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Von Neumann's inequality for noncommuting contractions

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Abstract

Let $T_1, \dots, T_d$ be linear contractions on a complex Hilbert space and p a complex polynomial in d variables which is a sum of d single variable polynomials. We show that the operator norm of $p(T_1, \dots, T_d)$ is bounded by

$$d \sin\left(\frac{\pi}{2d}\right) \sup_{|z_1|,\dots,|z_d| \leqslant 1} |p(z_1, \dots, z_d)|$$

and that this is the best possible inequality of this type.

AMS classification: 47L30; 15A45

Keywords: Linear contraction; Hilbert space; Von Neumann's inequality; Set theoretic sums

1. Introduction

For $z \in \mathbb{C}$ and $r \geqslant 0$, we denote $D(z, r) = \{w \in \mathbb{C}; |w - z| \leqslant r\}$, $D = D(0, 1)$, $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$, $A(D)$ denotes the disk algebra of functions continuous on D and holomorphic in Δ and similarly, $A(D^d)$ denotes the polydisk algebra for the d-dimensional polydisk.

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The von Neumann inequality [1, Chapter 1, Proposition 8.3] can be stated as follows:

Theorem 1. *Let T be a linear contraction on a complex Hilbert space H and let p be a polynomial of one variable. Then*

$$|||p(T)||| \leqslant \sup_{|z|\leqslant 1} |p(z)|. \tag{1}$$

The quantity $|||S|||$ denotes the operator norm of S. It follows from von Neumann's result that the symbolic calculus of linear contractions on a complex Hilbert space can be extended to the case where $p \in A(D)$ with the inequality (1) continuing to hold.

We investigate the following question. Let $T_1, \ldots, T_d$ be linear contractions on a complex Hilbert space H (which are not assumed to commute). Let f be an analytic function of the form

$$f(z_1, \ldots, z_d) = \sum_{k=1}^{d} f_k(z_k), \quad \tilde{z} \in \Delta^d,$$

where $\tilde{z}$ denotes $(z_1, \ldots, z_d)$ and the f_k are analytic functions of one variable defined in Δ. We assume that f is continuous on the closed polydisk D^d and it then follows easily that the f_k are also continuous on D. Is it true that

$$|||f(T_1, \ldots, T_d)||| \leqslant C \sup_{\tilde{z}\in D^d} |f(z_1, \ldots, z_d)| \tag{2}$$

for some constant C depending only on d? The point of taking only functions of the specified form is that this is the most general function that allows the operator $f(T_1, \ldots, T_d)$ to be defined unambiguously. It is very easy to see that the answer is yes and a proof can be given simply by writing

$$f(z_1, \ldots, z_d) = \sum_{k=1}^{d} f(0, \ldots, 0, z_k, 0, \ldots, 0) - (d-1)f(0, \ldots, 0)$$

and applying the von Neumann inequality. This method gives $C = 2d - 1$ and the only issue is that of finding the best constant.

Theorem 2. *The best possible constant in (2) is $d \sin\left(\frac{\pi}{2d}\right)$.*

2. An example

We show first that one cannot do better than $C = d \sin\left(\frac{\pi}{2d}\right)$. The example that we will use is on a 2-dimensional Hilbert space. Let

$$U_k = \begin{pmatrix} \cos\left(\frac{k\pi}{2d}\right) & \sin\left(\frac{k\pi}{2d}\right) \\ -\sin\left(\frac{k\pi}{2d}\right) & \cos\left(\frac{k\pi}{2d}\right) \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and set

$$T_k = U_k^{\star} J U_k = \begin{pmatrix} \cos\left(\frac{k\pi}{d}\right) & \sin\left(\frac{k\pi}{d}\right) \\ \sin\left(\frac{k\pi}{d}\right) & -\cos\left(\frac{k\pi}{d}\right) \end{pmatrix}.$$

Then since U_k is unitary, it follows that T_k is a normal contraction.

Now, let $\delta > 0$ be small and consider the eye-shaped domain Ω_δ between the circle passing through -1, $\mathrm{i}\delta$ and 1 and the circle passing through -1, $-\mathrm{i}\delta$ and 1. By the Riemann Mapping Theorem, there is an analytic function g mapping the open unit disk onto Ω_δ with $g'(0)$ real and positive and indeed, this function can be computed explicitly and it can be verified that g extends continuously to the closed unit disk and that $g(\pm 1) = \pm 1$. For f we take

$$f(z_1, z_2, \ldots, z_d) = \sum_{k=1}^{d} \omega^k g(z_k),$$

where $\omega = \mathrm{e}^{\frac{\pi i}{d}}$. We claim that $\sup_{\underline{z} \in D^d} |f(z_1, \ldots, z_d)| \leqslant d\delta + \operatorname{cosec}(\frac{\pi}{2d})$. Since $\Omega_\delta \subseteq [-1, 1] + D(0, \delta)$, it suffices to show that

$$\left|\sum_{k=1}^{d} \omega^k \epsilon_k\right| \leqslant \operatorname{cosec}\left(\frac{\pi}{2d}\right)$$

for all choices of $\epsilon_k = \pm 1$. However, the numbers $\omega^k \epsilon_k$ are distinct $2d$th roots of unity, so our conclusion will follow if we can show that

$$\left|\sum_{k \in \Lambda} \omega^k\right| \leqslant \operatorname{cosec}\left(\frac{\pi}{2d}\right)$$

for all subsets Λ of $\{1, 2, \ldots, 2d\}$ with d elements. An elementary rearrangement theorem shows that the sum on the left takes its largest value when the set $\{\omega^k, k \in \Lambda\}$ is a block of d consecutive $2d$th roots of unity and the claim follows from summing a geometric series.

On the other hand, since the eigenvalues of T_k are 1 and -1, we have $g(T_k) = T_k$, so that

$$f(T_1, T_2, \ldots, T_d) = \sum_{k=1}^{d} \omega^k T_k = \frac{1}{2} d\omega \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}.$$

It follows that $|||f(T_1, T_2, \ldots, T_d)||| = d$. Thus the optimal constant in (2) can be no smaller than $d(\delta + \operatorname{cosec}(\frac{\pi}{2d}))^{-1}$ and letting δ tend to zero, we have our result.

3. Operator theoretic issues

To complete the proof of Theorem 1 we establish the following result.

Proposition 3. *Let $T_1, \ldots, T_d$ be linear contractions on a complex Hilbert space. Let $f \in A(D^d)$ be of the form*

$$f(z_1, \ldots, z_d) = \sum_{k=1}^{d} f_k(z_k),$$

where $f_k \in A(D)$. Then $|||f(T_1, \ldots, T_d)||| \leqslant d \sin(\frac{\pi}{2d}) \sup_{\underline{z} \in D^d} |f(z_1, \ldots, z_d)|$.

We break the proof up into steps.

Lemma 4. *It suffices to prove Proposition 3 in the case when $T_1, \ldots, T_d$ are unitary with finite spectrum.*

Proof. We start by showing that the case when $T_1, \ldots, T_d$ unitary will suffice. Towards this, we will show that we may construct *separate* unitary dilations U_k $(k = 1, \ldots, d)$ to the T_k. This involves finding a complex Hilbert space K, an isometric inclusion $J : H \longrightarrow K$ and unitary operators U_k on K such that $T_k^n = J^\star U_k^n J$ for all $n \in \mathbb{Z}^+$ and all $k = 1, \dots, d$.

Achieving this is routine, it suffices to dilate the operators in turn. Suppose that, at the kth stage we have $T_1, \dots, T_{k-1}$ unitary and $T_k, \dots, T_d$ contractions on a space H. Then by the Sz. Nagy dilation theorem [1, Chapter 1, Theorem 4.2], we may dilate the operator T_k to a unitary operator S_k. This involves a complex Hilbert space K, an isometric inclusion $J : H \longrightarrow K$ and the identity $T_k^n = J^\star S_k^n J$ for all $n \in \mathbb{Z}^+$. We set $S_j = JT_jJ^\star + (I - JJ^\star)$ for $j \neq k$. Then it is routine to verify that $T_j^n = J^\star S_j^n J$ for all $n \in \mathbb{Z}^+$ and all $j = 1, \dots, d$, that $S_1, \dots, S_{k-1}$ are unitary and that $S_{k+1}, \dots, S_d$ are contractions.

At this point, we may assume that $T_1, \dots, T_d$ are unitary. We write the spectral resolution of each such unitary as

$$T_k = \int_{\mathbb{T}} z_k \mathrm{d}P_k(z_k),$$

where the spectral measures P_k do not necessarily commute. We have

$$f(T_1, \dots, T_d) = \sum_{k=1}^{d} \int_{\mathbb{T}} f_k(z_k) \mathrm{d}P_k(z_k).$$

Let ξ be a unit vector. Then denoting the probability measure $\Omega \mapsto \langle \xi, P_k(\Omega)\xi \rangle$ by μ_k we have

$$\begin{aligned} \langle \xi, f(T_1, \dots, T_d)\xi \rangle &= \sum_{k=1}^{d} \int_{\mathbb{T}} f_k(z_k) \mathrm{d}\mu_k(z_k) \\ &= \int_{\mathbb{T}^d} f(\tilde{z}) \mathrm{d}(\mu_1 \times \cdots \times \mu_d)(\tilde{z}). \end{aligned}$$

Let $\delta > 0$ and let $\beta : \mathbb{T} \longrightarrow \mathbb{T}$ be a Borel map with finite image which moves points a distance at most δ. For $k = 1, \dots, d$, let $Q_k = \check{\beta}(P_k)$, the image of P_k by β (i.e. $Q_k(\Omega) = P_k(\beta^{-1}(\Omega))$. Further, let

$$R_k = \int_{\mathbb{T}} z_k \mathrm{d}Q_k(z_k)$$

unitaries with finite spectrum. Then denoting

$$\nu_k(\Omega) = \langle \xi, Q_k(\Omega)\xi \rangle = \langle \xi, P_k(\beta^{-1}(\Omega))\xi \rangle = \check{\beta}(\mu_k)(\Omega),$$

we have

$$\langle \xi, f(R_1, \dots, R_d)\xi \rangle = \sum_{k=1}^{d} \int_{\mathbb{T}} f_k(z_k) \mathrm{d}\nu_k(z_k) = \int_{\mathbb{T}^d} f(\tilde{z}) \mathrm{d}(\nu_1 \times \cdots \times \nu_d)(\tilde{z}).$$

This leads to

$$|\langle \xi, (f(T_1, \dots, T_d) - f(R_1, \dots, R_d))\xi \rangle| \leqslant \omega_f(\delta) \|\xi\|^2,$$

where ω_f is the modulus of continuity of f on $\mathbb{T}^d$ for the maximum metric on $\mathbb{T}^d$. Using the standard result that the norm of a Hilbert space operator is bounded by twice its numerical radius, this gives

$$|||f(T_1, \dots, T_d) - f(R_1, \dots, R_d)||| \leqslant 2\omega_f(\delta).$$

Since f is uniformly continuous on $\mathbb{T}^d$, we see that the result for unitaries with finite spectrum implies that for general unitaries. $\square$

The remainder of the proof hinges on the following geometrical proposition which is of some interest in its own right and which we state in greater generality than we need.

Proposition 5. *Let $S_1, \dots, S_d$ be subsets of $\mathbb{C}$ such that $S_1 + S_2 + \cdots + S_d \subseteq D$. Then there exist $w_k \in \mathbb{C}$ and $r_k \geqslant 0$ for $k = 1, \dots, d$ such that $\sum_{k=1}^d w_k = 0$, $S_k \subseteq D(w_k, r_k)$ for each k and $\sum_{k=1}^d r_k \leqslant d\sin(\frac{\pi}{2d})$.*

There is no loss in assuming also that the S_k are compact and convex.

Proof of Proposition 3 given Proposition 5. We can assume without loss of generality that $\sup_{\underline{z}\in D^d} |f(z_1, \dots, z_d)| = 1$. By Lemma 4 for each k with $1 \leqslant k \leqslant d$, there is an integer m_k such that we may write $T_k = \sum_{j=1}^{m_k} z_{k,j} E_{k,j}$ where $E_{k,j}$ are orthogonal projections with $I = \sum_{j=1}^{m_k} E_{k,j}$ for each k and $|z_{k,j}| = 1$. We therefore have for ξ and η arbitrary unit vectors that

$$\langle \eta, f(T_1, \dots, T_d)\xi\rangle = \sum_{k=1}^{d}\sum_{j=1}^{m_k} \langle \eta, E_{k,j}\xi\rangle f_k(z_{k,j}) = \sum_{k=1}^{d}\sum_{j=1}^{m_k} \alpha_{k,j} f_k(z_{k,j})$$

with the notation $\alpha_{k,j} = \langle \eta, E_{k,j}\xi\rangle$. We have $\sum_{j=1}^{m_k} \alpha_{k,j} = \langle \eta, \xi\rangle$ for all $k = 1, \dots, d$ and

$$\begin{aligned}\sum_{j=1}^{m_k} |\alpha_{k,j}| &= \sum_{j=1}^{m_k} |\langle \eta, E_{k,j}\xi\rangle| = \sum_{j=1}^{m_k} |\langle E_{k,j}\eta, E_{k,j}\xi\rangle| \leqslant \sum_{j=1}^{m_k} \|E_{k,j}\eta\|\|E_{k,j}\xi\| \\ &\leqslant \left\{\sum_{j=1}^{m_k} \|E_{k,j}\eta\|^2\right\}^{\frac{1}{2}} \left\{\sum_{j=1}^{m_k} \|E_{k,j}\xi\|^2\right\}^{\frac{1}{2}} = \|\eta\|\|\xi\| = 1.\end{aligned}$$

Let $w_{k,j} = f_k(z_{k,j})$. Then for all mappings $\lambda : \{1, \dots, d\} \longrightarrow \mathbb{N}$, with $1 \leqslant \lambda(k) \leqslant m_k$ we have

$$\left|\sum_{k=1}^{d} w_{k,\lambda(k)}\right| = \left|\sum_{k=1}^{d} f_k(z_{k,\lambda(k)})\right| = |f(z_{1,\lambda(1)}, \dots, z_{d,\lambda(d)})| \leqslant \|f\|_\infty = 1.$$

Thus, denoting $S_k = \{w_{k,j};\ j = 1, \dots, m_k\}$, we have $S_1 + S_2 + \cdots + S_d \subseteq D$. By Proposition 5 there exist w_k summing to zero, with

$$\sum_{k=1}^{d} \max_{1\leqslant j\leqslant m_k} |w_k - w_{k,j}| \leqslant d\sin\left(\frac{\pi}{2d}\right)\|f\|_\infty.$$

Then

$$\begin{aligned}|\langle \eta, f(T_1, \dots, T_d)\xi\rangle| &= \left|\sum_{k=1}^{d}\sum_{j=1}^{m_k} \alpha_{k,j} w_{k,j}\right| = \left|\sum_{k=1}^{d}\sum_{j=1}^{m_k} \alpha_{k,j}(w_{k,j} - w_k)\right| \\ &\leqslant \sum_{k=1}^{d}\sum_{j=1}^{m_k} |\alpha_{k,j}||w_k - w_{k,j}| \leqslant \sum_{k=1}^{d} \max_{1\leqslant j\leqslant m_k} |w_k - w_{k,j}| \\ &\leqslant d\sin\left(\frac{\pi}{2d}\right)\|f\|_\infty.\end{aligned}$$

This completes the proof. □

We remark that in the above proof, it is possible with a little extra work to specialize to the case where $m_k \leqslant 3$ for all k, but not in general to the case $m_k = 2$ for all k.

4. Geometric issues

In the balance of this paper we offer a proof of Proposition 5.

We start by remarking that we may assume without loss of generality that the S_k are finite sets. To see this, we choose $\epsilon > 0$ and select from each S_k an ϵ-net $S_k(\epsilon)$. Applying the result for finite sets implies the existence of $w_k(\epsilon)$ such that $\sum_{k=1}^{d} w_k(\epsilon) = 0$, $S_k(\epsilon) \subseteq D(w_k(\epsilon), r_k(\epsilon))$ and $\sum_{k=1}^{d} r_k(\epsilon) \leqslant d\sin(\frac{\pi}{2d})$. Thus $S_k \subseteq D(w_k(\epsilon), r_k(\epsilon) + \epsilon)$. We may choose a sequence ϵ_m decreasing to zero such that all the $w_k(\epsilon_m)$ and $r_k(\epsilon_m)$ converge. The full result then follows. By perturbing the S_k we may also assume that they are generic sets.

It is also possible to eliminate the case that one or more of the S_k is a singleton by using an induction on d. The case $d = 1$ of the proposition is evident. Suppose now that $d \geqslant 2$ and that (for example) S_d is a singleton. We set $S'_{d-1} = S_{d-1} + S_d$, and use an induction hypthesis to handle the situation

$$S_1 + S_2 + \cdots + S_{d-2} + S'_{d-1} \subseteq D,$$

yielding numbers $w_1, w_2, \ldots, w_{d-2}, w'_{d-1} \in \mathbb{C}$ summing to zero and nonnegative $r_1, r_2, \ldots, r_{d-2}, r_{d-1}$ with $S_k \subseteq D(w_k, r_k)$ for $k = 1, 2, \ldots, d-2$, $S'_{d-1} \subseteq D(w'_{d-1}, r_k)$ and $\sum_{k=1}^{d-1} r_k \leqslant (d-1)\sin(\frac{\pi}{2(d-1)})$. If $S_d = \{w_d\}$, it now suffices to set $r_d = 0$ and $w_{d-1} = w'_{d-1} - w_d$ by virtue of the fact that the bound $d\sin(\frac{\pi}{2d})$ is increasing with d.

Now, enumerating S_k as $(w_{k,j})_{j=1}^{s_k}$, with s_k denoting $\operatorname{card}(S_k)$, consider the problem of finding $(w_k)_{k=1}^{d}$ that minimizes

$$f((w_k)_{k=1}^{d}) \equiv \sum_{k=1}^{d} \max_{1 \leqslant j \leqslant s_k} |w_k - w_{k,j}|$$

subject to the constraint

$$g((w_k)_{k=1}^{d}) \equiv \sum_{k=1}^{d} w_k = 0.$$

It can be shown that f is a convex function on the linear subspace given by $g(w) = 0$. The minimum is then uniquely determined except in truly exceptional cases which are avoided by the assumptions $d \geqslant 2$ and S_k generic. We assume in the sequel that $(w_k)_{k=1}^{d}$ denotes this minimum point. Let Λ_k be the set of j such that the maximum value of $\max_{1 \leqslant j \leqslant s_k} |w_k - w_{k,j}|$ is attained. Then we can replace S_k by $\{w_{k,j};\ j \in \Lambda_k\}$ without any loss. In fact since S_k are generic (we can suppose that no four distinct points of S_k lie on a circle) we can assume that Λ_k and hence that S_k have one, two or three points. Since the case that S_k is a singleton can also be eliminated, we may always assume that $w_k \neq w_{k,j}$ for all $j \in \Lambda_k$.

Ideally, one would like to work with the case $s_k = 2$ for all k. However, there are examples that show that one cannot dispose of the case $s_k = 3$ at least in this formulation of the problem.

Lemma 6. *Let $(w_k)_{k=1}^{d}$ be the minimum point of f described above subject to the constraint $g = 0$. Then there is a complex number w such that*

$$w \in \bigcap_{k=1}^{d} \operatorname{co}\{\operatorname{sgn}(w_k - w_{k,j});\ j = 1, \ldots, s_k\}.$$

In the lemma, we have denoted by $\operatorname{co}(A)$ the convex hull of the set A in $\mathbb{C}$. We have also denoted the sign of the complex number z by $\operatorname{sgn}(z)$. Specifically, the definition is

$$\operatorname{sgn}(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1}z & \text{otherwise.} \end{cases}$$

Proof. Let $u_k \in \mathbb{C}$ satisfy $\sum_{k=1}^{d} u_k = 0$ and consider making a variation $w_k(t) = w_k + tu_k$ for $t \geqslant 0$. We find

$$\sum_{k=1}^{d} \max_{1 \leqslant j \leqslant s_k} |w_k(t) - w_{k,j}| = \sum_{k=1}^{d} r_k + t \sum_{k=1}^{d} \max_{1 \leqslant j \leqslant s_k} \Re(\overline{u_k} \operatorname{sgn}(w_k - w_{k,j})) + \mathrm{O}(t^2),$$

where $r_k = \max_{1 \leqslant j \leqslant s_k} |w_k - w_{k,j}|$. It follows that for all such $(u_k)_{k=1}^{d}$,

$$\sum_{k=1}^{d} \max_{1 \leqslant j \leqslant s_k} \Re(\overline{u_k} \operatorname{sgn}(w_k - w_{k,j})) \geqslant 0$$

or equivalently, there exist $v_k \in \operatorname{co}\{\operatorname{sgn}(w_k - w_{k,j});\ j = 1, \ldots, s_k\}$ such that $\sum_{k=1}^{d} \Re(\overline{u_k} v_k) \geqslant 0$. Let Q be the quotient vector space of $\mathbb{C}^d$ by the relation that identifies two vectors that differ by a constant vector (i.e. one in which all the entries are identical). The dual space of Q is identified to the set of all $(u_k)_{k=1}^{d} \in \mathbb{C}^d$ that satisfy $\sum_{k=1}^{d} u_k = 0$ in a natural way. We now invoke the Separation Theorem for Convex Sets [2] or [3] to see that the image of the product set $P = \prod_{k=1}^{d} \operatorname{co}\{\operatorname{sgn}(w_k - w_{k,j});\ j = 1, \ldots, s_k\}$ by the quotient map taking $\mathbb{C}^d$ to Q contains the zero vector of Q. An equivalent statement is that the set P contains a constant vector. But this is precisely the conclusion of the lemma. □

Using Lemma 6, we see that Proposition 5 can be reformulated as the following Proposition by replacing the original $w_{k,j}$ with $w_{k,j} - w_k$.

Proposition 7. *Let $r_k > 0$ for $k = 1, \ldots, d$ and $w_{k,j} \in \mathbb{C}$ satisfy $|w_{k,j}| = r_k$ for $k = 1, \ldots, d$ and $j \in \Lambda_k$ where Λ_k is either $\{1, 2\}$ or $\{1, 2, 3\}$. Suppose also that*

$$\left| \sum_{k=1}^{d} w_{k,\lambda(k)} \right| \leqslant 1$$

for all mappings $\lambda : \{1, \ldots, d\} \longrightarrow \mathbb{N}$ with $\lambda(k) \in \Lambda_k$ for all k. Suppose further that there is a complex number w such that

$$r_k w \in \operatorname{co}\{w_{k,j};\ j \in \Lambda_k\} \quad \textit{for } k = 1, 2, \ldots, d.$$

Then $\sum_{k=1}^{d} r_k \leqslant d \sin(\frac{\pi}{2d})$.

We now proceed by using the following trick. We may always assume by rotational symmetry that w is real and nonnegative. In case $\operatorname{card}(\Lambda_k) = 3$, the point $r_k w$ lies in a triangle and it follows that at least one side of the triangle meets the nonnegative real axis. Choose such a side. The strategy is to throw away the vertex of the triangle which is not on this side. This leads to a harder problem, but one that involves only intervals.

Proposition 8. *Let* $r_k > 0$ *for* $k = 1, \dots, d$, *let* α_k, β_k *also be given such that*

$$|\alpha_k| \leqslant \beta_k \leqslant \frac{\pi}{2}. \tag{3}$$

Suppose further that

$$\left|\sum_{k=1}^{d} r_k \mathrm{e}^{\mathrm{i}(\alpha_k+\epsilon_k\beta_k)}\right| \leqslant 1$$

for all choices $\epsilon_k = \pm 1$. *Then* $\sum_{k=1}^{d} r_k \leqslant d\sin(\frac{\pi}{2d})$.

We remark that the condition (3) encodes the fact that the line segment from $\mathrm{e}^{\mathrm{i}(\alpha_k-\beta_k)}$ to $\mathrm{e}^{\mathrm{i}(\alpha_k+\beta_k)}$ meets the nonnegative real axis.

Before proving Proposition 8, it may be helpful to describe some of the situations that lead to equality.

- $\beta_k = \frac{\pi}{2}$, $r_k = 2\sin(\frac{\pi}{2d})$ for $k = 1, \dots, d$; $-\frac{\pi}{2} \leqslant \alpha_1 \leqslant -\frac{(d-2)\pi}{2d}$, $\alpha_k = \alpha_{k-1} + \frac{\pi}{d}$, $k = 2, \dots, d$.
- $\alpha_1 = \beta_1 = 0$; $r_k = 2\sin(\frac{\pi}{2d})$ for $k = 1, \dots, d$; $\beta_k = \frac{\pi}{2}$ and $\alpha_k = \frac{(2k-d-2)\pi}{2d}$ for $k = 2, \dots, d$.

Proof. The proof of the proposition is very messy and is split up into steps.
Step 1. The case when $d = 2$.

Suppose without loss of generality that $\alpha_1 \leqslant \alpha_2$. Using the hypotheses explicitly, we find that

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\alpha_2 - \alpha_1 + \beta_1 - \beta_2) \leqslant 1, \tag{4}$$

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\alpha_2 - \alpha_1 - \beta_1 - \beta_2) \leqslant 1, \tag{5}$$

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\alpha_2 - \alpha_1 - \beta_1 + \beta_2) \leqslant 1. \tag{6}$$

We claim that at least one of the cosines is nonnegative. If not, then suppose that $\beta_1 \geqslant \beta_2$, (a similar argument involving (5) and (6) works in case $\beta_2 \geqslant \beta_1$). We must have $\alpha_2 - \alpha_1 + \beta_1 - \beta_2 > \pi/2$ since this quantity is known to be in the interval $[0, 3\pi/2]$ and similarly $\alpha_2 - \alpha_1 - \beta_1 - \beta_2 < -\pi/2$ since this quantity is known to be in the interval $[-\pi, 0]$. Subtracting off, we get $2\beta_1 > \pi$ a contradiction. Thus by either (4) or (5), we must have $r_1^2 + r_2^2 \leqslant 1$ and our result follows from the Cauchy–Schwarz inequality.
Step 2. The case when $4r_k^2 + \sin^2(\beta_k) \leqslant 4$ for all $k = 1, \dots, d$.
We can rewrite $\sum_{k=1}^{d} r_k \mathrm{e}^{\mathrm{i}(\alpha_k+\epsilon_k\beta_k)} = a + \sum_{k=1}^{d} \epsilon_k b_k$ where

$$a = \sum_{k=1}^{d} r_k \mathrm{e}^{\mathrm{i}\alpha_k}\cos(\beta_k) \quad \text{and} \quad b_k = i r_k \mathrm{e}^{\mathrm{i}\alpha_k}\sin(\beta_k).$$

Let $\sin(\gamma_0) = |a|$ and $\sin(\gamma_k) = |b_k| = r_k\sin(\beta_k)$, then we assert that

$$\sum_{k=0}^{d} \gamma_k \leqslant \frac{\pi}{2}.$$

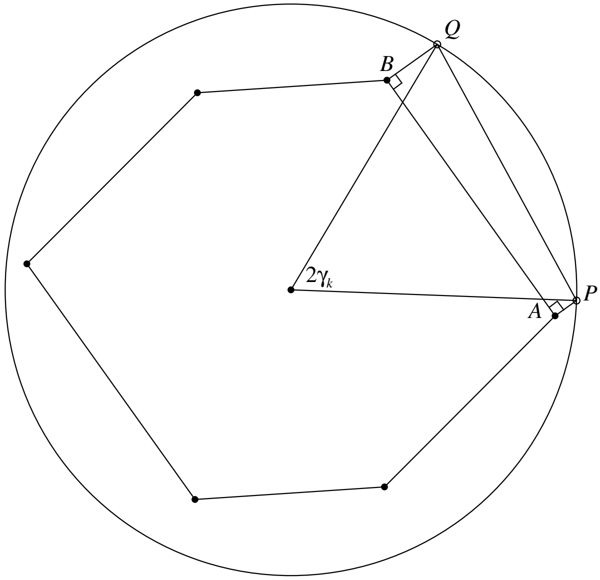

Fig. 1. A side AB of the convex polygon and its associated chord PQ.

This is because the boundary polygon of the convex hull of the points $\{\epsilon_0 a + \sum_{k=1}^d \epsilon_k b_k; \epsilon_k = \pm 1, k = 0, \ldots, d\}$ lies inside the unit disk and has pairs of sides with length $2|a|$ and $2|b_k|$, $k = 1, \ldots, d$. For a given side of the polygon, to each endpoint we consider the intersection of the unit circle with the outward normal to the side at the endpoint. This defines a chord in the unit circle. As illustrated in Fig. 1, the chord PQ is associated to the side AB of the polygon. It is clear that the length of PQ is at least that of AB, that it subtends an angle $2\gamma_k$ at the centre of the unit circle. Chords corresponding to different sides do not meet.

We note that $|a| \geqslant \Re a = \sum_{k=1}^d r_k \cos(\alpha_k)\cos(\beta_k) \geqslant \sum_{k=1}^d r_k \cos^2(\beta_k)$. We deduce that

$$\arcsin\left(\sum_{k=1}^d r_k \cos^2(\beta_k)\right) + \sum_{k=1}^d \arcsin(r_k \sin(\beta_k)) \leqslant \frac{\pi}{2}.$$

Now if t_k are nonnegative numbers with $\sum_{k=1}^d t_k \leqslant 1$, then it can be shown by induction that

$$\sum_{k=1}^d \arcsin(t_k) \leqslant \arcsin\left(\sum_{k=1}^d t_k\right).$$

Therefore, we obtain

$$\sum_{k=1}^d \arcsin(r_k \cos^2(\beta_k)) + \sum_{k=1}^d \arcsin(r_k \sin(\beta_k)) \leqslant \frac{\pi}{2}. \tag{7}$$

The condition $4r_k^2 + \sin^2(\beta_k) \leqslant 4$ ensures that

$$\arcsin(r_k) \leqslant \arcsin(r_k \cos^2(\beta_k)) + \arcsin(r_k \sin(\beta_k))$$

for each $k = 1, \ldots, d$ and hence we obtain $\sum_{k=1}^d \arcsin(r_k) \leqslant \frac{\pi}{2}$. We now get from the fact that sin is concave down on $[0, \pi/2]$ that

$$d^{-1}\sum_{k=1}^{d}\sin(\arcsin(r_k)) \leqslant \sin\left(d^{-1}\sum_{k=1}^{d}\arcsin(r_k)\right) \leqslant \sin\left(\frac{\pi}{2d}\right),$$

as required.

Step 3. Reduction to the case of a single large r_k.

Step 2 establishes the result unless there exists k with $4r_k^2 + \sin^2(\beta_k) > 4$. In this step, we show that there cannot be two such k. Otherwise, we can assume after reordering that $4r_1^2 + \sin^2(\beta_1) > 4$ and $4r_2^2 + \sin^2(\beta_2) > 4$. According to hypothesis, and averaging over $(\epsilon_k)_{k=3}^d$, we have

$$\left|z + r_1 e^{i\alpha_1}\cos(\beta_1) + r_2 e^{i\alpha_2}\cos(\beta_2) + i(\epsilon_1 r_1 e^{i\alpha_1}\sin(\beta_1) + \epsilon_2 r_2 e^{i\alpha_2}\sin(\beta_2))\right| \leqslant 1,$$

where $z = \sum_{k=3}^{d} r_k e^{i\alpha_k}\cos(\beta_k)$. Squaring and averaging over $(\epsilon_k)_{k=1}^2$ yields

$$|z + r_1 e^{i\alpha_1}\cos(\beta_1) + r_2 e^{i\alpha_2}\cos(\beta_2)|^2 + r_1^2\sin^2(\beta_1) + r_2^2\sin^2(\beta_2) \leqslant 1. \tag{8}$$

Now $\Re z \geqslant 0$, so

$$\begin{aligned} &|z + r_1 e^{i\alpha_1}\cos(\beta_1) + r_2 e^{i\alpha_2}\cos(\beta_2)| \\ &\quad \geqslant \Re(z + r_1 e^{i\alpha_1}\cos(\beta_1) + r_2 e^{i\alpha_2}\cos(\beta_2)) \\ &\quad \geqslant r_1\cos(\alpha_1)\cos(\beta_1) + r_2\cos(\alpha_2)\cos(\beta_2) \geqslant r_1\cos^2(\beta_1) + r_2\cos^2(\beta_2). \end{aligned} \tag{9}$$

Thus we obtain, substituting (9) into (8) and throwing away the cross term in the square, that

$$\sum_{k=1}^{2} r_k^2(\cos^4(\beta_k) + \sin^2(\beta_k)) \leqslant 1,$$

leading to

$$\sum_{k=1}^{2}\left(1 - \frac{1}{4}\sin^2(\beta_k)\right)(\cos^4(\beta_k) + \sin^2(\beta_k)) < 1.$$

However, the minimum value of

$$\left(1 - \frac{1}{4}\sin^2(\theta)\right)(\cos^4(\theta) + \sin^2(\theta))$$

as θ runs over $[0, \frac{\pi}{2}]$ is seen to be approximately .6458745. This contradiction establishes the conclusion of this step.

Step 4. The remaining case of a single large r_k.

After relabelling, we can assume that $r_1 > \frac{\sqrt{3}}{2}$ and that $4r_k^2 + \sin^2(\beta_k) \leqslant 4$ for $k = 2, \dots, d$. The idea is that the $(r_k)_{k=1}^d$ are so unbalanced that the situation is far from extremal. Let

$$\arcsin(R_1) = \arcsin(r_1\cos^2(\beta_1)) + \arcsin(r_1\sin(\beta_1)),$$

then starting from (7) and repeating the argument of step 2 with r_1 missing gives

$$\sum_{k=2}^{d} r_k \leqslant (d-1)\sin\left(\frac{\arccos(R_1)}{d-1}\right).$$

Now the equation

$$R_1 = r_1\left(\sin(\beta_1)\sqrt{1 - r_1^2\cos^4(\beta_1)} + \cos^2(\beta_1)\sqrt{1 - r_1^2\sin^2(\beta_1)}\right)$$

leads to $R_1 \geqslant \mu r_1$ where $\mu = \frac{10(3\sqrt{3}-5)}{(3-\sqrt{3})^3} \approx 0.96225$, the smallest value that may be taken by $\sin(\beta_1)\sqrt{1-\cos^4(\beta_1)} + \cos^2(\beta_1)\sqrt{1-\sin^2(\beta_1)}$. Thus, it remains to prove that

$$r_1 + (d-1)\sin\left(\frac{\arccos(\mu r_1)}{d-1}\right) \leqslant d\sin\left(\frac{\pi}{2d}\right) \tag{10}$$

provided that $r_1 > \frac{\sqrt{3}}{2}$ and $d \geqslant 3$. The derivative of the left hand side of (10) with respect to r_1 is

$$1 - \frac{\mu}{\sqrt{1-\mu^2 r_1^2}}\cos\left(\frac{\arccos(\mu r_1)}{d-1}\right)$$

and this quantity is seen to be negative for $r_1 > \frac{\sqrt{3}}{2}$ and $d \geqslant 2$. Thus it remains to show that

$$\frac{\sqrt{3}}{2} + (d-1)\sin\left(\frac{\arccos(\mu\sqrt{3}/2)}{d-1}\right) \leqslant d\sin\left(\frac{\pi}{2d}\right)$$

for $d \geqslant 3$. This is easy to verify for large values of d using the bound

$$|u^{-1}\sin(tu) - t| \leqslant t^3u^2/6 \quad \text{for } 0 \leqslant tu \leqslant \pi/2$$

using the pairs $(u,t) = (d^{-1}, \pi/2)$ and $(u,t) = ((d-1)^{-1}, \arccos(\mu\sqrt{3}/2))$ and for smaller values of $d \geqslant 3$ numerically. It fails for $d = 2$ which explains the necessity of providing a separate proof of this case. □

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