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Remarks on A Simple Fractional-Calculus Approach to the Solutions of the Bessel Differential Equation of General Order and Some of Its Applications

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Abstract—In a remarkably large number of recent works, one can find the emphasis upon (and demonstrations of) the usefulness of fractional calculus operators in the derivation of (explicit) particular solutions of significantly general families of linear ordinary and partial differential equations of the second and higher orders. The main object of the present paper is to continue our investigation of this simple fractional-calculus approach to the solutions of the classical Bessel differential equation of general order and to show how it would lead naturally to several interesting consequences which include (for example) an alternative derivation of the complete power-series solutions obtainable usually by the Frobenius method. The underlying analysis presented here is based chiefly upon some of the general theorems on (explicit) particular solutions of a certain family of linear ordinary fractional differintegral equations with polynomial coefficients. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND DEFINITIONS

During the past three decades or so, the widely-investigated subject of *fractional calculus* (that is, calculus of derivatives and integrals of any *arbitrary* real or complex order) has remarkably gained importance and popularity due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details, [1–4]). Recently, by applying the following definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of order $\nu \in \mathbb{R}$, many authors have *explicitly* derived particular solutions of a large number of families of homogeneous (as well as nonhomogeneous) linear ordinary and partial fractional differintegral equations (see, for details, [5–21], and the references cited in *each* of these earlier works).

DEFINITION (CF. [22–28]). *If the function $f(z)$ is analytic (regular) inside and on $\mathcal{C}$, where*

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}, \tag{1.1}$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\Im(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\Im(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\nu(z) = (f(z))_\nu := \frac{\Gamma(\nu+1)}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta \qquad \left(\nu \in \mathbb{R} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1,-2,-3,\dots\}\right) \tag{1.2}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\} \qquad (n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.3}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi \qquad \text{for } \mathcal{C}^-, \tag{1.4}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi \qquad \text{for } \mathcal{C}^+, \tag{1.5}$$

then $f_\nu(z)$ $(\nu > 0)$ is said to be the fractional derivative of $f(z)$ of order ν and $f_\nu(z)$ $(\nu < 0)$ is said to be the fractional integral of $f(z)$ of order $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in \mathbb{R}). \tag{1.6}$$

Here, as well as in many of the aforecited earlier works, we simply write f_ν for $f_\nu(z)$ whenever the argument of the differintegrated function f is clearly understood by the surrounding context. Moreover, in case f is a many-valued function, we tacitly consider the *principal value* of f in our investigation. For the sake of convenience in dealing with their various (known or new) special cases, we choose also to state one of the fundamental results (Theorem 1 below) for homogeneous (as well as nonhomogeneous) linear ordinary fractional differintegral equations of a *general* order $\mu \in \mathbb{R}$.

Some of the recent contributions on the subject of explicit particular solutions of linear ordinary and partial fractional differintegral equations with polynomial coefficients are those given by Tu *et* *al.* [5] who presented unification and generalization of a significantly large number of widely scattered results on this subject (see also the many relevant earlier works cited by Tu *et al.* [5]). For the sake of ready reference, we choose to recall here one of the *main* results of Tu *et al.* [5], involving a family of linear ordinary fractional differintegral equations, as Theorem 1 below. For analogous treatments of some closely related families of integro-differential equations with polynomial coefficients, the interested reader may be referred also to the recent works by Ali and Kalla [6] (and also by Odibat and Shawagfeh [17]).

THEOREM 1. *(See [5 Theorem 1, p. 295; Theorem 2, p. 296].) Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$P(z;p) := \sum_{k=0}^{p} a_k z^{p-k} = a_0 \prod_{j=1}^{p} (z - z_j) \qquad (a_0 \neq 0;\ p \in \mathbb{N}) \tag{1.7}$$

and

$$Q(z;q) := \sum_{k=0}^{q} b_k z^{q-k} \qquad (b_0 \neq 0;\ q \in \mathbb{N}). \tag{1.8}$$

Suppose also that $f_{-\nu}(\neq 0)$ exists for a given function f.

Then, the following nonhomogeneous linear ordinary fractional differintegral equation:

$$\begin{gathered} P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q)\right] \phi_{\mu-k}(z) \\ + \binom{\nu}{q} q!\, b_0 \phi_{\mu-q-1}(z) = f(z) \\ (\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N}) \end{gathered} \tag{1.9}$$

has a particular solution of the form:

$$\phi(z) = \left(\left(\frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)} \right)_{-1} \cdot e^{-H(z;p,q)} \right)_{\nu-\mu+1} \qquad (z \in \mathbb{C} \setminus \{z_1, \ldots, z_p\}), \tag{1.10}$$

where, for convenience,

$$H(z;p,q) := \int^{z} \frac{Q(\zeta;q)}{P(\zeta;p)}\, d\zeta \qquad (z \in \mathbb{C} \setminus \{z_1, \ldots, z_p\}), \tag{1.11}$$

provided that the second member of (1.10) exists.

Furthermore, the following homogeneous linear ordinary fractional differintegral equation:

$$\begin{gathered} P(z;p)\,\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q)\right] \phi_{\mu-k}(z) \\ + \binom{\nu}{q} q!\, b_0 \phi_{\mu-q-1}(z) = 0 \\ (\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N}) \end{gathered} \tag{1.12}$$

has solutions of the form:

$$\phi(z) = K \left(e^{-H(z;p,q)} \right)_{\nu-\mu+1}, \tag{1.13}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (1.11), it being provided that the second member of (1.13) exists.

REMARK 1. As already pointed out in conclusion by Tu *et al.* [5, p. 301], it is fairly straightforward to observe that either or both of the polynomials $P(z;p)$ and $Q(z;q)$, involved in Theorem 1, can be of degree 0 as well. Thus, in the definitions (1.7) and (1.8), and in analogous situations appearing elsewhere in this paper, $\mathbb{N}$ may easily be replaced (if and where needed) by $\mathbb{N}_0$. The definitions (1.7) and (1.8) do serve the *main* purpose in most (if not all) situations including (for example) those occurring in recent works to which this paper is essentially a sequel.

For various interesting applications of Theorem 1, we choose to refer the interested reader to the earlier works [7–16] and also [5,6,17–21], in *each* of which numerous further references on this subject can be found. The main object of the present paper is to continue our investigations of the solutions of some general families of second-order linear ordinary differential equations, which are associated with the familiar Bessel differential equation of general order ν (cf. [29, Chapter 7; 30; 31, Chapter 17])

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\nu^2\right)w=0, \tag{1.14}$$

which is named after Friedrich Wilheim Bessel (1784-1846). More precisely, just as in the earlier work [8], we aim here at demonstrating how the underlying simple fractional-calculus approach to the solutions of the classical differential equation (1.14) would lead us naturally to several interesting consequences including (for example) an alternative investigation of the power-series solutions of (1.14) in terms of the familiar Bessel function $J_\nu(z)$ defined by

$$\begin{aligned} J_\nu(z) &:= \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \\ &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-;\nu+1;-\frac{1}{4}z^2\right) \\ &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\exp(\pm iz)\,{}_1F_1\left(\nu+\frac{1}{2};2\nu+1;\mp 2iz\right), \end{aligned} \tag{1.15}$$

which are derived usually by appealing to the standard method attributed to Ferdinand Georg Frobenius (1849-1917) (cf., e.g., [32, Chapter 16]).

The last hypergeometric ${}_1F_1$ representation in (1.15) follows readily from the usual hypergeometric ${}_0F_1$ representation by means of a familiar hypergeometric transformation which is popularily known as *Kummer's second theorem* (see, for example, [33]).

REMARK 2. It is fairly obvious that the Bessel differential equation (1.14) remains *unaltered* when z is replaced by $-z$ (and also when ν is replaced by $-\nu$), so the functions $J_{\pm\nu}(-z)$ are solutions of the equation (1.14) satisfied by $J_{\pm\nu}(z)$.

2. A FAMILY OF GENERALIZED BESSEL DIFFERENTIAL EQUATIONS

Motivated essentially by the *celebrated* Bessel differential equation (1.14), Lin *et al.* [9] presented a systematic investigation of the following general family of second-order nonhomogeneous *non-Fuchsian* linear ordinary differential equations:

$$Az^2\frac{d^2\varphi}{dz^2}+(Bz+C)\frac{d\varphi}{dz}+\left(Dz^2+Ez+F\right)\varphi(z)=f(z), \tag{2.1}$$

which obviously corresponds to (1.14) when the parameters $A\neq 0$, B, C, $D\neq 0$, E, and F are specialized as follows:

$$A=B=D=1, \qquad C=E=0, \qquad \text{and} \qquad F=-\nu^2. \tag{2.2}$$

Indeed, by applying Theorem 1 in order to find (*explicit*) particular solutions of the nonhomogeneous non-Fuchsian differential equation (2.1), Lin *et al.* [9] deduced the following result.

THEOREM 2. *(See [9, Theorem 3, p. 39].) If the given function f satisfies the constraint (1.6) and $f_{-\nu} \neq 0$, then, the following nonhomogeneous linear ordinary differential equation:*

$$Az^2\frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2 + Ez + F\right)\varphi(z) = f(z) \qquad (A \neq 0;\ D \neq 0), \tag{2.3}$$

has a particular solution in the form:

$$\varphi(z) = z^{\rho}e^{\lambda z}\left(\left(A^{-1}z^{-\nu-1+(2A\rho+B)/A}\cdot e^{2\lambda z}\left(z^{-\rho-1}\cdot e^{-\lambda z}\cdot f(z)\right)_{-\nu}\right)_{-1} \cdot z^{\nu-(2A\rho+B)/A}\cdot e^{-2\lambda z}\right)_{\nu-1} \qquad (A \neq 0; D \neq 0; z \in \mathbb{C}\setminus\{0\}), \tag{2.4}$$

where ρ and λ are given by

$$\rho = \frac{A - B \pm \sqrt{(A-B)^2 - 4AF}}{2A} \qquad \text{and} \qquad \lambda = \pm i\sqrt{\frac{D}{A}}, \tag{2.5}$$

and

$$\nu = \frac{(2A\rho + B)\lambda + E}{2A\lambda}, \tag{2.6}$$

it being provided that the second member of (2.4) exists.

Furthermore, the following homogeneous linear ordinary differential equation:

$$Az^2\frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2 + Ez + F\right)\varphi(z) = 0 \tag{2.7}$$

has solutions of the form:

$$\varphi(z) = Kz^{\rho}e^{\lambda z}\left(z^{\nu-(2A\rho+B)/A}\cdot e^{-2\lambda z}\right)_{\nu-1} \qquad (A \neq 0; D \neq 0; z \in \mathbb{C}\setminus\{0\}), \tag{2.8}$$

where K is an arbitrary constant, ρ and λ are given by (2.5), and ν is given by (2.6), it being provided that the second member of (2.8) exists.

REMARK 3. By *first* setting $\nu \longmapsto \nu + (1/2)$ and *then* specializing the involved parameters A, B, D, E, and F as in (2.2), Theorem 2 would immediately yield the following special case involving the Bessel differential equation (1.14).

THEOREM 3. *(See Nishimoto [34, Theorem 1, p. 27, Theorem 2, p. 29]; see also [9, Corollary 1, p. 40].) Under the hypotheses of Theorem 2, the following nonhomogeneous linear ordinary differential equation:*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \nu^2\right)\varphi(z) = f(z) \tag{2.9}$$

has a particular solution in the form:

$$\varphi(z) = z^{\nu}e^{\lambda z}\left(\left(z^{\nu-(1/2)}\cdot e^{2\lambda z}\left(z^{-\nu-1}\cdot e^{-\lambda z}\cdot f(z)\right)_{-\nu-(1/2)}\right)_{-1}\cdot z^{-\nu-(1/2)}\cdot e^{-2\lambda z}\right)_{\nu-(1/2)} \qquad (\nu \in \mathbb{R}; \lambda = \pm i; z \in \mathbb{C}\setminus\{0\}), \tag{2.10}$$

provided that the second member of (2.10) exists.

Furthermore, the following homogeneous linear ordinary differential equation:

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \nu^2\right)\varphi(z) = 0 \tag{2.11}$$

has solutions of the form:

$$\varphi(z) = Kz^{\nu}e^{\lambda z}\left(z^{-\nu-(1/2)}\cdot e^{-2\lambda z}\right)_{\nu-(1/2)} \qquad (\nu\in\mathbb{R};\lambda=\pm i; z\in\mathbb{C}\setminus\{0\}), \tag{2.12}$$

where K is an arbitrary constant, it being provided that the second member of (2.12) exists.

3. SOLUTIONS OF THE BESSEL DIFFERENTIAL EQUATION (1.14) WHEN $\nu = n$ $(n\in\mathbb{Z})$

In their aforementioned earlier work, Lin *et al.* [8, Sections 3 and 4] applied the assertions of Theorem 3 in order to provide the *complete* solutions of the Bessel differential equation (1.14) in the two cases when

$$\nu = n+\frac{1}{2} \qquad (n\in\mathbb{N}_0) \quad \text{and} \quad \nu\notin\mathbb{Z}.$$

In particular, when $\nu\notin\mathbb{Z}$, by appealing appropriately to some known fractional differintegral formulas, Lin *et al.* [8, Section 4] deduced two *linearly independent* solutions of (1.14) as follows:

$$\begin{aligned}W_{\nu}^{(1)}(z) = K\frac{2^{\nu+1}}{\sqrt{2z}}&\left[\cos\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\&\left.-\sin\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right]\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}W_{\nu}^{(2)}(z) = K\frac{2^{\nu+1}}{\sqrt{2z}}&\left[\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\&\left.-\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right].\end{aligned} \tag{3.2}$$

Thus, by comparing (3.1) and (3.2) with the following known results [30, Equations 7.21 (1) and 7.21 (3), p. 199]:

$$\begin{aligned}J_{\nu}(z) \sim \sqrt{\frac{2}{\pi z}}&\left[\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\&\left.-\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}J_{-\nu}(z) \sim \sqrt{\frac{2}{\pi z}}&\left[\cos\left(z+(1/2)\nu\pi-(1/4)\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\&\left.-\sin\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right],\end{aligned} \tag{3.4}$$

each of which is valid for *large* values of $|z|$ provided that

$$|\arg(z)| \leqq \pi-\varepsilon \qquad (0<\varepsilon<\pi), \tag{3.5}$$

Lin *et al.* [8] eventually arrived at the following *general solution* of (1.14):

$$w(z) = K_1 J_{-\nu}(z) + K_2 J_\nu(z) \qquad (\nu \notin \mathbb{Z}) \tag{3.6}$$

at least for large values of $|z|$ under the constraint (3.5).

With a view to complementing the work by Lin *et al.* [8] by providing a *second* linearly independent solution of the Bessel differential equation (1.14) also in the *exceptional* case when $\nu = n$ ($\nu \in \mathbb{Z}$), we first set

$$\Pi_\nu^{(j)}(z) := \frac{\partial}{\partial \nu}\left\{W_\nu^{(j)}(z)\right\} \qquad (j = 1, 2), \tag{3.7}$$

where $W_\nu^{(j)}(z)$ $(j = 1, 2)$ are given by (3.1) and (3.2), respectively. In terms of the Psi (or Digamma) function $\psi(z)$ defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\, dt, \tag{3.8}$$

we thus find from (3.1) and (3.2) that

$$\begin{aligned}
\Pi_\nu^{(1)}(z) &= K\frac{2^{\nu+1}}{\sqrt{2z}}(\log 2)\left[\cos\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\
&\quad\left.-\sin\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right]\\
&\quad+K\frac{2^{\nu+1}}{\sqrt{2z}}\left[\cos\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}\right.\\
&\quad\cdot\left[\psi\left(\nu+2k+\frac{1}{2}\right)-\psi\left(\nu-2k+\frac{1}{2}\right)\right](2z)^{-2k}\\
&\quad-\sin\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}\\
&\quad\left.\cdot\left[\psi(\nu+2k+(3/2))-\psi(\nu-2k-(1/2))\right](2z)^{-2k-1}\right]\\
&\quad-K\frac{2^{\nu+1}}{\sqrt{2z}}\frac{\pi}{2}\left[\sin\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\
&\quad\left.+\cos\left(z + \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right]
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\Pi_\nu^{(2)}(z) &= K\frac{2^{\nu+1}}{\sqrt{2z}}(\log 2)\left[\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.\\
&\quad\left.-\sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right]
\end{aligned} \tag{3.10}$$

$$+K\frac{2^{\nu+1}}{\sqrt{2z}}\left[\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}\right.$$

$$\cdot\left[\psi\left(\nu+2k+\frac{1}{2}\right)-\psi(\nu-2k+(1/2))\right](2z)^{-2k}$$

$$-\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}$$

$$\left.\cdot\left[\psi\left(\nu+2k+\frac{3}{2}\right)-\psi\left(\nu-2k-\frac{1}{2}\right)\right](2z)^{-2k-1}\right] \qquad (3.10)(\text{cont.})$$

$$+K\frac{2^{\nu+1}}{\sqrt{2z}}\frac{\pi}{2}\left[\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(1/2))}{(2k)!\Gamma(\nu-2k+(1/2))}(2z)^{-2k}\right.$$

$$\left.+\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+(3/2))}{(2k+1)!\Gamma(\nu-2k-(1/2))}(2z)^{-2k-1}\right].$$

Now, since the function emerging from

$$\left((-1)^{\nu}\,\Pi_{\nu}^{(2)}(z)-\Pi_{\nu}^{(1)}(z)\right)\Big|_{\nu\to n} \qquad (n\in\mathbb{Z})$$

is easily seen to be a solution of the Bessel differential equation (1.14) of order $\nu = n$ ($n \in \mathbb{Z}$), upon taking the limit as $\nu \to n$ ($n \in \mathbb{Z}$), we deduce from (3.9) and (3.10) that

$$\Pi_n(z) := \lim_{\nu\to n}\left[(-1)^{\nu}\,\Pi_{\nu}^{(2)}(z)-\Pi_{\nu}^{(1)}(z)\right]$$

$$= K\frac{2^{n+1}}{\sqrt{2z}}\left[\sin\left(z-\frac{1}{2}n\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+(1/2))}{(2k)!\Gamma(n-2k+(1/2))}(2z)^{-2k}\right. \qquad (3.11)$$

$$\left.+\cos\left(z-\frac{1}{2}n\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+(3/2))}{(2k+1)!\Gamma(n-2k-(1/2))}(2z)^{-2k-1}\right] \qquad (n\in\mathbb{Z}).$$

Finally, for the Bessel function of the second kind, it is known that [30, Equation 3.54 (2), p. 64, Equation 7.21 (5), p. 199]

$$Y_n(z) := \lim_{\nu\to n}\left[\frac{\cos(\pi\nu)\,J_{\nu}(z)-J_{-\nu}(z)}{\sin(\pi\nu)}\right]$$

$$= \frac{1}{\pi}\left(\frac{\partial}{\partial\nu}\{J_{\nu}(z)\}-(-1)^n\frac{\partial}{\partial\nu}\{J_{-\nu}(z)\}\right)\Big|_{\nu\to n} \qquad (3.12)$$

$$\sim\sqrt{\frac{2}{\pi z}}\left[\sin\left(z-\frac{1}{2}n\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+(1/2))}{(2k)!\Gamma(n-2k+(1/2))}(2z)^{-2k}\right.$$

$$\left.+\cos\left(z-\frac{1}{2}n\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+(3/2))}{(2k+1)!\Gamma(n-2k-(1/2))}(2z)^{-2k-1}\right] \qquad (n\in\mathbb{Z}),$$

which evidently provides a second linearly independent solution in (3.6) in the *exceptional* case when $\nu = n$ ($n \in \mathbb{Z}$) by means of our simple *fractional-calculus* approach, that is, *without* using the classical Frobenius method for finding power-series solutions of the Bessel differential equation (1.14) of a *general* order ν.

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