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Gluino zero-modes for calorons at finite temperature

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ABSTRACT

We study the solutions of the Dirac equation in the adjoint representation(gluinos) in the background field of SU(2) unit charge calorons. Our solutions are forced to be antiperiodic in thermal time and would occur naturally in a semiclassical approach to $\mathcal{N}=1$ supersymmetric Yang–Mills theory at finite temperature.

1. Introduction

In this Letter we derive analytic expressions for the finite temperature gluino zero modes of the Dirac operator in the background field of the $Q=1$ SU(2) calorons. These are self-dual configuration in $R^3 \times S_1$ including the well-known Harrington–Shepard (HS) solution [1] as well as the non-trivial holonomy calorons [2–4]. The periodicity in one direction, to be referred as thermal-time, occurs naturally in a path-integral approach to finite temperature Yang–Mills theory and, with the inclusion of spinor fields in the adjoint representation (gluinos), in its minimal supersymmetric extension. Calorons are thus the natural objects to be considered in a semiclassical approach to these theories at finite temperature. They smoothly interpolate between instantons and BPS monopoles at zero and high temperature respectively [2–5], providing a very interesting link between them. One of the required ingredients for such semiclassical analysis is the knowledge of fermionic zero modes in the background of the caloron field. Although those in the fundamental representation of the gauge group have been known for quite some time [6–10], this is not the case for the gluino zero modes. They have been derived only recently by two of the present authors [11], and just for the case of periodic boundary conditions in S_1. These are the relevant modes for supersymmetric compactifications but not what is needed when studying $\mathcal{N}=1$ SUSY Yang–Mills fields at finite temperature. Antiperiodicity in thermal-time has to be required in that case. The goal of this Letter is to obtain analytic expressions for the antiperiodic solutions, derived here for the first time even for the trivial holonomy, Harrington–Shepard, case. This requires a different approach than the one employed in [11] which was based on the relation between zero modes and self-dual deformations of the gauge field, providing only periodic solutions.

The Letter is organized as follows. In Section 2 we will describe the strategy followed to obtain the antiperiodic zero modes and present the analytic expressions for the solutions. In Section 3 we analyze their properties in several relevant limits, paying particular attention to the one in which the caloron dissociates into a pair of static BPS constituent monopoles. The trivial-holonomy HS zero mode solution and the equal mass constituent monopole cases are also discussed in some detail. Conclusions and a brief summary of results are presented in Section 4.

2. Formalism

As mentioned previously, our goal is that of solving the massless covariant Dirac equation in the adjoint representation of the group

$$\not{D}\Psi = 0 \tag{1}$$

in the background field of a $Q=1$ caloron [2–4]. This problem has been partially addressed in Ref. [11]. The approach that was followed in that Letter was based on the well-known relation between self-dual deformations of the gauge field and the zero-modes of the Dirac operator in the adjoint representation. However, the solutions obtained in this way are periodic in thermal-time with the same period β (to be taken equal to 1 in what

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follows) as the gauge field itself. Thus, a different strategy has to be set up to derive the antiperiodic modes relevant for finite temperature. In what follows we will present the basic idea behind our procedure and the results obtained with it. In all technical aspects we will rely strongly in the notation and derivations done in Ref. [11].

The observation that leads to our solution is the fact that antiperiodic solutions turn out to be periodic in the double period. Thus, the method of attack developed in Ref. [11] for periodic zero-modes can be carried over if the whole problem is seen as living in this duplicated space–time. This *replica trick* has been used by some of the authors in other works [12,13] and is an important source of information when dealing with periodic gauge fields. In our case, the problem becomes that of finding self-dual deformations of the $Q=2$ caloron obtained by the replica procedure. Notice that the topological charge is 2 in this case, so we expect 4 (CP-pairs) of self-dual deformations. Since the gauge field is periodic in the original period, they can be split into those which are periodic and those that are antiperiodic in the original period. The former were studied in our previous paper and correspond to the ordinary deformations of the $Q=1$ caloron. Since there are 2 pairs of those, which are periodic in the *small* torus, we expect to find two pairs of antiperiodic zero-modes. Unfortunately, although some particular solutions are known [14], there is no analytic general expression for the $Q=2$ caloron which would reduce the study of deformations to the differentiation of the general solution with respect to the parameters of the moduli space. In this Letter we will thus follow an alternative strategy. Incidentally our results could well prove useful in achieving the goal of obtaining the most general $Q=2$ caloron solution.

The general formula relating deformations to zero-modes in the adjoint representation is:

$$\Psi = \frac{1}{2}\delta A_\mu \gamma_\mu (\mathbf{I} \pm \gamma_5) V, \tag{2}$$

with the + or − sign depending on whether the solution is self-dual or antiself-dual. V is an arbitrary constant spinor and hence, the zero-modes that we are looking for can be arranged into two-dimensional complex vector spaces. These spaces are generated by any solution Ψ and its euclidean CP transform

$$\Psi \to \Psi^c \equiv \gamma_5 C \Psi^*. \tag{3}$$

Our formula can easily be shown to satisfy the Dirac equation provided the deformation satisfies the background Lorentz gauge condition

$$D_\mu \delta A_\mu = 0. \tag{4}$$

Using the general ADHM construction one can obtain formulas for the self-dual deformations in terms of those for the Nahm-ADHM data. The ADHM construction for SU(2) tells us that a self-dual gauge field can be constructed as [15]

$$A_\mu(x) = \frac{i}{F}\big(u^\dagger \partial_\mu u\big)', \tag{5}$$

where u is a vector in quaternions, $F = 1 + u^\dagger u$, and the prime denotes the traceless part. The vector u is obtained as the solution of the following equation

$$\big(\tilde{A}^\dagger - x_\mu \bar\sigma_\mu\big) u = q, \tag{6}$$

where the quaternionic matrix $\tilde{A}^\dagger$ and the vector q are x-independent. We have introduced the Weyl matrix $\bar\sigma_\mu = (\mathbf{I}, i\vec\tau)$ whose adjoints are σ_μ ($\vec\tau$ are the Pauli matrices). In the proof of self-duality one must demand that the following matrix

$$R \equiv \big(\tilde{A}^\dagger - x_\mu \bar\sigma_\mu\big)(\tilde{A} - x_\mu \sigma_\mu) + q \otimes q^\dagger \tag{7}$$

is real and invertible. For our purpose it is interesting to write down the expression of the adjoint zero-modes in terms of the deformations of the ADHM data

$$\delta A_\mu = \frac{-i}{2}\big(\delta q^\dagger - u^\dagger \delta\tilde{A}\big)\bar\sigma^\mu \sigma^\nu \partial_\nu \omega + \text{h.c.}, \tag{8}$$

where $\omega = R^{-1} q$. To guarantee that the deformations δq and $\delta\tilde{A}$ provide a self-dual deformation δA_μ satisfying the background field gauge condition, one must impose certain conditions. These are best expressed in terms of the matrix with quaternionic entries $\mathcal{F} \equiv M^\dagger \delta M \equiv \mathcal{F}_\mu \sigma_\mu$, where $M^\dagger = (q, \tilde{A}^\dagger - \bar\sigma_\mu x_\mu)$. The condition then reduces to the hermiticity of $\mathcal{F}_\mu$ ($\mathcal{F}_\mu = \mathcal{F}_\mu^\dagger$).

The previous formulas apply for $Q=1$ calorons by extending the vector q to become a delta-like functional over the periodic functions in one-variable z, while $2\pi i\tilde{A}$ is a covariant Weyl operator with respect to a 1-dimensional Abelian gauge field $\hat{A}_\mu(z)$, the Nahm-dual gauge field [16]. After suitable rotations, translations and gauge transformations the caloron Nahm data can be taken to be [2,3]

$$q^{(0)}(z) = \rho\big(P_+\delta(z-\delta_1) + P_-\delta(z+\delta_1)\big), \tag{9}$$

where $P_\pm = (1 \pm \tau_3)/2$. The parameter δ_1 parametrizes the holonomy, becoming trivial for 0 and $\frac{1}{2}$. Without loss of generality we will assume in what follows that $\delta_1 \leqslant \delta_2 \equiv \frac{1}{2} - \delta_1$. In the previous formula the delta functions have to be taken as periodic functions in z with unit period. The Nahm-dual gauge field of the caloron is given by

$$\hat{A}^{(0)}_\mu(z) = -2\pi\delta_{\mu 3}\big(X_3^1 \chi_1(z) + X_3^2 \chi_2(z)\big), \tag{10}$$

where X_3^a is the position of the ath constituent monopole on the z-axis. They can be obtained from the relations $m_1 X_3^1 + m_2 X_3^2 = 0$, and $X_3^2 - X_3^1 = \pi\rho^2$, where $m_a = 4\pi\delta_a$ are proportional to the constituent monopole masses. The function χ_1 is the characteristic function of the interval $[-\delta_1, \delta_1]$ and χ_2 that of its complementary.

Eq. (7) implies that the Nahm-dual gauge field is self-dual at all but a finite number of points. Eq. (6) is then the solution of the Weyl equation except at those isolated points. As we will see later the conditions on the deformations $\delta\tilde{A}$ that enter Eq. (8), are precisely equivalent to requiring that $\delta\hat{A}_\mu$ is again a self-dual deformation satisfying the background gauge condition. Thus, they can be obtained as the solution of the adjoint Weyl equation of the Nahm-dual field, up to delta functions.

Now we should apply this scheme to the replicated caloron taken as a self-dual solution in the double torus with period 2β (remember β is fixed to 1). Since this caloron now has charge $Q=2$ its corresponding Nahm-dual gauge field is now a matrix. Using the general construction of Nahm-dual replicas given in [12] we obtain

$$\hat{A}^R_\mu(z) = \begin{pmatrix} \hat{A}^{(0)}_\mu(z) & 0 \\ 0 & \hat{A}^{(0)}_\mu(z+\frac{1}{2}) \end{pmatrix}, \tag{11}$$

where $\hat{A}^{(0)}_\mu(z)$ is the Nahm data of the ordinary caloron, and $\hat{A}^R_\mu(z)$ is the Nahm data of the replicated caloron.

One may now wonder which is the corresponding q for such a replica solution. We will argue that the solution is actually given by

$$q^R(z) = \begin{pmatrix} q^{(0)}(z) \\ q^{(0)}(z+\frac{1}{2}) \end{pmatrix}. \tag{12}$$

Notice that each of the components of q and $\hat{A}$ are periodic with unit period, but the whole set is periodic with period 1/2 with a twist matrix given by τ_1:

$$\hat{A}^R_\mu\left(z+\frac{1}{2}\right)=\tau_1\hat{A}^R_\mu(z)\tau_1. \qquad (13)$$

The quantity q transforms by periodicity as follows:

$$q^R(z+1/2)=\tau_1 q^R(z). \qquad (14)$$

From here it is possible to use the general formulas of the ADHM construction to verify that indeed we obtain a replicated solution. In particular we have that $u^R(z)$ is given by:

$$u^R(z)=\begin{pmatrix} u^{(0)}(z) \\ u^{(0)}(z+\frac{1}{2})\end{pmatrix}. \qquad (15)$$

Now

$$F^R-1=\int_0^{\frac{1}{2}} dz\, u^{R\dagger}(z)u^R(z)=\int_0^1 dz\, u^{(0)\dagger}(z)u^{(0)}(z), \qquad (16)$$

which coincides with $F-1$ for the caloron. The replicated gauge potential follows from

$$A^R_\mu(x)=\frac{i}{F^R}\int_0^{\frac{1}{2}} dz\left(u^{R\dagger}(z)\partial_\mu u^R(z)\right)'=A^{(0)}_\mu(x). \qquad (17)$$

One might wonder whether the choice of q and $\hat{A}^R_\mu$ are consistent with the condition of self-duality in Nahm-dual space, namely that $R=M^\dagger M$ commutes with the quaternions. To verify that this is so, one must realize that the condition of self-duality should hold only in the domain of these operators. These are two-component vectors $\psi(z)$ satisfying

$$\psi\left(z+\frac{1}{2}\right)=\tau_1\psi(z). \qquad (18)$$

Thus, they should be of the form

$$\psi(z)=\begin{pmatrix}\phi(z)\\ \phi(z+\frac{1}{2})\end{pmatrix}. \qquad (19)$$

Thus, $qq^\dagger$ acting on this vector yields:

$$2\rho\begin{pmatrix} q^{(0)}(z)\\ q^{(0)}(z+\frac{1}{2})\end{pmatrix}\left(P_+\phi(\delta_1)+P_-\phi(-\delta_1)\right)$$
$$=2\rho^2\begin{pmatrix} P_+\phi(\delta_1)\delta(z-\delta_1)+P_-\phi(-\delta_1)\delta(z+\delta_1)\\ P_+\phi(\delta_1)\delta(z+\delta_2)+P_-\phi(-\delta_1)\delta(z-\delta_2)\end{pmatrix}. \qquad (20)$$

The imaginary part of the upper component coincides with

$$\rho^2\tau_3\left(\delta(z-\delta_1)-\delta(z+\delta_1)\right)\phi(z), \qquad (21)$$

which is what is needed to cancel the self-duality violation.

Now we proceed to study the self-dual deformations of this replicated caloron satisfying the background field condition. We will make use of our general formula Eq. (8). The conditions following that equation when translated to our case become

$$\hat{\hat{D}}\psi^R\equiv\frac{d\psi^R}{dz}-i\bar{\sigma}_\mu\left[\hat{A}^R_\mu,\psi^R\right]=4\pi^2 i\left(q^R_\mu\delta q^{\dagger R}_\nu-\delta q^R_\nu q^{\dagger R}_\mu\right)\bar{\sigma}_\mu\sigma_\nu, \qquad (22)$$

where $\psi^R=\delta\hat{A}^R_\mu\sigma_\mu=-\delta\tilde{A}^R/(2\pi)$. The quantities δq^R_ν are two-component column vectors whose elements are linear combinations of delta functions with complex coefficients. The holonomy fixes that the argument of the delta functions must be $z\pm\delta_1$ and $z\pm\delta_1+\frac{1}{2}$. Notice that, as anticipated previously, up to the delta functions in the right-hand side, the equation adopts the form of the Weyl equation for adjoint zero-modes in Nahm dual space.

Our next step will then be that of finding the solution of Eq. (22). Notice that both ψ^R and $\delta q^R\equiv\delta q^R_\nu\bar{\sigma}_\nu$ are the unknowns. Without much effort one can demonstrate that given a solution one can obtain other solutions by the operation $\psi^R\to\psi^R Q$, $\delta q^R\to Q^\dagger\delta q^R$, with Q an arbitrary constant quaternion. This transformation is associated to the double degeneracy of adjoint zero-modes. We must also point out certain subtleties necessary to understand Eq. (22) and their solutions. The main idea is that the equation must be understood as one relating two operators acting on the space of two-component functions of the form Eq. (19). The right-hand side of Eq. (22) acts by multiplication. Thus, the left-hand side must be equivalent, when acting over our space of functions, to the multiplication by a linear combination of delta functions. This imposes non-trivial conditions on the form of δq^R. In what follows we will give the possible values for δq^R that follow from the previous analysis, as well as the resulting form for the equation for ψ^R, skipping all the details of the derivation.

Before showing the equations, we recall that ψ^R is a 2×2 matrix in (Nahm-dual) colour space

$$\psi^R(z)=\begin{pmatrix}\psi_{11}(z) & \psi_{12}(z)\\ \psi_{21}(z) & \psi_{22}(z)\end{pmatrix}. \qquad (23)$$

The boundary conditions specify that it is enough to know the form of ψ_{11} and ψ_{12} (the other components can be obtained by translating in z by 1/2). The equations for ψ_{11} coincide with those for the $Q=1$ caloron, and therefore can be associated with deformations that are periodic in time. Thus, our sought time-antiperiodic zero-modes should follow from the equation

$$\partial_z\psi_{12}+\tau_3(\Delta\hat{A})\psi_{12}=4\pi^2\rho\left\{P_+\left(\delta(z-\delta_1)-\delta(z-\delta_2)\right)\right.$$
$$\left.+P_-\left(\delta(z+\delta_2)-\delta(z+\delta_1)\right)\right\}Q, \qquad (24)$$

where the function $\Delta\hat{A}$ is given by

$$\Delta\hat{A}\equiv\hat{A}^{(0)}_3(z)-\hat{A}^{(0)}_3(z+1/2)$$
$$=2\pi^2\rho^2\left(\chi(-\delta_1,\delta_1)-\chi(\delta_2,1-\delta_2)\right). \qquad (25)$$

The arbitrary quaternion Q reflects the degeneracy of solutions mentioned earlier. Keeping that in mind one only needs to solve the equation for $Q=0$ and $Q=1$. A particular solution is all that is needed, since the general solution can be obtained by linear combinations of these ones with quaternionic coefficients. The counting matches the predictions of the index theorem. As for the periodic case there are essentially two CP-pairs of zero-modes.

After these considerations we proceed to show the two particular solutions that we will need. The first one corresponds to the inhomogeneous equation ($Q=1$) and is given by

$$\psi_{12}=4\pi^2\rho\left(P_+\chi(\delta_1,\delta_2)+P_-\chi(1-\delta_2,1-\delta_1)\right). \qquad (26)$$

The value of δq^R associated to it is

$$\delta q^R=iP_+\begin{pmatrix}\delta(z+\delta_2)\\ \delta(z-\delta_1)\end{pmatrix}-iP_-\begin{pmatrix}\delta(z-\delta_2)\\ \delta(z+\delta_1)\end{pmatrix}. \qquad (27)$$

These expressions can now be introduced into the general formula Eq. (8) to obtain the first solution

$$\delta A^{(1)}_\mu=-\frac{1}{2}\left(P_+\bar{\sigma}_\mu\hat{\partial}\omega(-\delta_2)-P_-\bar{\sigma}_\mu\hat{\partial}\omega(\delta_2)\right)$$
$$-i\pi\rho\left(\int_{\delta_1}^{\delta_2}u^\dagger\left(z+\frac{1}{2}\right)P_-\bar{\sigma}_\mu\hat{\partial}\omega(z)\right.$$
$$\left.+\int_{1-\delta_2}^{1-\delta_1}u^\dagger\left(z+\frac{1}{2}\right)P_+\bar{\sigma}_\mu\hat{\partial}\omega(z)\right)+\text{h.c.} \qquad (28)$$

The quantities u and ω are the ones associated to the $Q=1$ caloron. The analytic expressions needed to do the calculation were explicitly given in our previous paper [11].

Now we investigate the other solution, associated to $\delta q^R=0$. One has to solve the homogeneous equation (24) for vanishing right-hand side. A particular solution is given by

$$\psi_{12}(z)=\exp\left\{-\tau_3\int_0^z dz'\,\Delta\hat{A}(z')\right\}\equiv\phi_s(z)-\tau_3\phi_a(z). \tag{29}$$

Since $\Delta\hat{A}(z')$ is constant at intervals, the integral in the exponent is trivial to perform. We leave the explicit form of $\phi_s(z)$ and $\phi_a(z)$ to the reader. It is interesting to point out nonetheless, that $\phi_s(z)$ is periodic in z with period $\frac{1}{2}$ and $\phi_a(z)$ antiperiodic.

From the previous expression we can compute the corresponding self-dual deformation using Eq. (8). The result is given by

$$\delta A_\mu^{(2)}=\frac{-i}{4\pi}\int_0^1 dz\left(u^\dagger\left(z+\frac{1}{2}\right)\left(\phi_s(z)+\tau_3\phi_a(z)\right)\bar{\sigma}_\mu\hat{\partial}\omega(z)\right)+\text{h.c.} \tag{30}$$

Again, the integration over z can be performed analytically using the formulas of our previous paper [11].

We have arrived to the general solution of our problem. The adjoint zero-modes of the (self-dual) caloron which are antiperiodic in time are

$$\Psi=\frac{1}{2}\delta A_\mu^{(1)}\gamma_\mu(\mathbf{I}+\gamma_5)V_1+\frac{1}{2}\delta A_\mu^{(2)}\gamma_\mu(\mathbf{I}+\gamma_5)V_2, \tag{31}$$

where V_a are arbitrary constant spinors and $\delta A_\mu^{(a)}$ are given in Eqs. (28)–(30).

It is interesting to mention that the general investigation of the possible values of δq^R has led to another solution having a fairly simple form. The expression of the left-handed Weyl spinor, $\Psi^{(3)}\equiv\Psi_a^{(3)}\tau_a$, is:

$$\Psi_a^{(3)}=\sigma_\mu\partial_\mu T^a\sigma_a V, \tag{32}$$

where a labels a colour component, σ_α acts on the spin indices and V denotes an arbitrary constant 2-spinor. The functions T_a depend on the colour index as: $T^1=T^2=-1/F$ and $T^3=P_+\chi+P_-\bar{\chi}$. The function χ is essentially the function with the same name given in Refs. [2,3]. Curiously this solution interpolates between the non-supersymmetric periodic adjoint zero-mode for $m_1=0$ ($\delta_1=0$) and one of our antiperiodic solutions (Eq. (28)) for $m_1=m_2$ ($\delta_1=1/4$). Using the formulas of the next section it can be proven that the solution is neither periodic non-antiperiodic for other values of the mass m_1.

3. Properties of the solutions

In this section we will investigate the general properties of the solutions found in the previous section.

3.1. Periodicity in time

Here we will explicitly verify the required antiperiodicity in time of our general solution. In our gauge the caloron vector potential satisfies

$$A_\mu^{(0)}(x_0+1)=e^{i\frac{m_1\tau_3}{2}}A_\mu^{(0)}(x_0)e^{-i\frac{m_1\tau_3}{2}}. \tag{33}$$

Thus, the required antiperiodicity of the adjoint zero-modes amounts to:

$$\Psi(x_0+1)=-e^{i\frac{m_1\tau_3}{2}}\Psi(x_0)e^{-i\frac{m_1\tau_3}{2}}. \tag{34}$$

This property follows easily from the form of our solutions and the periodicity behaviour of u:

$$u(z,x_0+1)=e^{i2\pi z}u(z,x_0)e^{-i\frac{m_1\tau_3}{2}} \tag{35}$$

and an identical relation for ω and q.

3.2. Far-field limit and normalization

The reader might question whether our general solution Eq. (31) is normalizable. One can investigate the behaviour at points whose distance to the location of the constituent monopoles (r_1 and r_2) is much larger that β and that $\pi\rho^2$. For the unequal mass case the zero-mode density goes to zero exponentially as $e^{-(m_2-m_1)r_2}$. The equal mass case ($m_1=m_2=\pi$) is more subtle since both solutions decay in power-like fashion. The non-homogeneous solution Eq. (28) coincides with the additional solution Eq. (32) in this case. In the limit under consideration χ goes to zero exponentially and $F=(r_1+r_2+\pi\rho^2)/(r_1+r_2-\pi\rho^2)$. Thus the density behaves as $1/r^4$.

An alternative approach to normalizability of the solutions is to compute the norm of the solutions. In fact there exist a general formula [2,19] which allows one to compute the norm and the scalar products of the solutions in terms of Nahm-data directly. This is also useful in checking if the real dimensionality of the space of solutions is 8 (4 complex dimensions, 2 quaternionic dimensions), as indicated by the index theorem. Using this formula we obtain

$$\left|\delta A_\mu^{(1)}\right|^2=4\pi^2+8\pi^4\rho^2(\delta_2-\delta_1), \tag{36}$$

$$\left|\delta A_\mu^{(2)}\right|^2=\frac{\sinh(4\pi^2\rho^2\delta_1)}{2\pi^2\rho^2}+(\delta_2-\delta_1)\cosh\left(4\pi^2\rho^2\delta_1\right), \tag{37}$$

$$\left\langle\delta A_\mu^{(1)},\delta A_\mu^{(2)}\right\rangle=2\pi^2\rho e^{-2\pi^2\rho^2\delta_1}(\delta_2-\delta_1). \tag{38}$$

3.3. Profile of the zero-mode density

In this subsection we will describe the qualitative properties of the zero-mode densities. For that purpose we developed two independent programs to draw these profiles. Both programs give matching results. In Fig. 1 we give the contour plot in a z–y plane of the solution $\delta A_\mu^{(2)}$ (top) and an orthogonal CP-pair (bottom) for $\rho=1$ (giving an intermediate size caloron separation) and two representative values of the masses. The z axis is the line joining the constituent monopoles and is represented horizontally. The vertical axis denotes the y axis (the density is axially symmetric).

For the equal mass case ($m_1=m_2=\pi$) the mode following from Eq. (30) has an approximately constant higher density along the line joining both calorons (top left). This can be interpreted as a string. In contrast, the other solution associated to Eq. (28) has a region of small density located along the line joining the two monopoles (bottom left). As the masses become unequal, the most massive monopole dominates the densities. The right contour plots show the situation for $\delta_1=0.23$.

3.4. Limiting cases

The caloron is an interesting solution which interpolates between the gauge potential of an instanton and that of a BPS monopole. It is interesting then to see how our antiperiodic zero-modes behave in these extreme cases. We will first concentrate in the situation corresponding to the trivial holonomy, Harrington–Shepard, caloron: $\delta_1=0$. In that case one of the constituent monopoles is massless and pushed to infinity. The ρ parameter of the solution does no longer control the separation between the monopoles but is still a free parameter. For small ρ the HS caloron approaches an ordinary, zero temperature, instanton. From

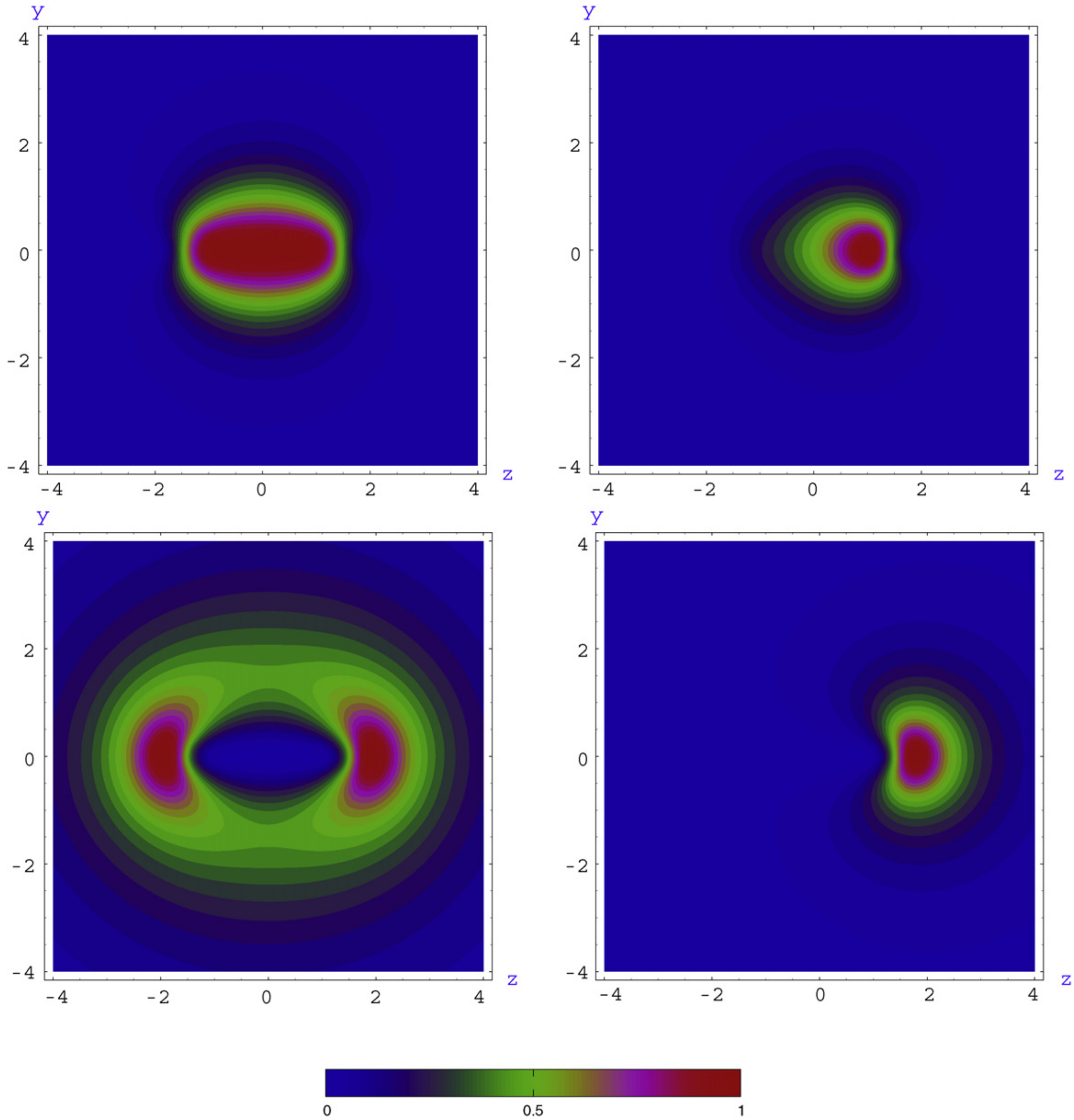

Fig. 1. Contour plots of the density of the two antiperiodic zero modes in the y–z plane. Constituent monopoles are localized at $y = 0$ and separated along the z-axis which is drawn horizontally. Left: For $m_1 = m_2 = \pi$ and $\rho = 1$. Right: For $\delta_1 = 0.23$ and $\rho = 1$.

our general formulas, it is easy to check that in that limit and close to the center of the caloron the time periodicity becomes irrelevant and the two zero mode CP-pairs approach the periodic zero modes of the instanton. In the opposite, $\rho \to \infty$, limit the HS caloron becomes a BPS monopole with time independent action density. Despite the time independence of the background there are still 4 non-trivial antiperiodic zero-modes. They can be easily derived from Eqs. (28) and (30) by taking the appropriate $\delta_1 = 0$ and $\rho \to \infty$ limits. Up to a gauge transformation we obtain:

$$\delta A_\mu^{(1)'}(x) = \eta^\alpha_{3\mu}\pi\rho\big(\bar{e}_1^2(x)E_\alpha^{\rm bps}(r) - \bar{e}_2^2(x)\tilde{E}_\alpha(r)\big) + \text{h.c.}, \tag{39}$$

$$\delta A_\mu^{(2)}(x) = \frac{1}{4\pi}\big(\bar{e}_2^2(x)E_\mu^{\rm bps}(r) + \bar{e}_1^2(x)\tilde{E}_\mu(r)\big) + \text{h.c.}, \tag{40}$$

where $\delta A_\mu^{(2)}$ is directly derived from Eq. (30) and $\delta A_\mu^{(1)'}$ is the combination of Eqs. (28) and (30) orthogonal to $\delta A_\mu^{(2)}$. In the expression above, $E_\alpha^{\rm bps}$ is the electric field of the BPS monopole:

$$E_\alpha^{\rm bps}(x) = -i\frac{g^2(2\pi r) - 1}{2r^2}P_\alpha^+ - i\frac{\pi g'(2\pi r)}{r}P_\alpha^-, \tag{41}$$

and we have introduced the time independent quantity:

$$\tilde{E}_\alpha(x) = \frac{\tanh(\pi r)}{2\cosh(\pi r)}\left(i\frac{g(\pi r) - \cosh(\pi r)}{r^2}P_\alpha^+ - i\frac{\pi g'(\pi r)}{r}P_\alpha^-\right), \tag{42}$$

with $g(u) = u/\sinh(u)$, $g'(u)$ its derivative with respect to u, and $P_\mu^\pm = (\bar{\sigma}_\mu \pm \hat{n}\bar{\sigma}_\mu\hat{n})/2$, $\hat{n} = x_i\tau_i/r$. The antiperiodicity of the solution is encoded in the time dependent quaternions $\bar{e}_1^2$ and $\bar{e}_2^2$ defined through:

$$e^{-i\pi x_\mu\bar{\sigma}_\mu} = i\big(e_1^2(x) + ie_2^2(x)\big). \tag{43}$$

For non-trivial holonomy there is also an interesting limit in which the caloron solution tends to the BPS monopole. It corresponds to making the separation of the constituent monopoles tend to infinity ($\rho \to \infty$). Our adjoint zero-modes lead to those of the BPS monopole if the appropriate limit is taken ($r_1 \ll \pi\rho^2$, $\rho \gg 1$). For example, for the equal mass case ($m_1 = m_2 = \pi$) the first solution Eq. (28) follows quite simply by applying the appropriate limit to Eq. (32). Computing the density we obtain:

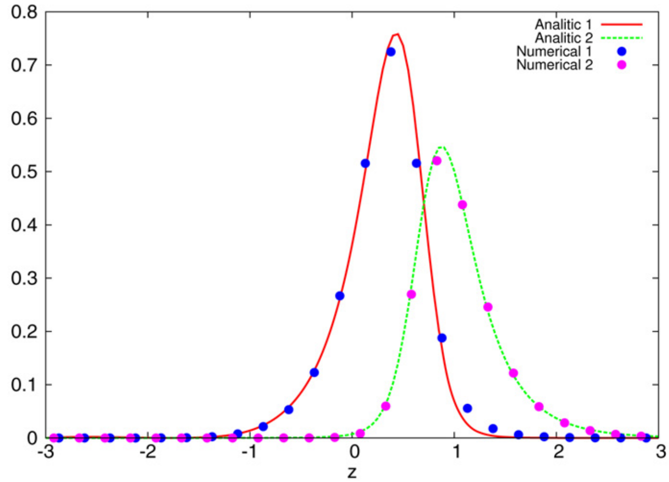

Fig. 2. Comparison between numerical (circles) and analytic (lines) zero modes for $\rho = 0.79$ and $\delta_1 = 0.172$. We display the density of the zero modes, integrated in time, along the line joining the two monopoles.

$$2\big(h'^2(\pi r) + 1 - 2h'(\pi r)\cos\theta\big) + g^2(\pi r) + g'^2(\pi r) + 2g(\pi r)g'(\pi r)\cos\theta, \quad (44)$$

where $h'(u)$ is the derivative of $h(u) \equiv u\coth(u)$. This profile has axial symmetry depending explicitly on the azimuthal angle θ. Notice also that the solution is non-normalizable.

3.5. *Comparison with numerical results*

We have crosschecked our results with a direct evaluation of adjoint zero-modes on the torus obtained by lattice methods using Neuberger's overlap operator [17] in the adjoint representation. One expects that the spatial profile of the torus solutions approaches our analytical formulas as the box size becomes much larger than all scales of the problem (β and $\pi\rho^2$). To make a quantitative comparison we computed the zero-mode density integrated in time along the line $x = y = 0$ joining both constituent monopoles. It is not possible a priori to construct numerical zero-modes with a prescribed value of ρ and δ_1, although some tuning is possible [18]. For the numerical comparison displayed in Fig. 2 we slightly tuned by hand these parameters to improve the agreement ($\rho = 0.79$, $\delta_1 = 0.172$). A technical point which one has to address is how to guarantee that the same linear combinations are selected for the numerical and analytical data. We chose to define the two linearly independent modes by imposing that at the center of mass ($x = y = z = 0$) one has maximal and the other minimal density (integrated over time).

4. Conclusions

In this Letter we have obtained analytic formulas for the zero-modes of the Dirac equation for gluinos in the background field of $Q = 1$ SU(2) calorons with antiperiodic boundary conditions in thermal-time. Our formulas are valid for non-trivial holonomy as well as for the Harrington–Shepard caloron and include as a limiting case those of BPS monopoles. The solutions have finite norm and decay exponentially with distance if the masses of the constituent monopoles differ. Their density profile contrasts with the case of periodic zero-modes. For example, as the monopoles are pulled apart the density does not decouple into independent lumps centered at the monopoles, but rather describes a string joining the monopoles. Nonetheless, the number of normalizable zero-modes matches in both cases.

Our work has methodological interest since our approach is applicable to other cases including the extension to SU(N), and might be instrumental in finding formulas for calorons of higher charge. From a physical viewpoint our work provides a first step towards a semiclassical study of $N = 1$ SUSY Yang–Mills at finite temperature. There are interesting issues at stake such as that of supersymmetry breaking at finite temperature, which has been a subject of debate since early times [20–26]. It is our intention to address these questions in future work.

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