A neighborhood condition which implies the existence of a complete multipartite subgraph

Ralph J. Faudree
Memphis State University, USA

Debra J. Knisley
Mathematics Department, East Tennessee State University, Box 22390A, Johnson City, TN 37614, USA

Received 22 June 1990
Revised 17 September 1991

Abstract
Faudree, R.J. and D.J. Knisley, A neighborhood condition which implies the existence of a complete multipartite subgraph, Discrete Mathematics 118 (1993) 57–68.

Given a graph G and u∈V(G), the neighborhood N(u)= {v∈V(G) | uv∈E(G)}.

We define NCd(G) = \min \left\{ \left| \bigcup_{i=1}^{k} N(u_i) \right| \right\}

where the minimum is taken over all k-independent sets \{u_1, \ldots, u_k\} of vertices in V(G).

We shall show that if G is a graph of order n that satisfies the neighborhood condition

\[ NC_d(G) > \frac{d-2}{d-1} n + cn^{1-\varepsilon} \]

for some real number \( c = c(m, d, k, r) \) then for sufficiently large n, G contains at least one copy of \( K(r, m, \ldots, m_d) \) where \( m_i = m \) for each \( i \) and \( r > m \). When \( r = 1, 2 \) or 3, this result is best possible.

Definitions and notations
The complete multipartite graph with \( d \) partite sets, each containing \( m \) vertices, will be denoted by \( K(d; m) \). If all of the partite sets are not of the same cardinality, the graph \( K(m_1, m_2, \ldots, m_d) \) is the complete \( d \)-partite graph with \( m_i \) vertices in each partite set, \( 1 \leq i \leq d \). The special case where \( m_i = 1 \) for every \( i \), yields the complete graph on \( d \) vertices and is denoted by \( K_d \). The join of graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 + G_2 \), has \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{ uv | u \text{ is an element of } V(G_1), v \text{ is an element of } V(G_2) \} \). Thus, for example, if \( G = K_2 + K(3; m) \),
then \( G = K(1, 1, m, m, m) \). If a graph \( G \) consists of \( t \) vertex disjoint copies of a graph \( H \), then \( G = tH \). Given a graph \( G \) and \( u \in V(G) \), the neighborhood \( N(u) = \{v \in V(G) \mid uv \in E(G)\} \). We define \( NC_k(G) = \min \{ \sum_{i=1}^{k} |N(u_i)| \} \) where the minimum is taken over all \( k \) independent sets \( \{u_1, \ldots, u_k\} \) of vertices in \( V(G) \). We shall frequently want to consider the cardinality of the union of the neighborhoods of an arbitrary collection of \( k \) independent vertices in a subset \( V \subseteq V(G) \). Let \( V^k \) denote an arbitrary collection of \( k \) independent vertices in \( V \). Then the inequality \( |N(V^k)| > s \) means that the cardinality of the union of the neighborhoods of any \( k \) independent vertices in \( V \) exceeds \( s \). Let \( N_V(x) \) denote the set of all vertices in \( V \) that are adjacent to \( x \). Note that since \( |N(x)| = d(x) \), by \( d_V(x) \) we mean \( |N_V(x)| \). Let \( H \) be a graph, \( v \in V(H) \). The graph, \( H_v(t) \), \( t \) a positive integer, is the graph obtained from \( H \) by replacing \( v \) with \( t \) copies of itself, \( v_1, \ldots, v_t \), such that \( N_H(v) = N_H(v_i) \) for every \( i \). Given that \( H \) is a subgraph of \( G \), let \( n_G(H) \) denote the number of isomorphic copies of \( H \) in \( G \). The smallest integer \( p \) such that any graph on \( p \) vertices either contains a graph \( G \) or its complement contains \( H \) is denoted by \( r(G, H) \). The number \( r(G, H) \) is the Ramsey number of the pair of graphs \( G \) and \( H \).

Other notations and definitions not found in the preceding discussion generally follow that of [3].

**Introduction**

The concept of a neighborhood condition is not new. However, until recently, there were few neighborhood condition type results. In extremal theory in particular, most results are either based on edge density or minimum degree requirements. One of the earliest results of this nature was established by Turán in 1941 [10]. He established the sufficient number of edges a graph \( G \) of order \( n \) must have to ensure that \( G \) contains a complete graph on \( p \) vertices. In fact, he characterized those graphs which contain the maximum number of edges and fail to contain such a subgraph. Alon, Faudree and Furedi [1] recently showed that if \( G \) is a graph of order \( n \) that satisfies the neighborhood condition

\[
NC_k(G) > \frac{d-2}{d-1} n,
\]

then for sufficiently large \( n \), \( G \) contains a \( K_d \) as a subgraph. The Turán Graphs verify that this result is best possible. Their theorem can be viewed as a neighborhood analogue to Turán's Theorem. However, it differs from Turán's in several significant ways. One, it is asymptotic in nature. Let \( k = 2 \) and let \( G = C_5 \). Then any pair of nonadjacent vertices are collectively adjacent to \( \frac{2}{5} n > \frac{1}{2} n \), but a \( C_5 \) obviously does not contain a \( K_3 \) as a subgraph. The expression for \( n \) sufficiently large means that there exist an \( n_0 \) such that if \( n \geq n_0 \), then given any graph of order \( n \), the conclusion follows. Thus, \( n_0 > 5 \) in the above illustration. There have been other recent results involving
neighborhood conditions. Gould and Jacobson introduced a neighborhood condition that was similar to the Ore type degree condition.

They, along with Faudree and Schelp established the following result [7].

**Theorem 1.** If G is a 2-connected graph of order n > 2 such that the union of the neighborhoods of each pair of nonadjacent vertices is of cardinality at least \((2n-1)/3\), then G is hamiltonian.

For a survey on other recent results involving neighborhood unions, see [9]. In this paper we are interested in the Alon, Faudree, Füredi result and in the following result due to Erdős and Stone. Five years after Turán’s theorem, Erdős and Stone [5] proved that for every natural number \(d\) and every \(\varepsilon > 0\), if \(n\) is sufficiently large and

\[
m \geq \left(1 - \frac{1}{d-1}\right) \frac{n^2}{2} + \varepsilon n^2 + O(n),
\]

then every graph on \(n\) vertices with \(m\) edges contains a complete \(d\)-partite graph with arbitrarily large vertex classes. Considering the fact that the Alon, Faudree, Füredi result determines the least value of \(s\) such that if \(G\) is any graph of sufficiently large order \(n\) that satisfies the neighborhood condition \(NC_k > s\) implies \(G\) contains a \(K_d\) as a subgraph, we would now like to find the least value of \(q\) such that if \(G\) is any graph of sufficiently large order \(n\) that satisfies the neighborhood condition \(NC_k > q\), then \(G\) contains a \(K(d; m)\) as a subgraph. Since \(K_d\) is a subgraph of \(K(d; m)\) we conclude that

\[
q \geq \frac{d-2}{d-1} n.
\]

The first question we might ask is, does the coefficient of \(n\) increase? Is \(q\) perhaps \(((d-2)/(d-1) + \varepsilon)n\) for some \(\varepsilon > 0\). The answer is no. This was answered by Faudree, Gould, Jacobson and Lesniak in [6] in the following theorem. Here, \(S(H)\) is the order of the largest color class of \(H\) in a \(\chi(H)\)-coloring of the vertices of \(H\).

**Theorem 2.** Let \(k\) and \(d\) be fixed positive integers \(d \geq 2\). Let \(G\) be a graph of order \(n\), and \(H\) a graph with chromatic number \(d\), such that \(S(H) = S\). If \(G\) satisfies the neighborhood condition

\[
NC_k \geq \frac{d-2}{d-1} n + \frac{1}{d-1} n^x \quad \text{where} \quad 1 - \frac{dS}{S_d} < x < 1,
\]

then for sufficiently large \(n\), \(G\) contains \(H\).

Let \(H = K(d; m)\). Then \(H\) has chromatic number \(d\) and \(S(H) = m\). By the above theorem, if \(G\) is any graph of sufficiently large order \(n\) such that \(G\) satisfies the
neighborhood condition
\[ NC_k \geq \frac{d-2}{d-1} n + \frac{1}{d-1} n^a, \quad \text{where } 1 - \frac{d}{m^{d-1}} < \alpha < 1, \]

then \( G \) contains \( H \). However, let \( H = K(1; r) + K(d-1; m) \) where \( r \leq m \) and note that \( \alpha \) depends on \( d \) and \( m \), the order of the largest color class, and not on \( r \). They were not able to conclude at that time that their result is best possible. But it gives us the following bounds for \( q \). We conclude at this point that
\[ \frac{d-2}{d-1} n \leq q \leq \frac{d-2}{d-1} n + cn^a \quad \text{where } 1 - \frac{d}{m^{d-1}} < \alpha < 1. \]

Throughout this paper we shall frequently claim that for \( n \) sufficiently large, there exists a real number \( c \) such that if \( G \) satisfies the neighborhood condition
\[ NC_k \geq \frac{d-2}{d-1} n + cn^a, \]

then for some integer \( i \), \( |D| \geq cn^a \) where \( D \subset V(G) \). It is understood that \( c \) and \( c_i \) depend on many values such as the value of \( k \) in the \( k \)-neighborhood condition, the values of \( d, m \) and \( r \) or other constants specified in the hypothesis. These values, however, are always fixed and the conclusion only follows when \( n \) is sufficiently large.

Our first objective is to verify the claim that our result is best possible if \( r = 1, 2 \) or 3. In general, very little is known about the extremal graphs of the \( K(r, s) \). However, Kovári, Sós, and Turán [8] determined an upper bound for \( ex(n, K(r, s)) \). They prove the following.

**Theorem 3.** For \( r \leq s \),
\[ ex(n, K(r, s)) \leq \frac{1}{2(s-1)^{1/r}} n^{2-1/r} + O(n). \]

When \( r = 1, 2, \) or 3 this bound is sharp. Erdős, Renyi and Sós provided a geometrical construction in [4] which shows the result is sharp for the \( K(2, 2) \) and Brown [2] provided a geometrical construction which shows it is sharp for the \( K(3, 3) \). We now consider the following modification of Turán’s graph. Let \( G \) be a graph on \( n \) vertices. Partition \( V(G) \) into \( d-1 \) classes which we will denote by \( V_i \), \( 1 \leq i \leq d-1 \). For some constant \( c \), let
\[ |V_i| = \frac{1}{d-1} n - cn^{2/3} \quad \text{when } 2 \leq i \leq d-1 \]
and let
\[ |V_1| = \frac{1}{d-1} n + (d-2)cn^{2/3}. \]
Using the vertices in $V_1$, construct Brown's graph. Then, for some constant $c_1$, this graph satisfies the neighborhood condition

$$NC_k \geq \frac{d-2}{d-1} n + c_1 n^{2/3},$$

but it clearly contains no $K(1, 3) + K(d-1; m)$. This implies that in the case when $r = 3$,

$$q \geq \frac{d-2}{d-1} n + cn^{2/3}.$$

It is generally believed that the bound in [S] is sharp for all $r$. If this is indeed the case, then

$$q = \frac{d-2}{d-1} n + cn^{1-1/r}$$

is the best possible neighborhood condition that would insure that a graph contain a $K(1; r) + K(d-1; m)$ as a subgraph. The fact that this neighborhood condition is sufficient in our main result. The theorem is stated below.

**Theorem A.** Let $G$ be a graph of order $n$, and let $k, m, d \geq 2$ and $r, r \leq m$ be positive integers. There exist a constant $c = c(r, m, k, d)$ such that if $G$ satisfies the neighborhood condition

$$NC_k > \frac{d-2}{d-1} n + cn^{1-1/r},$$

then for $n$ sufficiently large, $G$ contains a $K(1; r) + K(d-1; m)$. Furthermore, if $r = 1, 2$ or 3 this result is best possible.

**Preliminary lemmas**

**Lemma 1** ([1]). Let $t$ be a fixed positive integer and $H$ a fixed graph of order $p$. If $G$ is any graph of sufficiently large order $n$ with

$$n(G)(H) = h \quad \text{where} \quad h \geq c_1 n^{r-1}, \quad \text{for some} \quad c_1 = c_1(H),$$

then there is a constant $c = c(p, t)$ such that

$$n_G(H_v(t)) \geq \left\lfloor \frac{ch^t}{n^{(p-1)(t-1)}} \right\rfloor$$

for any vertex of $v$ of $H$.

**Lemma 2** ([1]). Let $k, d \geq 2$ be integers and let $G$ be a graph of sufficiently large order $n$ that satisfies

$$NC_k > \frac{d-2}{d-1} n.$$
Then, there exist positive constants $c = c(d, k)$ and $c' = c'(d, k)$ such that:

1. $n_G(K_{d-1}) \geq \lceil c n^{d-1} \rceil$
2. $n_G(K(d-1; t)) \geq \lceil c'n^{(d-1)t} \rceil$

The following lemma shows that the copies of the complete multipartite graphs $K(d-1; t)$ in $G$ guaranteed by the preceding lemma, are induced copies if $G$ does not contain a $K(1; r) + K(d-1; m)$ for some integers $m$ and $r$.

**Lemma 3.** Let $t, k, d > 1$ and $m > r > 0$ be integers. Let $G$ be a graph of order $n$ that satisfies the neighborhood condition

$$NC_k \geq \frac{d-2}{d-1} n$$

and does not contain a $K(1; r) + K(d-1; m)$. Then, for $n$ sufficiently large, there exists a real number $c = c(r, m, d, t)$ such that $G$ contains at least $cn^{(d-1)t}$ induced copies of a $K(d-1; t)$.

**Proof.** Let $G$ be a graph that satisfies all the conditions of the lemma. Note that $r(K(r, m), K_r)$ is bounded above by some integer $s = s(m, t, r)$. By Lemma 2, for $n$ sufficiently large, $G$ contains $c_1 n^{(d-1)s}$ copies of a $K(d-1; s)$. Let $K$ be a fixed copy of a $K(d-1; s)$ and let $V_i$, $1 \leq i \leq d-1$, denote the partite sets of $K$. Clearly, if the graph induced by $V_i$ for any $i$ contains a $K(r, m)$, then $G$ contains a $K(1; r) + K(d-1; m)$. Hence, the graphs induced by the $V_i$'s cannot contain a $K(r, m)$ as a subgraph. By our choice of $s$, each $V_i$ must contain a collection of $t$ independent vertices. Hence $K$ contains an induced $K(d-1; t)$ which we will denote by $K'$. For $n$ sufficiently large, the number of ways $K'$ can appear as a subgraph of some $K(d-1; s)$ is at most

$$\binom{n-(d-1)t}{s-t} \binom{n-(d-1)t-(s-t)}{s-t} \cdots \binom{n-(d-1)t-(d-2)(s-t)}{s-t} \geq c_2 n^{(d-1)(s-t)}$$

Since $G$ contains at least $c_1 n^{(d-1)s}$ copies of $K(d-1; s)$, dividing out multiplicities we have that there are at least

$$\frac{c_1 n^{(d-1)s}}{c_2 n^{(d-1)(s-t)}} = cn^{(d-1)t}$$

induced copies of $K(d-1; t)$ in $G$. \qed

**Lemma 4.** Let $d, k \geq 2$ and $m \geq r > 0$ be positive integers and let $G$ be a graph of order $n$ that does not contain a $K(1; r) + K(d-1; m)$. Let $c = c(r, m, d, k)$ be a real number such that $0 \leq c < 1$. Then there exists a real number $c = c(r, m, d, k)$ such that if $G$ satisfies the neighborhood condition

$$NC_k \geq \frac{d-2}{d-1} n + cn^k,$$
then for sufficiently large \( n \) we may choose an integer \( t \geq k \) such that \( G \) contains an \( K(d-1; t) \) with partite sets \( \{ V_i \mid 1 \leq i \leq d-1 \} \), and a set \( C \subseteq V(G) \) where \( |C| \geq c_1 n^a \) for some real number \( c_1 \). Furthermore, each vertex in \( C \) has at least one adjacency in each of the \( V_i \)'s and

\[
\left| \bigcap_{i=j+1}^{d-1} N_C(V_i^+) \right| \geq \frac{j}{d-1} |C| \quad \text{where } 1 \leq j \leq d-2.
\]

**Proof.** Let \( G \) be a graph of order \( n \) that satisfies the conditions of the lemma. Thus, by Lemma 3, \( G \) contains \( c'n^{(d-1)t} \) copies of an induced \( K(d-1; t) \) for some real number \( c' \).

Let \( K \) denote a fixed copy with partite sets \( \{ V_i \mid 1 \leq i \leq d-1 \} \). Partition \( V(G - K) \) as follows. Let

\[ A_i = \{ a \in V(G - K) \mid a \notin E(G) \text{ for any } v_i \in V_i \}. \]

Let \( C \) denote the collection of all remaining vertices. Since the labeling of the \( V_i \)'s is completely arbitrary, without loss of generality we may assume that \( |A_1| \leq |A_2| \leq \cdots \leq |A_{d-1}| \). We claim

\[ |A_{d-1}| \leq \frac{1}{d-1} n - cn^a. \]

Suppose not. Then

\[ |A_{d-1}| \geq \frac{1}{d-1} n - cn^a. \]

Thus

\[ \sum_{i=1}^{d-2} |A_i| + |C| + (d-1)t \leq n - \left( \frac{1}{d-1} n - cn^a \right). \]

Therefore, we can conclude that

\[ |N(V_{d-1}^+)\| \leq \sum_{i=1}^{d-2} |A_i| + (d-2)t + |C| \leq \frac{d-2}{d-1} n + cn^a, \]

which contradicts our neighborhood assumption. Hence, the supposition that

\[ |A_{d-1}| \geq \frac{1}{d-1} n - cn^a \]

is false and we have

\[ |A_1| \leq \cdots \leq |A_{d-1}| \leq \frac{1}{d-1} n - cn^a. \]

Therefore,

\[ |C| > n - (d-1) \left( \frac{1}{d-1} n - cn^a \right) - (d-1)t \geq (d-1) \left[ cn^a - t \right]. \]
Thus $|C| \geq c_1 n^x$ for some real number $c_1$.

Let

$$A = \bigcup_{i=1}^{d-1} A_i;$$

thus, $|A| = n - |C| - (d-1)t$. For each $i$, $1 \leq i \leq d-1$, let $B_i$ denote $\tilde{N}_c(V^k_i) = C - N_c(V^k_i)$. Then

$$n - \left(\frac{d-2}{d-1} n + cn^x\right) > |\tilde{N}(V^k_i)| \geq |A_i| + |B_i| \quad \text{for each } i.$$ 

Thus

$$|B_i| < \frac{1}{d-1} n - cn^x - |A_i|.$$ 

This implies that

$$\sum_{i=j+1}^{d-1} |B_i| < \frac{d-1-j}{d-1} n \cdot (d-1)cn^x \cdot \sum_{i=j+1}^{d-1} |A_i|. $$

Since

$$\sum_{i=1}^{j} |A_i| \leq \frac{j}{d-1} |A|,$$

we have

$$\sum_{i=j+1}^{d-1} |A_i| \geq \frac{d-1-j}{d-1} |A|. $$

Therefore

$$\sum_{i=j+1}^{d-1} |B_i| < \frac{d-1-j}{d-1} n - (d-1-j)cn^x - \frac{d-1-j}{d-1} |A| = \frac{d-1-j}{d-1} n - (d-1-j)cn^x - \frac{d-1-j}{d-1} \left( n - (d-1)\right)(d-1)-|C|.$$ 

Thus

$$\sum_{i=j+1}^{d-1} |B_i| < \frac{d-1-j}{d-1} |C| + (d-1-j)\cdot t - (d-1-j)cn^x.$$ 

Finally, this implies that

$$\left| \bigcap_{i=j+1}^{d-1} N_c(V^k_i) \right| = |C| - \left| \bigcup_{i=j+1}^{d-1} B_i \right| \geq |C| - \sum_{i=j+1}^{d-1} |B_i| > |C| - \left[ \frac{d-1-j}{d-1} |C| + (d-1-j)\cdot t - (d-1-j)cn^x \right] = \frac{j}{d-1} |C| - (d-1-j)[t - cn^x] \geq \frac{j}{d-1} |C| \quad \text{if } cn^x \geq t.$$
Clearly, if $x > 0$, then for $n$ sufficiently large, $cn^x \geq t$. If $x = 0$, then since $t$ is fixed, there exist a real number $c$ such that $c \geq t$. This completes the proof of Lemma 4. □

**Lemma 5.** Let $p$ and $m$ be positive integers. Let $C$ be a set, and $\mathcal{S}$ a collection of subsets of $C$ with the property that given any $S \in \mathcal{S}$, $|S| \geq \frac{1}{p}|C|$. Then, given any subcollection $\mathcal{S}'$ of $\mathcal{S}$ of order $pm$, there exist a set $C'$ in $C$ that is the intersection of $m$ sets in $\mathcal{S}'$. Furthermore,

$$|C'| \geq \frac{1}{\left(\begin{array}{c} pm \\ m \end{array}\right)(pm-mm+1)} |C|.$$

**Proof.** Let $\mathcal{S}'$ be any collection of $pm$ sets in $\mathcal{S}$. Partition the elements in $C$ as follows. Given $c \in C$, let $c \in A_i$, $0 \leq i \leq pm$, if $c$ is contained in exactly $i$ of the $pm$ sets in $\mathcal{S}'$. Since $|S| \geq \frac{1}{p}|C|$ for each $S$ in $\mathcal{S}'$,

$$\frac{pm}{p} |C| \leq pm |A_{pm}| + \cdots + 2 |A_2| + |A_1|.$$

Let $A = \bigcup_{i=m}^{pm} A_i$ and $B = \bigcup_{i=0}^{m-1} A_i$. Then $m|C| \leq pm |A| + (m-1)|B|$, which implies $|C| \leq (pm - m + 1)|A|$, since $|B| = |C| - |A|$. Thus,

$$|A| \geq \frac{1}{pm-mm+1} |C|.$$

The number of ways the elements of $A$ can be contained in the intersection of some collection of $m$ sets of $\mathcal{S}'$ is given by

$$\left(\begin{array}{c} pm \\ m \end{array}\right)|A_{pm}| + \left(\begin{array}{c} pm-1 \\ m \end{array}\right)|A_{pm-1}| + \cdots + |A_m|.$$

Hence, there exist

$$\left(\begin{array}{c} pm \\ m \end{array}\right)|A_{pm}| + \left(\begin{array}{c} pm-1 \\ m \end{array}\right)|A_{pm-1}| + \cdots + |A_m|$$

elements in $C$, which we shall denote by $C'$, that are contained in the intersection of a fixed collection of $m$ sets of $\mathcal{S}'$. Therefore, we have

$$\frac{1}{\left(\begin{array}{c} pm \\ m \end{array}\right)(pm-mm+1)} |C| \leq \frac{\left(\begin{array}{c} pm \\ m \end{array}\right)|A_{pm}| + \cdots + |A_m|}{\left(\begin{array}{c} pm \\ m \end{array}\right)}.$$

□
Lemma 6. Let $d, k \geq 2$, and $m \geq r > 0$ be positive integers and $x$ a real number such that $0 \leq x < 1$. Let $G$ be a graph of order $n$ that does not contain a $K(1; r) + K(d - 1; m)$. There exist a $c = c(m, d, r, k)$ such that if $G$ satisfies the neighborhood condition
\[ NC_k \geq \frac{d-2}{d-1} n + cn^x, \]
then for any integer $t \leq c$ and for sufficiently large $n$, $G$ contains a $K(1; t) + K(d - 2; m)$ as a subgraph of each fixed copy of a $K(d - 1; t)$ and a set $D \subseteq V(G)$ where $|D| \geq c_1 n^x$ for some real number $c_1$ and each vertex in $D$ is completely joined to the $K(d - 2; m)$ and has at least one adjacency in the $K(1; t)$.

Proof. Let $G$ be a graph of order $n$ that satisfies the neighborhood condition $NC_k \geq \frac{d-2}{d-1} n + cn^x$ and does not contain a $K(1; r) + K(d - 1; m)$. By Lemma 3, for sufficiently large $n$, $G$ contains $\alpha(n^{d-1}t)$ induced copies of a $K(d - 1; t)$ for some real number $\alpha$ and any fixed integer $t$. Let $K$ denote a fixed $K(d - 1; t)$ with partite sets $\{V_i, i = 1, \ldots, d - 1\}$. By Lemma 4 with $j = 1$, $V(G)$ contains a set $C$ such that $|C| \geq \alpha_2 n^x$ for some real number $\alpha_2$

Let $f$ be the collection of all $J_i$ where $1 \leq i \leq (d - 1)m$. By Lemma 5, there is a subcollection $\mathcal{F}'$ of $\mathcal{F}$ of size $m$ such that
\[ \left| \bigcap_{i} J_i \right| \geq \delta |C| \]
where $\delta = \frac{1}{m(d-2)+1}$.

Let $\mathcal{F}' = \{I_1, \ldots, I_m\}$ and let $I = \bigcap_{j=1}^{m} I_j$.

Associated with each $I_j$ in $\mathcal{F}'$, there are $k$ vertices in each $V_i$, $2 \leq i \leq d - 1$, whose neighborhood union covers $I_j$. For each $I_j$ in $\mathcal{F}'$, denote those $k$ vertices in $V_i$ by $W_{ji}$. Let $V'_j$ be the subset of $V_i$ of order $mk$ such that $V'_j = \bigcup_{j=1}^{m} W_{ji}$. Let $K'$ be the graph induced by $V'_j$ and $\{V'_j, i = 2, \ldots, d - 1\}$. Thus $K' = K(1; t) + K(d - 2; km)$

Let $u \in I$ and make the following observations. Since $u \in I_j$, $u \in I_j$ for every $j, 1 \leq j \leq m$. For each fixed $j$, $u$ has at least one adjacency among the $k$ vertices in $W_{ji}$ for every $i, 2 \leq i \leq d - 1$. Since this is true for every $j, 1 \leq j \leq m$, $u$ has at least $m$ adjacencies in each $V'_j$. Thus, associated with each $u \in I$, there is at least one $K(d - 2; m)$ contained in
the $K(d-2; km)$ of $K'$ to which $u$ is completely joined. However, there are at most
\[
\binom{km}{m}^{d-2}
\]
distinct $K(d-2; m)$ in the $K(d-2; km)$ of $K'$.

Thus, there are at least
\[
\frac{|I|}{\binom{km}{m}^{d-2}} \geq \frac{1}{\binom{(d-1)m}{m} (m(d-2)+1) \binom{km}{m}^{d-2}} |C|.
\]
vertices in $I$ which are completely joined to a fixed $K(d-2; m)$ in the $K(d-2; km)$ of $K'$. Let $D$ denote this subset of $I$ of size
\[
|D| \geq c_1 n^a.
\]
Since $|C| \geq a_2 n^a$ and $d, m, k$ are fixed, $|D| \geq c_1 n^a$ for some real number $c_1$.
Furthermore, since $D \subset C$, by the definition of $C$, each vertex in $D$ has at least one adjacency in $V_1 = K(1; t)$.  

**Proof of theorem**

**Theorem A.** Let $G$ be a graph of order $n$, and let $k, m, d \geq 2$ and $r, r < m$ be positive integers. There exist a real number $c = c(m, r, k, d)$ such that if $G$ satisfies the neighborhood condition
\[
NC_k > \frac{d-2}{d-1} n + cn^{1-1/r},
\]
then for $n$ sufficiently large, $G$ contains a $K(1; r) + K(d-1; m)$ as a subgraph. Furthermore, if $r = 1, 2$ or $3$, this result is best possible.

**Proof.** Let $G$ be a graph of order $n$ that satisfies the neighborhood condition
\[
NC_k > \frac{d-2}{d-1} n + cn^{1-1/r},
\]
and suppose that $G$ does not contain a $K(1; r) + K(d-1; m)$. We will show that for $n$ sufficiently large, this leads us to a contradiction. By Lemma 6 with $\alpha = 1 - \frac{1}{r}$, associated with each $K(d-1; t)$ in $G$, $G$ contains a $K(1; t) + K(d-2; m)$ and a set $D \subseteq V(G)$ such that every vertex in $D$ is completely joined to the $K(d-2; m)$ and has at least one adjacency in the $K(1; t)$. Furthermore,
\[
|D| \geq c_1 n^{1-1/r}
\]
for some real number $c_1$. Hence, there exists a real number $c_2$ such that $D$ contains a subset $D'$ where

$$|D'| \geq c_2 n^{1 - 1/r}$$

and every vertex in $D'$ is adjacent to a fixed vertex in the $K(1; t)$. Hence there are at least

$$\binom{|D'|}{r} \geq \binom{c_2 n^{1 - 1/r}}{r} \geq c_3 n^{r - 1}$$

copies of $K(1, r, m_1, \ldots, m_{d-2})$ associated with each fixed copy of a $K(d-1; t)$ in $G$ for some real number $c_3$. Since by Lemma 3 there are at least $c_4 n^{(m-1)t}$ copies of a $K(d-1; t)$ in $G$, taking into account the multiplicities in $G$, there are at least

$$c_4 c_5 n^{(m-1)(r - 1) + (d-1)t} \binom{n}{t-1} \binom{n}{t-m} d^{d-2} \geq c_6 n^m(d-2)^r$$

copies of a $K(1, r, m_1, \ldots, m_{d-2})$ in $G$ for some real numbers $c_5$ and $c_6$.

Applying Lemma 1, we have that $G$ contains at least

$$c_7(n^m(d-2)+r)m n^m(d-2)+r \binom{n}{m(d-2)+r(m-1)} = c_8 n^{m(d-2)+r}$$

copies of a $K(1; r)+K(d-1; m)$ for some real number $c = c(m, r, k, d)$. \(\square\)

It still remains to be determined how many copies, either distinct or disjoint, the graph $G$ of order $n$ contains when it satisfies the conditions of Theorem A. A fairly straightforward argument shows that if $s > 0$, there must be at least $n^s$ disjoint copies of a $K(1; r)+K(d-1; m)$ contained in $G$. Since neighborhood condition type results are distinct from edge density type results, there are many other interesting open problems.

References