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On the self-dual maximal Cohen–Macaulay modules

Akira Ooishi

Department of Mathematics, Faculty of Education, Yokohama National University, Yokohama 240, Japan

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Abstract

We study the duality for maximal Cohen–Macaulay modules (MCM modules for short) over Cohen–Macaulay local rings. We characterize (low dimensional) rings over which any MCM module is self-dual, and establish a correspondence between the isomorphism classes of a class of self-dual MCM modules (called "orientable" *Auslander modules*) and the even linkage classes of Gorenstein ideals of height two over Gorenstein normal domains. An application is given to the complete intersection ideals of height two.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen–Macaulay local ring with the canonical module K_R. A d-dimensional Cohen–Macaulay R-module is called a maximal Cohen–Macaulay R-module (an MCM R-module for short). For a given MCM R-module M, the R-module $M^* = \mathrm{Hom}_R(M, K_R)$ is called the (canonical) *dual* of M. Then M^* is also an MCM R-module, and the duality $M^{**} \cong M$ holds (see Proposition 1.1). We say that M is *self-dual* if M is isomorphic to its dual. The aim of this paper is to study some fundamental questions on self-dual MCM modules.

First, in Section 1, we examine elementary properties of the dual of an MCM module. In Section 2, we consider the problem of characterizing the local rings over which any MCM module is self-dual, especially in the cases of low dimensional local rings. In Section 3, we introduce and examine a class of self-dual MCM modules which we call *Auslander modules*. These are self-dual MCM R-modules obtained as extensions of Gorenstein ideals of height two by the canonical module of R. Special cases of these modules are studied by many authors in various contexts. For Gorenstein normal domains, we establish a correspondence (Rao correspondence) between the set of isomorphism classes of "orientable" Auslander modules and the set of even linkage classes of Gorenstein ideals of height two (this is essentially done in [4, 8]). As an application, we show that for an isolated hypersurface singularity of dimension

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greater than five, any Gorenstein ideal of height two is a complete intersection. This gives a slight generalization of a famous theorem of Serre.

0.1. Notations and terminology

Throughout this paper, $(R, \mathfrak{m}, k)$ stands for a Cohen–Macaulay local ring with $\dim(R) = d$. We denote by $E_R(k)$ the injective envelope of the R-module k. We denote by $\ell(M)$ and $\mu(M)$ the length and the minimal number of generators of an R-module M, respectively. For an MCM R-module M, we denote by $e(M)$ and $r(M)$, the multiplicity and the Cohen–Macaulay type $\dim_k \mathrm{Ext}^d(k, M)$ of M, respectively. We say that R is a *hypersurface* if $\mathrm{emb}(R) = d + 1$, i.e., $R\hat{\ } \cong S/(f)$ with a regular local ring S and a non-zero element f of S.

1. The dual of an MCM module

Let M be an MCM R-module. A finitely generated R-module N is said to be a (*canonical*) *dual* of M if its completion $N\hat{\ }$ is isomorphic to $\mathrm{Hom}_R(H^d_{\mathfrak{m}}(M), E_R(k))$ as an $R\hat{\ }$-module. A dual of M does not necessarily exist, but if it exists it is unique up to isomorphisms. (This follows, for example, from [9, p. 48, Lemma 5.8].) Hence we often say that N is *the dual* of M, and we denote it by K_M. The dual K_R of R is called the *canonical module* of R, and it is well-known that it exists if and only if R is a residue ring of a Gorenstein ring (theorem of Foxby and Reiten). We recall the following well-known facts for completeness.

Proposition 1.1. (1) *Assume that the canonical module K_R of R exists. Then $M^* := \mathrm{Hom}_R(M, K_R)$ is a dual of M.*

(2) *If N is the dual of M, then N is also an MCM R-module and M is the dual of N.*

Proof. We may assume that R is complete. (1) follows from the isomorphisms

$$\begin{aligned}\mathrm{Hom}_R(H^d_{\mathfrak{m}}(M), E_R(k)) &\cong \mathrm{Hom}_R(M \otimes H^d_{\mathfrak{m}}(R), E_R(k))\\ &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R H^d_{\mathfrak{m}}(R), E_R(k)))\\ &\cong \mathrm{Hom}_R(M, K_R).\end{aligned}$$

(2) It is enough to show that M^* is an MCM R-module and $M^{**} \cong M$. This is easy and well known (see [8]). $\square$

If $d = 0$, then $K_R \cong E_R(k)$ and M^* is the *Matlis dual* of M. If R is a Gorenstein local ring, then the dual of M is isomorphic to the usual R-dual $\mathrm{Hom}_R(M, R)$ of M. More generally, if R is a finite extension of a Gorenstein local ring S, then $K_R \cong \mathrm{Hom}_S(R, S)$ and $M^* = \mathrm{Hom}_R(M, K_R) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(R, S)) \cong \mathrm{Hom}_S(M, S)$ as R-modules.

In the rest of this section, for simplicity, we assume the existence of the canonical module K_R of R.

Proposition 1.2. *For any $\mathfrak{m}$-primary ideal I of R and any parameter ideal J contained in I, we have*

$$\ell(M^*/IM^*) = \ell((JM:I)_M/JM) = \ell(\mathrm{Ext}_R^d(R/I, M)).$$

Proof. This follows from the isomorphisms

$$\begin{aligned} M^*/IM^* &\cong M^*/JM^* \otimes_{R/J} R/I \cong (M/JM)^* \otimes_{R/J} R/I \\ &\cong \mathrm{Hom}_{R/J}(M/JM, E_{R/J}(k)) \otimes_{R/J} R/I \\ &\cong \mathrm{Hom}_{R/J}(\mathrm{Hom}_{R/J}(R/I, M/JM), E_{R/J}(k)), \end{aligned}$$

$$\mathrm{Hom}_R(R/I, M/JM) \cong (JM:I)_M/JM \cong \mathrm{Ext}_R^d(R/I, M)$$

and the Matlis duality. □

Corollary 1.3. *We have $e(M^*) = e(M)$, $\mu(M^*) = r(M)$ and $r(M^*) = \mu(M)$. If R is a hypersurface, then $\mu(M) = r(M)$.*

Proof. We may assume that k is an infinite field. Take a minimal reduction I of $\mathfrak{m}$. Then by the Matlis duality,

$$e(M^*) = \ell(M^*/IM^*) = \ell((M/IM)^*) = \ell(M/IM) = e(M),$$

and $\mu(M^*) = \ell(M^*/\mathfrak{m}M^*) = \ell(\mathrm{Ext}_R^d(R/\mathfrak{m}, M)) = r(M)$. By the duality $M^{**} \cong M$, we get $r(M^*) = \mu(M)$. Finally, the last assertion follows from [9, Lemma 1.5]. □

In general, we have the inequalities $\mu(M) \le e(M)$ and $r(M) \le e(M)$. The equality $e(M) = \mu(M)$ holds if and only if $e(M) = r(M)$, and in this case M is called a *Ulrich* R-module (cf. [16, Proposition 1.1])

Corollary 1.4 *M is a Ulrich R-module if and only if M^* is a Ulrich R-module.*

Corollary 1.5 (cf. Kirby [10, Theorems 4.3 and 4.4]). *For any $\mathfrak{m}$-primary ideal I of R, the function*

$$f(n) = \ell(\mathrm{Ext}_R^d(R/I^{n+1}, M))$$

is a polynomial function.

Proof. By Proposition 1.2, we have $f(n) = \ell(M^*/I^{n+1}M^*)$, and this is the Hilbert–Samuel function of M^* with respect to I. □

We say that an MCM R-module M is *self-dual* if it is isomorphic to its dual.

Example 1.6. (1) Let $R \subset S$ be a finite extension of Cohen–Macaulay local rings. If S is Gorenstein, then S is a self-dual MCM R-module. In fact, we may assume that R is complete, and then we have $\mathrm{Hom}_R(S, K_R) \cong K_S \cong S$ as R-modules. For example, if a finite group G acts on a Gorenstein local ring S and the order of G is a unit in S, then the invariant subring $R = S^G$ is Cohen–Macaulay and S is a self-dual MCM R-module. The converse does not hold in general. But, if S is contained in the total quotient ring of R, then S is Gorenstein if and only if S is a self-dual MCM R-module, since two torsion-free S-modules are isomorphic if and only if they are isomorphic as R-modules.

(2) Let $R = \mathbb{C}[[X, Y, Z]]/(f)$ be a normal hypersurface singularity. Then Martsinkovsky shows that the module of derivations $\Theta_R = \mathrm{Der}_{\mathbb{C}}(R)$ is a self-dual MCM R-module (cf. [13, Corollary 5.2]).

2. Rings over which the self-duality property holds

In this section, we investigate the problem of characterizing Cohen–Macaulay local rings over which any MCM module is self-dual, especially in the cases of low dimensional local rings. Note that such rings are necessarily Gorenstein rings.

Proposition 2.1. *Let R be an artinian Gorenstein local ring and I an ideal of R. Then I is self-dual if and only if I is a principal ideal.*

Proof. If I is self-dual, then by Corollary 1.4, we have $\mu(I) = \mu(I^*) = r(I) \le r(R) = 1$. Hence I is a principal ideal. Conversely, assume that I is principal ideal, and put $J = \mathrm{ann}(I)$. From the exact sequence $0 \to J \to R \to I \to 0$, we get the exact sequence $0 \to I^* \to R \to J^* \to 0$. Since $J^* \cong (R/I)^{**} \cong R/I$, we have $I^* \cong \mathrm{Ker}(R \to R/I) \cong I$. $\square$

Theorem 2.2 (cf. Ooishi [14, Theorem 1.7]). *Let R be an artinian local ring. Then the following conditions are equivalent:*

(1) *Any finitely generated R-module M is self-dual.*

(2) *R is Gorenstein and $\mathfrak{m}^* \cong \mathfrak{m}$.*

(3) *m is a principal ideal.*

(4) *$R \cong S/(\pi^r)$ for some complete discrete valuation ring and some integer $r \ge 1$, where π is a prime element of S.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) follows from Proposition 2.1. The equivalence of (3) and (4) follows from Cohen's structure theorem for complete local rings.

(4) $\Rightarrow$ (1): It is easy to see that $\mathrm{ann}(\mathfrak{m}^i) = \mathfrak{m}^{r-i}$, $0 \le i \le r-1$. Since M is a direct sum of cyclic R-modules, we may assume that $M \cong R/\mathfrak{m}^s$ for some integer $s \ge 0$. Then, we have

$$M^* \cong \mathrm{ann}(\mathfrak{m}^s) = \mathfrak{m}^{r-s} \cong R/\mathrm{ann}(\mathfrak{m}^{r-s}) = R/\mathfrak{m}^s \cong M.$$

Proposition 2.3. *Let R be a one-dimensional Gorenstein local ring and I an $\mathfrak{m}$-primary ideal of R such that $e(I) = \ell(I/I^2) = 2\ell(R/I)$. Then I is self-dual.*

Proof. Put $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, $R(I) = \bigoplus_{n\geq 0} I^n$ and $S = \bigcup_{n\geq 1}(I^n : I^n)$. Then, under our assumption, $G(I)$ is Gorenstein by [15], which implies that $\operatorname{Proj}(G(I))$ is Gorenstein. Hence $\operatorname{Spec}(S) \cong \operatorname{Proj}(R(I))$ is Gorenstein, i.e., S is a Gorenstein ring. Therefore $I = IS \cong S$ is self-dual. $\square$

Corollary 2.4 (Ooishi [15]). *Let R be a one-dimensional Cohen–Macaulay local ring.*

(1) *If* $\operatorname{emb}(R) = 2$, *then $\mathfrak{m}^n$ is self-dual for any $n \geq e(R) - 1$.*

(2) *Assume that R is Gorenstein and* $\operatorname{emb}(R) = e(R) - 1$. *Then $\mathfrak{m}^n$ is self-dual for any $n \geq 2$.*

Proposition 2.5. *Let R be a one-dimensional Cohen–Macaulay local ring with infinite residue field. Then $\mathfrak{m}^{-1} \cong \mathfrak{m}$ if and only if* $\operatorname{emb}(R) = e(R)$.

Proof. "If": We may assume that R is not a discrete valuation ring. Take an element x of $\mathfrak{m}$ such that $x\mathfrak{m} = \mathfrak{m}^2$. Then $\mathfrak{m} \subset (xR:\mathfrak{m}) \subsetneq R$. Hence $\mathfrak{m} = (xR:\mathfrak{m}) = x\mathfrak{m}^{-1} \cong \mathfrak{m}^*$.

"Only if": Take an element x such that $x\mathfrak{m}^{-1} = \mathfrak{m}$. Then $R \supset \mathfrak{m}^{-1}\mathfrak{m} = x^{-1}\mathfrak{m}^2 \supset \mathfrak{m}$. Hence $R = \mathfrak{m}^{-1}\mathfrak{m}$ or $x^{-1}\mathfrak{m}^2 = \mathfrak{m}$. Therefore R is a discrete valuation ring or $x\mathfrak{m} = \mathfrak{m}^2$. In any case, we have $\operatorname{emb}(R) = e(R)$. $\square$

Theorem 2.6. *Let R be a one-dimensional Cohen–Macaulay local ring with infinite residue field which is not a discrete valuation ring. Then the following conditions are equivalent:*

(1) *R is Gorenstein and $\mathfrak{m}^* \cong \mathfrak{m}$.*

(2) $\operatorname{emb}(R) = 2$ *and $\mathfrak{m}^* \cong \mathfrak{m}$.*

(3) $e(R) = 2$.

Moreover, if R is an analytically unramified local domain, these conditions are also equivalent to the following condition:

(4) *Any finitely generated torsion-free R-module is self-dual.*

Proof. For a one-dimensional Cohen–Macaulay local ring which is not a discrete valuation ring, the following facts are well known: in general, $\operatorname{emb}(R) \leq e(R)$; if $\operatorname{emb}(R) = 2$, then R is Gorenstein; $e(R) = 2$ if and only if R is Gorenstein and $\operatorname{emb}(R) = e(R)$. Hence the equivalence of (1)–(3) follows from Proposition 2.5.

(4) $\Rightarrow$ (1): Since R is self-dual, R is Gorenstein.

(3) $\Rightarrow$ (4): Let I be an $\mathfrak{m}$-primary ideal and put $S = (I:I)$. Then S is a Gorenstein semilocal ring (cf. [11]). After completion, $S \cong S_1 \times \cdots \times S_r$ with S_i one-dimensional Gorenstein local rings. Since $S_i^* = \operatorname{Hom}_R(S_i, R)$ is the canonical module of S_i, we have $S_i^* \cong S_i$. Hence $S^* \cong S$. Since by [2] (see also [18]) $I \cong S$, I is self-dual. Since any

finitely generated torsion-free R-module is a direct sum of (m-primary) ideals (cf. [2, 18]), it is self-dual. $\square$

Theorem 2.7. *Let R be a two-dimensional complex analytic normal local domain which is not a regular local ring. Then any reflexive R-module is self-dual if and only if R is isomorphic to one of the following singularities*:

(A_1) $x^2 + y^2 + z^2 = 0$,

(D_n) $x^2y + y^{n-1} + z^2 = 0$,

where n is even and $n \geq 4$,

(E_7) $x^3 + xy^3 + z^2 = 0$,

(E_8) $x^3 + y^5 + z^2 = 0$.

Proof. Since $K_R \cong K_R^* \cong R$, R is Gorenstein. Let I be any divisorial ideal of R. Then, by the assumption, I is isomorphic to $I^* \cong \mathrm{Hom}(I, R) \cong I^{-1}$. Hence any non-zero element of the ideal class group $\mathrm{Cl}(R)$ of R has order two. Therefore R is a rational double point by [17, Section 6, Satz 1], and $\mathrm{Cl}(R)$ is isomorphic to a finite direct sum of $\mathbb{Z}/2\mathbb{Z}$. Thus our assertion follows from the well-known list of the rational double points and their class groups (cf. [12; 19, p. 123]).

The converse follows from the well-known classification of MCM modules over the rational double points. Namely, we only have to look the Dynkin diagrams consisting of indecomposable MCM R-modules and to consider the existence of irreducible homomorphisms from the already known self-dual MCM R-modules successively starting from R or divisorial ideals (which are self-dual by the assumption), cf. [19, pp. 95–96]. (This proof was pointed out to the author by Professor Y. Yoshino and Professor K.-i. Watanabe.) $\square$

The author would like to thank Professor S. Goto who communicated the author the following:

Theorem 2.8. *If any MCM R-module is self-dual, then R is a hypersurface.*

Proof. Let M and N be the dth and $(d+1)$th syzygy module of the R-module k, respectively, and let $0 \to N \to F_d \to M \to 0$ be an exact sequence, where F_d is a free R-module. Then its dual sequence $0 \to M^* \to F_d^* \to N^* \to 0$ is exact, and by the assumption, this is isomorphic to the exact sequence $0 \to M \to F_d \to N \to 0$. Hence M is $(d+2)$th syzygy module of k. Continuing this process to higher syzygies, we see that the Betti number $\beta_i(k)$ is constant for any $i \geq d$. Therefore by a theorem of Tate, R is a hypersurface [19, Lemma 8.18]. $\square$

Remark. It is a natural speculation that a local ring of positive dimension satisfying the property of Theorem 2.8 is either a regular local ring or a quadratic hypersurface.

This is certainly true for dimension one and two by the results of this section. But the author could not verify this speculation in general.

3. Auslander modules and Gorenstein ideals of height two

In this section, we assume that $\dim(R) \geq 2$ and R has a canonical module K. Here we introduce a special class of self-dual MCM R-modules which can be obtained as extensions of Gorenstein ideals of height two by K. For Gorenstein normal domains, we establish a correspondence between the set of isomorphism classes of these modules which are "orientable" and the set of even linkage classes of Gorenstein ideals of height two.

Recall that an ideal I of R is called a *Gorenstein ideal* if the residue ring R/I is Gorenstein. We say that an R-module M is an *Auslander R-module* if it is a self-dual MCM R-module with $e(M) = 2e(R)$. Note that if R is an integral domain, the latter condition is equivalent to the condition that $\operatorname{rank}(M) = 2$.

Proposition 3.1 (Brennen et al. [4, Corollary 4.4]). *Let I be a Gorenstein ideal of R with $\operatorname{ht}(I) = 2$. Then there exists an Auslander R-module $A(I)$ which is characterized by the following non-split exact sequence*:

$$0 \to K \to A(I) \to I \to 0.$$

Proposition 3.2. *For any Gorenstein ideal I of height two, we have $\mu(I) \leq 2e(R)$ and $\mu(A(I)) \leq \mu(I) + r(R)$. If $\mu(I) = 2e(R)$, then $A(I)$ is a Ulrich R-module.*

Proof. This follows immediately from the exact sequence in Proposition 3.1. □

Corollary 3.3. *Let I be a Gorenstein ideal of a normal hypersurface singularity with* $\operatorname{ht}(I) = 2$ and $\mu(I) = 3$. *Then we have $\mu(A(I)) = 4$.*

Proof. By Proposition 3.2, $\mu(A(I)) = 3$ or 4. On the other hand, under our assumption, $\mu(A(I))$ is even by [9, Theorem 3.1]. Hence $\mu(A(I)) = 4$. □

We call the R-module $A(I)$ the Auslander module of I. If $\dim(R) = 2$, the Auslander module $A_R = A(\mathfrak{m})$ of $\mathfrak{m}$ is called the Auslander module (or *the fundamental module*) of R [1, 3, 7, 13, 19, 20].

Remark. (1) Let R be a two-dimensional normal hypersurface. Then $\mu(A_R) = 4$ by Corollary 3.3.

(2) Let R be a two-dimensional complete normal local domain with $R/\mathfrak{m} \cong \mathbb{C}$. Then, by [20], A_R is decomposable if and only if R is a cyclic quotient singularity.

(3) Let R be a complex normal surface singularity. If R is quasi-homogeneous, then it is well known that $A_R \cong \mathrm{Hom}_R(\mathrm{Hom}_R(\Omega_R^1, R), R)$, the module of Zariski differentials of R. (Hence $A_R \cong \Theta_R$ if R is a hypersurface.) Martsinkovsky [13] conjectures that the converse is also true, and this conjecture is verified for some classes of singularities [3, 13].

Assume that R is normal. For a finitely generated R-module M of rank r, we define the *determinant* $\det(M)$ of M by $\det(M) = \mathrm{Hom}_R(\mathrm{Hom}_R(\wedge^r M, R), R)$. We denote the divisor class of a divisorial ideal J of R by $\mathrm{cl}(J)$. We say that M is *orientable* if $\det(M) \cong R$ [12].

Lemma 3.4. (Yoshino [19, Lemma 1.2]). *If R is normal, then $\det(A(I)) \cong K$ for any Gorenstein ideal I of R with $\mathrm{ht}(I) = 2$. In particular, if R is Gorenstein, $A(I)$ is orientable.*

Proof. From the exact sequence $0 \to K \to A(I) \to I \to 0$, we get $\mathrm{cl}(\det(A(I))) = \mathrm{cl}(\det(K)) + \mathrm{cl}(\det(I)) = \mathrm{cl}(K)$. Hence $\det(A(I)) \cong K$. □

Proposition 3.5. *For a Gorenstein ideal of height two, the following conditions are equivalent*:

(1) *$A(I)$ is a free module.*

(2) *I is a complete intersection, i.e., I is generated by a regular sequence.*

Moreover, if R is Gorenstein and normal, these conditions are also equivalent to the following condition:

(3) *$A(I)$ has a free direct summand.*

Proof. (1) ⇒ (2): If $A(I) \cong R^2$, then $2 = \mathrm{ht}(I) \le \mu(I) \le \mu(A) = 2$. Hence $\mu(I) = \mathrm{ht}(I)$ and I is a complete intersection.

(2) ⇒ (1): If I is generated by a regular sequence x, y, then the Koszul complex associated to x, y gives a non-split exact sequence $0 \to R \to R^2 \to I \to 0$. Hence $A(I) \cong R^2$.

(3) ⇒ (1): Assume that $A(I) \cong R \oplus J$, where J is a divisorial ideal of R. Then, by Lemma 3.4, $\mathrm{cl}(J) = \mathrm{cl}(\det(A(I))) = \mathrm{cl}(R) = 0$. Hence $J \cong R$ and $A(I)$ is a free R-module. □

Proposition 3.6. *Suppose that R is a Gorenstein normal domain. Then any orientable Auslander R-module A is isomorphic to $A(I)$ for some Gorenstein ideal I of height two.*

Proof. By [8, Proposition 1.8(b)], there exists an exact sequence $0 \to F \to A \to I \to 0$, where F is free and I is a Cohen–Macaulay ideal of height two or $I = R$. If $I = R$, A is free and the assertion is clear.

Otherwise, $F \cong R$ and we get the exact sequence

$$R \simeq \mathrm{Hom}(F, R) \to \mathrm{Ext}^1(I, R) \to \mathrm{Ext}^1(A, R) = 0.$$

Hence $K_{R/I} \cong \mathrm{Ext}^2(R/I, R) \cong \mathrm{Ext}^1(I, R)$ is a cyclic and R/I is Gorenstein.

Proposition 3.7. *Suppose that R is Gorenstein normal local domain. Then two Auslander R-modules are stably isomorphic if and only if they are isomorphic.*

Proof. Assume that A and A' are Auslander R-modules and $A \oplus R^m \cong A' \oplus R^n$ with $m \geq n$. Then $A \oplus R^{m-n} \cong A'$ by cancellation [16, Proposition 1]. If $m = n$, then $A \cong A'$. Assume that $m > n$. Then A' has a free direct summand and A' is free by Proposition 3.5. Hence A is also free and we get $A \cong A'$. □

Theorem 3.8 (Rao correspondence). *Suppose that R is Gorenstein and normal. Then there is a one-to-one correspondence between the set of isomorphism classes of orientable Auslander R-modules and the set of even linkage classes of Gorenstein ideals of height two.*

Proof. This follows from [8, Corollary 2.3], Propositions 3.6 and 3.7. □

For a finitely generated R-module M, the *non-free locus* $\mathrm{Nf}(M)$ of M is defined by

$$\mathrm{Nf}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not a free } R_{\mathfrak{p}}\text{-module}\}$$

Theorem 3.9 (Bruns [5]). *Assume that R is a hypersurface which is an integral domain. Then for any MCM R-module M, we have*

$$\mathrm{codim}(\mathrm{Nf}(M)) \leq 2\,\mathrm{rank}(M) + 1.$$

Proof. By [9, Lemma 1.1], M is isomorphic to its second syzygy module. Hence our assertion follows from [5, Corollary 2]. □

Corollary 3.10. *Let R be an isolated hypersurface singularity. Then any MCM R-module with* $\mathrm{rank}(M) < (d-1)/2$ is free.

Proof. Suppose that M is not free. Then by Theorem 3.9, we have $d = \mathrm{codim}(\mathrm{Nf}(M)) \leq 2\,\mathrm{rank}(M) + 1$. This contradicts our hypothesis. □

As is well known, by Serre, any Gorenstein ideal of height two of a regular local ring is a complete intersection. Using our previous results, we are able to show a slight generalization of this theorem:

Theorem 3.11. *Let R be a hypersurface singularity which is regular in codimension* ≤ 5 *(e.g., an isolated hypersurface singularity with* $\dim(R) \geq 6$*). Then any Gorenstein ideal of height two of R is a complete intersection.*

Proof. Let I be a Gorenstein ideal of height two of R. If R is not a complete intersection, then $M := A(I)$ does not have a free direct summand by Proposition 3.5. Hence by Theorem 3.9, we get $6 \leq \operatorname{codim}(\operatorname{Nf}(M)) \leq 2\operatorname{rank}(M) + 1 = 5$, which is a contradiction. (The authors would like to thank Professor K. Kurano for comments on Theorem 3.11. In particular, he showed by an example that the same conclusion in Theorem 3.11 does not necessarily hold for local rings with dimension less than or equal to five.) □

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