii). Using [14, Proposition 3.7], implication (ii) $\Rightarrow$ (i) can be obtained with an argument similar to [17, Corollary 7.20]. \qed

At the end of this section we state the following lemma which is used frequently in the next section:

**Lemma 2.2.** (See [2, Lemma 3].) Let $X$ be a locally compact Hausdorff space, $A$ be a subalgebra of $C_0(X)$ and $x \in \bigcap_{f \in M_f} M_f$. If $U$ is an open subset of $X$ containing $\bigcap_{x \in M_f} M_f$, then there exists $g \in A$ such that $\|g\|_X = 1 = g(x)$ and $|g(z)| < 1$ for all $z \in X \setminus U$.

3. **Surjective multiplicatively range-preserving maps**

As it was noted before, surjective multiplicatively spectrum-preserving maps were first studied by Molnár in [13]. His characterization shows, in particular, that each surjective multiplicatively spectrum-preserving map $T: C(X) \to C(X)$, where $X$ is a first countable compact Hausdorff space, is automatically linear. Later on, the result has been extended in [6,7,15,16]. In this section we extend the result for the case where such maps are defined between (non-unital) Banach function algebras.

In this section we assume that $A$ and $B$ are Banach function algebras on locally compact Hausdorff spaces $X$ and $Y$, respectively. Defining an equivalence relation $\sim$ on appropriate subsets $\overline{X}$ and $\overline{Y}$ of $X$ and $Y$, respectively, we show that each surjective multiplicatively range-preserving map $T: A \to B$ induces a homeomorphism between two quotient spaces $\overline{X}/\sim$ and $\overline{Y}/\sim$. There are some situations under which $\overline{X} = X$, $\overline{Y} = Y$ and each equivalence class consists of just one element, and so in these cases $T$ induces a homeomorphism between $X$ and $Y$.

For an element $x \in \bigcup_{f \in A} M_f$ set $I_x = \bigcap_{f \in F_x} M_f = \bigcap_{x \in M_f} M_f$. Since a non-empty intersection of peak sets intersects $c(A) = c(\overline{A})$ and for each $f \in F_x$, $M_f$ contains the peak set $\{t \in X : f(t) = f(x)\}$, the compact subset $I_x$ of $X$ intersects $c(A)$. Let $\mathcal{F} = \{I_x : x \in \bigcup_{f \in A} M_f\}$ and define the equivalence relation $\sim$ on $\overline{X} = \{x \in \bigcup_{f \in A} I_x : I_x$ is a minimal element of $(\mathcal{F}, \subseteq)\}$ as follows: For $x, z \in \overline{X}$, $x \sim z$ if and only if $F_x = F_z$ (or equivalently $I_x = I_z$). Since for each $x \in \overline{X}$, $I_x$ is minimal we get $[x] = I_x$. For $f \in A$, $\|f\|_{|x|}$ denotes the sup-norm of $f$ on $|x|$ as a compact subset of $X$. If each point in $c(A)$ is a strong boundary point for $A$ (this holds, for example, when $A$ is either a uniform algebra or a completely regular Banach function algebra on $X$), then $\overline{X} = c(A)$ and $[x] = I_x = [x]$ for all $x \in c(A)$.

We use the same notations for the subsets associated to the points in $Y$ and to the functions in $B$. We also consider similarly the subset $\overline{Y}$ of $Y$ with an analogous equivalence relation on $\overline{Y}$.

**Theorem 3.1.** Let $T : A \to B$ be a surjective multiplicatively range-preserving map. Then $T$ is injective, homogeneous and there is a homeomorphism $\psi$ from $\overline{X}/\sim$ onto $\overline{Y}/\sim$ such that for each $x \in \overline{X}$ and $f \in A$, $\|f\|_{|x|} = \|Tf\|_{\psi(|x|)}$. Moreover, if the points in $c(A)$ and $c(B)$ are all strong boundary points, then $T$ has the following representation

$$Tf(y) = h(y)f(\varphi(y)) \quad (f \in A, \ y \in c(B),$$

where $h$ is a continuous real-valued function on $c(B)$ with $h^2 = 1$ and $\varphi$ is a homeomorphism from $c(B)$ onto $c(A)$.

It should be noted that the above theorem has been proved in [7] for the case where $A$ and $B$ are unital with maximal ideal spaces equal to $X$ and $Y$, respectively, without any additional assumption on the Choquet boundaries.

In the following we assume that $T : A \to B$ satisfies the hypotheses of the theorem. We conclude the theorem from the following lemmas.

**Lemma 3.2.** For $f, g \in A, |f| \leq |g|$ on $c(A)$ if and only if for every $c \geq 0$ and $h \in A, |gh| \leq c$ implies $|fh| \leq c$.

**Proof.** The “only if” part is trivial. For the converse we modify the proof of [15, Remark 2]. Let there exist $x_0 \in c(A)$ such that $|f(x_0)| > |g(x_0)|$. If $\gamma = \frac{1}{2}(|f(x_0)| + |g(x_0)|)$, then $|g(x_0)| < \gamma < |f(x_0)|$ and hence there exists a neighborhood $V$
of $x_0$ such that $|g| < \gamma$ on $V$. Since $\tilde{A}$ is a uniform algebra on $X$ and $c(A) = c(\tilde{A})$, there exists a function $h \in \tilde{A}$ such that

\[
h(x_0) = 1 = \|h||x|, \quad |h| < \gamma/\|g||x| \quad \text{on $X \setminus V$}.
\]

Choose $n \in \mathbb{N}$ such that $1/n < ((f(x_0) - \gamma)/\beta$ and $n \leq \beta$, where $\beta = 2\gamma + 1 + \|g||x$, and then choose $k \in A$ such that $\|h - k||x| < 1/n$. Thus for $c = \max(\gamma/\gamma/n, \gamma + \|g||/n)$, $|g|| \leq c$ on $X$ while $|f(k(x_0))| > c$. □

Now we can easily deduce the following corollaries.

**Corollary 3.3.** For $f, g \in A$, if $(fgh)(X) = (gh)(X)$ for every $h \in A$, then $|f| = |g|$ on $c(A)$.

**Corollary 3.4.** For $f, g \in A$, $|f| \leq |g|$ on $c(A)$ if and only if $|Tf| \leq |Tg|$ on $c(B)$.

The following lemma shows that the map $T$ is, indeed, injective.

**Lemma 3.5.** For $f, g \in A$, $f = g$ if and only if $(fgh)(X) = (gh)(X)$ for every $h \in A$.

**Proof.** Let $f, g \in A$ and $(fgh)(X) = (gh)(X)$ for every $h \in A$, but $f \neq g$. By Corollary 3.3, there exists $x_0 \in c(A) = c(\tilde{A})$ such that $f(x_0) \neq 0$, $g(x_0) \neq 0$ and $f(x_0) \neq g(x_0)$. Considering each continuous complex-valued function on the locally compact Hausdorff space $X$ vanishing at infinity as an element of $C(X\infty)$, we have $(fgh)(X\infty) = (gh)(X\infty)$, for all $h \in A$. Since $x_0$ is a strong boundary point for $\tilde{A}$, for an arbitrary neighborhood $V$ of $x_0$ we can find a peaking function $h \in \tilde{A}$ such that $h(x_0) = 1$ and $|h| < 1$ on $X \setminus V$. If $E = \{x \in X: h(x) = 1\}$, by modifying $h$, we may assume that for all $z \in X \setminus E$,

\[
|\{(fgh)(z)\}| < \|f\|_E = \|fgh\|_X.
\]

Let $(h_n)$ be a sequence in $A$ converging uniformly to $h$ and let $x \in E$ be such that $|f(x)| = \|f\|_E = \|fgh\|_X$. By the assumption, for each $n \in \mathbb{N}$ there exists $y_n \in X$ such that $(fgh_n)(x) = (gh_n)(y_n)$. Hence for each $n$,

\[
|\{(fgh)(x) - (gh)(y_n)\}| \leq |\{(fgh)(x) - (fh_n)(x)\}| + |\{(gh_n)(y_n) - (gh)(y_n)\}|
\]

\[
\leq (\|f\|_X + \|g\|_X)\|h_n - h\|_X.
\]

The above inequality and the compactness of $(gh)(X\infty)$ imply that $(fgh)(x) = (gh)(X\infty)$. Since $(fgh)(x) \neq 0$ it follows that $(fgh)(x) \in (gh)(X\infty)$, hence $f(x) = f(xh)(x) = g(h)(x)$ for some $z \in X$. If $z \in X \setminus E$, then $|g(h)(z)| < \|g\|_E \|g\|_X = \|fgh\|_X = |f(x)|$ which is a contradiction. Therefore, $z \in E$ and consequently $f(x) = g(z)$. Since $x, z \in V$ and $V$ is an arbitrary neighborhood of $x_0$, it follows that $f(x_0) = g(x_0)$ which is again a contradiction. □

**Lemma 3.6.** $T$ is homogeneous, i.e., $T(cf) = cTf$, for all $f \in A$ and $c \in \mathbb{C}$.

**Proof.** Let $f \in A$ and $c \in \mathbb{C}$. For every $h \in A$, $T(cfh)(Y) = (cfh)(X) = c(Tfh)(Y)$. Since $T$ is surjective there exists an element $g$ in $A$ with $Tg = Tf$. Hence $(cfh)(X) = (gh)(X)$ and by Lemma 3.5, $cf = g$, i.e. $T(cf) = cT(f)$. □

**Lemma 3.7.** For each $x \in \tilde{X}$ there exists $y \in \tilde{Y}$ such that $\bigcap_{f \in F_x} M_{Tf} = [y]$.

**Proof.** Let $x \in \tilde{X}$. An argument similar to [16, Remark 3.5] shows that $\bigcap_{f \in F_x} M_{Tf} \neq \emptyset$. By Zorn’s Lemma, the family $\{T_Y: y \in \bigcap_{f \in F_x} M_{Tf}\}$ has a minimal element, say $T_y$, which is, in fact, a minimal element of $\{T_z: z \in \bigcup_{g \in B} M_g\}$. Hence $y \in \tilde{Y}$ and moreover, $[y] = T_y \subseteq \bigcap_{f \in F_x} M_{Tf}$. Now assume that there exists an element $z \in \bigcap_{f \in F_x} M_{Tf} \setminus [y]$. Let $W$ be a neighborhood of $[y]$ in $Y$ which does not contain $z$. By Lemma 2.2, we can find a function $g \in F_Y$ such that $|g| < 1$ on $Y \setminus W$, in particular, $|g(z)| < 1$. Surjectivity of $T$ implies that $g = Tf$ for some $f \in A$. We claim that $f \in F_x$. If $f \neq F_x$ then $|f(\xi)| < 1$ for every $\xi \in [x]$. Hence there exists a neighborhood $V$ of $[x]$ in $X$ such that $|f| < 1$ on $V$. Using Lemma 2.2 once again we deduce that there exists a function $h \in F_x$ such that $|h| < 1$ on $X \setminus V$. Hence $|Tfh(y)| = 1$ and consequently $\|fgh\|_X = \|Tfh\|_Y = 1$. Therefore, there is $x_0 \in X$ with $|f(x_0)| = 1 = |h(x_0)|$ which is impossible. Hence, $f \in F_x$ and so $|g(z)| = |Tf(z)| = 1$ which is a contradiction. Consequently $\bigcap_{f \in F_x} M_{Tf} = [y]$. □

By the above lemma we can now define $\psi: \tilde{X} \sim \rightarrow \tilde{Y} \sim$ by $\psi([x]) = \bigcap_{f \in F_x} M_{Tf} = [y]$. Then clearly we have the following corollary:

**Corollary 3.8.** For each $x \in \tilde{X}$ and $z \in \psi([x])$, $T(F_x) \subseteq F_z$. Conversely, if $g \in F_z$ and $Tf = g$, then $f \in F_x$.

**Lemma 3.9.** For each $x \in \tilde{X}$ and $f \in A$, $\|f||_X| = \|Tf||\psi([x])||$. 
**Proof.** Let $x \in \tilde{X}$, $f \in A$ and $\psi([x]) = \cap_{f \in F, M_{Tf} = [y]}$, where $y \in Y$. If $\|Tf\|_{\psi([x])} = 0$, then for a given $\epsilon > 0$, by using Lemma 2.2, we can easily choose $g \in A$ such that $\|Tg\|_Y = 1 = \|Tg\|_Y$ and $\|TfTg\|_Y < \epsilon$. Thus by Corollary 3.8, $g \in F_x$ and so for each $x' \in [x]$, 
\[
|f(x')| = |f(x')| = \|fg\|_x = \|TfTg\|_Y < \epsilon.
\]
Since $\epsilon$ is arbitrary we conclude that $\|f\|_x = 0 = \|Tf\|_{\psi([x])}$. We now consider the case where $\|Tf\|_{\psi([x])} \neq 0$. Let $V$ be an arbitrary neighborhood of $[y]$ in $Y$ and choose a function $g \in A$ such that $\|Tg\|_Y = 1 = \|Tg\|_Y$ and $\|Tg\| < 1$ on $Y \setminus V$. By considering $(Tg)^n$ for a sufficiently large $n \in \mathbb{N}$, we can assume that $TfTg$ attains its maximum modulus at a point $y_0 \in V$. By the preceding corollary $g \in F_x$, it follows that $|g(x')| = 1$, for all $x' \in [x]$. Hence 
\[
|f(x')| = |f(g(x'))| \leq \|fg\|_x = \|TfTg\|_Y = |(TfTg)(y_0)| \leq |Tf(y_0)|
\]
for all $x' \in [x]$, that is $\|f\|_x \leq \|Tf\|_{\psi([x])}$. Now the continuity of $Tf$ implies that $\|f\|_x \leq \|Tf\|_{\psi([x])}$. The same argument as in the first part (for $f$ instead of $Tf$) shows that we can assume $\|f\|_x \neq 0$ and so similar arguments conclude that $\|Tf\|_{\psi([x])} \leq \|f\|_x$. Hence $\|f\|_x = \|Tf\|_{\psi([x])}$.

We now complete the proof of the first part of Theorem 3.1 by proving the next lemma.

**Lemma 3.10.** The function $\psi$ is a homeomorphism from $\tilde{X}/\sim$ onto $\tilde{Y}/\sim$, where both $\tilde{X}/\sim$ and $\tilde{Y}/\sim$ are equipped with quotient topologies.

**Proof.** Let $x, x' \in \tilde{X}$ be such that $\psi([x]) = \psi([x'])$. Then by the preceding lemma, $\|f\|_x = \|f\|_{x'}$ for all $f \in A$. If $[x] \neq [x']$, then there exists an open neighborhood $U$ of the compact subset $[x] = X_x$ of $X$ with $\cap U \neq \emptyset$. Hence by Lemma 2.2 we can find a function $f \in A$ such that $\|f\|_x = 1 > \|f\|_{x'}$ which is impossible. Thus $[x] = [x']$, that is, $\psi$ is injective. For the continuity of $\psi$, let $x_0 \in X$ and $y_0 \in Y$ with $\psi([x_0]) = [y_0]$. Let $W$ be a neighborhood of $[y_0]$ in the quotient topology on $Y/\sim$. Then there is a neighborhood $W_0$ of $y_0$ in $Y$ with $\pi^{-1}(W) = Y \cap W_0$, where $\pi : Y \rightarrow Y/\sim$ is the quotient map. Clearly $W_0 \subseteq W_0$, hence by Lemma 2.2 there exists a function $g \in A$ such that $Tg(y_0) = 1 = \|Tg\|_Y$ and $\|Tg\| < 1/2$ on $Y \setminus W_0$, in particular $\|Tg\| < 1/2$ on $Y/\sim$. Using the preceding lemma, $V = \{x \in \tilde{X}/\sim : \|Tg\|_{\psi([x])} > 1/2\}$ is a neighborhood of $[x_0]$ in $\tilde{X}/\sim$ such that $\psi(V) \subseteq W$, that is, $\psi$ is continuous. Since $T$ is injective and our conditions are symmetric with respect to $Y$ and $T^{-1}$, there exists a continuous map $\varphi$ from $Y/\sim$ into $X/\sim$ associated to $T^{-1}$ with the same properties as $\psi$. Thus $\|f\|_x = \|f\|_{\psi([x])}$ for all $x \in X$ and $f \in A$. Similarly $\|Tf\|_y = \|Tf\|_{\psi([y])}$ for all $y \in Y$ and $f \in A$ which implies, as the beginning of the proof, that $\psi$ is the inverse of $\varphi$, i.e. $\psi$ is a homeomorphism.

**Remark.** If the points in the Choquet boundaries of $A$ and $B$ are all strong boundary points for $A$ and $B$, respectively, then $\tilde{X} = (c(A), \tilde{Y} = (c(B)$ and the equivalence classes considered earlier consist of just one element. In this case $\psi$ is a homeomorphism between $c(A)$ and $c(B)$. In particular, if both $A$ and $B$ are completely regular, then $T$ induces a homeomorphism between $X$ and $Y$.

We now state the following lemma to prove the second part of Theorem 3.1. We note that if all points in $c(A)$ and $c(B)$ are strong boundary points for $A$ and $B$, respectively, then for a given element $y \in c(B)$ there exists a peak function $f \in B$ such that $f(y) = 1$. Let $\varphi = \psi^{-1}$. Since $f^2(y) = (T^{-1}f)^2(x)$ and $|T^{-1}f(\varphi(y))| = |f(y)| = 1$, by Lemma 3.9, it follows that $T^{-1}f(\varphi(y)) \in [1, -1]$. Let $h$ be the function defined on $c(B)$ by $h(y) = T^{-1}f(\varphi(y))$. We can easily check that the definition of $h(y)$ is independent of the choice of $f$.

**Lemma 3.11.** If all points in $c(A)$ and $c(B)$ are strong boundary points, then for all $y \in c(B)$ and $f \in A$
\[
Tf(y) = h(y)f(\varphi(y)),
\]
where $\varphi$ and $h$ are defined as above.

**Proof.** Let $f \in A$ and $y \in c(B)$. Take $\alpha = f(\varphi(y))$ and $\beta = Tf(y)$. Since $|Tf(y)| = |f(\varphi(y))|$, we can assume that $\alpha \neq 0$ and $\beta \neq 0$. Since each point in $c(A)$ is assumed to be a strong boundary point for $A$, considering the closed subsets $F_0 \subseteq X : |f(x) - \alpha| \geq |\alpha|/2$ and 
\[
F_n = \{x \in X : \frac{|\alpha|}{2^n+1} \leq |f(x) - \alpha| \leq \frac{|\alpha|}{2^n}\} \quad (n \in \mathbb{N})
\]
in $X$ we can find the peak functions $u_0, u_1, u_2, \ldots, u_n, \ldots$ in $A$ such that $u_n(\varphi(y)) = 1$, $n \geq 0$, and 
\[
\left\{\begin{array}{lcl}
|u_0(x)| & \leq \frac{|\alpha|}{\|f\|_X} & (x \in F_0), \\
|u_n(x)| & \leq \frac{1}{2^n+1} & (x \in F_n, \quad n = 1, 2, \ldots).
\end{array}\right.
\]
Set \( v_n = u_0 \sum_{k=1}^{n} \frac{u_k}{2^k} \) and \( z_n = T(v_n)(y) \), \( n \in \mathbb{N} \). Then for each \( n \in \mathbb{N} \)

\[
\beta = z_n^{-1}(Tf)(y)z_n = z_n^{-1}(fv_n)(x_n)
\]

for some \( x_n \in X \). Clearly the sequence \( \{v_n\} \) converges uniformly to a function \( u \in C_0(X) \). Using the above inequalities one can see that \( f(u)(X) \subseteq \{\lambda \in \mathbb{C}: |\lambda| < |\alpha| \} \cup \{0\} \). Passing to a subsequence we can assume that \( z_n^{-1} \) converges to a scalar \( \lambda \) with \( |\lambda| = 1 \). Since for each \( n \in \mathbb{N} \)

\[
|\beta - \lambda(f(u)(x_n))| \leq |z_n^{-1}(fv_n)(x_n) - \lambda(fv_n)(x_n)| + |\lambda(fv_n)(x_n) - \lambda(f(u)(x_n))|
\]

\[
\leq \|f\| \|x_n(z_n^{-1} - \lambda) + |\lambda| \|v_n - u\| \|x\|
\]

the compactness of \( \lambda(f(u)(X)) \) implies that \( \beta = \lambda(f(u)(X)) \). Since \( \beta \neq 0 \) and \( |\beta| = |\alpha| \) it follows that \( \beta = \lambda \alpha \). One can now easily check that \( \lambda = h(y) \) as desired. \( \square \)

**Remark.** Let \( A \) and \( B \) satisfy the hypotheses of the above lemma. Then since \( c(B) \) is a boundary for \( B \), the representation given in this lemma shows that \( T \) is linear. Moreover, since the norm of each element in a Banach function algebra is at least its sup-norm, the Closed Graph theorem yields the continuity of \( T \). Using a similar argument to [8, Lemma 4.7], the functions \( h \) and \( \varphi \) can be extended, respectively, to a continuous function on the maximal ideal space \( M_B \) of \( B \) with values in \((-1, 1) \) and to a homeomorphism from \( M_B \) onto \( MA \) such that for all \( y \in M_B \) and \( f \in A \), \( y(T(f)) = h(y)\varphi(y)(f) \). These extensions can be obtained directly when \( X \) and \( Y \) are compact. Indeed in this case, \( h = T1 \in B \) and moreover, \( T1f = T1f \) defines a homomorphism from \( A \) onto \( B \), so it suffices to consider the Gelfand transformation of \( h \) and the restriction of \( T1 \), the adjoint of \( T1 \), to \( M_B \) as the desired extensions of \( h \) and \( \varphi \), respectively. Thus if \( A \) and \( B \) are as in Lemma 3.11, then every multiplicatively range-preserving map from \( A \) onto \( B \) is multiplicatively spectrum-preserving.

For a locally compact group \( G \), the Figà–Talamanca–Herz algebra \( A_p(G) \), \( 1 < p < \infty \) is a Banach function algebra on \( G \) [4, Theorem 4.5.30]. Moreover, \( A_p(G) \) is completely regular. Indeed, for \( x_0 \in \mathbb{G} \) and arbitrary neighborhood \( W \) of \( x_0 \), we can choose a compact symmetric neighborhood \( U \) of \( x_0 \), the identity of \( G \), such that \( UUx_0 \subseteq W \). Then since \( \chi_U \in L_p(\lambda) \) and \( x_n = \chi_{U_i} U \in L_q(\lambda) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \lambda \) is the Haar measure on \( G \), the function \( f = \lambda(U)^{-1} \chi_U * \chi_{Ux_0} \) is an element of \( A_p(G) \) with \( f(x_0) = 1 \), \( f = 0 \) on \( G \setminus W \) and \( \|f\| \leq \lambda(U)^{-1} \|\chi_U\|_{L_p} \|\chi_{Ux_0}\|_q = 1 \). So we have the following corollary:

**Corollary 3.12.** Let \( p \in (1, \infty) \) and let \( G_1 \) and \( G_2 \) be locally compact groups. If \( T : A_p(G_1) \to A_p(G_2) \) is a surjective multiplicatively range-preserving map, then there exist a homeomorphism \( \varphi \) from \( G_2 \) onto \( G_1 \) and a continuous function \( h : G_2 \to [-1, 1] \) such that

\[
Tf(a) = h(a)\varphi(a) \quad (f \in A_p(G_1), \ a \in G_2).
\]

**Remark.** Let \( G \) be a locally compact group and \( B_p(G) \) be the multiplier algebra of \( A_p(G) \), \( 1 < p < \infty \), where by a multiplier on \( A_p(G) \) we mean a complex-valued function \( u \) on \( G \) such that \( uv \in A_p(G) \), for all \( v \in A_p(G) \). In the above corollary, for every \( f \in A_p(G_1) \) we have \( f \circ \varphi \in B_p(G_2) \), because for each \( g \in A_p(G_2) \) there exists \( k \in A_p(G_1) \) with \( g = Tk \), hence for every \( a \in G_2 \), \( g(a) \varphi(a) = Tk(a) \varphi(a) = h(a)k(\varphi(a)) = h(a)k(\varphi(a)) = h(k)(\varphi(a)) = T(k)(\varphi(a)) \), i.e., \( g \circ (f \circ \varphi) = T(k) \in A_p(G_2) \).

As an another application of the results we state the following example.

**Example 3.13.** Let \( (X, d) \) be a locally compact metric space and \( 0 < \alpha < 1 \). Let \( \text{Lip}(X, \alpha) \) be the algebra of all bounded continuous complex-valued functions on \( X \) satisfying the Lipschitz condition of order \( \alpha \). Then \( \text{Lip}(X, \alpha) \) equipped with the norm

\[
\|f\| = \|f\|_X + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} \quad (f \in \text{Lip}(X, \alpha))
\]

is a Banach function algebra on \( X \). Set \( A = C_0(X) \cap \text{Lip}(X, \alpha) \). Clearly \( A \) is a (non-unital) closed subalgebra of \( \text{Lip}(X, \alpha) \). It is easy to see that for each \( x_0 \in X \) and compact neighborhood \( V \) of \( x_0 \), the function \( f \) defined by \( f(x) = \max(0, 1 - \frac{d(x, x_0)}{d(x, x_0, V)}) \), \( x \in X \), is an element of \( A \) such that \( f(x_0) = 1 \), \( \|f\|_X \leq 1 \) and \( f = 0 \) on \( V^c \). Hence \( A \) is a completely regular Banach function algebra on \( X \). Therefore, by Theorem 3.1, if \( Y \) is a locally compact metric space and \( 0 < \beta < 1 \), then each multiplicatively range-preserving map \( T \) from \( A \) onto \( B = C_0(Y) \cap \text{Lip}(Y, \beta) \) is of the form

\[
Tf(y) = h(y)f(\varphi(y)) \quad (f \in A, \ y \in Y),
\]

where \( h : Y \to [-1, 1] \) is a continuous function and \( \varphi \) is a homeomorphism from \( Y \) onto \( X \).
4. 2-Local multiplicatively range-preserving maps

We recall that for two Banach algebras $A$ and $B$ and a given family $\mathcal{G}$ of mappings from $A$ into $B$, an arbitrary map $\varphi : A \to B$ belongs 2-locally to $\mathcal{G}$ if for each $x, y \in A$ there exists an element $\psi \in \mathcal{G}$ with $\varphi(x) = \psi(x)$ and $\varphi(y) = \psi(y)$. A map belonging 2-locally to the family of all surjective multiplicatively range-preserving (respectively spectrum-preserving) maps between two Banach function algebras will be referred as a 2-locally multiplicatively range-preserving (respectively spectrum-preserving) map.

In this section, by generalizing a result of Kowalski and Słodkowski [10, Lemma 2.1], we give a representation for 2-local multiplicatively range-preserving maps between certain Banach function algebras. We first state some notations. For a Banach algebra $A$ and an element $x \in A$, $\sigma(x)$ and $r(x)$ denote the spectrum and the spectral radius of $x$, respectively. When $A$ is commutative and unital by $M_A$ we mean the maximal ideal space of $A$ and $\lambda$ denotes the Gelfand transformation of $x \in A$. Via the Gelfand transformation each unital semisimple commutative Banach algebra can be considered as a Banach function algebra on its maximal ideal space.

The main result of this section is based on the following lemma:

**Lemma 4.1.** Let $A$ be a unital complex Banach algebra and let $\varphi : A \to \mathbb{C}$ be an $\mathbb{R}$-linear map such that $\varphi(i) = i \varphi(1)$. If there exists a real-valued function $f$ on $A$ which is away from zero and $\varphi(x) = f(x)\sigma(x)$ for all $x \in A$, then $\varphi$ is $\mathbb{C}$-linear.

**Proof.** We first note that the hypotheses implies that $f(i) = f(1)$. Consider the $\mathbb{R}$-linear functionals

$$\varphi_1(x) = \Re \varphi(x) + i \Im(-i \varphi(ix)) = \Re \varphi(x) - i \Re \varphi(ix),$$

$$\varphi_2(x) = \Re(-i \varphi(ix)) + i \Im \varphi(x) = \Im \varphi(ix) + i \Im \varphi(x),$$

on $A$. Since $\varphi_j(ix) = i \varphi_j(x)$, $j = 1, 2$, it follows that $\varphi_1$ and $\varphi_2$ are $\mathbb{C}$-linear. It is easy to verify that for a fixed $x \in A$, $\Re \varphi(x) + i \Im(-i \varphi(ix))$ and $\Re(-i \varphi(ix)) + i \Im \varphi(x)$ are both in the set $\mathcal{T}_x = \{e^{it} \varphi(e^{-it}x) : t \in \mathbb{R}\}$ [see the proof of [10, Lemma 2.1]]. Moreover, since for every $r \in \mathbb{R}$, $e^{it} \varphi(e^{-it}x) \in e^{it} f(e^{-it}x)\sigma(e^{-it}x) = f(e^{-it}x)\sigma(x)$ it follows that $\mathcal{T}_x \subseteq \{f(e^{-it}x)\sigma(x) : t \in \mathbb{R}\}$. Hence $\varphi_1(x), \varphi_2(x) \in \{f(e^{-it}x)\sigma(x) : t \in \mathbb{R}\}$. In particular, for all invertible elements $x$ in $A$, $\varphi_1(x) \neq 0$ and $\varphi_2(x) \neq 0$. Since $\varphi_1(1) = f(1)$ and $\varphi_2(1) = f(1) = f(i)$, Gleason–Kahane–Zelazko theorem implies that $\frac{1}{f(1)} \varphi_1$ and $\frac{1}{f(1)} \varphi_2$ are multiplicative on $A$. We shall show that $\varphi_1 = \varphi_2$. Assume on the contrary that there is $a \in A$ such that $\varphi_1(a) = 1$ and $\varphi_2(a) = 0$. For every $\alpha > 0$ we define the entire function $h_{\alpha}$ by $h_{\alpha}(z) = \frac{1}{f(1)}(ae^{\frac{\pi i \alpha z}{2}} - 1)$, $z \in \mathbb{C}$. We see that for each $\alpha$

$$\varphi(h_{\alpha}(a)) = \Re \varphi(h_{\alpha}(a)) + i \Im \varphi(h_{\alpha}(a)) = \Re \varphi_1(h_{\alpha}(a)) + i \Im \varphi_2(h_{\alpha}(a))$$

$$= f(1) \Re h_{\alpha}\left(\frac{1}{f(1)} \varphi_1(a)\right) + i f(1) \Im h_{\alpha}\left(\frac{1}{f(1)} \varphi_2(a)\right) = -1.$$  

Hence if there exists a scalar $\alpha > 0$ with $f(h_{\alpha}(a)) = f(1)$, then $-1 \in f(1)\sigma(h_{\alpha}(a)) = f(1)h_{\alpha}(\sigma(a))$, which is impossible. Thus for each $\alpha > 0$, $f(h_{\alpha}(a)) \neq f(1)$. Since $f$ is away from zero there exists $s > 0$ such that $|f| > s$ on $A$. Now choose a scalar $\beta > 0$ such that $\frac{\ln|f(1)|}{\beta} < 1$ and $|\ln(1 + |f(1)|s/\beta)| > \pi s |f(1)|$. By the above argument $\varphi(h_{\beta}(a)) = -1 \in f(h_{\beta}(a))h_{\beta}(\sigma(a))$. Thus there exists $z \in \sigma(a)$ such that

$$-1 = \frac{f(h_{\beta}(a))}{f(1)} \left(\beta e^{\frac{\pi i \alpha z}{2}} - 1\right).$$

and hence

$$|z| > \frac{2}{\pi |f(1)|} \left|\ln\left(\frac{1}{\beta} |1 - \frac{f(1)}{f(h_{\beta}(a))}|\right)\right|.$$  

Since

$$\frac{1}{|\beta|} |1 - \frac{f(1)}{f(h_{\beta}(a))}| \leq \frac{1}{\beta} \left(1 + \frac{|f(1)|s}{\beta}\right) < 1,$$

it follows that $|z| > r(a)$, which is impossible. Therefore, $\varphi_1 = \varphi_2$ and hence $\varphi$ is $\mathbb{C}$-linear on $A$. \hfill $\square$

**Theorem 4.2.** Let $A$ and $B$ be Banach function algebras on compact Hausdorff spaces $X$ and $Y$, respectively. Let $\mathcal{G}$ be the family of all operators $S$ from $A$ into $B$ of the form

$$Sf(y) = h(y)f(\varphi(y)) \quad (f \in A, y \in Y),$$

where $h : Y \to [-1, 1]$ and $\varphi : Y \to X$ are continuous functions. If $T : A \to B$ is an arbitrary map belonging 2-locally to $\mathcal{G}$, then $T$ is a continuous linear map and $f \mapsto T1Tf$ is a homomorphism from $A$ into $B$. If, furthermore, $M_A = X$, then $T \in \mathcal{G}$. 

Proof. Let $y \in Y$ and let $\delta_y$ be the evaluation homomorphism at $y$ defined on $B$. We first claim that $\delta_y \circ T$ is linear and a scalar multiplication of a complex homomorphism on $A$. Take $\varphi = \delta_y \circ T$. Then since $T$ belongs $2$-locally to $G$, $\varphi(0) = 0$ and $\varphi(1) \in \{1, -1\}$. By the hypothesis for each $f, g \in A$ there exists $T_{f,g} \in G$ such that $T_{f,g}(f) = T(f)$ and $T_{f,g}(g) = T(g)$. Since the norm of each element in a Banach function algebra is at least its sup-norm it follows that

$$|\varphi(f) - \varphi(g)| = |T_{f,g}(f - g)(y)| \leq \|f - g\|_X \leq \|f - g\|,$$

i.e. $\varphi$ satisfies the Lipschitz condition. As in the proof of [10, Theorem 1.2] we can assume that $A$ is separable. Thus by [12] $\varphi$ has real differentials except for some zero set (in the sense of [10, Definition 2.2]). Let $f_0 \in A$ such that $\varphi$ has real differential at $f_0$. Since $T$ belongs $2$-locally to $G$ and since each element in $G$ is linear it follows that for each $r \in \mathbb{R}\setminus\{0\}$ and $f \in A$, $\varphi(\frac{f_0 + rf}{r}) \in \pm f(X)$. Hence

$$(D\varphi)f_0(f) = \lim_{r \to 0} \frac{\varphi(f_0 + rf) - \varphi(f_0)}{r} \in \pm f(X) \subseteq \pm \sigma(f).$$

Moreover, the $\mathbb{R}$-linear map $(D\varphi)f_0$ satisfies the condition $(D\varphi)f_0(i) = i(D\varphi)f_0(1)$ of Lemma 4.1. For suppose that $(D\varphi)f_0(i) \neq i(D\varphi)f_0(1)$. As before, $\frac{\varphi(f_0 + ri) - \varphi(f_0)}{i} \notin \{1, -1\}$ and $\frac{\varphi(f_0 + ri) - \varphi(f_0)}{i} \in \{1, -1\}$. Hence $(D\varphi)f_0(i) \in \{i, -i\}$ and $(D\varphi)f_0(1) \in \{1, -1\}$. If $(D\varphi)f_0(i) = -i$ and $(D\varphi)f_0(1) = 1$, then for small enough $r$, $\frac{\varphi(f_0 + ri) - \varphi(f_0)}{i} = -i$ and $\varphi(f_0 + ri) - \varphi(f_0) = 1$, hence $\varphi(f_0 + ri) - \varphi(f_0) = -i - 1$. On the other hand, since for each $r$, $T(f_0 + ri) = T_{f_0, f_0 + ri}(f_0 + ri)$ and $T(f_0 + ri) = T_{f_0 + ri, f_0 + ri}(f_0 + ri)$, it follows that for small enough $r$, $-i - 1 = \frac{\varphi(f_0 + ri) - \varphi(f_0)}{i} = (i - 1)\delta_y \circ T_{f_0, f_0 + ri}(1) \in \{i - 1, -1 + i\}$ which is a contradiction. Similarly the other possibility for the values of $(D\varphi)f_0(i)$ and $(D\varphi)f_0(1)$ gives a contradiction. Consequently $(D\varphi)f_0(i) = i(D\varphi)f_0(1)$ and hence by Lemma 4.1, $(D\varphi)f_0$ is $C$-linear. Now applying a similar argument to [10, Theorem 1.2] we can conclude that $\varphi \in \sigma(C)$. Since $\varphi(f) \in \pm \sigma(f)$, $f \in A$, it follows from the Gleason–Kahane–Zelazko theorem that $\frac{1}{\varphi(f)} \varphi \in \sigma(C)$ is multiplicative on $A$. More precisely, for each $y \in Y$, the linear functional $\frac{1}{\varphi(y)} \delta_y \circ T = T(1(y))\delta_y \circ T$ is multiplicative on $A$.

The above argument shows that for every $y \in Y$ there exists an element $\psi(y) \in M_A$, such that

$$Tf(y) = T(1(y))f(\psi(y)) \quad (f \in A).$$

Using the Closed Graph theorem we deduce that $T$ is continuous. Since $T(1(y)) \in \{1, -1\}$, $y \in Y$, the above representation for $T$ implies that $\psi : Y \to M_A$ is continuous and $f \mapsto T(1)f$ is a homomorphism from $A$ into $B$.

If $M_A = X$, i.e., each element in $M_A$ is an evaluation homomorphism at a point of $X$, then since $x \mapsto \delta_x$ is a homeomorphism from $X$ onto $M_A$ it follows that $\psi$ is a continuous map from $Y$ into $X$ and hence $T \in G$. □

Corollary 4.3. Let $A$ and $B$ be Banach function algebras on compact Hausdorff spaces $X$ and $Y$, respectively, and let $T : A \to B$ be a $2$-locally multiplicatively spectrum-preserving map. Then there exist a continuous map $\varphi$ from $M_B$ into $M_A$ and a function $h \in B$ such that

$$\widehat{T}(y) = \widehat{h}(y) \hat{f}(\varphi(y)) \quad (f \in A, \ y \in M_B).$$

In particular, $T$ is a continuous linear operator.

Proof. Since $T$ is $2$-locally multiplicatively spectrum-preserving it follows from [7, Theorem 3.2] that for every $f, g \in A$ there exist a homeomorphism $\varphi_{f,g}$ from $M_B$ onto $M_A$ and an element $h_{f,g} \in B$ whose spectrum is contained in $\{1, 1\}$ such that

$$\widehat{T}(y) = \widehat{h_{f,g}}(y) \hat{f}(\varphi_{f,g}(y)) \text{ and } \widehat{T}(y) = \widehat{h_{g,f}}(y) \hat{g}(\varphi_{f,g}(y)) \text{ for all } y \in M_B.$$ Hence, considering $A$ and $B$ as Banach function algebras on their maximal ideal spaces, Theorem 4.2 implies that $T$ is a continuous linear map and there exists a continuous map $\varphi$ from $M_B$ into $M_A$ such that

$$\widehat{T}(y) = \widehat{h}(y) \hat{f}(\varphi(y)) \quad (f \in A, \ y \in M_B).$$

□

Corollary 4.4. Let $A$ and $B$ be Banach function algebras on compact Hausdorff spaces $X$ and $Y$, respectively. If all points in $c(A)$ and $c(B)$ are strong boundary points and $T : A \to B$ is a $2$-locally multiplicatively range-preserving map, then there exist a continuous map $\varphi$ from $M_B$ into $M_A$ and a function $h \in B$ such that

$$\widehat{T}(y) = \widehat{h}(y) \hat{f}(\varphi(y)) \quad (f \in A, \ y \in M_B).$$

Proof. This immediately follows from the preceding corollary, since by the remark after Lemma 3.11, $T$ is $2$-locally multiplicatively spectrum-preserving. □

References