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On the one-sided crossing minimization in a bipartite graph with large degrees

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Abstract

Given a bipartite graph $G = (V, W, E)$, a 2-layered drawing consists of placing nodes in the first node set V on a straight line L_1 and placing nodes in the second node set W on a parallel line L_2. For a given ordering of nodes in W on L_2, the one-sided crossing minimization problem asks to find an ordering of nodes in V on L_1 so that the number of arc crossings is minimized. A well-known lower bound LB on the minimum number of crossings is obtained by summing up $\min\{c_{uv}, c_{vu}\}$ over all node pairs $u, v \in V$, where c_{uv} denotes the number of crossings generated by arcs incident to u and v when u precedes v in an ordering. In this paper, we prove that there always exists a solution whose crossing number is at most $(1.2964 + 12/(\delta - 4))LB$ if the minimum degree δ of a node in V is at least 5.

Keywords: Approximation algorithm; Bipartite graph; Edge crossing; Graph drawing; 2-layered drawing; Randomized algorithm

1. Introduction

Given a bipartite graph $G = (V, W, E)$, a 2-layered drawing consists of placing nodes in the first node set V on a straight line L_1 and placing nodes in the second node set W on a parallel line L_2. The problem of minimizing the number of crossings between arcs

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in a 2-layered drawing was first introduced by Harary and Schwenk [6,7]. The one-sided crossing minimization problem asks to find an ordering of nodes in V to be placed on L_1 so that the number of arc crossings is minimized (while the ordering of the nodes in W on L_2 is given and fixed). Applications of the problem can be found in VLSI layouts [14] and hierarchical drawings [1].

However, the two-sided and one-sided problems are shown to be NP-hard by Garey and Johnson [5] and by Eades and Wormald [4], respectively. Muñoz et al. [11] have proven that the one-sided problem remains to be NP-hard even for sparse graphs such as forests of 4-stars. Dujmović and Whitesides [3] have given an $O(\phi^k \cdot n^2)$ time algorithm to the one-sided problem, where k is the number of crossings to be checked, $n = |V| + |W|$ and $\phi = (1 + \sqrt{5})/2$, thus showing that the problem is Fixed Parameter Tractable. Recently Dujmović et al. [2] gave an $O(1.4656^k + k|V|^2)$ time algorithm for this problem.

There are several heuristics that deliver theoretically or empirically good solutions. The so-called barycenter heuristic finds an $O(\sqrt{n})$-approximation solution or a $(\Delta - 1)$-approximation solution, where Δ is the maximum degree of nodes in the free side V (see [9] for the analysis). Eades and Wormald [4] proposed a simple and theoretically better heuristic, the median heuristic which delivers a 3-approximation solution. They have also proved that the performance guarantee of the median heuristic approaches to 1 as the density $|E|/(|V||W|)$ of G becomes 1. Yamaguchi and Sugimoto [16] gave a 2-approximation algorithm if $\Delta \leqslant 4$. All these algorithms are *key based heuristics*, which determine an ordering of V with respect to some key values $\kappa(u)$, $u \in V$, and the performance guarantees of these heuristics are based on a conventional lower bound LB that is obtained by summing up $\min\{c_{uv}, c_{vu}\}$ over all node pairs $u, v \in V$, where c_{uv} denotes the number of crossings generated by arcs incident to u and v when u precedes v in an ordering. An extensive computational experiment of several heuristics has been conducted by Jünger and Mutzel [8] and by Mäkinen [10]. Jünger and Mutzel [8] reported that most of the heuristics gave good solutions whose crossing numbers are nearly equal to the lower bound. Recently Nagamochi [12,13] has proposed a randomized key based heuristic, and has proved that there always exists a solution whose crossing number is at most $1.4664LB$.

In this paper, we analyze the performance of the randomized key based heuristic [12,13] in terms of the minimum degree δ of nodes in V, and by designing an appropriate probabilistic distribution for the heuristic, we prove that there always exists a solution whose crossing number is at most $(1.2964 + 12/(\delta - 4))LB$ if $\delta \geqslant 5$. Note that the performance guarantee approaches to 1.2964 as the minimum degree δ becomes large (even if graphs remain sparse).

The paper is organized as follows. In Section 2, we introduce basic definitions on 2-layered drawing and a geometric representation for crossing numbers c_{uv} and c_{vu} for two nodes $u, v \in V$. In Section 3, we review the probabilistic algorithm for determining a 2-layered drawing and some basic properties for analyzing the algorithm. In Section 4, we show that the algorithm can deliver a solution whose crossing number is at most $(1.2964+12/(\delta-4))$ times of the lower bound. In Section 5, we, however, show that our approach cannot prove that the gap between the optimal and the lower bound is less than 1.2698. In Section 6, we describe some concluding remarks.

2. Preliminaries

Let $G = (V, W, E)$ be a bipartite graph with a partition V and W of a node set. Assume that G has no isolated node. Let π denote a permutation of $\{1, 2, \ldots, |V|\}$ and σ denote a permutation of $\{1, 2, \ldots, |W|\}$. A pair of π and σ defines a *2-layered drawing* of G in the plane in such a way that, for two parallel horizontal lines L_1 and L_2, the nodes in V (resp., in W) are arranged on L_1 (resp., L_2) according to π (resp., σ) and each arc is depicted by a straight line segment joining the end-nodes, where the directions for traversing L_1 and L_2 are taken as the same one (see Fig. 1a). For any choice of coordinates of points for nodes in $V \cup W$ in a 2-layered drawing of G defined by (π, σ), two arcs $(v, w), (v', w') \in E$ intersect properly (or create a *crossing*) if and only if $(\pi(v) - \pi(v'))(\sigma(w) - \sigma(w'))$ is negative. So we simply call a pair (π, σ) a 2-layered drawing of G. In this paper, we consider the following problem.

One-sided crossing minimization: Given a bipartite graph $G = (V, W, E)$ and a permutation σ on W, find a permutation π on V that minimizes the number of crossings in a 2-layered drawing (π, σ) of G.

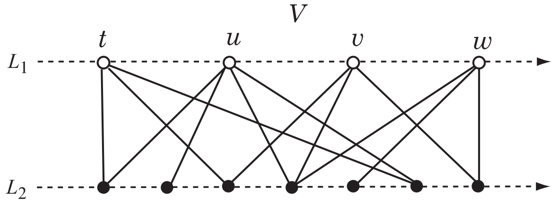
Since the permutation σ on $W = \{1, 2, \ldots, |W|\}$ is fixed, we assume throughout the paper that $\sigma(i) = i$ for all $i \in W$. For each node u in G, let $\Gamma(u)$ denote the set of nodes adjacent to u, and let $d_u = |\Gamma(u)|$. For two nodes $u, v \in V$, let $\gamma_{uv} = |\Gamma(u) \cap \Gamma(v)|$. The *crossing number* c_{uv} for an ordered pair of two nodes $u, v \in V$ is the number of crossing generated by an arc incident to u and an arc incident to v when $\pi(u) < \pi(v)$ holds in a 2-layered drawing (π, σ). (Fig. 1b shows the crossing numbers in the graph in Fig. 1a.) Let δ denote the minimum degree of nodes in V. It is a simple matter to see that for two nodes $u, v \in V$,

$$d_u d_v = c_{uv} + c_{vu} + \gamma_{uv},$$

$$\min\{c_{uv}, c_{vu}\} \geqslant \frac{\gamma_{uv}(\gamma_{uv} - 1)}{2}.$$

For a permutation π on V, let

$$cross(u, v; \pi) := \begin{cases} c_{uv} & \text{if } \pi(u) < \pi(v), \\ c_{vu} & \text{otherwise.} \end{cases}$$

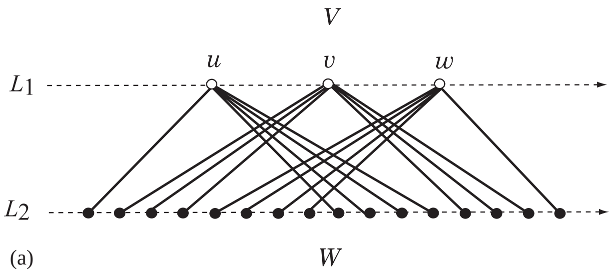


(a)

	t	u	v	w
t	-	5	2	2
u	5	-	3	2
v	6	8	-	2
w	7	9	5	-

(b)

Fig. 1. (a) A 2-layered drawing of a bipartite graph. (b) Crossing numbers for each pair of nodes in the top layer.



	u	v	w
u	-	12	16
v	18	-	12
w	9	18	-

(b)

Fig. 2. (a) A 2-layered drawing of a bipartite graph. (b) Crossing numbers for each pair of nodes in the top layer.

Define

$$cross(\pi) := \sum_{u,v \in V: \pi(u)<\pi(v)} c_{uv} = \sum_{u,v \in V} cross(u, v; \pi).$$

The optimal to the problem is denoted by $opt = \min\{cross(\pi) \mid \text{permutation } \pi \text{ on } V\}$. For $LB = \sum_{u,v \in V} \min\{c_{uv}, c_{vu}\}$, it holds

$$opt \geqslant LB.$$

In this paper, we prove the next results.

Theorem 1. *For a bipartite graph $G = (V, W, E)$ with $\delta \geqslant 5$ and a given permutation σ on W, there exists a permutation π on V such that $cross(\pi) \leqslant (1.2964 + 12/(\delta - 4))LB$.*

Theorem 2. *For a bipartite graph $G = (V, W, E)$ such that $d_w = 1$, $w \in W$ and a given permutation σ on W, there exists a permutation π on V such that $cross(\pi) \leqslant 1.2964LB$.*

Fig. 2 shows an example such that $opt = 39$ and $LB = 33$. Hence the maximum ratio LB/opt over all bipartite graphs is at least $13/11 \simeq 1.1818$.

We here review a geometric representation [12,13] that illustrates how two sets $\Gamma(u)$ and $\Gamma(v)$ determine crossing numbers c_{uv} and c_{vu} in a bipartite graph G. Rectangles that we treat here are axis-parallel in the *xy*-coordinate, and they are denoted by the coordinates of the lower-left corner and the upper-right corner, where the *x*-coordinate increases in the right direction and the *y*-coordinate increases in the upward direction. For example, $[(0, 0), (0.5, 1)]$ represents the square with four corners $(0, 0)$, $(0, 1)$, $(0.5, 0)$ and $(0.5, 1)$.

Let S denote a unit square $[(0, 0), (1, 1)]$. For a connected region R in S, we may use R to denote the sets of points in the region R, and let $a(R)$ denote the area size of R. For two points $b, b' \in S$, a line segment connecting b and b' is denoted by bb'. A part of the boundary of a region R may be called an *edge* if it is a line segment. For a line segment (or an edge) e, its length is denoted by $\ell(e)$. We say that edge e *overlaps* with another edge e' if the intersection of e and e' is a line segment of a positive length.

For two integers $d, d' \geqslant 1$, the square $S = [(0, 0), (1, 1)]$ is called (d, d')-*sliced* if it is sliced by $(d - 1)$ horizontal line segments and $(d' - 1)$ vertical segments so that these line segments give rise to $d \times d'$ congruent rectangles (see Fig. 3). Each of such rectangles is called a *block*, which has four edges.

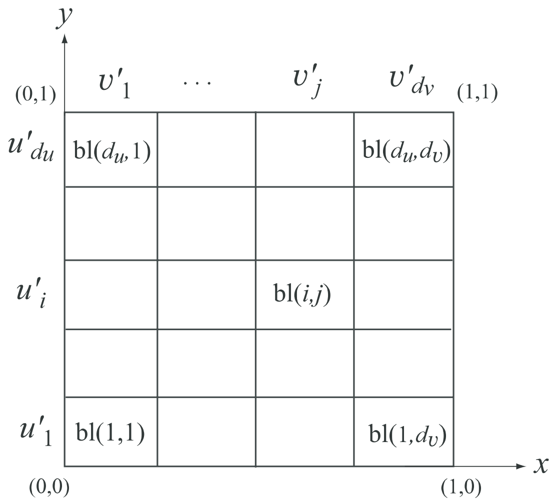


Fig. 3. Illustration for blocks in a (d_u, d_v)-sliced square S.

We represent the positions of nodes in $\Gamma(u)$ and $\Gamma(v)$ in the permutation σ by using the unit square S in the *xy*-coordinate. Let $\Gamma(u) = \{u'_1, u'_2, \ldots, u'_{d_u}\}$ and $\Gamma(v) = \{v'_1, v'_2, \ldots, v'_{d_v}\}$. For an ordered pair (u, v) of nodes in V, we consider $d_u d_v$ blocks in the (d_u, d_v)-sliced square S. We denote these blocks by

$$bl(i, j) = \left[\left(\frac{j-1}{d_v}, \frac{i-1}{d_u}\right), \left(\frac{j}{d_v}, \frac{i}{d_u}\right)\right], \ 1 \leqslant i \leqslant d_u \text{ and } 1 \leqslant j \leqslant d_v$$

(see Fig. 3). We let $bl(i, j)$ correspond to a pair of arcs (u, u'_i) and (v, v'_j). Note that arcs (u, u'_i) and (v, v'_j) create a crossing in a permutation π with $\pi(u) < \pi(v)$ or $\pi(u) > \pi(v)$ if $u'_i \neq v'_j$, but generate no crossing in any permutation π otherwise. We call a block $bl(i, j)$ with $u'_i \neq v'_j$ an *up-block* if arcs (u, u'_i) and (v, v'_j) creates a crossing in a permutation π with $\pi(u) < \pi(v)$ and an *down-block* otherwise. We call a block $bl(i, j)$ with $u'_i = v'_j$ a *neutral-block*. Observe that the number of up-blocks (resp., down-blocks and neutral-blocks) is equal to c_{uv} (resp., c_{vu} and $\gamma_{uv} = \gamma_{vu}$). We here partition the set of these blocks into two groups *UP* and *DWN* as follows (where a neutral-block may be split into two half blocks in the partitioning).

Definition 1. For each node $u \in V$, where $\Gamma(u) = \{w_1, w_2, \ldots, w_{d_u}\} \subseteq \{1, 2, \ldots, |W|\}$ $(w_1 < w_2 < \cdots < w_{d_u})$, we define the *median index* $\mu(u)$ of its neighbors by

$$\mu(u) := \begin{cases} w_{\frac{d_u+1}{2}} & \text{if } d_u \text{ is odd,} \\ \frac{1}{2}\left(w_{\frac{d_u}{2}} + w_{\frac{d_u}{2}+1}\right) & \text{if } d_u \text{ is even.} \end{cases}$$

(i) If $\mu(u) < \mu(v)$, then let *UP* be the set of all up-blocks, and *DWN* be the set of down-blocks and neutral-blocks (see Fig. 4).

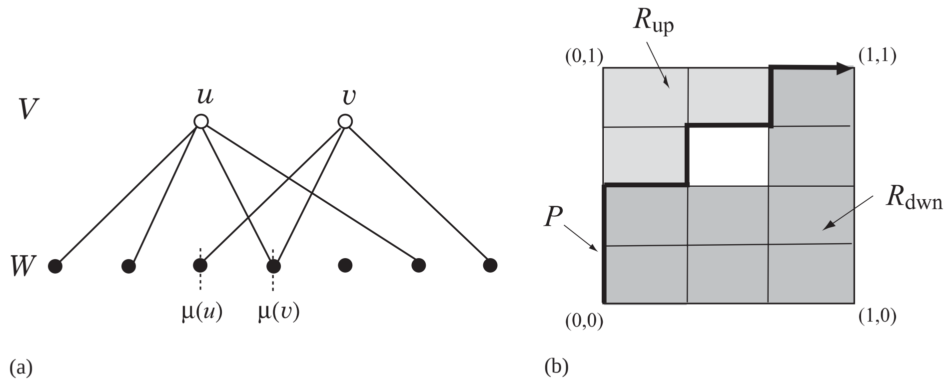


Fig. 4. (a) Two nodes *u* and *v* in the top layer, where $c_{uv} = 3$ and $c_{vu} = 8$. (b) A (u, v)-path *P* of a (4, 3)-sliced square *S* in the case of (i).

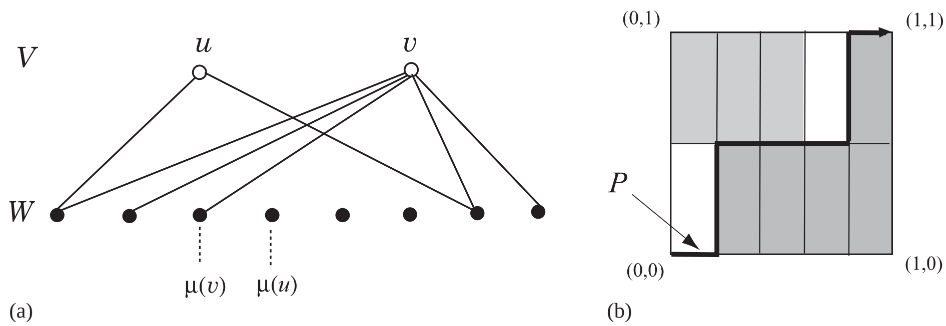


Fig. 5. (a) Two nodes *u* and *v* in the top layer. (b) A (u, v)-path *P* of a $(2, 5)$-sliced square *S* in the case of (ii).

(ii) If $\mu(u) > \mu(v)$, then let *UP* be the set of all up-blocks and neutral-blocks, and *DWN* be the set of down-blocks (see Fig. 5).
(iii) If $\mu(u) = \mu(v)$, then split each neutral-block $[p, q]$ into two parts by the line segment *pq*, and put the upper-left part into *UP* and the other in *DWN*. Then put all up-blocks in the *UP*, and all down-blocks in the *DWN* (see Fig. 6).

The set of all points in the blocks in *UP* forms a connected region, which we denoted by R_{up}. Similarly R_{dwn} is defined by *DWN*.

A path *P* between points $(0, 0)$ and $(1, 1)$ in *S* is called *monotone* if none of the *x*- and *y*-coordinates of the point on *P* decreases when we traverse points on *P* from $(0, 0)$ to $(1, 1)$. (In general a monotone path is not necessarily piecewise linear.) From Definition 1, we easily observe the next property.

Lemma 1 (*Nagamochi [12,13]*). *Let R_{up} and R_{dwn} be the regions defined for an ordered pair of nodes u and v in V. Then there is a monotone path P that separates S into R_{up} and*

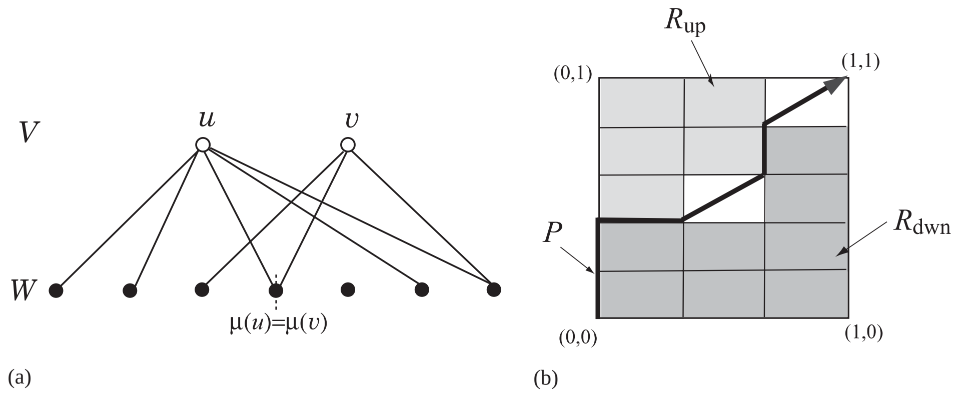


Fig. 6. (a) Two nodes u and v in the top layer. (b) A (u, v)-path P of a (5, 3)-sliced square S in the case of (iii).

R_{dwn}, and it holds

$$a(R_{up}) = \begin{cases} \dfrac{c_{uv}}{d_u d_v} & \textit{if } \mu(u) < \mu(v), \\ \dfrac{c_{uv} + \frac{\gamma_{uv}}{2}}{d_u d_v} & \textit{if } \mu(u) = \mu(v), \\ \dfrac{c_{uv} + \gamma_{uv}}{d_u d_v} & \textit{if } \mu(u) > \mu(v). \end{cases}$$

Moreover, R_{up} contains point $(0.5, 0.5)$ if $\mu(u) \geqslant \mu(v)$.

Such a path P in the lemma is called the (u, v)*-path* with respect to G and σ.

Lemma 2. *For two node $u, v \in V$ such that $d_u, d_v \geqslant 3$, $\mu(u) \geqslant \mu(v)$ and $c_{uv} \neq c_{vu}$, it holds $0 < a(R_{up})d_u d_v - \gamma_{uv} \leqslant c_{uv}$.*

Proof. By Lemma 1, it holds $a(R_{up}) \leqslant (c_{uv} + \gamma_{uv})/(d_u d_v)$, from which we have $d_u d_v a(R_{up}) - \gamma_{uv} \leqslant c_{uv}$. Thus, it suffices to show that $d_u d_v a(R_{up}) - \gamma_{uv} > 0$. Again by Lemma 1, R_{up} contains point $(0.5, 0.5)$, implying that $a(R_{up}) \geqslant 1/4$. Obviously $\gamma_{uv} \leqslant \min\{d_u, d_v\}$. Note that $d_u = d_v = \gamma_{uv}$ cannot occur since otherwise $c_{uv} = c_{vu}$ would hold. Hence $\max\{d_u, d_v\} \geqslant \gamma_{uv} + 1$. Therefore, $d_u d_v a(R_{up}) \leqslant \gamma_{uv}$ can hold only when $\max\{d_u, d_v\} = 4$, $\gamma_{uv} = \min\{d_u, d_v\} = 3$ and $a(R_{up}) = 1/4$. However, this is impossible since $a(R_{up}) \geqslant 1/3$ if $\max\{d_u, d_v\} = 4$ and $\min\{d_u, d_v\} = 3$. $\square$

We close this section by reviewing some technical lemmas.

Lemma 3 (*Nagamochi [12,13]*). *For constants $a > 0$, $b, c > 0$ and d such that $ad - bc \geqslant 0$, function $f(x) = (ax + b)(1/(cx + d) - 2)$ takes the maximum $(\sqrt{a} - \sqrt{2(ad - bc)})^2/c$ over x with $cx + d > 0$.*

Lemma 4 (*Nagamochi [12,13]*). *For four positive constants a, b, c and d with $b/a < d \leqslant 1/\sqrt{2c}$, function $f(x) = (ax - b)^2(1/(cx^2) - 2)$ $(b/a < x \leqslant d)$ takes the maximum at* $x = \min\{d, (b/(2ac))^{\frac{1}{3}}\}$.

3. Randomized key based heuristic

In this section, we review a randomized key based heuristic [12,13]. Let $\theta : V \to (0, 1]$ be a function from V to the set of reals in $(0, 1]$, where $\theta(u)$ is called the *real key* of node u. Given a real-key function θ, we construct a permutation π_θ of $\{1, 2, \ldots, |V|\}$ by the next procedure.

PERMUTE(θ; π_θ):

Step 1. For each node $u \in V$, compute $j = \lceil \theta(u) d_u \rceil$, and define an *integer key* $\kappa(u)$ of u by

$$\kappa(u) := w_j \text{ for the } j\text{th neighbor } w_j \in \Gamma(u),$$

where $\Gamma(u) = \{w_1, w_2, \ldots, w_{d_u}\}$ $(w_1 < w_2 < \cdots < w_{d_u})$.

Step 2. Sort nodes $u \in V$ in the lexicographical order with respect to $(\kappa(u), \mu(u))$, where the ties among nodes u with the same key $(\kappa(u), \mu(u))$ are broken randomly. We denote by π_θ the resulting permutation of $\{1, 2, \ldots, |V|\}$.

We easily observe the following property.

Lemma 5 (*Nagamochi [12,13]*). *For two nodes $u, v \in V$, let R_{up} and R_{dwn} be the regions in Definition* 1. *Then for a given real-key function θ, $\pi_\theta(u) < \pi_\theta(v)$ if point $(\theta(u), \theta(v))$ is inside R_{dwn} and $\pi_\theta(u) > \pi_\theta(v)$ if point $(\theta(u), \theta(v))$ is inside R_{up}.*

A *scheme* based on which we choose a real-key function θ probabilistically is defined by a set of tuples of reals $\mathcal{S} = \{(s_i, t_i, p_i) \mid i = 1, 2, \ldots, h\}$, such that $0 < s_i \leqslant t_i < 1$ and $0 \leqslant p_i$ for $i = 1, 2, \ldots, h$ and $\sum_{1 \leqslant i \leqslant h} p_i = 1$, where we call each (s_i, t_i, p_i) a *subscheme*. Given a scheme $\mathcal{S}$, we choose a real-key function θ in the following manner.

RANDOM-KEY($\mathcal{S}$; θ):

Step 1. Choose a subscheme $(s_i, t_i, p_i) \in \mathcal{S}$ with probability p_i.

Step 2. For each node $u \in V$, choose a real key $\theta(u)$ from $(s_i, t_i]$ uniformly.

We denote by $E_{\mathcal{S}}[cross(u, v; \pi_\theta)]$ and $E_{\mathcal{S}}[cross(\pi_\theta)]$ respectively, the expectations of $cross(u, v; \pi_\theta)$ and $cross(\pi_\theta)$ over all real-key functions θ resulting from RANDOM-KEY. In this paper, we prove the next result.

Theorem 4. *For a bipartite graph $G = (V, W, E)$ with $\delta \geqslant 3$ and a permutation σ on W, there is a scheme $\mathcal{S}$ such that*

$$E_{\mathcal{S}}[cross(\pi_\theta)] \leqslant \left(1.2964 + \max_{u,v \in V}\left\{\frac{12\gamma_{uv}}{d_u d_v - 4\gamma_{uv}}\right\}\right) LB.$$

Theorem 4 implies Theorem 2 since $\gamma_{uv} = 0$, $u, v \in V$ if $d_w = 1$, $w \in W$. Also by noting that $12\gamma_{uv}/(d_u d_v - 4\gamma_{uv}) = 12/(d_u d_v/\gamma_{uv} - 4) \leqslant 12/(\delta - 4)$ if $\gamma_{uv} \neq 0$, we see that Theorem 1 follows from Theorem 4. As observed in [13], algorithm PERMUTE with keys generated by RANDOM-KEY can be derandamized, and a permutation π of V with the bounds stated in Theorems 4 and 2 can be constructed by a deterministic polynomial time algorithm.

By the linearity of expectations, if we have a constant $\alpha \geqslant 1$ such that

$$E_{\mathcal{S}}[cross(u, v; \pi_\theta)] \leqslant \alpha \min\{c_{uv}, c_{vu}\}, \quad u, v \in V,$$

then it holds $E_{\mathcal{S}}[cross(\pi_\theta)] \leqslant \alpha LB$.

In the rest of this paper, we fix two nodes $u, v \in V$, and analyze $E_{\mathcal{S}}[cross(u, v; \pi_\theta)]$ for a given scheme $\mathcal{S}$. Without loss of generality we assume that $c_{uv} \neq c_{vu}$ (the case of $c_{uv} = c_{vu}$ needs no special consideration to prove Theorem 4). Moreover, we can assume that $\min\{c_{uv}, c_{vu}\} \geqslant 1$ since otherwise (i.e., $\min\{c_{uv}, c_{vu}\} = 0$) $\pi_\theta(u) < \pi_\theta(v)$ holds in any permutation π_θ computed by PERMUTE due to the comparison of $\mu(u)$ and $\mu(v)$.

For a given scheme $\mathcal{S}$ and a region $R \subseteq S$, let $p_{\mathcal{S}}(R)$ denote the probability that point $(\theta(u), \theta(v))$ falls inside R. By Lemma 5, we observe the next formula.

Lemma 6 (*Nagamochi [12,13]*). $E_{\mathcal{S}}[cross(u, v; \pi_\theta)] = p_{\mathcal{S}}(R_{dwn})c_{uv} + p_{\mathcal{S}}(R_{up})c_{vu}$.

We are ready to derive an important inequality.

Lemma 7. *Assume that $d_u, d_v \geqslant 3$ and $1 \leqslant \min\{c_{uv}, c_{vu}\} < \max\{c_{uv}, c_{vu}\}$ hold. Then it holds*

$$\frac{E_{\mathcal{S}}[cross(u, v; \pi_\theta)]}{\min\{c_{uv}, c_{vu}\}} \leqslant 1 + p_{\mathcal{S}}(R_{up}) \left(\frac{1}{a(R_{up})} - 2 \right) + \frac{12\gamma_{uv}}{d_u d_v - 4\gamma_{uv}}.$$

Proof. Let $c_{uv} = \min\{c_{uv}, c_{vu}\}$ without loss of generality. By Lemma 6, we get

$$\begin{aligned}
\frac{E_{\mathcal{S}}[cross(u, v; \pi_\theta)]}{\min\{c_{uv}, c_{vu}\}} &= \frac{p_{\mathcal{S}}(R_{dwn})c_{uv} + p_{\mathcal{S}}(R_{up})c_{vu}}{c_{uv}} \\
&= \frac{(1 - p_{\mathcal{S}}(R_{up}))c_{uv} + p_{\mathcal{S}}(R_{up})(d_u d_v - c_{uv} - \gamma_{uv})}{c_{uv}} \\
&= 1 + p_{\mathcal{S}}(R_{up}) \left(\frac{d_u d_v - \gamma_{uv}}{c_{uv}} - 2 \right).
\end{aligned}$$

First consider the case of $\mu(u) < \mu(v)$. By Lemma 1, we have $a(R_{up}) = c_{uv}/(d_u d_v)$. Hence

$$\frac{d_u d_v - \gamma_{uv}}{c_{uv}} - 2 = \frac{1}{c_{uv}} \left(\frac{c_{uv}}{a(R_{up})} - \gamma_{uv} \right) - 2 \leqslant \frac{1}{a(R_{up})} - 2.$$

Next consider the case of $\mu(u) \geqslant \mu(v)$. By Lemma 2, we have $1/c_{uv} \leqslant 1/(a(R_{up})d_u d_v - \gamma_{uv})$. Then

$$\begin{aligned}
\frac{d_u d_v - \gamma_{uv}}{c_{uv}} - 2 &\leqslant \frac{d_u d_v - \gamma_{uv}}{a(R_{up})d_u d_v - \gamma_{uv}} - 2 \\
&= \frac{1}{a(R_{up})} - 2 + \frac{d_u d_v - \gamma_{uv}}{a(R_{up})d_u d_v - \gamma_{uv}} - \frac{1}{a(R_{up})}
\end{aligned}$$

$$= \frac{1}{a(R_{up})} - 2 + \frac{(1-a(R_{up}))\gamma_{uv}}{a(R_{up})(a(R_{up})d_u d_v - \gamma_{uv})}$$
$$\leqslant \frac{1}{a(R_{up})} - 2 + \frac{12\gamma_{uv}}{d_u d_v - 4\gamma_{uv}} \quad (\text{since } a(R_{up}) \geqslant 1/4 \text{ by Lemma 1}).$$

Hence by $p_{\mathcal{S}}(R_{up}) \leqslant 1$, we have

$$1 + p_{\mathcal{S}}(R_{up})\left(\frac{d_u d_v - \gamma_{uv}}{c_{uv}} - 2\right) \leqslant 1 + p_{\mathcal{S}}(R_{up})\left(\frac{1}{a(R_{up})} - 2\right) + \frac{12\gamma_{uv}}{d_u d_v - 4\gamma_{uv}}.$$

This completes the proof. □

We wish to find an optimal scheme $\mathcal{S}$ that minimizes $\max_{u,v \in V} E_{\mathcal{S}}[cross(u, v; \pi_\theta)]/\min\{c_{uv}, c_{vu}\}$. For this, we consider an arbitrary monotone path P between points $(0, 0)$ and $(1, 1)$ in the unit square S (not necessarily a (u, v)-path for particular nodes $u, v \in V$). Define $R_{up}(P)$ and $R_{dwn}(P)$ be the regions obtained by splitting S with P, where we assume that $R_{up}(P)$ is above $R_{dwn}(P)$. Let

$$\beta(\mathcal{S}, P) := p_{\mathcal{S}}(R_{up}(P))\left(\frac{1}{a(R_{up}(P))} - 2\right)$$

and $\beta(\mathcal{S}) := \max\{\beta(\mathcal{S}, P) \mid \text{ monotone path } P\}$. Given a scheme $\mathcal{S}$, a monotone path P from $(0, 0)$ to $(1, 1)$ in the unit square S is called $\mathcal{S}$-*maximal* if $\beta(\mathcal{S}, P) = \beta(\mathcal{S})$.

Since the choice of monotone paths P is relaxed, we obtain $E_{\mathcal{S}}[cross(\pi_\theta)] \leqslant (1+\beta(\mathcal{S}))LB$. Let $\beta^* = \min\{\beta(\mathcal{S}) \mid \text{schemes } \mathcal{S}\}$. Therefore, to prove Theorem 4, it suffices to show that $\beta^* < 0.2964$, i.e., there exists a scheme $\mathcal{S}$ such that $\beta(\mathcal{S}) < 0.2964$.

4. A scheme $\mathcal{S}$

In this section, we present a scheme $\mathcal{S}$ that achieves Theorem 4. Let

$$\mathcal{S} = \{(s_1 = 0.014, t_1 = 0.221, p_1 = 0.087), (s_2 = 0.221, t_2 = 0.402, p_2 = 0.229),$$
$$(s_3 = 0.402, t_3 = 0.598, p_3 = 0.368), (s_4 = 0.598, t_4 = 0.779, p_4 = 0.229),$$
$$(s_5 = 0.779, t_5 = 0.986, p_5 = 0.087)\}$$

(see Fig. 7), where the values for s_i, t_i, p_i have been determined by a computational experiment). We denote the squares in the subschemes in $\mathcal{S}$ by

$$S_i = [(s_i, s_i), (t_i, t_i)], \quad i = 1, 2, 3, 4, 5,$$

where corners of these squares are denoted by $A_1, \ldots, A_6$, $B_1, \ldots, B_5$ and $C_1, \ldots, C_5$ as shown in Fig. 7.

Now consider a pair of arbitrary nodes u and v in V. It is not difficult to see that an $\mathcal{S}$-maximal monotone path P consists of axis-parallel line segments, and that the resulting

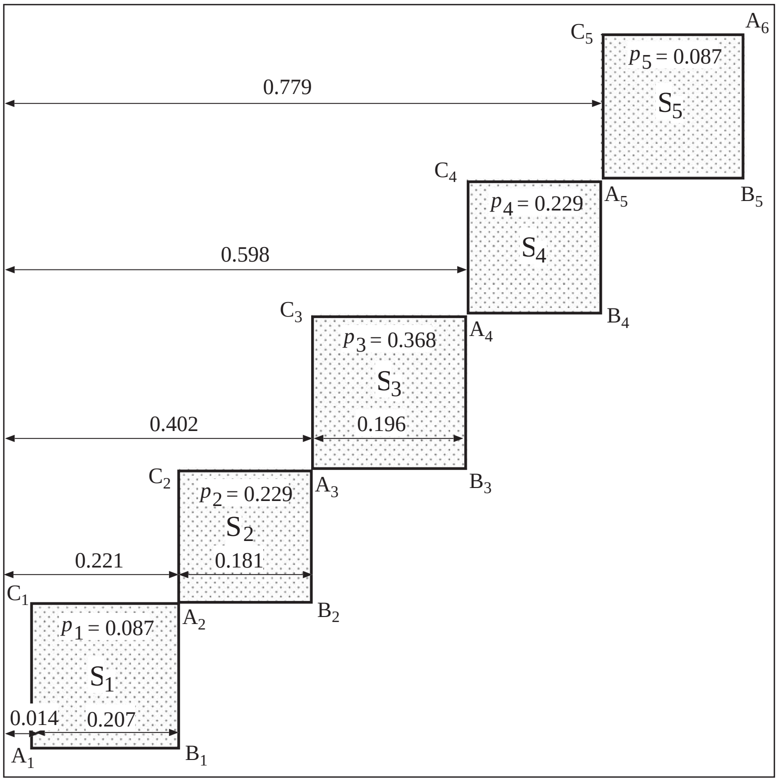


Fig. 7. A scheme $\mathcal{S}$ that attains Theorem 4.

region $R_{up}(P)$ contains at most one convex corner in each subscheme S_i $(i = 1, 2, 3, 4, 5)$. For simplicity, we consider a single subscheme S_i. As shown in Fig. 8a, if a monotone path P does not satisfy these properties, then we can modify the path P into another monotone path P' such that $a(S_i \cap R_{up}(P')) = a(S_i \cap R_{up}(P))$ and $a(R_{up}(P')) \leqslant a(R_{up}(P))$. Thus we only have to treat an axis-parallel piecewise linear monotone path P, which we denote the sequence of the corner points by

$$b_0 = (0, 0), b_1, \ldots, b_k = (1, 1),$$

and the sequence of the edges by

$$e_1 = b_0 b_1, e_2 = b_1 b_2, \ldots, e_k = b_{k-1} b_k$$

(see Fig. 9). Let e be an edge on a path P, where e may be a partial segment of some edge e_i. Without loss of generality we further assume that an $\mathcal{S}$-maximal monotone path P is chosen so that the number of edges of squares in subschemes or of the entire unit square that are overlapped by the edges in P is maximized among all $\mathcal{S}$-maximal monotone paths.

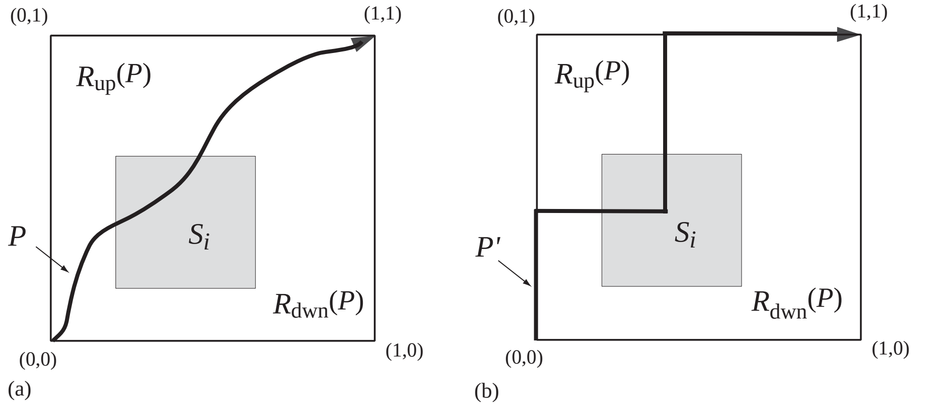


Fig. 8. Two monotone paths P and P' that pass through a square S_i such that $a(S_i \cap R_{up}(P')) = a(S_i \cap R_{up}(P))$ and $a(R_{up}(P')) < a(R_{up}(P))$.

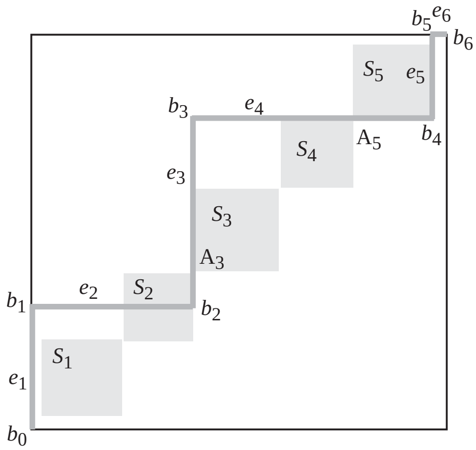


Fig. 9. Illustration of a piecewise linear monotone path P.

We define the *gain* of edge e with respect to a subscheme $S_i = (s_i, t_i, p_i) \in \mathcal{S}$ as follows. Consider how much amount of $p_{\mathcal{S}}(R_{up})$ changes if we move the line segment e in its orthogonal direction by an infinitely small amount ε. The change in $p_{\mathcal{S}}(R_{up})$ is

$$\frac{\varepsilon \ell(e \cap S_i) p_i}{(t_i - s_i)^2},$$

where $\ell(e \cap S_i)$ means the length of the intersection of e and S_i. On the other hand, the change in $a(R_{up}(P))$ is

$$\varepsilon \ell(e).$$

The *gain* of edge e with respect to a subscheme S_i is defined by the ratio of these two, i.e.,

$$g(e; S_i) = \frac{\ell(e \cap S_i)p_i}{(t_i - s_i)^2 \ell(e)}.$$

For a subscheme S_i, a vertical line segment e on a path P is called S_i*-incrementable* (resp., S_i*-decrementable*) if

- There is a real $\delta > 0$ such that gain $g(e; S_i)$ remain unchanged after translating it rightward (resp., leftward) by any amount $\delta' \in [0, \delta]$ (i.e., e remains to be intersecting S_i),
- For the rectangle R formed between e and the translated edge e' and the current path P, there is a monotone path P' such that $R_{up}(P') = R_{up}(P) \cup R$ (resp., $R_{up}(P') = R_{up}(P) - R$).

Analogously, the S_i-incrementability (resp., S_i-decrementability) of a horizontal line segment e is defined by replacing "rightward" with "downward" (resp., "leftward" with "upward"). In Fig. 9, for example, edge e_4 is S_4-incrementable but not S_4-decrementable, and e_4 is S_5-decrementable but not S_5-incrementable.

An edge e_i between two corners in a path P is called a *free edge* if it does not overlap with any edge of square S_i in a subscheme or of the entire unit square S. A free edge is S_i-incrementable and S_i-decrementable for some S_i. For example, e_2 in Fig. 9 is a free edge.

By definition, we observe the following.

Lemma 8. *For an $\mathcal{S}$-maximal monotone path P, let e and e' be respectively an S_i-incrementable edge and an S_j-decrementable edge. Then if e and e' are not adjacent, then $g(e; S_i) < g(e'; S_j)$. If e and e' are adjacent, then $g(e; S_i) = g(e'; S_j)$.*

Proof. Otherwise we would have another monotone path P' such that $\beta(\mathcal{S}, P') > \beta(\mathcal{S}, P)$ or such that $\beta(\mathcal{S}, P') = \beta(\mathcal{S}, P)$ and P' overlaps with more edges of the squares than P does. $\square$

In particular, there is no pair of non-adjacent free edges in an $\mathcal{S}$-maximal monotone path P.

In the sequel, P is assumed to be an $\mathcal{S}$-maximal monotone path. For simplicity, we may write $R_{up}(P)$, $p^{up}_{\mathcal{S}}(P)$ and $\beta(\mathcal{S}, P)$ as R_{up}, p^{up} and β, respectively. To prove that $\beta \leqslant 0.2964$ holds for our scheme $\mathcal{S}$, we distinguish the following cases:

Case 1: For $i = 1$ or $i = 5$, $R_{up} \cap S_i \neq \emptyset$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 2, 3, 4, 5\} - \{i\}$.

Case 2: For $i = 2$ or $i = 4$, $R_{up} \cap S_i \neq \emptyset$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 2, 3, 4, 5\} - \{i\}$.

Case 3: $R_{up} \cap S_3 \neq \emptyset$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 2, 4, 5\}$.

Case 4: For $\{i, i'\} = \{2, 3\}$ or $\{i, i'\} = \{3, 4\}$, $R_{up} \cap S_i \neq \emptyset \neq R_{up} \cap S_{i'}$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 2, 3, 4, 5\} - \{i, i'\}$.

Case 5: $R_{up} \cap S_i \neq \emptyset$, $i \in \{2, 4\}$ and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 3, 5\}$.

Case 6: $R_{up} \cap S_i \neq \emptyset$, $i \in \{2, 3, 4\}$ and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 5\}$.

Case 7: $R_{up} \cap S_i \neq \emptyset$, $i \in \{1, 5\}$, and $R_{up} \cap S_j = \emptyset$, $j \in \{2, 3, 4\}$.

We now consider the case where $R_{up} \cap S_1 \neq \emptyset$ or $R_{up} \cap S_5 \neq \emptyset$ (otherwise one of the above cases holds). We assume without loss of generality that $R_{up} \cap S_1 \neq \emptyset$ and that, in

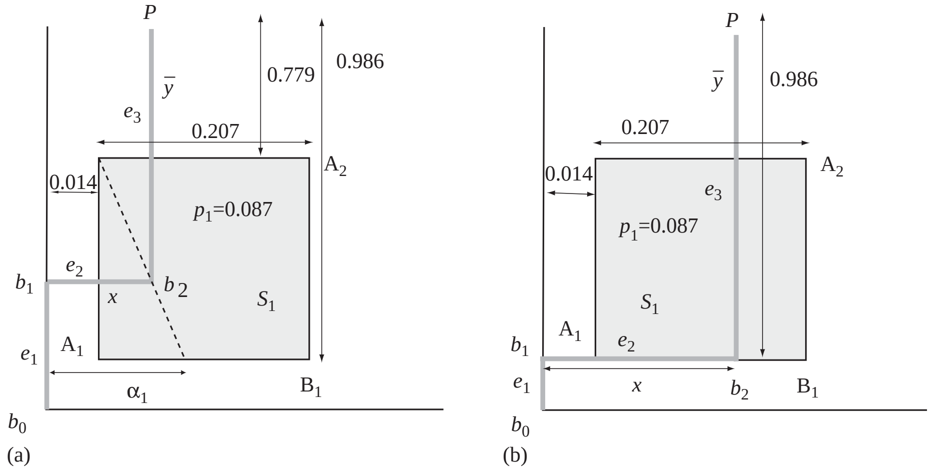


Fig. 10. Illustration for Case 1, where (a) indicates the case where e_2 is a free edge, and (b) indicates the case where b_2 is on edge A_1B_1.

addition, if $R_{up} \cap S_5 \neq \emptyset$ then $a(R_{up} \cap S_1) \leqslant a(R_{up} \cap S_5)$ holds.

Case 8 : $R_{up} \cap S_1 \neq \emptyset \neq R_{up} \cap S_2$.

Case 9 : $R_{up} \cap S_1 \neq \emptyset \neq R_{up} \cap S_3$.

Case 10 : $R_{up} \cap S_1 \neq \emptyset \neq R_{up} \cap S_4$.

Each of the above ten cases will be discussed in the following subsections.

4.1. Case 1

Assume without loss of generality that $R_{up} \cap S_1 \neq \emptyset$, and $R_{up} \cap S_j = \emptyset$, $j \in \{2, 3, 4, 5\}$. Consider edges $e_2 = b_1b_2$ and $e_3 = b_2b_3$ in P. Let $x = \ell(e_2) \in (0.014, 0.221]$ and $\bar{y} = \ell(e_3) \in (0.779, 0.986]$. We consider the following two subcases (a) and (b).

(a) Edge e_2 does not overlap with A_1B_1, i.e., e_2 is a free edge (see Fig. 10a): Then

$$g(e_2; S_1) = \frac{0.087}{(0.207)^2} \times \frac{x - 0.014}{x}, \quad g(e_3; S_1) = \frac{0.087}{(0.207)^2} \times \frac{\bar{y} - 0.779}{\bar{y}}.$$

Since P is $\mathcal{S}$-maximal, it must hold $g(e_2; S_1) = g(e_3; S_1)$ for two free edges. Thus we have $\bar{y} = 0.779x/0.014$, from which $\bar{y} - 0.779 = 0.779x/0.014 - 0.779 = 0.779(x - 0.014)/0.014$. By $\bar{y} \leqslant 0.986$, $x < \alpha_1$, where $\alpha_1 = 0.986 \times 0.014/0.779 < 0.018$. We have $a(R_{up}) = x\bar{y}$ and

$$p^{up} = 0.087 \times \frac{(x - 0.014)(\bar{y} - 0.779)}{(0.207)^2} = \frac{0.087 \times 0.779}{(0.207)^2 \times 0.014}(x - 0.014)^2.$$

Then

$$\beta = p^{up}\left(\frac{1}{a(R_{up})} - 2\right) = \frac{0.087 \times 0.779}{(0.207)^2 \times 0.014}(x - 0.014)^2 \left(\frac{1}{\frac{0.779}{0.014}x^2} - 2\right).$$

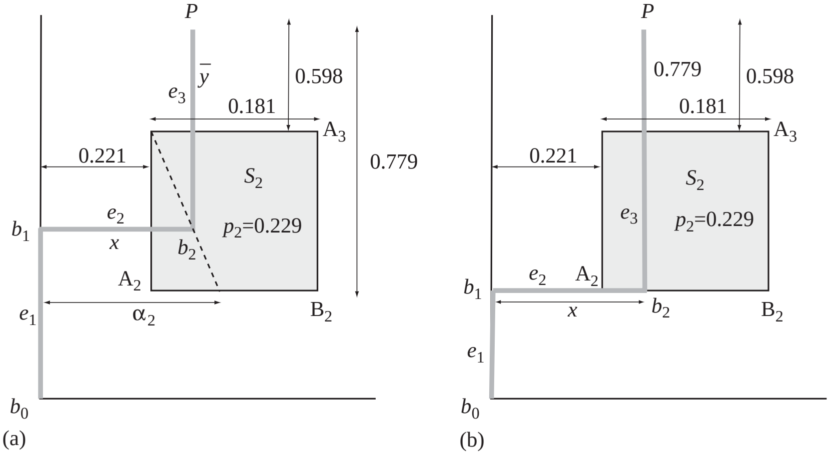


Fig. 11. Illustration for Case 2, where (a) indicates the case where e_2 is a free edge, and (b) indicates the case where b_2 is on edge A_2B_2.

By Lemma 4 with $a = 1, b = 0.014$ and $c = 0.779/0.014$, the function $\beta(x)$, $x \in (0.014, \alpha_1]$ takes the maximum at

$$x = \min\left\{\alpha_1, \left(\frac{0.014}{2 \times \frac{0.779}{0.014}}\right)^{1/3}\right\} = \alpha_1.$$

Since the maximum is attained at $x = \alpha_1$, we only have to consider the second case (b) where b_2 is on edge A_1B_1.

(b) b_2 is on edge A_1B_1 (see Fig. 10b): Then $a(R_{up}) = 0.986x$, $p^{up} = 0.087(x - 0.014)/0.207$, and

$$\beta = p^{up}\left(\frac{1}{a(R_{up})} - 2\right) = \frac{0.087}{0.207}(x - 0.014)\left(\frac{1}{0.986x} - 2\right).$$

By Lemma 3 with $a = 1, b = -0.014, c = 0.986$ and $d = 0$, we have

$$\beta \leqslant \frac{0.087}{0.207} \times \frac{1}{c}\left(\sqrt{a} - \sqrt{2(ad - bc)}\right)^2 < 0.2964.$$

4.2. Case 2

Assume without loss of generality that $R_{up} \cap S_2 \neq \emptyset$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 3, 4, 5\}$. Consider edges $e_2 = b_1b_2$ and $e_3 = b_2b_3$ in P. Let $x = \ell(e_2) \in (0.598, 0.779]$ and $\bar{y} = \ell(e_3) \in (0.779, 0.986]$. We consider the following two subcases (a) and (b).

(a) Edge e_2 does not overlap with A_1B_1, i.e., e_2 is a free edge (see Fig. 11a). Then

$$g(e_1; S_2) = \frac{0.229}{(0.181)^2} \times \frac{x - 0.221}{x}, \quad g(e_2; S_2) = \frac{0.229}{(0.181)^2} \times \frac{\bar{y} - 0.598}{\bar{y}}.$$

Since P is $\mathcal{S}$-maximal, it must hold $g(e_2; S_2) = g(e_3; S_2)$ for two free edges. Thus we have $\bar{y} = 0.598x/0.221$, from which

$$\bar{y} - 0.598 = \frac{0.598}{0.221}(x - 0.221).$$

By $\bar{y} \leqslant 0.779$, $x < \alpha_2$, where $\alpha_2 = 0.779 \times 0.221/0.598 < 0.29$. We have $a(R_{up}) = x\bar{y}$ and

$$p^{up} = 0.229 \times \frac{(x - 0.221)(\bar{y} - 0.598)}{(0.181)^2} = \frac{0.229 \times 0.598}{(0.181)^2 \times 0.221}(x - 0.221)^2.$$

Then

$$\beta = p^{up}\left(\frac{1}{a(R_{up})} - 2\right) = \frac{0.229 \times 0.598}{(0.181)^2 \times 0.221}(x - 0.221)^2 \left(\frac{1}{\frac{0.598}{0.221}x^2} - 2\right).$$

By Lemma 4 with $a = 1, b = 0.221$ and $c = 0.598/0.221$, the function $\beta(x)$, $x \in (0.221, \alpha_2]$ takes the maximum at

$$x = \min\left\{\alpha_2, \left(\frac{0.221}{2 \times \frac{0.598}{0.221}}\right)^{\frac{1}{3}}\right\} = \alpha_2.$$

Since the maximum is attained at $x = \alpha_2$, we only have to consider the second case (b) where b_2 is on edge A_2B_2.

(b) b_2 is on edge A_2B_2 (see Fig. 11b): Then $a(R_{up}) = 0.779x$, $p^{up} = 0.229(x - 0.221)/0.181$, and

$$\beta = p^{up}\left(\frac{1}{a(R_{up})} - 2\right) = \frac{0.229}{0.181}(x - 0.221)\left(\frac{1}{0.779x} - 2\right).$$

By Lemma 3 with $a = 1, b = -0.221, c = 0.779$ and $d = 0$, we have

$$\beta \leqslant \frac{0.229}{0.181} \times \frac{1}{c}\left(\sqrt{a} - \sqrt{2(ad - bc)}\right)^2 < 0.28.$$

4.3. Case 3

Consider edges $e_2 = b_1b_2$ and $e_3 = b_2b_3$ in P. Let $x = \ell(e_2) \in (0.402, 0.598]$ and $\bar{y} = \ell(e_3) \in (0.402, 0.598]$. Since P is $\mathcal{S}$-maximal, it must hold $g(e_2; S_3) = g(e_3; S_3)$ for two free edges. Thus $\bar{y} = x$ by symmetry (see Fig. 12). We have $a(R_{up}) = x^2$,

$$p^{up} = 0.368 \times \frac{(x - 0.402)^2}{(0.196)^2}, \quad \beta = \frac{0.368}{(0.196)^2}(x - 0.402)^2\left(\frac{1}{x^2} - 2\right).$$

By Lemma 4 with $a = 1, b = 0.402$ and $c = 1$, this takes the maximum at $x = \alpha_3$, where $\alpha_3 = (0.402/2)^{1/3} \in (0.402, 0.598)$. For the $x = \alpha_3$, we have

$$\beta = \frac{0.368}{(0.196)^2}(x - 0.402)^2\left(\frac{1}{x^2} - 2\right) < 0.296.$$

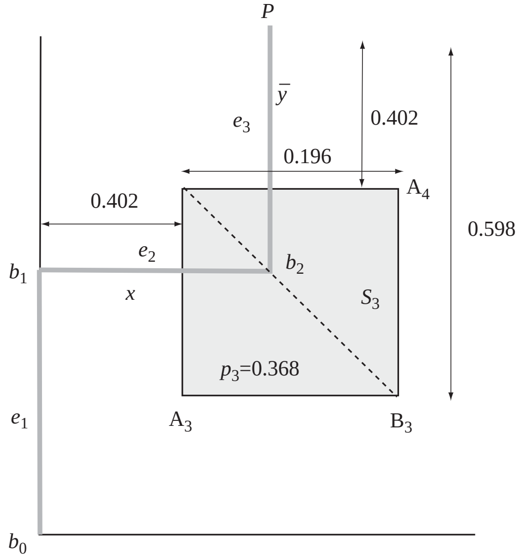


Fig. 12. Illustration for Case 3.

4.4. Case 4

Assume without loss of generality that $R_{up} \cap S_2 \neq \emptyset \neq R_{up} \cap S_3$, and $R_{up} \cap S_j = \emptyset$, $j \in \{1, 4, 5\}$. Note that edge e_2 overlaps with edge A_2B_2 or edge e_5 overlaps with edge B_3A_4 (otherwise both would be free edges). We consider the following five subcases (a)–(e).

(a) Edge e_5 overlaps with edge B_3A_4, and b_2 is on edge A_2B_2 (but $b_2 \neq B_2$): (see Fig. 13a.) Since

$$g(e_3; S_2) \geqslant g(B_2C_3; S_2) = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.181 + 0.196} > 3.14$$

and

$$g(e_5; S_3) \leqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.598} < 3.14,$$

it holds $g(e_3; S_2) > g(e_5; S_3)$ for S_2-incrementable edge e_3 and S_3-decrementable edge e_5, contradicting the $\mathcal{S}$-maximality of P.

(b) Edge e_5 overlaps with edge B_3A_4, and b_2 is not on edge A_2B_2 or B_2A_3 (see Fig. 13b): Since e_2 is a free edge, e_4 must overlap with A_3B_3 (otherwise we would have two nonadjacent free edges e_2 and e_4). We have

$$g(e_3; S_2) = \frac{0.229}{(0.181)^2} > 3.14 \text{ and } g(e_5; S_3) = \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.598} < 3.14.$$

Then it holds $g(e_3; S_2) > g(e_5; S_3)$ for S_2-incrementable edge e_3 and S_3-decrementable edge e_5, contradicting the $\mathcal{S}$-maximality of P.

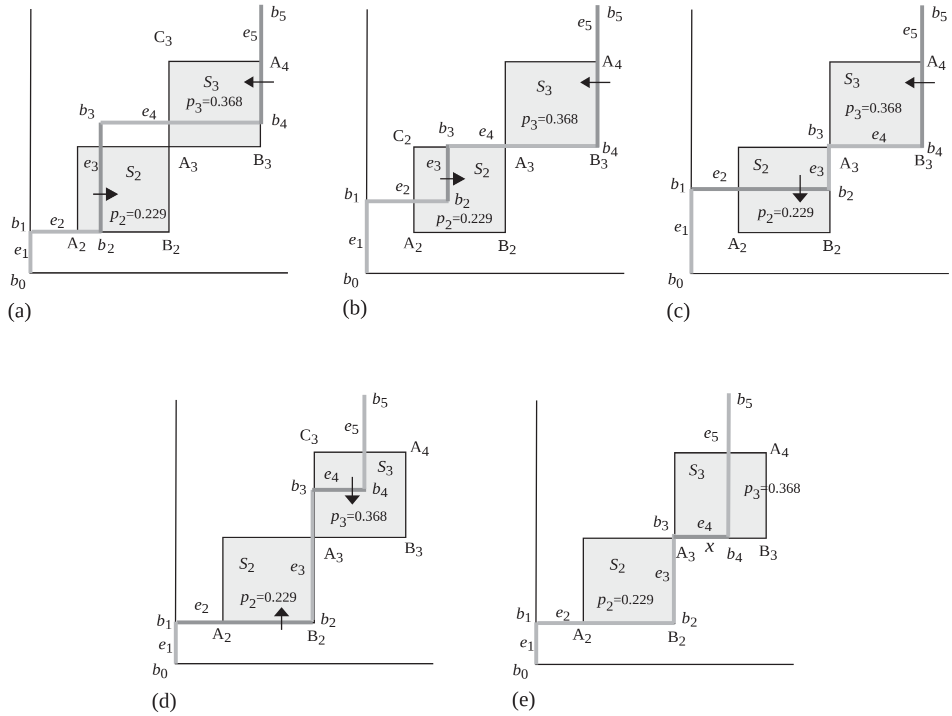


Fig. 13. Illustration for five subcases (a)–(e) in Case 4.

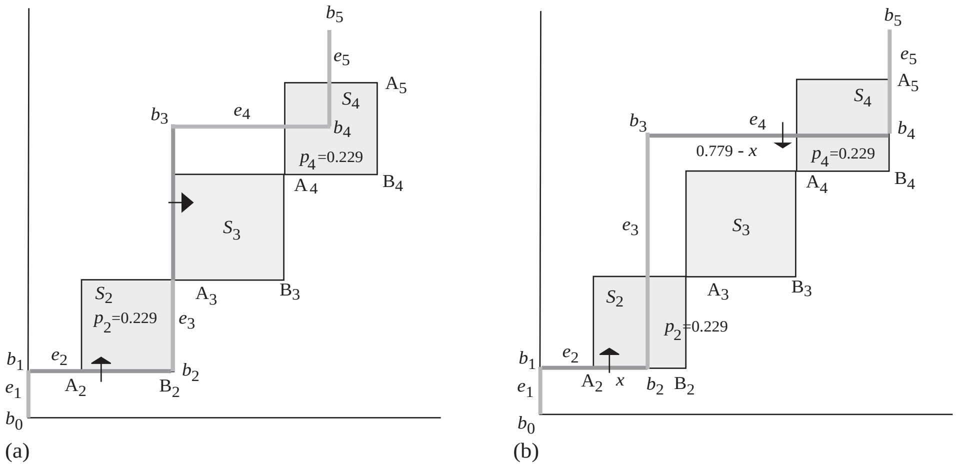


Fig. 14. Illustration for Case 5, where (a) indicates the case where e_3 overlaps with edge A_3B_2, and (b) indicates the case where e_5 overlaps with edge B_4A_5.

(c) Edge e_5 overlaps with edge B_3A_4, and b_2 is on edge B_2A_3 (see Fig. 13c): Since

$$g(e_2; S_2) = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} > 3.14 \text{ and } g(e_5; S_3) = \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.598} < 3.14,$$

it holds $g(e_2; S_2) > g(e_5; S_3)$ for S_2-incrementable edge e_2 and S_3-decrementable edge e_5, contradicting the $\mathcal{S}$-maximality of P.

(d) Edge e_2 overlaps with edge A_2B_2, and b_4 is not on edge A_2B_2 or B_2A_3 (see Fig. 13d): Since e_4 is a free edge, e_3 must overlap with B_2A_3. We have

$$g(e_4; S_3) = \frac{0.368}{(0.196)^2} > 9 \text{ and } g(e_2; S_2) = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} < 3.15.$$

Then it holds $g(e_4; S_3) > g(e_2; S_2)$ for S_3-incrementable edge e_4 and S_2-decrementable edge e_2, contradicting the $\mathcal{S}$-maximality of P.

(e) Edge e_2 overlaps with edge A_2B_2, and b_4 is on edge A_3B_3 (see Fig. 13e): Then $a(R_{up}) = 0.598x + 0.402 \times 0.779$,

$$p^{up} = 0.368 \times \frac{x}{0.196} + 0.229,$$

$$\beta = \left(\frac{0.368x}{0.196} + 0.229\right)\left(\frac{1}{0.598x + 0.402 \times 0.779} - 2\right).$$

By Lemma 3 with $a = 0.368/0.196$, $b = 0.229$, $c = 0.598$ and $d = 0.402 \times 0.779$, we have $\beta = (\sqrt{a} - \sqrt{2(ad - bc)})^2/c < 0.296$.

4.5. Case 5

Note that edge e_2 overlaps with edge A_2B_2 or edge e_5 overlaps with edge B_4A_5 (otherwise both would be free edges); we assume without loss of generality that e_2 overlaps with edge A_2B_2. Similarly edge e_3 overlaps with edge A_3B_2 or edge e_5 overlaps with edge A_5B_4. We consider the following two subcases (a) and (b).

(a) Edge e_3 overlaps with edge A_3B_2, i.e., $b_2 = B_2$ (see Fig. 14a): For S_3-incrementable edge A_3b_3 and S_2-decrementable edge e_2, we have $g(A_3b_3; S_3) > g(e_2; S_2)$, since

$$g(A_3b_3; S_3) \geqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.196 + 0.181} > 4$$

and

$$g(e_2; S_2) = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} < 3.15.$$

This, however, contradicts that P is $\mathcal{S}$-maximal.

(b) Edge e_5 overlaps with edge B_4A_5 (see Fig. 14b): Let $x = \ell(e_2) \in [0.221, 0.402]$. Then

$$g(e_4; S_4) = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.779 - x}, \quad g(e_2; S_2) = \frac{0.229}{(0.181)^2} \times \frac{x - 0.221}{x},$$

$a(R_{up}) = (0.779)^2 - (0.779 - x)^2$ and $p^{up} = 2 \times 0.229 \times (x - 0.221)/0.181$. Since R_{up} contains no interior point from other S_i than S_2 and S_4, we have $p^{up} \leqslant 0.229 + 0.229 = 0.458$. From this, we see that if $a(R_{up}) \geqslant (0.2963/0.458 + 2)^{-1} (\leqslant 0.378)$ then

$$\beta = p^{up} \left(\frac{1}{a(R_{up})} - 2 \right) \leqslant 0.2963.$$

Hence assume $a(R_{up}) < 0.378$. From $a(R_{up}) = (0.779)^2 - (0.779 - x)^2 < 0.378$, we have $x \in [0.221, 0.3006]$. For such x, $0.181/(0.779 - x) > (x - 0.221)/x$ holds, and hence $g(e_4; S_4) > g(e_2; S_2)$ for S_4-incrementable edge e_4 and S_2-decrementable edge e_2. This contradicts the $\mathcal{S}$-maximality of P.

4.6. Case 6

Observe that one of B_2, B_3 and B_4 is a convex corner of R_{up} (otherwise P would have two nonadjacent free edges). We consider the following three subcases (a)–(c).

(a) At least two of B_2, B_3 and B_4 are convex corners of R_{up} at the same time (see Fig. 15a): In this case, $p^{up} \leqslant 1 - 0.087 \times 2$ and $a(R_{up}) \geqslant 0.779 \times 0.598 - 0.181 \times 0.196 > 0.43$ hold. From this,

$$\beta = p^{up} \left(\frac{1}{a(R_{up})} - 2 \right) < 0.269.$$

(b) B_3 is a convex corner of R_{up}, and neither B_2 nor B_4 is a convex corner of R_{up} (see Fig. 15b): In this case, we have two free edges each from S_2 and S_4, a contradiction to the $\mathcal{S}$-maximality of P.

(c) B_2 is a convex corner of R_{up}, neither B_3 nor B_4 is a convex corner of R_{up} (the case where B_4 is a convex corner of R_{up} can be treated symmetrically): (see Fig. 15c) There must be at least (hence exactly two) free edges, which must be adjacent edges e_5 and e_6. However, $g(e_5; S_3) > g(e_2; S_2)$ holds for S_3-incrementable edge e_5 and S_2-decrementable edge e_2, since

$$g(e_5; S_3) \geqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.196 + 0.181} > 4$$

and

$$g(e_2; S_2) \leqslant = \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} < 3.15,$$

contradicting the $\mathcal{S}$-maximality of P.

4.7. Case 7

Note that edge e_2 overlaps with edge A_1B_1 or edge e_5 overlaps with edge B_5A_6 (otherwise both would be free edges); we assume without loss of generality that e_2 overlaps with edge A_1B_1. We consider the following two subcases (a) and (b).

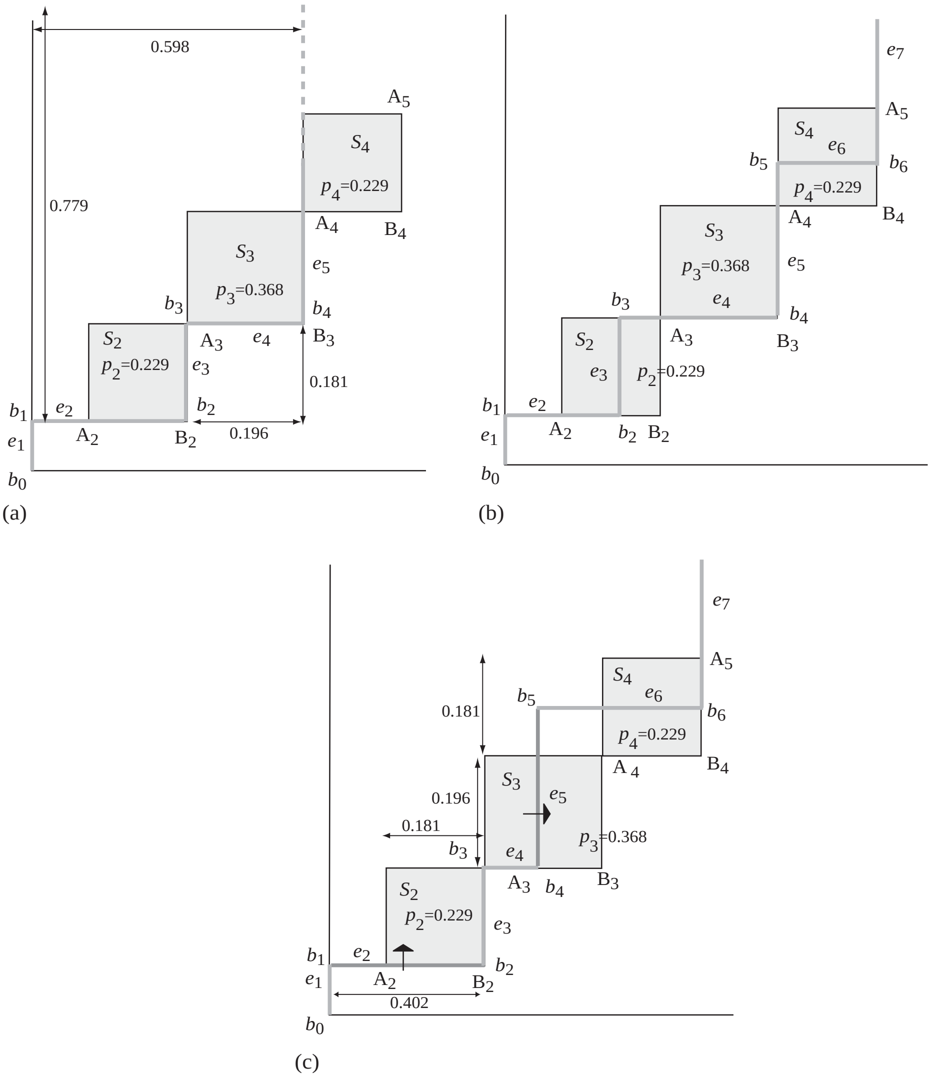


Fig. 15. Illustration for three subcases (a)–(c) in Case 6.

(a) $b_2 = B_1$ (see Fig. 16a): In this case, we have

$$g(e_4; S_5) \leqslant \frac{0.087}{(0.207)^2} \times \frac{0.207}{0.765} < 0.55$$

and

$$g(A_2 b_3; S_2) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.765} > 1.65.$$

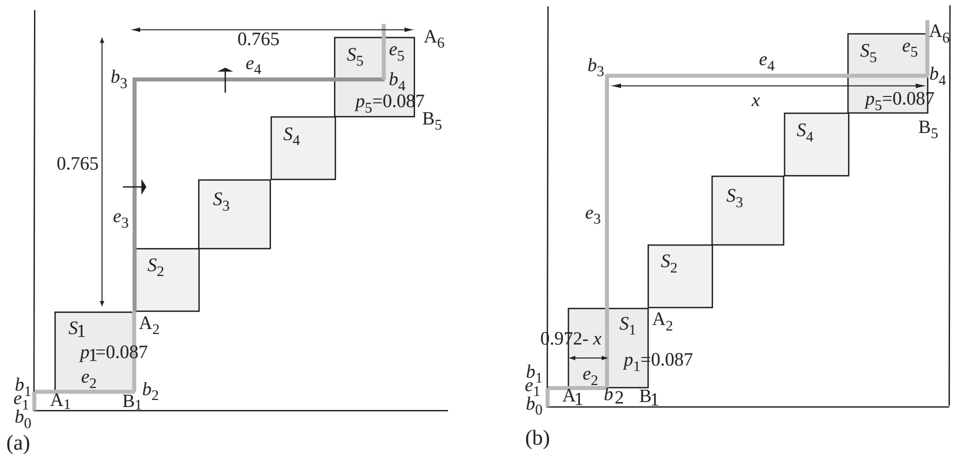


Fig. 16. Illustration for Case 7, where (a) indicates the case where $b_2 = B_1$, and (b) indicates the case where none of B_1 and B_5 is a convex corner of R.

Hence it holds $g(A_2b_3; S_2) > g(e_4; S_5)$ for S_2-incrementable edge A_2b_3 and S_5-decrementable edge e_4, contradicting the $\mathcal{S}$-maximality of P.

(b) None of B_1 and B_5 is a convex corner of R_{up} (see Fig. 16b): Since e_3 and e_4 are free edges, e_5 is not a free edge and overlaps with B_5A_6. Then $g(e_3; S_1) = g(e_4; S_5)$ must hold, implying $\ell(e_3) = \ell(e_4)$. Let $x = \ell(e_4) \in (0.765, 0.973)$. Then $a(R_{up}) = (0.986)^2 - x^2$, and

$$p^{up} = 2 \times \frac{0.087}{0.207} \times (0.972 - x) \quad (\leqslant 2 \times 0.087).$$

We can see that $\beta = p^{up}(1/a(R_{up}) - 2) < 0.2963$ holds for $x \in (0.765, 0.973)$. (for example, to see this, we repeat the following computation after initializing $p := 2 \times 0.087$:

$$R := \frac{1}{\frac{0.2963}{p} + 2}, \quad x := \sqrt{(0.986)^2 - R}, \quad p := 2 \times \frac{0.087}{0.207} \times (0.972 - x)$$

After a finite number of iterations, x becomes greater than 0.973, which implies that there is no $x \in (0.765, 0.973)$ such that $\beta \geqslant 0.2963$.)

4.8. Case 8

Observe that if $a(R_{up}) \geqslant 0.43806$ then $\beta = p^{up}_{\mathcal{S}}(1/a(R_{up}) - 2) \leqslant (1/a(R_{up}) - 2) \leqslant 0.2964$ holds. Hence we assume that $a(R_{up}) < 0.43806$. From this, we see that B_2, B_3 and B_4 cannot be convex corners of R at the same time since $a(R_{up})$ in such a case is at least $0.779^2 - 0.377^2 + 0.196^2 > 0.43806$. Note that edge e_2 overlaps with edge A_1B_1 or edge e_5 overlaps with edge B_2A_3 (otherwise both would be free edges). We consider the following subcases (1) and (2).

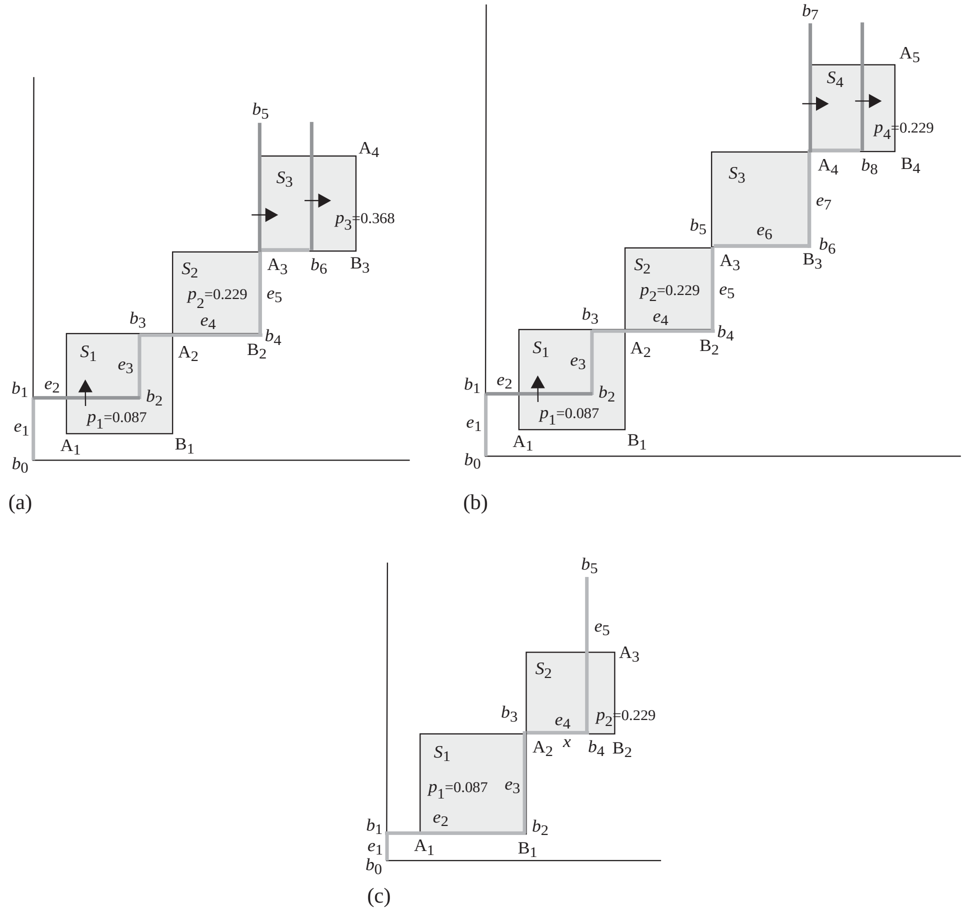


Fig. 17. Illustration for Case 8.

(1) Edge e_5 overlaps with edge B_2A_3 (see Fig. 17a): For S_1-decrementable edge e_2,

$$g(e_2; S_1) \leqslant \frac{0.087}{(0.207)^2} \times \frac{0.207}{0.221} < 2.$$

We show that P has an S_i-incrementable edge e' with $g(e'; S_i) > g(e_2; S_1)$, which contradicts the $\mathcal{S}$-maximality of P.

If $b_4 \neq B_2$, then $e' = e_4$ is S_2-incrementable and $g(e'; S_2) > g(e_2; S_1)$. Then assume $b_4 = B_2$. If $b_5 \neq A_3$ (resp., $b_5 = A_3$ and $b_6 \neq B_3$), then $e' = A_3b_5$ (resp., $e' = e_7 = b_6b_7$) is an S_3-incrementable edge of P with

$$g(e'; S_3) \geqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.598} > 3 > g(e_2; S_1).$$

Finally assume $b_5 = A_3$ and $b_6 = B_3$ (see Fig. 17b). Since B_2, B_3 and B_4 cannot be convex corners of R at the same time, either $b_7 \neq A_4$ or $b_7 = A_4$ and $b_8 \neq B_4$ holds.

If $b_7 \neq A_4$ (resp., $b_7 = A_4$ and $b_8 \neq B_4$), then $e' = A_4 b_7$ (resp., $e' = e_9 = b_8 b_9$) is an S_4-incrementable edge of P with

$$g(e'; S_4) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} > 3 > g(e_2; S_1).$$

(2) Edge e_2 overlaps with edge $A_1 B_1$ but edge e_5 does not overlap with edge $B_2 A_3$ (see Fig. 17c). Then e_5 is a free edge. Hence e_3 must overlap with $B_1 A_2$ (i.e., $b_2 = B_1$). If $R_{up} \cap S_i \neq \emptyset$ for some $i \in \{3, 4, 5\}$, then $\ell(e_5) \leqslant 0.181 + 0.196 + 0.181 = 0.558$ (since if $R_{up} \cap S_5 \neq \emptyset$ then $S_5 \subseteq R_{up}$ by the assumption $a(R_{up} \cap S_1) \leqslant a(R_{up} \cap S_5)$), implying

$$g(e_5; S_2) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.558} > 2.2.$$

By $g(e_2; S_1) < 1.9$, we have $g(e_5; S_2) > g(e_2; S_1)$ for S_2-incrementable edge e_5 and S_1-decrementable edge e_2, contradicting the $\mathcal{S}$-maximality of P.

Assume that $R_{up} \cap S_i = \emptyset$ $(i \neq 1, 2)$, that is, $\ell(e_5) = 0.779$. Let $x = \ell(e_4)$. Then $a(R_{up}) = 0.779x + 0.221 \times 0.986$, and $p^{up} = 0.229x/0.181 + 0.087$. By Lemma 3 with $a = 1.26519337$, $b = 0.087$, $c = 0.779$ and $d = 0.217906$, we have $\beta \leqslant (\sqrt{a} - \sqrt{2(ad - bc)})^2/c < 0.296$.

4.9. Case 9

We can assume that $R_{up} \cap S_2 = \emptyset$ (otherwise such a case is treated in Case 8). Then $p^{up} \leqslant 1 - 0.229 = 0.771$. Assume $a(R_{up}) \leqslant 1/(0.2963/p + 2) < 0.42$ (otherwise $\beta < 0.2963$). Note that B_3 and B_4 cannot be convex corners of R at the same time since $a(R_{up})$ in such a case is at least $0.779 \times 0.598 - 0.196 \times 0.181 > 0.42$ contradicting the assumption $a(R_{up}) < 0.42$ on R_{up}. Observe that edge e_2 overlaps with edge $A_1 B_1$ or edge e_5 overlaps with edge $B_3 A_4$ (otherwise both would be free edges). We consider the following two subcases (a) and (b).

(a) Edge e_5 overlaps with edge $B_3 A_4$ (see Fig. 18a):

For S_1-decrementable edge e_2,

$$g(e_2; S_1) \leqslant \frac{0.087}{(0.207)^2} \times \frac{0.207}{0.221} < 2.$$

We show that P has an S_i-incrementable edge e' with $g(e'; S_i) > g(e_2; S_1)$, which contradicts the $\mathcal{S}$-maximality of P.

If $b_4 \neq B_3$, then $e' = e_4$ is S_3-incrementable and

$$g(e'; S_3) \geqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.598 - 0.014} > 3.$$

Then assume $b_4 = B_3$. Since B_3 and B_4 cannot be convex corners of R_{up} at the same time, either $b_5 \neq A_4$ holds or $b_5 = A_4$ and $b_6 \neq B_4$ hold. If $b_5 \neq A_4$ (resp., $b_5 = A_4$ and $b_6 \neq B_4$), then $e' = A_4 b_5$ (resp., $e' = e_7 = b_6 b_7$) is an S_4-incrementable

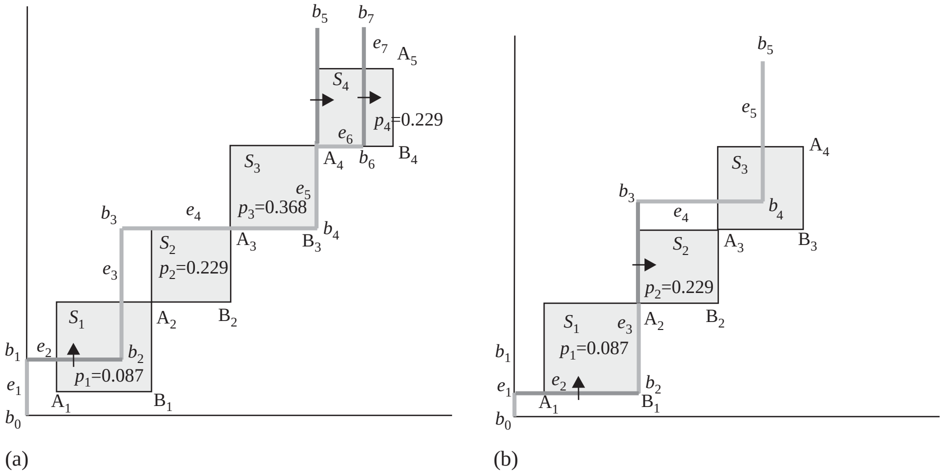


Fig. 18. Illustration for Case 9, where (a) indicates the case where e_5 overlaps with edge B_3A_4, and (b) indicates the case where e_2 overlaps with edge A_1B_1 but e_5 does not overlap with edge B_2A_3.

edge of P with

$$g(e'; S_4) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.402} > 3.$$

(b) Edge e_2 overlaps with edge A_1B_1 but edge e_5 does not overlap with edge B_2A_3 (see Fig. 18b): Then $b_2 = B_1$ (otherwise e_3 and e_5 are free edges). Hence $e' = A_2b_3$ is an S_2-incrementable edge with

$$g(e'; S_2) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.377} > 4.$$

Note that e_2 is an S_1-decrementable edge with $g(e_2; S_1) < 2$. Hence, $g(e'; S_2) > g(e_2; S_1)$, a contradiction to the $\mathcal{S}$-maximality of P.

4.10. Case 10

We can assume that $R_{up} \cap S_2 = R_{up} \cap S_3 = \emptyset$ (otherwise such a case is treated in Case 8 or Case 9). Then $p^{up} \leqslant 1 - 0.229 - 0.368 = 0.403$. Assume $a(R_{up}) \leqslant 1/(0.2963/p + 2) < 0.37$ (otherwise $\beta < 0.2963$). Note that A_2 and B_4 cannot be on the path P at the same time since $a(R_{up})$ in that case is at least $0.779 \times 0.402 + 0.221 \times (0.196 + 0.181) > 0.37$, contradicting the assumption $a(R_{up}) \leqslant 0.37$ on R_{up}. Observe that edge e_2 overlaps with edge A_1B_1 or edge e_5 overlaps with edge B_4A_5 (otherwise both would be free edges). We consider the following subcases (1) and (2).

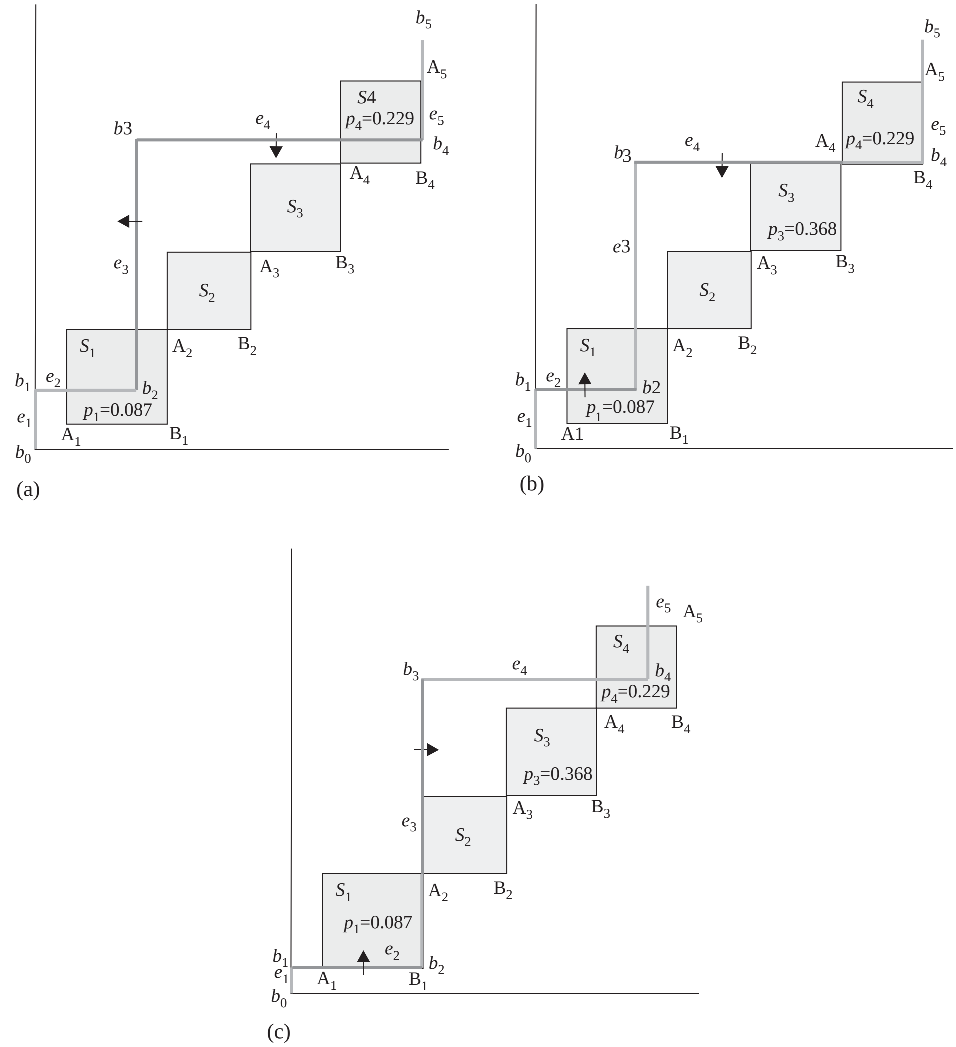


Fig. 19. Illustration for Case 10.

(1) Edge e_5 overlaps with edge B_4A_5. If $b_4 \neq B_4$, i.e., A_4 is not on P (see Fig. 19a), then e_4 is an S_4-incrementable edge and e_3 is an S_1-decrementable edge such that

$$g(e_4; S_4) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.765} > 1.6$$

and

$$g(e_3; S_1) \leqslant \frac{0.087}{(0.207)^2} \times \frac{0.207}{0.207 + 0.181 + 0.196} < 0.72,$$

a contradiction to the $\mathcal{S}$-maximality of P.

Assume $b_4 = B_4$, i.e., A_4 is on P (see Fig. 19b). Then $e' = b_3A_4$ is an S_3-incrementable edge and e_2 is an S_1-decrementable edge such that

$$g(e'; S_3) \geqslant \frac{0.368}{(0.196)^2} \times \frac{0.196}{0.584} > 3.2$$

and

$$g(e_2; S_1) \leqslant \frac{0.087}{(0.207)^2} \times \frac{0.207}{0.221} < 2,$$

again a contradiction to the $\mathcal{S}$-maximality of P.

(2) Edge e_2 overlaps with edge A_1B_1 but edge e_5 does not overlap with edge B_4A_5 (see Fig. 19c). Then $b_2 = B_1$ (otherwise e_3 and e_5 are free edges). Hence $e' = A_2b_3$ is S_2-incrementable, and

$$g(e'; S_2) \geqslant \frac{0.229}{(0.181)^2} \times \frac{0.181}{0.181 + 0.196 + 0.181} = 2.267371631 > g(e_2; S_1).$$

Since e_2 is S_1-decrementable, this contradicts the $\mathcal{S}$-maximality of P.

From the arguments in this section, we have shown that $\beta(\mathcal{S}) < 0.294$ and thereby Theorem 4 holds.

5. Lower bound on β^*

One may consider whether there is a scheme $\mathcal{S}'$ that has $\beta(\mathcal{S}')$ smaller than 0.2964. In this section, we, however, show that there is no scheme $\mathcal{S}'$ with $\beta(\mathcal{S}') < 0.2698$. That is, we prove the next result.

Theorem 5. $0.2698 < \beta^* < 0.2964$.

Since we have shown $\beta^* \leqslant 0.2964$ in the previous section, we now estimate β^* from below. Let $\mathcal{S}$ be an arbitrary scheme. For $x_1 = y_1 = 0.22183$, $x_2 = y_2 = 0.41285$, $x_3 = y_3 = 1 - x_2$, and $x_4 = y_4 = 1 - x_1$, we partition the unit square S into 25 blocks by three vertical lines with x-coordinates x_1, x_2, x_3 and x_4 and three horizontal lines with y-coordinates y_1, y_2, y_3 and y_4 (see Fig. 20). We consider the following 10 monotone piecewise linear paths:

$$P_1 = \langle(0, 0), (0, y_2), (x_3, y_2), (x_3, 1), (1, 1)\rangle, \quad P'_1 = \langle(0, 0), (x_2, 0), (x_2, y_3), (1, y_3), (1, 1)\rangle,$$
$$P_2 = \langle(0, 0), (0, y_1), (x_2, y_1), (x_2, 1), (1, 1)\rangle, \quad P'_2 = \langle(0, 0), (x_3, 0), (x_3, y_4), (1, y_4), (1, 1)\rangle,$$
$$P_3 = \langle(0, 0), (0, y_3), (x_4, y_3), (x_4, 1), (1, 1)\rangle, \quad P'_3 = \langle(0, 0), (x_1, 0), (x_1, y_2), (1, y_2), (1, 1)\rangle,$$
$$P_4 = \langle(0, 0), (0, y_4), (1, y_4), (1, 1)\rangle, \quad P'_4 = \langle(0, 0), (0, y_1), (1, y_1), (1, 1)\rangle,$$
$$P_5 = \langle(0, 0), (x_1, 0), (x_1, 1), (1, 1)\rangle, \quad P'_5 = \langle(0, 0), (x_4, 0), (x_4, 1), (1, 1)\rangle.$$

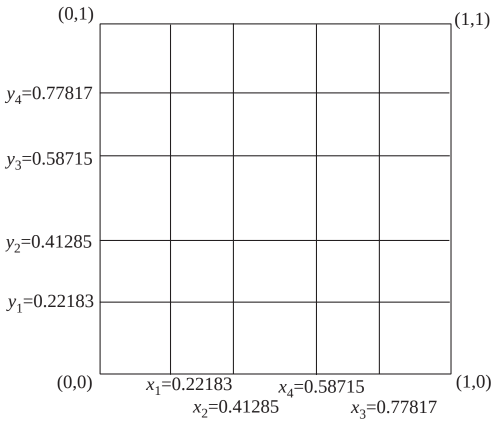


Fig. 20. A partition of a unit square *S*.

Let $a_1 = (1 - 0.412849) \times (1 - 0.412849)$, $a_2 = 0.412849 \times (1 - 0.22183)$, and $a_3 = 0.22183$. Then we have

$$a(R_{up}(P_1)) = a(R_{dwn}(P_1')) = a_1,$$
$$a(R_{up}(P_2)) = a(R_{dwn}(P_2')) = a(R_{up}(P_3)) = a(R_{dwn}(P_3')) = a_2,$$
$$a(R_{up}(P_4)) = a(R_{dwn}(P_4')) = a(R_{up}(P_5)) = a(R_{dwn}(P_5')) = a_3.$$

Observe that each block in S is contained in at least two regions from $\{R_{up}(P_1), \ldots, R_{up}(P_5), R_{dwn}(P_1'), \ldots, R_{dwn}(P_5')\}$. Therefore, it holds

$$\sum_{i=1}^{5} p_{\mathcal{S}}^{up}(P_i) + \sum_{i=1}^{5} p_{\mathcal{S}}^{dwn}(P_i') \geqslant 2. \tag{1}$$

By definition, β^* satisfies

$$p_{\mathcal{S}}^{up}(P_i)\left(\frac{1}{a(R_{up}(P_i))} - 2\right) \leqslant \beta^* \quad (i = 1, 2, 3, 4, 5).$$

Similarly, by considering path P_i' as a monotone path from $(1, 1)$ to $(0, 0)$, we have

$$p_{\mathcal{S}}^{dwn}(P_i')\left(\frac{1}{a(R_{dwn}(P_i'))} - 2\right) \leqslant \beta^* \quad (i = 1, 2, 3, 4, 5).$$

Hence it holds

$$\begin{aligned}\sum_{i=1}^{5} p_{\mathcal{S}}^{up}(P_i) + \sum_{i=1}^{5} p_{\mathcal{S}}^{dwn}(P_i') &\leqslant \beta^* \sum_{i=1}^{5} \frac{1}{1/[a(R_{up}(P_i))] - 2} \\ &\quad + \beta^* \sum_{i=1}^{5} \frac{1}{1/[a(R_{dwn}(P_i'))] - 2} \\ &= \beta^* \left(\frac{2}{1/a_1 - 2} + \frac{4}{1/a_2 - 2} + \frac{4}{1/a_3 - 2}\right).\end{aligned}$$

From this and (1), we have

$$\beta^* \geqslant \frac{\sum_{i=1}^{5} p_{\mathcal{S}}^{up}(P_i) + \sum_{i=1}^{5} p_{\mathcal{S}}^{dwn}(P'_i)}{\frac{2}{1/a_1-2} + \frac{4}{1/a_2-2} + \frac{4}{1/a_3-2}} \geqslant \frac{2}{\frac{2}{1/a_1-2} + \frac{4}{1/a_2-2} + \frac{4}{1/a_3-2}} > 0.2698,$$

as required. The current choice of 5×5 blocks over the unit square S and the values for a_1, a_2 and a_3 is based on some limited computer experiment, and there may exist a better choice of blocks in S for evaluating a lower bound on β^*.

6. Concluding remarks

In this paper, we have analyzed the performance of the randomized key based heuristic due to Nagamochi [12,13] in terms of the minimum degree δ of nodes in V, and have proved that, for the scheme $\mathcal{S}$ in Section 4, the heuristic delivers a solution whose average crossing number is at most $(1.2964+12/(\delta-4))LB$. For graphs with large δ, this is an improvement over the previous best bound 1.4664 [12,13]. On the other hand, we have shown in Section 5 that no scheme $\mathcal{S}'$ can achieve any better ratio than 1.2698. Note that this does not imply that the gap between the optimal and the lower bound is actually 1.2698. The currently known gap is $13/11 \simeq 1.1818$, as shown in Fig. 2. Determining $\max_G\{opt(G)/LB(G)\}$ is left for the future research.

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