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Positive Solutions to Superlinear Semipositone Periodic Boundary Value Problems with Repulsive Weak Singular Forces

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Abstract—This paper is devoted to study the existence of positive solutions to the second-order semipositone periodic boundary value problem $x'' + a(t)x = f(t,x)$, $x(0) = x(1)$, $x'(0) = x'(1)$. Here, $f(t,x)$ may be singular at $x = 0$ and may be superlinear at $x = +\infty$. Our analysis relies on a fixed-point theorem in cones. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Periodic boundary value problem, Singular, Positive solutions, Fixed-point theorem in cones.

1. INTRODUCTION

In this paper, we are devoted to study the existence of positive solutions to periodic boundary value problem,

$$\begin{aligned} x'' + a(t)x = f(t,x), &\qquad 0 \le t \le 1, \\ x(0) = x(1), &\qquad x'(0) = x'(1); \end{aligned} \tag{1.1}$$

here, $a(t) \in L^1[0,1]$ satisfies the conditions under which the corresponding linear system,

$$\begin{aligned} x'' + a(t)x = 0, &\qquad 0 \le t \le 1, \\ x(0) = x(1), &\qquad x'(0) = x'(1); \end{aligned} \tag{1.2}$$

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has a Green function $G(t,s) > 0$ for all $(t,s) \in [0,1] \times [0,1]$. In particular, our nonlinear term $f(t,x)$ may be singular at $x = 0$ and may be superlinear at $x = \infty$.

Generally speaking, problem (1.1) is called *singular* if $f(t,x)$ tends to infinity when $x \to 0^+$. This research work was first opened by Lazer and Solimini [1], in which the model equations,

$$x'' \pm \frac{1}{x^\alpha} = p(t),$$

were studied. Since then, many researchers have been devoted to study the existence of periodic solutions for this type of problems and there are many papers (see, for instance, [2–6] and their references) in literature.

It is said that problem (1.1) has an *attractive singularity* if

$$\lim_{x \to 0^+} f(t,x) = -\infty, \qquad \text{uniformly for } t \in [0,1],$$

and has a *repulsive singularity* if

$$\lim_{x \to 0^+} f(t,x) = +\infty, \qquad \text{uniformly for } t \in [0,1].$$

In some systems like the N-body problem problem, the singularities are of attractive type. The classical technique for proving existence of periodic solutions is the lower and upper solution method for attractive singularities (see [7]). When the singularities are of repulsive type, for the scalar singular equation,

$$x'' + g(t,x) = 0, \qquad x > 0, \tag{1.3}$$

we mention the following results. Let $g(t,x) = g(x) - h(t)$, where $h \in C(\mathbf{R},\mathbf{R})$ is T-periodic and $g \in C((0,\infty),\mathbf{R})$ satisfies the following strong force condition at $x = 0$,

$$\lim_{x \to 0^+} g(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} G(x) = \infty,$$

and g is superlinear at $x = \infty$,

$$\lim_{x \to \infty} \frac{g(x)}{x} = \infty;$$

here, $G(x) = \int^x g(x)\,dx$, Fonda, Manásevich and Zanolin [8] used the Poincaré-Birkhoff theorem to obtain the existence of positive periodic solutions, including all subharmonics. Similarly, del Pino and Manásevich proved in [9] the existence of infinitely many periodic solutions to (1.3), when $g(t,x)$ is superlinear at $x = \infty$ and satisfies the following strong force condition at $x = 0$. There are positive constants c, c', μ, such that $\mu \geq 1$ and

$$c'x^{-\mu} \leq -g(t,x) \leq cx^{-\mu}, \tag{1.4}$$

for all t and all x sufficiently small.

When $g(t,x)$ is semilinear at $x = \infty$, del Pino, Manásevich and Montero [3] proved the existence of at least one positive periodic solution of (1.3) if $g(t,x)$ satisfies (1.4) near $x = 0$ and the following nonresonance conditions at $x = \infty$. There is an integer $k \geq 0$ and a small constant $\varepsilon > 0$, such that

$$\left(\frac{k\pi}{T}\right)^2 + \varepsilon \leq \frac{g(t,x)}{x} \leq \left(\frac{(k+1)\pi}{T}\right)^2 - \varepsilon, \tag{1.5}$$

for all t and all $x \gg 1$. We note that conditions (1.5) are the standard uniform nonresonance conditions with respect to the antiperiodic boundary condition, not with respect to the periodic boundary condition. For example,

$$x'' + \mu x = \frac{1}{x^3} + h(t), \tag{1.6}$$

where $\mu > 0$ and $h \in C(\mathbf{R}, \mathbf{R})$ is 2π-periodic. Nonresonance holds when

$$\mu \neq \left(\frac{k}{2}\right)^2, \qquad k = 1, 2, \dots,$$

i.e., μ is not an eigenvalue of the antiperiodic boundary value problem. Moreover, the author in [5,6] used the coincidence degree theory of Mawhin to study the existence of positive 2π-periodic solutions to the following scalar singular semilinear equations,

$$\begin{aligned} x'' + f(x)x' + g(t,x) = 0, \qquad & 0 \le t \le 2\pi, \\ x(0) = x(2\pi), \qquad & x'(0) = x'(2\pi); \end{aligned} \tag{1.7}$$

here, $g \in C(\mathbf{R} \times (0,\infty), \mathbf{R})$ satisfies the strong force condition at $x = 0$.

In the references mentioned above, two most common techniques have frequently been employed:

(1) the obtention of *priori* bounds for the possible solutions and then the applications of topological degree arguments [10] and
(2) the theory of upper and lower solutions [7].

These two techniques have often been interconnected and have proved to be very strong and fruitful and became very popular in this research area. However, the above two techniques have their own limitations and in fact, for practical purposes, serious difficulties arise frequently in the search for *priori* bounds or upper and lower solutions.

On the other hand, some fixed-point theorems in a cone for completely continuous operators have been extensively employed in the related literature, specially to study several kinds of separated boundary value problems (see for instance in [11,12] and their references), while for the periodic boundary value problems, it is more difficult to find references, and only very recently, papers [13, 14] are known to us. The reason for this contrast may be the fact that it is more difficult to perform a study of the sign of Green's function for the corresponding linear periodic problems. In paper [14], the author succeeded in overcoming this difficulty by using a new L^p−maximum principle developed in [15] and obtained some new existence results to problem (1.1).

In this paper, we will exploit some results developed in [14], together with a fixed-point theorem in cones, to study the existence of positive solutions to problem (1.1).

REMARK 1.1. By a positive solution of problem (1.1) we understand a function $x \in C[0,1]$, $x' \in AC[0,1]$ with $x(t) > 0$ for all $t \in [0,1]$ and satisfying (1.1) for a.e., $t \in [0,1]$.

This paper is organized as follows. In Section 2, some preliminary results will be given, which will be used in Section 3. In Section 3, we are devoted to the existence results for the singular semipositone case, i.e., $f(t,x) : [0,1] \times (0,\infty) \to \mathbf{R}$ is continuous, $f(t,x) \to +\infty$ when $x \to 0^+$ and there exists a $M > 0$ such that $f(t,x) + M \ge 0$ for all $(t,x) \in [0,1] \times (0,\infty)$. In this case, we prove that the weak singularity of $f(t,x)$ at $x = 0$ is allowed, as revealed in [13,14]. In the context of repulsive singularities, it is usual to assume some kind of strong force condition, which means roughly that the potential in zero is infinity. Typically, this condition is employed to obtain *priori* bounds of the solutions. In paper [1], it is proved that the strong singulary condition cannot be dropped without further assumptions, and in fact such a condition has become standard in the related literature. Recently, Rachunková *et al.* [13] have obtained for the first time existence results in the presence of weak singularities, by using topological degree arguments. In our case, we are able to deal also with weak singularities because the strong force conditions are not needed in Theorem 3.1.

To conclude this section, we state here a well-known fixed-point theorem in cones [16], which will be used in Section 3 and Section 4.

THEOREM 1.1. *Let X be a Banach space, and $K(\subset X)$ be a cone. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and compact operator such that either

(i) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. SOME PRELIMINARY RESULTS

In this section, we present some preliminary results which will be needed in Sections 3 and 4.

First, we fix some notations to be used in the following: Given $a \in L^1[0,1]$, we write $a \succ 0$ if $a \geq 0$ for a.e. $t \in [0,1]$ and it is positive in a subset of positive measure. The usual L^p-norm is denoted by $\|.\|_p$, whereas $\|.\|$ is used for the norm of the supermum.

Now let us consider the linear periodic boundary value problem,

$$\begin{aligned} x'' + a(t)x = 0, \qquad & 0 \leq t \leq 1, \\ x(0) = x(1), \qquad & x'(0) = x'(1). \end{aligned} \tag{2.1}$$

Throughout this paper, we assume the conditions under which the only solution of problem (2.1) is the trivial one. As a consequence of Fredholm's alternative, we have the following result.

LEMMA 2.1. *Suppose $h : [0,1] \to [0,\infty)$ is continuous. Then, the boundary value problem.*

$$\begin{aligned} x'' + a(t)x = h(t), \qquad & 0 \leq t \leq 1, \\ x(0) = x(1), \qquad & x'(0) = x'(1); \end{aligned} \tag{2.2}$$

has a unique solution that can be written as

$$x(t) = \int_0^1 G(t,s)\,h(s)\,ds, \tag{2.3}$$

where $G(t,s)$ is the Green's function of problem (2.1).

In order to state the next result, the following best Sobolev constants will be used,

$$\mathbf{K}(q) = \begin{cases} \dfrac{2\pi}{q}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{if } 1 \leq q < \infty, \\ 4, & \text{if } q = \infty, \end{cases} \tag{2.4}$$

where Γ is the Gamma function. For a given p, let us define

$$p^* = \begin{cases} \dfrac{p}{p-1}, & \text{if } 1 \leq p < \infty, \\ 1, & \text{if } p = \infty. \end{cases}$$

Now, the following result follows immediately from [14].

LEMMA 2.2. *Assume that $a(t) \succ 0$ and $a \in L^p[0,1]$ for some $1 \leq p \leq \infty$. If*

$$\|a\|_p < K(2p^*), \tag{2.5}$$

then $G(t,s) > 0$ for all $(t,s) \in [0,1] \times [0,1]$.

REMARK 2.1. If $p = +\infty$, then hypothesis (2.5) is equivalent to $\|a\|_\infty < \pi^2$, which is a well-known criterion for the maximum principle yet used in the related literature.

In order to present our results briefly, let us define the set of functions,

$$\Lambda = \left\{a \in L^1[0,1] : a \succ 0,\ \|a\|_p < K(2p^*) \text{ for some } 1 \leq p \leq \infty\right\}.$$

It follows from Lemma 2.2 that problem (2.1) has a Green's function $G(t,s) > 0$, for all $(t,s) \in [0,1] \times [0,1]$ if $a \in \Lambda$. In particular, if $A = \min_{0\le s,t\le 1} G(t,s)$ and $B = \max_{0\le s,t\le 1} G(t,s)$. then $B > A > 0$ for $a \in \Lambda$.

REMARK 2.2. As we all know, we can compute the maximum B and the minimum A of the Green's function $G(t,s)$ when $a(t) = m^2 (0 < m < \pi)$, obtaining

$$A = \frac{1}{2m} \cot\left(\frac{m}{2}\right), \qquad B = \frac{1}{2m \sin(m/2)}, \qquad \text{and} \qquad \sigma = \frac{A}{B} = \cos\left(\frac{m}{2}\right).$$

These explicit values will be employed in Sections 3 and 4.

Let $X = C[0,1]$ and define

$$K = \left\{ x \in X : x(t) \ge 0 \text{ and } \min_{0\le t\le 1} x(t) \ge \sigma \|x\| \right\}, \tag{2.6}$$

here $\sigma = A/B$ and $\|x\| = \max_{0\le t\le 1} |x(t)|$.

One may readily verify that K is a cone in X. Finally, we define an operator $T: X \to K$ by

$$(Tx)(t) = \int_0^1 G(t,s)F(s,x(s))\,ds \tag{2.7}$$

for $x \in X$ and $t \in [0,1]$, where $F : [0,1] \times \mathbf{R} \to [0,\infty)$ is continuous and $G(t,s)$ is the Green function to problem (2.1).

LEMMA 2.3. *$T : X \to K$ is well defined.*

PROOF. Let $x \in X$, then we have

$$\|Tx\| \le B \int_0^1 F(s,x(s))\,ds \qquad \text{and} \qquad (Tx)(t) \ge A \int_0^1 F(s,x(s))\,ds.$$

Therefore,

$$(Tx)(t) \ge \frac{A}{B} \|Tx\|, \qquad \text{i.e., } Tx \in K.$$

This completes the proof. ∎

Finally, it is easy to prove the following.

LEMMA 2.4. *$T: X \to K$ is continuous and completely continuous.*

3. SEMIPOSITONE CASE

In this section, we establish the existence of positive solutions to the periodic boundary value problem,

$$\begin{aligned} x'' + a(t)x &= f(t,x), & 0 \le t \le 1, \\ x(0) &= x(1), & x'(0) = x'(1); \end{aligned} \tag{4.1}$$

here, $a(t) \in \Lambda$ and $f(t,x)$ may be singular at $x = 0$. In particular, our nonlinear term $f(t,x)$ may be superlinear at $x = +\infty$ and may take on negative values. We are interested in working out what weak force conditions of $f(t,x)$ at $x = 0$ and what superlinear growth conditions of $f(t,x)$ at $x = +\infty$ are needed to obtain the existence of positive solutions to problem (3.1). Throughout this section, we assume the following conditions hold.

(B_1) $a(t) \in \Lambda$.

(B_2) $f : [0,1]\times(0,\infty) \to \mathbf{R}$ is continuous and there exists a constant $M > 0$ with $f(t,x)+M \ge 0$ for all $t \in [0,1]$ and $x \in (0,\infty)$.

(B_3) $F(t,x) = f(t,x) + M \le g(x) + h(x)$ for $(t,x) \in [0,1] \times (0,\infty)$ with $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \ge 0$ continuous on $(0,\infty)$ and h/g nondecreasing on $(0,\infty)$.

(B_4) There exists

$$r > \frac{M\|\omega\|}{\sigma},$$

such that

$$\frac{r}{g(\sigma r - M\|\omega\|)\{1 + h(r)/g(r)\}} \geq \|\omega\|,$$

here $\sigma = A/B$, $\|\omega\| = \max_{0\leq t\leq 1} |\omega(t)|$, and $\omega(t)$ is a unique solution to problem,

$$\begin{gathered} x'' + a(t)\, x = 1, \\ x(0) = x(1), \qquad x'(0) = x'(1). \end{gathered} \tag{3.2}$$

(B_5) $F(t,x) = f(t,x) + M \geq g_1(x) + h_1(x)$ for all $(t,x) \in [0,1] \times (0,\infty)$ with $g_1 > 0$ continuous and nonincreasing on $(0,\infty)$, $h_1 \geq 0$ continuous on $(0,\infty)$ and h_1/g_1 nondecreasing on $(0,\infty)$.

(B_6) There exists $R > r$, such that

$$\frac{R}{\sigma g_1(R)\{1 + (h_1(\sigma R - M\|\omega\|))/(g_1(\sigma R - M\|\omega\|))\}} \leq \|\omega\|,$$

here, σ and $\omega(t)$ are the same as in (B_4).

THEOREM 3.1. *Suppose Conditions (B_1)–(B_6) hold, then problem (3.1) has a solution $x \in C[0,1]$, $x' \in AC[0,1]$ with $x(t) > 0$ for $t \in [0,1]$ and $r \leq \|x + M\omega\| \leq R$.*

PROOF. To show (3.1) has a positive solution, we will show

$$\begin{aligned} x'' + a(t)\, x &= F(t, x(t) - M\omega(t)), && 0 \leq t \leq 1, \\ x(0) &= x(1), && x'(0) = x'(1); \end{aligned} \tag{3.3}$$

has a solution $x \in C[0,1], x' \in AC[0,1]$ with $x(t) > M\omega(t)$ for $t \in [0,1]$ and $r \leq \|x\| \leq R$.

If this is true, then $u(t) = x(t) - M\omega(t)$ is a positive solution of (3.1) and $r \leq \|u + M\omega\| \leq R$, since

$$\begin{aligned} u''(t) + a(t)\, u(t) &= x''(t) - M\omega''(t) + a(t)\, x(t) - Ma(t)\, \omega(t) \\ &= F(t, x(t) - M\omega(t)) - M \\ &= f(t, x(t) - M\omega(t)) \\ &= f(t, u(t)), \end{aligned}$$

for all $t \in [0,1]$.

As a result, we will only concentrate our study on (3.3).

Let $X = C[0,1]$ and K be a cone in X defined by (2.6). Let

$$\Omega_r = \{x \in X : \|x\| < r\}, \qquad \Omega_R = \{x \in X : \|x\| < R\},$$

and define the operator $T : K \cap (\bar{\Omega}_R \setminus \Omega_r) \to K$ by

$$(Tx)(t) = \int_0^1 G(t,s)\, F(s, x(s) - M\omega(s))\, ds, \qquad 0 \leq t \leq 1, \tag{3.4}$$

where $G(t,s)$ is the Green function to problem (2.1).

Since $r \leq \|x\| \leq R$ for any $x \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$, thus, $0 < \sigma r - M\|\omega\| \leq x(s) - M\omega(s) \leq R$. Since $F : [0,1] \times [\sigma r - M\|\omega\|, R] \to [0,\infty)$ is continuous, it follows from Lemma 2.3 and Lemma 2.4 that the operator $T : K \cap (\bar{\Omega}_R \setminus \Omega_r) \to K$ is well defined and is continuous and completely continuous.

First, we show

$$\|Tx\| \le \|x\|, \qquad \text{for } x \in K \cap \partial\Omega_r. \tag{3.5}$$

In fact, if $x \in K \cap \partial\Omega_r$, then $\|x\| = r$ and $x(t) \ge \sigma r > M\|\omega\|$ for $0 \le t \le 1$. So, we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\, F(s, x(s) - M\omega(s))\, ds \\
&\le \int_0^1 G(t,s)\, g(x(s) - M\omega(s)) \left\{1 + \frac{h(x(s) - M\omega(s))}{g(x(s) - M\omega(s))}\right\} ds \\
&\le \int_0^1 G(t,s)\, g(\sigma r - M\|\omega\|) \left\{1 + \frac{h(r)}{g(r)}\right\} ds \\
&= \omega(t)\, g(\sigma r - M\|\omega\|) \left\{1 + \frac{h(r)}{g(r)}\right\} \\
&\le \|\omega\|\, g(\sigma r - M\|\omega\|) \left\{1 + \frac{h(r)}{g(r)}\right\} \\
&\le r = \|x\|,
\end{aligned}$$

for $t \in [0,1]$, since $\sigma r - M\|\omega\| \le x(s) - M\omega(s) \le r$.

This implies $\|Tx\| \le \|x\|$, i.e., (3.5) holds.

Next, we show

$$\|Tx\| \ge \|x\|, \qquad \text{for } x \in K \cap \partial\Omega_R. \tag{3.6}$$

To see this, let $x \in K \cap \partial\Omega_R$, then $\|x\| = R$ and $x(t) \ge \sigma R > M\|\omega\|$ for $0 \le t \le 1$. As a result, it follows from (B_5) and (B_6) that, for $0 \le t \le 1$,

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\, F(s, x(s) - M\omega(s))\, ds \\
&\ge \int_0^1 G(t,s)\, g_1(x(s) - M\omega(s)) \left\{1 + \frac{h_1(x(s) - M\omega(s))}{g_1(x(s) - M\omega(s))}\right\} ds \\
&\ge \int_0^1 G(t,s)\, g_1(R) \left\{1 + \frac{h_1(\sigma R - M\|\omega\|)}{g_1(\sigma R - M\|\omega\|)}\right\} ds \\
&= \omega(t)\, g_1(R) \left\{1 + \frac{h_1(\sigma R - M\|\omega\|)}{g_1(\sigma R - M\|\omega\|)}\right\} \\
&\ge \sigma\|\omega\|\, g_1(R) \left\{1 + \frac{h_1(\sigma R - M\|\omega\|)}{g_1(\sigma R - M\|\omega\|)}\right\} \\
&\ge R = \|x\|,
\end{aligned}$$

since $\sigma R - M\|\omega\| \le x(s) - M\omega(s) \le R$.

This implies $\|Tx\| \ge \|x\|$, i.e., (3.6) holds.

Now, (3.5), (3.6), and Theorem 1.1 guarantee that T has a fixed point $x \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \le \|x\| \le R$. Clearly, this x is a positive solution of (3.3). ∎

EXAMPLE 3.1. Let us consider the following periodic boundary value problem,

$$\begin{aligned}
x'' + a(t)\,x &= \mu\left(x^{-\alpha} + x^{\beta} + k(t)\right), &\qquad 0 \le t \le 1, \\
x(0) &= x(1), & x'(0) = x'(1);
\end{aligned} \tag{3.7}$$

where $a(t) \in \Lambda, a^* = \operatorname{ess\,sup} a(t) < \infty$, $\alpha > 0$, $\beta > 1$, and $k : [0,1] \to \mathbf{R}$ is continuous, $\mu > 0$ is chosen such that

$$\mu < \sup_{x \in ((M\|\omega\|)/\sigma, \infty)} \frac{x(\sigma x - M\|\omega\|)^{\alpha}}{\|\omega\|\left\{1 + 2Hx^{\alpha} + x^{\alpha+\beta}\right\}}, \tag{3.8}$$

here $H = \|k\|$. Then, problem (3.7) has a positive solution $x \in C[0,1]$, $x' \in AC[0,1]$.

To see this, we will apply Theorem 3.1 with $M = \mu H$ and

$$g(x) = g_1(x) = \mu x^{-\alpha}, \qquad h(x) = \mu\left(x^{\beta} + 2H\right), \qquad h_1(x) = \mu x^{\beta}.$$

Clearly, (B_1)–(B_3) and (B_5) are satisfied.

Set

$$T(x) = \frac{x\left(\sigma x - M\|\omega\|\right)^{\alpha}}{\|\omega\|\left\{1 + 2Hx^{\alpha} + x^{\alpha+\beta}\right\}}, \qquad x \in \left(\frac{M\|\omega\|}{\sigma}, +\infty\right).$$

Since $T((M\|\omega\|)/\sigma) = 0$, $T(\infty) = 0$, then there exists $r \in ((M\|\omega\|)/\sigma, \infty)$, such that

$$T(r) = \sup_{x \in ((M\|\omega\|)/\sigma, \infty)} \frac{x\left(\sigma x - M\|\omega\|\right)^{\alpha}}{\|\omega\|\left\{1 + 2Hx^{\alpha} + x^{\alpha+\beta}\right\}}.$$

This implies that there exists

$$r \in \left(\frac{M\|\omega\|}{\sigma}, \infty\right),$$

such that

$$\mu < \frac{r\left(\sigma r - M\|\omega\|\right)^{\alpha}}{\|\omega\|\left\{1 + r^{\alpha+\beta}\right\}},$$

so (B_4) is satisfied.

Finally notice (B_6) is satisfied for R large enough since

$$\begin{aligned} &\frac{R}{\sigma g_1(R)\left\{1 + \left(h_1(\sigma R - M\|\omega\|)\right) / \left(g_1(\sigma R - M\|\omega\|)\right)\right\}} \\ &= \frac{R^{\alpha+1}}{\sigma\mu\left(1 + (\sigma R - M\|\omega\|)^{\alpha+\beta}\right)} \to 0 \qquad (\text{as } R \to \infty), \end{aligned}$$

since $\beta > 1$. Thus, all the conditions of Theorem 3.1 are satisfied, so the existence is guaranteed.

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