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Towards computability of elliptic boundary value problems in variational formulation

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Abstract

We present computable versions of the Fréchet–Riesz Representation Theorem and the Lax–Milgram Theorem. The classical versions of these theorems play important roles in various problems of mathematical analysis, including boundary value problems of elliptic equations. We demonstrate how their computable versions yield computable solutions of the Neumann and Dirichlet boundary value problems for a simple non-symmetric elliptic differential equation in the one-dimensional case. For the discussion of these elementary boundary value problems, we also provide a computable version of the Theorem of Schauder, which shows that the adjoint of a computably compact operator on Hilbert spaces is computably compact again.
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1. Introduction

In the theory of elliptic boundary value problems one is interested in solving differential equations such as

$$-u'' + u' + u = f \quad \text{on } [0, 1],$$

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with boundary conditions $u'(0) = u'(1) = 0$ or $u(0) = u(1) = 0$. Here, f is some given function. The solution u of such boundary value problems is best understood using Sobolev spaces that include information on the function u and its derivatives as square-integrable functions and which are, in particular, Hilbert spaces (see text books like [11] for the discussion of the mathematical background of elliptic boundary value problems and their variational formulations). Boundary value conditions regarding the derivative, such as $u'(0) = u'(1) = 0$ are often called *natural* or *Neumann* conditions, whereas conditions regarding the solution itself, such as $u(0) = u(1) = 0$, are called *essential* or *Dirichlet* conditions. The difference in the variational formulation of these problems is that natural boundary conditions appear implicitly in the variational formulation, whereas essential boundary conditions are explicitly included in the choice of the corresponding Sobolev spaces.

Here, we are interested in the question whether elliptic boundary value problems can be solved computably in the sense of computable analysis (the Turing machine-based theory of computability on reals, as developed in [20,15,23]). In particular, one might ask whether for any computable f there is a computable solution u (which is subject to the respective boundary conditions). But more than this, one can ask whether the solution map W that maps any function f to the respective (weak) solution u is computable as well. We will answer both questions in the affirmative for the specific elliptic boundary value problem above. For this purpose we have to provide some computable versions of the underlying theorems from functional analysis that are required for the variational formulation of boundary value problem.

In general, a variational formulation of a boundary value problem is discussed in the context of a Hilbert space H together with a Hilbert space V that is (densely) embeddable into H and a sesquilinear form $B : V \times V \to \mathbb{F}$ which is bounded and coercive. The problem is to find a $u \in V$ such that

$$B(u, v) = F(v)$$

for all $v \in V$, for a given $F \in V'$, where V' denotes the dual space of V. The sesquilinear form B and the functional F are determined depending on the differential equation of interest and F typically depends on the function f. As explained above, the boundary conditions either appear implicitly or they are incorporated in the choice of V.

In case that B is a conjugate symmetric sesquilinear form, the existence and uniqueness of the solution u can be established with the help of the Fréchet–Riesz Theorem [13,21] and in the asymmetric case, the Lax–Milgram Theorem [17] can be used for the same purpose. We will discuss computable versions of these theorems in Sections 4 and 5. But before we do this, we have to provide some preliminaries. In Section 2 we discuss computable Hilbert spaces and, in particular, those Sobolev spaces which we will use for examples of elliptic boundary value problems. In Section 3 we discuss the Fourier representation of Hilbert spaces, which is a very convenient tool for the study of computability on Hilbert spaces. Section 6 is devoted to a computable version of the Theorem of Schauder which shows that the adjoint of any computably compact operator is computably compact again. The Theorem of Schauder is not required in the classical discussion of the boundary value problems mentioned above, but it turns out to be very useful in the computable setting. Finally, in Section 7 we discuss the computability of the boundary value problem above with essential and natural boundary conditions.

2. Computable Hilbert spaces

In this section we briefly introduce the required tools from computable analysis, which we will use in the following. For a more comprehensive introduction, the reader is referred to [23] and the other cited references. We will not introduce notions from functional analysis here and the reader is referred to standard textbooks in this case. In the following we will discuss operators $T :\subseteq H \to H$ on Hilbert spaces H and we are in particular interested in computable Hilbert spaces, which we define below (the inclusion symbol "$\subseteq$" indicates that T might be partial). In general, we assume that H is defined over the field $\mathbb{F}$, which might either be $\mathbb{R}$ or $\mathbb{C}$. Throughout the paper, we assume that $H \neq \{0\}$.

Definition 2.1. A *computable Hilbert space* $(H, \langle . \rangle, e)$ is a separable Hilbert space $(H, \langle . \rangle)$ together with a fundamental sequence $e : \mathbb{N} \to H$ (i.e. the linear span of range(e) is dense in H) such that the induced normed space is a computable normed space.

The induced normed space is the normed space with the norm given by $\|x\| := \sqrt{\langle x, x\rangle}$. A *computable normed space* is a normed space such that the metric d induced by $d(x, y) := \|x - y\|$ together with the sequence $\alpha_e : \mathbb{N} \to H$, defined by $\alpha_e\langle k, \langle n_0, \ldots, n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$, form a computable metric space such that the linear operations (vector space addition and scalar multiplication) become computable. Here, $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and of $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_{\mathbb{F}}(n) = 0$. Without loss of generality, we can even assume that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H (see Lemma 3.1). Note that $\langle . \rangle$ is used in two meanings: if applied to natural numbers or sequences of natural numbers, it stands for the canonical *Cantor pairing function*; if applied to objects in a Hilbert space, it stands for the inner product. No ambiguity is to be expected here. A *computable metric space* X is a separable metric space together with a sequence $\alpha : \mathbb{N} \to X$ such that range(α) is dense in X and $d \circ (\alpha \times \alpha)$ is a computable (double) sequence of reals.

If not mentioned otherwise, then we assume that all computable Hilbert spaces H are represented by their Cauchy representation δ_H (of the induced computable metric space). The *Cauchy representation* $\delta :\subseteq \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$, which rapidly converges to x, where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. Here, Σ^ω denotes the set of infinite sequences over some finite set Σ (the *alphabet*) and Σ^ω is endowed with the product topology with respect to the discrete topology on Σ. All computability statements with respect to Hilbert spaces are to be understood with respect to the Cauchy representation. Given representations (i.e. surjective maps) $\delta :\subseteq \Sigma^\omega \to X$ and $\delta' :\subseteq \Sigma^\omega \to Y$, a map $f :\subseteq X \to Y$ is called *computable*, if there exists a computable $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $f\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(f\delta)$. Here, a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable* if there exists a Turing machine which computes F.

It is clear that the inner product of any computable Hilbert space is a computable map.

Proposition 2.2. *The inner product* $\langle . \rangle : H \times H \to \mathbb{F}, (x, y) \mapsto \langle x, y\rangle$ *of any computable Hilbert space* H *is computable.*

This can be directly concluded from the polar identities (see [9]). We now discuss a number of examples of computable Hilbert spaces. We say that a subspace V of a Hilbert space H is a *computable subspace*, if it is a Hilbert space with respect to some fundamental sequence that is computable in H. This implies that the inclusion $\iota : V \hookrightarrow H$ is computable.

Example 2.3. We let $I = (0, 1)$.

(1) The space ℓ_2 of sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ with $\sum_{i=0}^{\infty} |x_i|^2 < \infty$ together with the inner product

$$\langle x, y\rangle_{\ell_2} := \sum_{i=0}^{\infty} x_i y_i^*$$

and the fundamental sequence $(e_n)_{n\in\mathbb{N}}$ given by $e_{nk} := \delta_{nk}$ is a computable Hilbert space. Here, δ denotes the Kronecker symbol and y_i^* the conjugate of the complex number y_i.

(2) The space $L^2(I)$ of square-integrable functions over $\mathbb{F} = \mathbb{R}$ with the inner product

$$\langle f, g\rangle_{L^2} := \int_I fg \, \mathrm{d}x$$

and the fundamental sequence $(e_n)_{n\in\mathbb{N}}$ given by $e_0 = 1$, $e_{2n-1}(x) = \sqrt{2}\cos 2n\pi x$ and $e_{2n}(x) = \sqrt{2}\sin 2n\pi x$ for $n \geqslant 1$, is a computable Hilbert space. Here, $\mathrm{d}x$ denotes the Lebesgue measure.

(3) The Sobolev space $H^1(I)$ over $\mathbb{F} = \mathbb{R}$ is the space of all those $f \in L^2(I)$ such that $f' \in L^2(I)$, equipped with the inner product

$$\langle f, g\rangle_{H^1} := \int_I (fg + f'g') \, \mathrm{d}x = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}$$

and the fundamental sequence $(e_n)_{n\in\mathbb{N}}$ given by $e_0(x) = 1$ and $e_n(x) = \sqrt{2}\cos n\pi x$ for $n \geqslant 1$ is a computable Hilbert space. Here, f' denotes the weak derivative of f.

(4) The Sobolev space $H_0^1(I) = \{f \in H^1(I) : f(0) = f(1) = 0\}$ over $\mathbb{F} = \mathbb{R}$ is a computable subspace of $H^1(I)$. The sequence $(e_n)_{n\in\mathbb{N}}$ with $e_{n-1}(x) = \sqrt{2}\sin n\pi x$ for $n \geqslant 1$ is a computable fundamental sequence in this space.

The given fundamental sequences $(e_n)_{n\in\mathbb{N}}$ are computable orthonormal bases in case of ℓ_2 and $L^2(I)$. The fundamental sequences $(e_n)_{n\in\mathbb{N}}$ given under (3) and (4) are also computable orthonormal bases of $L^2(I)$ and sometimes it is convenient to work with one of these.

We recall that a function $f \in L^2(I)$ has a *weak derivative* in $L^2(I)$, if there exists a function $g \in L^2(I)$ such that

$$\langle g, \varphi\rangle_{L^2} = \int_I g\varphi \, \mathrm{d}x = -\int_I f\varphi' \, \mathrm{d}x = -\langle f, \varphi'\rangle_{L^2}$$

for all $\varphi \in \mathcal{D}(I) = C_0^{\infty}(I)$ (i.e. for all infinitely often differentiable $\varphi : I \to \mathbb{R}$ with compact support). In this case we write $f' = g$. By partial integration one can easily show that the so defined concept of a weak derivative generalizes the classical concept of a derivative (the boundary terms disappear in the equation above since φ has compact support). Since derivatives and integrals can easily be computed in the dense subset generated by $(e_n)_{n\in\mathbb{N}}$, it follows that the Sobolev space $H^1(I)$ is a computable Hilbert space. Computability on Sobolev spaces has also been studied in [27,18]. Note that strictly speaking the elements in $L^2(I)$ and $H^1(I)$ are equivalence classes of functions which coincide up to sets of measure zero. Thus, evaluation of functions in $L^2(I)$ is not a well-defined concept. If, however, for $f \in L^2(I)$ the weak derivative f' is in $L^2(I)$ as well, i.e. if $f \in H^1(I)$, then by Sobolev's Inequality it follows that f can be considered as a continuous function (more precisely, there is a continuous representative in its equivalence class). For a computable version, see also Theorem 3.4 in [27]. In particular, evaluation is a well-defined concept for functions in $H^1(I)$ and $H_0^1(I)$ is a well-defined subspace. The standard reference for Sobolev spaces is [2], introductions can be found in [11]. Computability of the space $L^2(I)$ has been discussed in Example 8.1.7 of [23] and in [29,16]. Further related results can be found in [28,25,26]. Computability and complexity properties of differential equations have been studied from different perspectives in [1,14,19,24]. We close this section by mentioning that (weak) differentiation, injection, integration and evaluation are computable on appropriate spaces as defined above.

Proposition 2.4. *The following operations are computable*:

(1) $D : H^1(I) \to L^2(I),\ f \mapsto f'$,
(2) $\mathrm{in} : H^1(I) \hookrightarrow L^2(I),\ f \mapsto f$,
(3) $I : L^2(I) \to \mathbb{R},\ f \mapsto \int_I f\,\mathrm{d}x$,
(4) $\mathrm{ev} : H^1(I) \times [0,1] \to \mathbb{R},\ (f,a) \mapsto f(a)$.

Proof. Here (1) follows directly from the fact that differentiation is computable in the dense subset and the fact that the information on $f' \in L^2(I)$ is included in a name for $f \in H^1(I)$. Similarly, it is clear that (2) is computable. For (3) one can use the fact that integration is obviously computable in the dense subset and $\|f\|_{L^1} \leqslant \|f\|_{L^2}$. The latter implies that $L^2(I) \hookrightarrow L^1(I)$ is computable. Computability of integration has been proved for $L^1(I)$ in Example 8.1.8 of [23]. For (4) we note that we can obtain the value $f(1)$ using

$$f(1) = \int_0^1 (tf(t))'\,\mathrm{d}t = \int_0^1 tf'(t)\,\mathrm{d}t + \int_0^1 f(t)\,\mathrm{d}t.$$

Then for any $a \in [0,1]$ we obtain

$$f(a) = f(1) - \int_a^1 f'(t)\,\mathrm{d}t.$$

Using (1)–(3) and the fact that integration is also computable with a variable lower bound a (see Exercises 8.1.11 and 8.1.12 in [23]), it follows that evaluation is computable. □

As a corollary of (4) we can conclude that the map $H^1(I) \hookrightarrow \mathcal{C}[0,1]$, which maps any $f \in H^1(I)$ to a continuous representative of the same equivalence class, is computable. Here, $\mathcal{C}[0,1]$ denotes the set of continuous functions $f : [0,1] \to \mathbb{R}$. We note that the main results in this paper will be formulated for Hilbert spaces over $\mathbb{F}$, whereas all examples and differential equations will be discussed over $\mathbb{F} = \mathbb{R}$.

3. The Fourier representation

It is natural to represent points of computable Hilbert spaces by their Fourier coefficients, which leads to a second natural representation besides the Cauchy representation (that can be defined for any computable metric space). For any (infinite-dimensional) Hilbert space H and $x \in H$ one obtains

$$x = \sum_{i=0}^{\infty} \langle x, e_i \rangle e_i$$

with respect to a fixed orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The series $\sum_{i=0}^{\infty}\langle x, e_i\rangle e_i$ is called *Fourier series* and the coefficients $\langle x, e_i\rangle$ are called *Fourier coefficients* of x. It is well-known that the Fourier series of any point $x \in H$ converges to x. In case that H is finite-dimensional, one can replace ∞ by $\dim(H) - 1$ here. Actually, it is easy to see that any computable Hilbert space has a computable orthonormal basis.

Lemma 3.1. *Let H be a computable Hilbert space. Then there exists a computable orthonormal basis of H.*

Proof. Given a computable fundamental sequence e one can find a computable function $f : \mathbb{N} \to \mathbb{N}$ such that the composition ef is a linearly independent computable fundamental sequence (see the Effective Independence Lemma in [20]). Starting from this sequence ef, one can use the Gram-Schmidt procedure to construct a computable orthonormal basis of H. □

The previous observations motivate the following definition (a similar idea has also been used in [16]).

Definition 3.2. Let H be a computable Hilbert space with some fixed computable orthonormal basis $(e_n)_{n\in\mathbb{N}}$. We define the *Fourier representation* δ_{Fourier} of H by

$$\delta_{\text{Fourier}}\langle p, q\rangle = x :\iff \delta_{\mathbb{F}}^{\mathbb{N}}(p) = (\langle x, e_n\rangle)_{n\in\mathbb{N}} \text{ and } \delta_{\mathbb{R}}(q) = \|x\|$$

for all $x \in H$ and $p, q \in \Sigma^{\omega}$ in case that H is infinite-dimensional (and similarly with finite tuples in case that H is finite-dimensional).

Here, $\delta_{\mathbb{F}}^{\mathbb{N}}$ denotes the canonical sequence representation induced by $\delta_{\mathbb{F}}$ (see [23]). That is, the Fourier representation denotes x by a standard name of the sequence $(\langle x, e_n\rangle)_{n\in\mathbb{N}}$ of its Fourier coefficients together with a standard name for the norm $\|x\|$. In the finite-dimensional case the information on the norm is redundant, as it can be computed from the finite tuple. In the infinite-dimensional case by Parseval's Identity

$$\|x\|^2 = \sum_{i=0}^{\infty} |\langle x, e_i\rangle|^2.$$

Thus, one can also read the Fourier representation such that any point $x \in H$ is considered as a point $x' \in \ell_2$. This is made precise by the following result. Here, a *computable isomorphism* T is an isomorphism T such that T as well as T^{-1} are computable. For the proof we use the computable Banach Inverse Mapping Theorem which states that any computable linear bounded operator $T : X \to Y$ on computable Banach spaces X, Y has a computable inverse $T^{-1} : Y \to X$ (see [7]). However, a direct proof can easily be found as well.

Theorem 3.3 (*Fourier representation theorem*). *Let H be an infinite-dimensional computable Hilbert space with some computable orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The map*

$$T : H \to \ell_2, x \mapsto (\langle x, e_n\rangle)_{n\in\mathbb{N}}$$

is a computable isometrical isomorphism (where H and ℓ_2 are represented by their Cauchy representations).

Proof. Let the input $x \in H$ be represented with respect to the Cauchy representation. Since the scalar product is computable it is clear that the sequence $(\langle x, e_n\rangle)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}}$ can be computed. Since the norm is computable, it follows by Parseval's Identity that $\sum_{i=0}^{\infty} |\langle x, e_i\rangle|^2 = \|x\|^2$ can be computed. By Proposition 2 in [10] it follows that a Cauchy name of the sequence $(\langle x, e_n\rangle)_{n\in\mathbb{N}} \in \ell_2$ can be computed. By the computable Banach Inverse Mapping Theorem one can conclude that T^{-1} is computable as well. □

As a corollary we obtain that any Fourier representation of a computable Hilbert space is computably equivalent to its Cauchy representation (the finite-dimensional case can easily be proved analogously). In general, we say that two representations δ, δ' of a set X are *computably equivalent*, if the identity $\mathrm{id} : (X, \delta) \to (X, \delta')$ and its inverse are computable.

Corollary 3.4. *Let H be a computable Hilbert space with some fixed computable orthonormal basis. The Fourier representation of H with respect to this basis and the Cauchy representation of H are computably equivalent.*

In particular, a point $x \in H$ is computable with respect to the Cauchy representation if and only if it is computable with respect to the Fourier representation which leads to a natural characterization of computable points in Hilbert spaces that we formulate separately.

Corollary 3.5. *Let H be an infinite-dimensional computable Hilbert space with induced norm $\|\ \|$ and computable orthonormal base $(e_n)_{n\in\mathbb{N}}$. Then x is computable if and only if the sequence $(\langle x, e_n\rangle)_{n\in\mathbb{N}}$ of Fourier coefficients is computable and $\|x\|$ is computable.*

In the finite-dimensional case the sequence can be replaced by a finite tuple again and the condition on the norm can be omitted. Since the equivalence of the Fourier representation and the Cauchy representation does not depend on the selected computable orthonormal basis, we can conclude that any two Fourier representations are equivalent.

Corollary 3.6. *Let H be some computable Hilbert space. Any two Fourier representations of H with respect to any computable orthonormal bases are equivalent to each other.*

Here, the orthonormal bases are meant to be computable with respect to the Cauchy representation of H. With the last corollary of the Fourier Representation Theorem we formulate an obvious observation.

Corollary 3.7. *Any infinite-dimensional computable Hilbert space H is computably isometrically isomorphic to ℓ_2.*

That is, ℓ_2 is a universal infinite-dimensional computable Hilbert space. One should note that computable Hilbert spaces are separable by definition and thus Hilbert spaces with an uncountable orthonormal basis cannot be computable.

4. The representation theorem of Fréchet–Riesz

In this section we present a computable version of the Fréchet–Riesz Representation Theorem. Firstly, we recall the classical formulation of the theorem.

Theorem 4.1 (*Fréchet–Riesz*). *Let H be a Hilbert space. The map*

$$R : H \to H', y \mapsto (x \mapsto \langle x, y \rangle)$$

is a surjective conjugate linear isometry.

Here *conjugate linear* means $R(x + y) = R(x) + R(y)$ and $R(\alpha x) = \alpha^* R(x)$ for any $x, y \in H$ and $\alpha \in \mathbb{F}$ and that R is an *isometry* means that $\|R(x)\| = \|x\|$ for any $x \in H$. In particular, this implies that R is injective and thus bijective altogether.

For the following it is convenient to denote the functional $R(y)$ by f_y, i.e.

$$f_y : H \to \mathbb{F}, x \mapsto \langle x, y \rangle.$$

The Fréchet–Riesz Representation Theorem states in particular that $\|f_y\| = \|y\|$ for all $y \in H$ and for any $f \in H'$ there is some $y \in H$ such that $f_y = f$.

In order to formulate a computable version of this theorem we need a representation of the dual space H'. We define the following ad hoc representation. For this purpose we use the function space representation $[\delta \to \delta']$ which is canonically defined for any two representations δ of X and δ' of Y (see [23]). In case that δ, δ' are so-called *admissible* representations of separable normed spaces X, Y, it follows that $[\delta \to \delta']$ is an admissible representation of the set $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$.

Definition 4.2. Let H be a computable Hilbert space. We define a representation $\delta_{H'}$ of the dual space H' by

$$\delta_{H'}\langle p, q \rangle = f :\Longleftrightarrow [\delta_H \to \delta_{\mathbb{F}}](p) = f \text{ and } \delta_{\mathbb{R}}(q) = \|f\|.$$

Thus, a name for a functional $f \in H'$ includes information on $f : H \to \mathbb{F}$ as a continuous function and information on the norm $\|f\|$. An analogous representation can be defined in the general context of computable normed spaces (see also [6]). Note that f is computable as a point in H' if and only if $f : H \to \mathbb{F}$ is computable as a function and $\|f\|$ is computable.

The following computable version of the Fréchet–Riesz Representation Theorem is (similarly as the computable Weierstraß Approximation Theorem) a direct corollary of the classical version and does not require an effectivization of the classical proof. As a technical tool we will only use the Fourier representation.

Theorem 4.3 (*computable theorem of Fréchet–Riesz*). *Let H be a computable Hilbert space. Then*

$$R : H \to H', y \mapsto (x \mapsto \langle x, y \rangle)$$

is a computable surjective conjugate linear isometry and the inverse R^{-1} is computable as well.

Proof. We have to show that R and R^{-1} are computable. Since H is a computable Hilbert space, it follows that the scalar product and $(y, x) \mapsto R(y)(x) = \langle x, y \rangle$ are computable. By type conversion it follows that R is $(\delta_H, [\delta_H \to \delta_{\mathbb{F}}])$–computable. Since R is an isometry, we can conclude that $\|R(y)\| = \|y\|$ can be computed from y, i.e. R is even $(\delta_H, \delta_{H'})$-computable.

Now given some functional $f \in H'$, we have to compute $y := R^{-1}(f)$ as well. Let $(e_n)_{n \in \mathbb{N}}$ be some computable orthonormal basis of H. By Corollary 3.4 it is sufficient to determine y

with respect to the Fourier representation of this basis. The sequence of Fourier coefficients $(\langle y, e_n\rangle)_{n\in\mathbb{N}} = (f(e_n)^*)_{n\in\mathbb{N}}$ of y can easily be computed from f by evaluation and $\|y\| = \|f\|$ is available in any $\delta_{H'}$–name of f. □

Note that the computable Fréchet–Riesz Representation Theorem does not imply that for any $f : H \to \mathbb{F}$ that is computable as a function there is some computable y with $f_y = f$. In fact, we get the following characterization.

Corollary 4.4. *Let H be a computable Hilbert space and $y \in H$. Then y is computable if and only if $f_y : H \to \mathbb{F}$ and $\|f_y\|$ are computable.*

A corresponding version of the Riesz Theorem is known in constructive analysis [3,12]. We note that by a formula, known as the Lemma of Ascoli,

$$|f(x)| = \|f\| \cdot \mathrm{dist}_{\mathrm{kern}(f)}(x)$$

for all $f \in H'$ and $x \in H$. Since a set with a computable distance function is also called *located*, it follows that $\|f\|$ is computable if and only if $\mathrm{kern}(f)$ is located, provided that f is computable. In particular, we can express the previous corollary as follows.

Corollary 4.5. *Let H be a computable Hilbert space and $y \in H$. Then y is computable if and only if $f_y : H \to \mathbb{F}$ is computable and* $\mathrm{kern}(f_y)$ *is located.*

It is clear that there are computable functionals whose norm is not computable. We adapt an example from [8].

Example 4.6. Let $a = (a_k)_{k\in\mathbb{N}}$ be a computable sequence of reals such that

$$\|a\|_{\ell_2} = \sqrt{\sum_{k=0}^{\infty} |a_k|^2}$$

exists, but is not computable. One can, for instance, choose $a_k = 2^{-\frac{g(k)}{2}}$ for some computable injective function $g : \mathbb{N} \to \mathbb{N}$ that enumerates some r.e. but non-recursive set $K \subseteq \mathbb{N}$. We use ℓ_2 over $\mathbb{F} = \mathbb{R}$ and define

$$f : \ell_2 \to \mathbb{R}, (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k=0}^{\infty} a_k x_k = \langle (a_k)_{k\in\mathbb{N}}, (x_k)_{k\in\mathbb{N}}\rangle_{\ell_2}.$$

Then f is bounded and obviously the sequence $(f(e_n))_{n\in\mathbb{N}}$ is computable. Altogether, f is computable. We obtain $\|f\| = \|a\|_{\ell_2}$ and consequently $\|f\|$ is not computable.

Another important consequence of the computable Fréchet–Riesz Theorem is the fact that the dual space of a computable Hilbert space is a computable Hilbert space again. This does not hold true for computable Banach spaces in general, as their dual spaces need not even be separable. This result also justifies our choice of the dual space representation.

Theorem 4.7. *Let H be a computable Hilbert space with computable orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Then the dual space H' with the fundamental sequence $(f_{e_n})_{n\in\mathbb{N}}$ is a computable Hilbert space as* *well. The fundamental sequence is a computable orthonormal basis of this space and the corresponding Fourier representation of H' is computably equivalent to the dual space representation of H'.*

Proof. By definition, the norm on H' is computable with respect to the dual space representation. Moreover, the linear operations and the limit map $\mathrm{Lim} :\subseteq H' \to H'$, restricted to rapidly converging sequences, are computable with respect to the dual space representation since these operations are computable on H and computability can be transferred with the help of the map R from the computable Fréchet–Riesz Theorem 4.3. For instance, in case of addition one obtains $f + g = R(R^{-1}(f) + R^{-1}(g))$, where R and R^{-1} are computable. Finally, $(f_{e_n})_{n\in\mathbb{N}}$ is a computable sequence in H' with respect to the dual space representation. Altogether, this implies that the dual space representation of H' is computably equivalent to the Cauchy representation with respect to $(f_{e_n})_{n\in\mathbb{N}}$ (see the Stability Theorem 10.7 and Theorem 11.8 in [5]) and thus, by Theorem 3.3 also to the corresponding Fourier representation. □

Note that in this case the inner product of H' is given by

$$\langle f_x, f_y\rangle = \langle y, x\rangle = \sum_{i=0}^{\infty} \langle y, e_i\rangle\langle e_i, x\rangle = \sum_{i=0}^{\infty} f_y(e_i)^* f_x(e_i).$$

5. The Lax–Milgram theorem

In this section we want to prove a computable version of the Lax–Milgram Theorem. We start with the classical version of this result and we recall that a function $B : H \times H \to \mathbb{F}$ is called *sesquilinear* if it is linear in the first component and conjugate linear in the second component (in the real case $\mathbb{F} = \mathbb{R}$ these functions are called *bilinear* for obvious reasons). Moreover, B is called *bounded* if there exists a constant $c \geqslant 0$ such that

$$|B(x, y)| \leqslant c \cdot \|x\| \cdot \|y\|$$

for all $x, y \in H$. A sesquilinear function B is bounded if and only if it is continuous. Finally, B is called *coercive* or *elliptic*, if there exists a constant $m > 0$ such that

$$|B(x, x)| \geqslant m \cdot \|x\|^2$$

holds for all $x \in H$. Now we are prepared to formulate the classical Lax–Milgram Theorem.

Theorem 5.1 (*Lax–Milgram*). *Let H be a Hilbert space over $\mathbb{F}$. Let $B : H \times H \to \mathbb{F}$ be a bounded sesquilinear form. Then there exists exactly one bounded linear operator $T : H \to H$ with*

$$B(x, y) = \langle Tx, y\rangle$$

for all $x, y \in H$. If, additionally, B is coercive with constant $m > 0$, then T is bijective and $\|T^{-1}\| \leqslant \frac{1}{m}$.

The analogous result for an operator $S : H \to H$ with $B(x, y) = \langle x, Sy\rangle$ holds true as well. According to the Fréchet–Riesz Theorem, for any functional $f \in H'$ there exists a $z \in H$ such that $f(x) = f_z(x) = \langle x, z\rangle$. If we choose $y := S^{-1}z$, then we obtain the following corollary.

Corollary 5.2. *Let H be a Hilbert space over $\mathbb{F}$ and $B : H \times H \to \mathbb{F}$ a coercive bounded sesquilinear form. Then for any bounded linear functional $f : H \to \mathbb{F}$ there exists exactly one $y \in H$ with $f(x) = B(x, y)$ for all $x \in H$.*

Analogously, there exists exactly one $x \in H$ with $f(y) = B(x, y)^*$. For any bounded sesquilinear functional $B : H \times H \to \mathbb{F}$ we define

$$B_1 : H \to H', x \mapsto (y \mapsto B(x, y)^*) \quad \text{and} \quad B_2 : H \to H', y \mapsto (x \mapsto B(x, y)).$$

We note that a sesquilinear B is bounded if and only if $B_1(x)$ and $B_2(y)$ are continuous for all $x, y \in H$ (which is a consequence of the Uniform Boundedness Theorem). Therefore, B_1 and B_2 are well-defined. Using these maps, we can define two representations of the space of bounded sesquilinear forms with the required computability properties.

Definition 5.3. Let H be a computable Hilbert space over $\mathbb{F}$. We define a representation $\delta_{\mathcal{S}_1(H)}$ of the set $\mathcal{S}(H)$ of bounded sesquilinear forms $B : H \times H \to \mathbb{F}$ by

$$\delta_{\mathcal{S}_1(H)}(p) = B :\iff [\delta_H \to \delta_{H'}](p) = B_1.$$

Analogously, we can define $\delta_{\mathcal{S}_2(H)}$ with B_2 in place of B_1. We write $\mathcal{S}_1(H)$ or $\mathcal{S}_2(H)$ in order to indicate which representation is to be used.

Obviously, the representation $\delta_{\mathcal{S}_1(H)}$ is defined such that $\mathcal{S}_1(H) \to \mathcal{C}(H, H'), B \mapsto B_1$ and its inverse are computable. Using this representation we can now define a computable version of the Lax–Milgram Theorem. The theorem roughly speaking states that given a sesquilinear form B in form of B_1, we can compute T, and vice versa. Moreover, given B_1 and the constant m of coercivity, we can even compute T^{-1}.

Theorem 5.4 (*computable theorem of Lax–Milgram*). *Let H be a computable Hilbert space over $\mathbb{F}$. Then the map*

$$L : \mathcal{S}_1(H) \to \mathcal{C}(H, H), B \mapsto T,$$

which maps each bounded sesquilinear form $B : H \times H \to \mathbb{F}$ to the linear bounded operator $T : H \to H$ with $B(x, y) = \langle Tx, y\rangle$ and its inverse L^{-1} are computable. Moreover,

$$M :\subseteq \mathcal{S}_1(H) \times \mathbb{R} \to \mathcal{C}(H, H), (B, m) \mapsto T^{-1},$$

with $\mathrm{dom}(M) = \{(B, m) \in \mathcal{S}_1(H) \times \mathbb{R} : m > 0$ *and* $(\forall x \in H)|B(x, x)| \geqslant m \cdot \|x\|^2\}$ *is computable as well. An analogous result holds for $\mathcal{S}_2(H)$ and S such that $B(x, y) = \langle x, Sy\rangle$.*

Proof. With a computable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H we obtain

$$Tx = \sum_{n=0}^{\infty} \langle Tx, e_n\rangle e_n = \sum_{n=0}^{\infty} B(x, e_n)e_n = \sum_{n=0}^{\infty} B_1(x)(e_n)^* e_n.$$

Given $B_1 : H \to H'$ and x, we can compute $(B_1(x)(e_n)^*)_{n\in\mathbb{N}}$ in $\mathbb{F}^{\mathbb{N}}$. In order to obtain T with the Fourier representation in the target space, one additionally needs $\|Tx\|$ which can be obtained from $B_1(x) \in H'$ by

$$\|Tx\|^2 = \sum_{n=0}^{\infty} |B_1(x)(e_n)|^2 = \|B_1(x)\|^2.$$

Thus, the map L is computable. Computability of M follows directly as the inversion map

$$\imath : \subseteq \mathcal{C}(H, H) \times \mathbb{R} \to \mathcal{C}(H, H), (T, b) \mapsto T^{-1},$$

defined for all (T, b) such that $T : H \to H$ is a bijective linear bounded operator and $\|T^{-1}\| \leqslant b$, is computable (see Theorem 6.9 in [4]). By the classical Lax–Milgram Theorem we can choose $b = \frac{1}{m}$, where $m > 0$ is the coercivity constant of T. Now, regarding the inverse L^{-1} we obtain

$$B_1(x)(y) = B(x, y)^* = \langle y, Tx \rangle,$$

which can easily be computed from T and by type conversion this allows to compute $\widehat{B_1} : H \to \mathcal{C}(H, \mathbb{F}), x \mapsto B_1(x)$. Now we additionally obtain $\|B_1(x)\| = \|Tx\|$, which can also be computed from T. Altogether, one can obtain $B_1 : H \to H'$, given $T : H \to H$. $\square$

Later, it will be helpful to use computability of the inverse L^{-1} in a slightly more general setting with two different Hilbert spaces. We formulate this result separately.

Lemma 5.5. *Let H_1, H_0 be computable Hilbert spaces. Then the map*

$$\Omega : \subseteq \mathcal{C}(H_1, H_0) \to \mathcal{C}(H_1, H_0'), T \mapsto (x \mapsto (y \mapsto \langle y, Tx \rangle)),$$

defined for all linear bounded $T : H_1 \to H_0$, is computable.

The proof is literally the same as the proof of computability of L^{-1}. Now we formulate a corollary of the computable Lax–Milgram Theorem 5.4 and the computable Theorem of Fréchet–Riesz 4.3.

Corollary 5.6. *Let H be a computable Hilbert space over $\mathbb{F}$. Then the map*

$$\Lambda : \subseteq \mathcal{S}_1(H) \times \mathbb{R} \times H' \to H, (B, m, f) \mapsto x,$$

which maps any sesquilinear form $B : H \times H \to \mathbb{F}$ with coercivity constant $m > 0$ and any functional $f : H \to \mathbb{F}$ to the unique $x \in H$ with $f(y) = B(x, y)^*$ for all $y \in H$ is computable.

Proof. Given f we can compute the unique $z \in H$ such that $f(y) = \langle y, z \rangle$ by the computable Fréchet–Riesz Theorem 4.3. Given B and m, we can compute $T^{-1} : H \to H$ such that $B(x, y) = \langle Tx, y \rangle$ for all $x, y \in H$. Thus, we can compute $x := T^{-1}z$ and we obtain

$$f(y) = \langle z, y \rangle^* = \langle Tx, y \rangle^* = B(x, y)^*. \qquad \square$$

An analogous result holds for $\mathcal{S}_2(H)$ and the unique $y \in H$ such that $f(x) = B(x, y)$ for all $x \in H$. Finally, we formulate the non-uniform version of the previous corollary.

Corollary 5.7. *Let H be a computable Hilbert space over $\mathbb{F}$ and $B : H \times H \to \mathbb{F}$ a coercive bounded sesquilinear form such that $B_1 : H \to H'$ is computable. Then for any computable linear functional $f : H \to \mathbb{F}$ with computable norm $\|f\|$ there exists exactly one computable $x \in H$ with $f(y) = B(x, y)^*$ for all $x \in H$.*

It is clear that the inner product induces a computable sesquilinear form B by $B(x, y) = \langle x, y\rangle$, which is bounded and coercive by the Cauchy–Schwarz Inequality and B_1 is computable by the computable Representation Theorem of Fréchet–Riesz 4.3. Thus, it follows from Example 4.6 that computability of $\|f\|$ cannot be eliminated in the previous corollary. The computability property of B used in the previous results, namely computability of B_1, is somewhat stronger than mere computability of B. The following proposition characterizes this notion in several ways.

Proposition 5.8. *Let H be a computable Hilbert space with computable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and $B : H \times H \to \mathbb{F}$ a bounded sesquilinear form. Then the following are equivalent*:

(1) $B_1 : H \to H', x \mapsto (y \mapsto B(x, y)^*)$ *is computable*,
(2) $B : H \times H \to \mathbb{F}$ *and* $N_1 : H \to \mathbb{R}, x \mapsto \|B_1(x)\|$ *are computable*,
(3) $B : H \times H \to \mathbb{F}$ *and* $(\sum_{k=0}^{\infty} |B(e_n, e_k)|^2)_{n\in\mathbb{N}}$ *are computable*,
(4) $T : H \to H$, *with* $B(x, y) = \langle Tx, y\rangle$ *for all* $x, y \in H$, *is computable*.

Proof. Computability of B_1 directly implies computability of N_1 and it implies computability of B by evaluation. Thus, (1) implies (2). Obviously, (2) implies (3) since

$$N_1(e_n) = \|B_1(e_n)\| = \sum_{k=0}^{\infty} |B_1(e_n)(e_k)|^2 = \sum_{k=0}^{\infty} |B(e_n, e_k)|^2.$$

Now we prove that (3) implies (1): in order to show that B_1 is computable, it is sufficient to show that $(B_1(e_n))_{n\in\mathbb{N}}$ is a computable sequence in H', since B_1 is bounded. Now, computability of B implies that $(B_1(e_n))_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{C}(H, \mathbb{F})$, but (3) also implies that $(\|B_1(e_n)\|)_{n\in\mathbb{N}}$ is a computable sequence in $\mathbb{R}$ such that, altogether, $(B_1(e_n))_{n\in\mathbb{N}}$ is actually a computable sequence in H'. The equivalence of (1) and (4) follows from the computable Lax–Milgram Theorem 5.4. □

Using this characterization, we can prove that the strong input information on B in the sense of $\mathcal{S}_1(H)$ in the computable Lax–Milgram Theorem 5.4 and its Corollary 5.6 is neither superfluous nor included in the corresponding information on B as a continuous function $B \in \mathcal{C}(H \times H, \mathbb{F})$. In particular, computability of $B : H \times H \to \mathbb{F}$ is in general a strictly weaker property than computability of $B_1 : H \to H'$.

Example 5.9. We use ℓ_2 over $\mathbb{F} = \mathbb{R}$. Let $a = (a_k)_{k\in\mathbb{N}}$ be a computable sequence of reals such that $\|a\|_{\ell_2}$ exists, but is not computable (e.g. as defined in Example 4.6). We can assume that $a_0 = 1$. Using this sequence we define a linear bounded operator $T : \ell_2 \to \ell_2$ using the matrix representation

$$T = \begin{pmatrix} 1 & -a_1 & -a_2 & -a_3 & \cdots \\ a_1 & 1 & 0 & 0 & \cdots \\ a_2 & 0 & 1 & 0 & \cdots \\ a_3 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Obviously, T is not computable as $T(e_0) = a$ is not computable in ℓ_2. However, the bilinear form $B : \ell_2 \times \ell_2 \to \mathbb{R}$, defined by $B(x, y) = \langle Tx, y\rangle$ is computable, as $B(e_0, e_j) = a_j$ and

$B(e_i, e_0) = -a_i$ for $i > 0$ and $B(e_i, e_j) = \delta_{ij}$ for $i, j > 0$. Since T is not computable, it follows by the previous proposition that $B_1 : \ell_2 \to \ell_2{}'$ is not computable. Moreover, B is also coercive, as for $x = (x_i)_{i\in\mathbb{N}} \in \ell_2$ we obtain

$$|B(x,x)| = |\langle Tx, x\rangle| = \left|\sum_{i=0}^{\infty} x_i^2 - x_0 \sum_{i=1}^{\infty} a_i x_i + x_0 \sum_{i=1}^{\infty} a_i x_i\right| = \|x\|_{\ell_2}^2.$$

We close this section with a result that shows that a symmetric and computable bilinear form that is coercive on a subspace induces a computable Hilbert space. This result can be applied to symmetric boundary value problems. We will only use it for Example 6.8.

Proposition 5.10. *Let H be a computable Hilbert space over $\mathbb{R}$ with a computable subspace V. If $B : H \times H \to \mathbb{R}$ is a computable bilinear form which is coercive on V,* i.e. *there is some $m > 0$ such that $|B(x,x)| \geqslant m \cdot \|x\|^2$ for all $x \in V$, then V with the inner product*

$$\langle x, y\rangle := B(x, y)$$

and some sequence $(e_n)_{n\in\mathbb{N}}$ whose linear span is dense in V and which is computable in H, is a computable Hilbert space V_B. The identity $\mathrm{id} : V \to V_B$ *is a computable isomorphism.*

Proof. It is known that V_B is a Hilbert space (see Proposition 2.5.3 in [11]). Moreover, as B is bounded and coercive we obtain

$$m \cdot \|x\|_H^2 \leqslant |B(x,x)| = \|x\|_{V_B}^2 \leqslant c\|x\|_H^2$$

for all $x \in V$ with some suitable constant $c \geqslant 0$. This shows that the identity $\mathrm{id} : V \to V_B$ is a computable isomorphism of computable normed spaces. That implies that the linear operations of V_B are computable and thus V_B is a computable Hilbert space. □

6. The theorem of Schauder

In this section we will prove a computable version of the Theorem of Schauder for Hilbert spaces. The classical theorem states that for Banach spaces X, Y the adjoint $T' : Y' \to X'$ of any compact linear operator $T : X \to Y$ is compact too. In this general form, there is no obvious way to formulate the theorem computably, since the dual space of a computable Banach space is not necessarily computable again. However, as we have seen, this obstacle does not exist for Hilbert spaces.

Before we start to formulate the theorem, we have to introduce computably compact operators. Therefore, we slightly generalize the approach presented in [9]. By $\mathcal{B}_\infty(H_1, H_0)$ we denote the set of compact operators $T : H_1 \to H_0$. We recall that an operator T is called *compact*, if the closure of the image $TS(0,1)$ of the unit sphere $S(0,1)$ is compact. First of all, we prove that the space $\mathcal{B}_\infty(H_1, H_0)$ is a computable normed space with the operator norm and the dense subset given by the numbering

$$\alpha\langle k, \langle n_0, \ldots, n_k\rangle, \langle l_0, \ldots, l_k\rangle\rangle(x) := \sum_{i=0}^{k} \langle x, \alpha_e(n_i)\rangle \alpha_{e'}(l_i).$$

Here, we assume that $(e_i)_{i\in\mathbb{N}}$ and $(e'_i)_{i\in\mathbb{N}}$ are computable orthonormal bases of H_1 and H_0, respectively, and $\alpha_e, \alpha_{e'}$ are the corresponding numberings of dense subsets, as defined in

Section 2. Then α is actually a numbering of certain finite rank operators $T_n : H_1 \to H_0$, defined by $T_n(x) := \alpha(n)(x)$, which form a dense subset of $\mathcal{B}_\infty(H_1, H_0)$ (see, for instance, Theorem 6.5 in [22]). By $\delta_{\mathcal{B}_\infty(H_1,H_0)}$ we denote the corresponding Cauchy representation. We start with a basic observation, which helps to handle the numbering α.

Lemma 6.1. *Let H_1 and H_0 be a computable Hilbert space with computable orthonormal bases $(e_i)_{i\in\mathbb{N}}$ and $(e'_i)_{i\in\mathbb{N}}$, respectively. There are computable functions $M : \mathbb{N} \to \mathbb{N}$ and $C : \mathbb{N} \to \mathbb{F}$ such that*

$$\alpha(n)(x) = \sum_{j=0}^{M(n)} \sum_{i=0}^{M(n)} C\langle n, i, j\rangle \langle x, e_i\rangle e'_j$$

for all $n \in \mathbb{N}$ and $x \in H_1$ (and such that $C\langle n, i, j\rangle = 0$ for i or $j > M(n)$).

The proof is literally the same as for the proof of the special case $H_1 = H_0$, which can be found in [9]. The same applies to the proof of the following result.

Proposition 6.2. *Let H_1, H_0 be computable Hilbert spaces. Then $(\mathcal{B}_\infty(H_1, H_0), \|\ \|, \alpha)$ is a computable normed space.*

We say that an operator $T : H_1 \to H_0$ is *computably compact* if it is a computable point in $\mathcal{B}_\infty(H_1, H_0)$. Correspondingly, as Lemma 17 in [9] for the special case $H_1 = H_0$, one can prove that any computably compact operator is computable. Now we are prepared to prove our computable version of the Theorem of Schauder for computable Hilbert spaces.

Theorem 6.3 (*computable theorem of Schauder*). *Let H_1, H_0 be computable Hilbert spaces. The map*

$$A : \mathcal{B}_\infty(H_1, H_0) \to \mathcal{B}_\infty(H'_0, H'_1), T \mapsto T'$$

which maps any compact linear operator $T : H_1 \to H_0$ to its compact linear adjoint $T' : H'_0 \to H'_1$, $f \mapsto fT$, is computable.

Proof. Let $(e_i)_{i\in\mathbb{N}}$ and $(e'_i)_{i\in\mathbb{N}}$ be computable orthonormal bases of H_1 and H_0, respectively. We first assume that $T = \alpha(n)$. We use the computable functions M and C from Lemma 6.1 and we define $m := M(n)$ and $c_{ij} := C\langle n, i, j\rangle$. That is,

$$Tx = \alpha(n)(x) = \sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij} \langle x, e_i\rangle e'_j.$$

Then

$$(T'f)(x) = fTx = \sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij} \langle x, e_i\rangle f(e'_j) = \sum_{j=0}^{m} \sum_{i=0}^{m} c_{ij} \langle f, f_{e'_j}\rangle f_{e_i}(x)$$

and thus

$$T'f = \sum_{j=0}^{m} \sum_{i=0}^{m} c_{ij} \langle f, f_{e'_j}\rangle f_{e_i}.$$

Note that by Theorem 4.7 $(f_{e_i})_{i\in\mathbb{N}}$ is a computable orthonormal basis of H_1' and $(f_{e_i'})_{i\in\mathbb{N}}$ is a computable orthonormal basis of H_0'. It follows that given $T = \alpha(n)$, one can compute $A(T) = T'$. Since for a general T

$$\|A(T) - A(\alpha(n))\| = \sup_{\|f\|=1} \|fT - f\alpha(n)\| \leqslant \|T - \alpha(n)\|,$$

we can conclude that A is computable. □

We directly obtain the following non-uniform result.

Corollary 6.4. *Let H_1, H_0 be computable Hilbert spaces and $T : H_1 \to H_0$ a computably compact operator. Then the adjoint operator $T' : H_0' \to H_1'$, $f \mapsto fT$ is computably compact as well.*

The next example shows that the adjoint T' of a computable operator T is not necessarily computable, not even if T is bijective. This shows that computable compactness is an important property which guarantees computability of the adjoint.

Example 6.5. We use ℓ_2 over $\mathbb{F} = \mathbb{R}$. Let $a = (a_k)_{k\in\mathbb{N}}$ be a computable sequence of positive reals such that $\|a\|_{\ell_2}$ exists, but is not computable (e.g. as defined in Example 4.6). We can assume that $a_0 = 1$ and $\|a\|_{\ell_2}^2 < 2$. Using this sequence we define a linear bounded operator $T : \ell_2 \to \ell_2$ using the matrix representation

$$T = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots \\ 0 & 1 & a_1 & a_2 & \cdots \\ 0 & 0 & 1 & a_1 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

This operator T is even computable, as $(T(e_i))_{i\in\mathbb{N}}$ is a computable sequence in ℓ_2. However, the Hilbert space adjoint operator $T^* : \ell_2 \to \ell_2$ that is uniquely defined by $\langle T^*x, y\rangle = \langle x, Ty\rangle$ and given by the transposed matrix is not computable, as $T^*(e_0) = a$ is not computable. The ordinary adjoint $T' : {\ell_2}' \to {\ell_2}'$ is related to the Hilbert space adjoint T^* by the equation $T^* = R^{-1}T'R$, where $R : \ell_2 \to {\ell_2}'$ is the map from the computable Fréchet–Riesz Representation Theorem 4.3. Thus, it follows that T' is not computable either. Now we show that the operator T defined above is even bijective. This follows from

$$\|\mathrm{id} - T\| = \sum_{i=1}^{\infty} a_i^2 = \|a\|_{\ell_2}^2 - 1 < 1$$

which implies that $T = \mathrm{id} - (\mathrm{id} - T)$ is invertible.

The example in particular shows that the maps

$$T \mapsto T^* \quad \text{and} \quad T \mapsto T'$$

are not computable in general (not even restricted to bijective operators T), provided that operators are represented as continuous functions.

The computable Theorem of Schauder 6.3 can be applied to boundary value problems. This will be demonstrated in the next section, where we will use the adjoint of the following map. By definition $H^1(I) \subseteq L^2(I)$ and the following result shows that this is a computably compact embedding.

Proposition 6.6. *The embeddings* $\text{in} : H^1(I) \hookrightarrow L^2(I)$ *and* $\text{in}_0 : H_0^1(I) \hookrightarrow L^2(I)$ *are computably compact.*

Proof. We choose $(e_n)_{n\in\mathbb{N}}$ with $e_0(x) = 1$ and $e_n(x) = \sqrt{2}\cos n\pi x$ for $n \geqslant 1$ as a computable orthonormal basis of $L^2(I)$. Consider the Fourier expansion of $f \in L^2(I)$:

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \sqrt{2}\cos n\pi x.$$

In this situation $\|f\|_{L^2}^2 = \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_n^2$ and

$$\|f\|_{H^1}^2 = \alpha_0^2 + \sum_{n=1}^{\infty}(1 + n^2\pi^2)\alpha_n^2$$

and $f \in H^1(I)$ if and only if $\|f\|_{H^1} < \infty$. If $\|f\|_{H^1} \leqslant 1$, then $\alpha_n^2 \leqslant \frac{1}{1+n^2\pi^2}$ follows for all $n \in \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}$ converges effectively, there is a computable modulus of convergence $m : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=m(k)+1}^{\infty} \frac{1}{1+n^2\pi^2} < 2^{-2k}$. Now consider the finite rank operators $I_n : H^1(I) \to L^2(I)$ given by $I_n(e_i) = e_i$ for $i = 0, \dots, n$ and $I_n(e_i) = 0$ for $i > n$. For f with $\|f\|_{H^1} \leqslant 1$ and Fourier expansion as above we obtain

$$\|I_{m(k)} f - \text{in} f\|_{L^2}^2 = \sum_{n=m(k)+1}^{\infty} \alpha_n^2 \leqslant \sum_{n=m(k)+1}^{\infty} \frac{1}{1+n^2\pi^2} < 2^{-2k}$$

and thus $\|I_{m(k)} - \text{in}\|_{\mathcal{B}_\infty} < 2^{-k}$. Since $(I_n)_{n\in\mathbb{N}}$ is obviously a computable sequence in $\mathcal{B}_\infty(H^1(I), L^2(I))$, we can conclude that $\text{in} : H^1(I) \hookrightarrow L^2(I)$ is a computable function in $\mathcal{B}_\infty(H^1(I), L^2(I))$. The statement for in_0 can be proved similarly. □

By an application of Corollary 6.4 and by using the fact that any computably compact operator is in particular computable, we obtain the following corollary.

Corollary 6.7. *The embeddings* $\text{in}' : L^2(I)' \hookrightarrow H^1(I)'$ *and* $\text{in}_0' : L^2(I)' \hookrightarrow H_0^1(I)'$ *are computably compact and, in particular, computable.*

We close this section with an example, which shows that even in the specific situation of computable embeddings $\iota : H_1 \hookrightarrow H_0$ of computable Hilbert spaces H_1, H_0 it is not in general the case that the dual embedding $\iota' : H_0' \hookrightarrow H_1'$ is computable again. Therefore, the compactness condition plays an important rôle also in case of computability of embeddings.

Example 6.8. We consider the computable bijective operator $T : \ell_2 \to \ell_2$ from Example 6.5 and we use the fact that the Hilbert space adjoint $T^* : \ell_2 \to \ell_2$ is not computable. We define a computable bilinear form $B : \ell_2 \times \ell_2 \to \mathbb{R}$ by

$$B(x, y) = \langle Tx, Ty \rangle.$$

Obviously, B is even symmetric and coercive as

$$|B(x,x)| = |\langle Tx, Tx\rangle| = \|Tx\|^2 \geqslant \|x\|^2.$$

Thus by Proposition 5.10 the Hilbert space H which consists of the set ℓ_2 endowed with the inner product induced by B is a computable Hilbert space and the identity $\imath : \ell_2 \to H$ is a computable isomorphism. We obtain for all $x \in H$ and $y \in \ell_2$

$$\langle x, \imath y\rangle_H = B(\imath^{-1}x, \imath^{-1}\imath y) = \langle T\imath^{-1}x, T\imath^{-1}\imath y\rangle_{\ell_2} = \langle T^*T\imath^{-1}x, y\rangle_{\ell_2}$$

which implies $\imath^* = T^*T\imath^{-1}$ and thus $T^* = \imath^*\imath T^{-1}$. Since by the computable Banach Inverse Mapping Theorem T^{-1} is computable, it follows that $\imath^*$ cannot be computable. Consequently, $\imath' : H' \to {\ell_2}'$ is not computable either.

7. Boundary value problems

In this section we want to study computability of some elementary boundary value problems in order to illustrate the application of the computable versions of the Fréchet–Riesz Theorem and, in particular, the Lax–Milgram Theorem. The first example is taken from [11] and translated to the computable setting here.

Proposition 7.1 (*non-symmetric Neumann boundary value problem*). *We consider the boundary value problem*

$$-u'' + u' + u = f \quad \text{on } [0,1], \quad u'(0) = u'(1) = 0.$$

The weak solution operator $W : L^2(I) \to H^1(I), f \mapsto u$, *which maps any* $f \in L^2(I)$ *to the uniquely determined weak solution* $u \in H^1(I)$ *of this problem, is computable.*

Proof. Let $I = (0, 1)$. In the variational formulation of this problem one can choose the Hilbert space $H = L^2(I)$ and the embedded space $V = H^1(I)$, the bilinear form $B : V \times V \to \mathbb{R}$ and the functional $F : H \to \mathbb{R}$, given by

$$B(u,v) = \int_I (u'v' + u'v + uv)\,\mathrm{d}x,$$
$$F(v) = \int_I fv\,\mathrm{d}x$$

for all $f \in H$. Continuity of B follows from the Cauchy–Schwarz Inequality by

$$|B(u,v)| \leqslant |\langle u,v\rangle_{H^1}| + \left|\int_I u'v\,\mathrm{d}x\right| \leqslant \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2} \leqslant 2\|u\|_{H^1}\|v\|_{H^1}.$$

Coercivity of B follows from

$$B(v,v) = \int_I ((v')^2 + v'v + v^2)\,\mathrm{d}x$$
$$= \frac{1}{2}\int_I (v'+v)^2\,\mathrm{d}x + \frac{1}{2}\int_I ((v')^2 + v^2)\,\mathrm{d}x \geqslant \frac{1}{2}\|v\|^2_{H^1}.$$

It is clear that $B : V \times V \to \mathbb{R}$ is computable. We also have to prove that

$$B_1 : V \to V', u \mapsto \left(v \mapsto \int_I (u'v' + u'v + uv)\,\mathrm{d}x\right)$$

is computable. On the one hand, it follows from the computable Fréchet–Riesz Theorem 4.3 that

$$C : V \to V', u \mapsto \left(v \mapsto \langle u, v \rangle_{H^1} = \int_I (u'v' + uv)\,\mathrm{d}x \right)$$

is computable. On the other hand, by Proposition 2.4, $D : V \to H, u \mapsto u'$ is computable and hence

$$E : V \to H', u \mapsto \left(v \mapsto \langle Du, v \rangle_{L^2} = \int_I u'v\,\mathrm{d}x \right)$$

is computable by the computable Lax–Milgram Theorem (more precisely, by the generalized inverse Lemma 5.5). By Corollary 6.7 it follows that $\mathrm{in}' : H' \hookrightarrow V'$ is computable and thus $E_V = \mathrm{in}' \circ E : V \to V'$ is computable as well. But that implies that $B_1 = C + E_V$ is computable as well. Now it remains to prove that the weak solution operator W is computable. By the computable Fréchet–Riesz Theorem 4.3 it is clear that

$$R : H \to H', \quad f \mapsto F = (v \mapsto \langle f, v \rangle_{L^2})$$

is computable. By Corollary 6.7 it is clear that $\mathrm{in}' : H' \hookrightarrow V'$ is computable. Thus,

$$R_V : H \to V', f \mapsto F|_V$$

is computable as well, as $R_V = \mathrm{in}' \circ R$. By Corollary 5.6 of the computable Lax–Milgram Theorem the map $S = \Lambda(B, \frac{1}{2}, .) : V' \to V$ which maps any functional $F|_V \in V'$ to the unique $u \in V$ with $B(u, v) = F|_V(v)$ is computable. Altogether, the solution operator

$$W = S \circ R_V = S \circ \mathrm{in}' \circ R : H \to V, f \mapsto u$$

which maps any $f \in H$ to the corresponding solution $u \in V$ is computable. □

For the solution of the Neumann boundary value problem in the previous proof we have used the computable Lax–Milgram Theorem only for a fixed computable bilinear form B and thus not in the fully uniform version. This is different for the following Dirichlet boundary value problem where the solution depends on a parameter a that is part of the bilinear form B which is used in the variational formulation of the problem.

Proposition 7.2 (*non-symmetric Dirichlet boundary value problem*). *We consider the boundary value problem*

$$-u'' + au' + u = f \quad \text{on } [0, 1], \quad u(0) = u(1) = 0.$$

The map $W : L^2(I) \times \mathbb{R} \to H_0^1(I), (f, a) \mapsto u$, which maps any $f \in L^2(I)$ and $a \in \mathbb{R}$ to the uniquely determined weak solution $u \in H_0^1(I)$ of this problem, is computable.

Proof. Let $I = (0, 1)$. In the variational formulation of this problem one can choose the Hilbert space $H = L^2(I)$ and the embedded space $V = H_0^1(I)$, the bilinear form $B : V \times V \to \mathbb{R}$ and the functional $F : H \to \mathbb{R}$, given by

$$B(u, v) = \int_I (u'v' + au'v + uv)\,\mathrm{d}x,$$

$$F(v) = \int_I fv\,\mathrm{d}x$$

for all $f \in H$. Continuity of B follows from the Cauchy–Schwarz Inequality by

$$|B(u,v)| \leqslant \|u\|_{H^1}\|v\|_{H^1} + |a| \cdot \|u'\|_{L^2}\|v\|_{L^2} \leqslant (1+|a|)\|u\|_{H^1}\|v\|_{H^1}.$$

Coercivity of B follows from

$$B(v,v) = \frac{1}{2}av^2\Big|_0^1 + \int_I ((v')^2 + v^2)\,\mathrm{d}x = \|v\|^2_{H^1},$$

which holds due to the boundary conditions. It is clear that $B : V \times V \to \mathbb{R}$ is computable. We also have to prove that

$$B_1 : V \to V', u \mapsto \left(v \mapsto \int_I (u'v' + au'v + uv)\,\mathrm{d}x\right)$$

is computable. On the one hand, the map $C : V \to V', u \mapsto \left(v \mapsto \langle u, v\rangle_{H^1}\right)$ (used already in the proof of the previous Proposition 7.1 with $V = H^1(I)$ instead of $V = H_0^1(I)$) is computable by the computable Fréchet–Riesz Theorem 4.3. On the other hand, by Proposition 2.4, $D : V \to H, u \mapsto u'$ is computable (as $\iota : H_0^1(I) \hookrightarrow H^1(I)$ is computable) and hence

$$E_a : V \to H', u \mapsto \left(v \mapsto \langle aDu, v\rangle_{L^2} = \int_I au'v\,\mathrm{d}x\right)$$

is computable by the computable Lax–Milgram Theorem (more precisely, by the generalized inverse Lemma 5.5). And more than this, by the same lemma even the map

$$A : \mathbb{R} \to \mathcal{S}_1(V), a \mapsto B \text{ with } B_1 = C + (\mathrm{in}_0' \circ E_a)$$

is computable, where $\mathrm{in}_0' : H' \hookrightarrow V'$ is the computable map from Corollary 6.7. This can be proved by evaluation and type conversion. Now it remains to prove that the weak solution operator W is computable. Similarly, as in the proof of the previous Proposition 7.1 we obtain computability of the map $R_V : H \to V', f \mapsto F|_V$. By Corollary 5.6 of the computable Lax–Milgram Theorem the map

$$\Lambda : \subseteq \mathcal{S}_1(V) \times \mathbb{R} \times V' \to V, (B, m, F|_V) \mapsto u,$$

which maps any sesquilinear $B : V \times V \to \mathbb{R}$ with coercivity constant $m > 0$ and any functional $F|_V \in V'$ to the unique $u = \Lambda(B, m, F|_V) \in V$ with $B(u,v) = F|_V(v)$ is computable. Here the coercivity constant is always $m = 1$, independently of a. Altogether, the solution operator W with

$$W(f,a) = \Lambda(A(a), 1, R_V(f))$$

which maps any $(f,a) \in H \times \mathbb{R}$ to the corresponding solution $u \in V$ is computable. □

We close this section with an example of a symmetric Dirichlet Boundary Value Problem that can be solved computably using the computable Fréchet–Riesz Theorem 4.3 and Proposition 5.10.

Proposition 7.3 (*symmetric Dirichlet boundary value problem*). *We consider the boundary value problem*

$$-u'' = f \quad \text{on } [0,1], \quad u(0) = u(1) = 0.$$

The map $W : L^2(I) \to H_0^1(I)$, $f \mapsto u$, *which maps any* $f \in L^2(I)$ *to the uniquely determined weak solution* $u \in H_0^1(I)$ *of this problem, is computable.*

Proof. Let $I = (0, 1)$. In the variational formulation of this problem one can choose the Hilbert space $H = L^2(I)$ and the embedded space $V = H_0^1(I)$, the bilinear form $B : V \times V \to \mathbb{R}$ and the functional $F : H \to \mathbb{R}$, given by

$$B(u, v) = \int_I u'v' \, \mathrm{d}x,$$

$$F(v) = \int_I f v \, \mathrm{d}x$$

for all $f \in H$. Continuity of B follows from the Cauchy–Schwarz Inequality as in the preceding two propositions (or even simpler). Coercivity of B follows from

$$B(v, v) = \sum_{n=1}^{\infty} n^2\pi^2\beta_n^2 \geqslant \frac{1}{2}\sum_{n=1}^{\infty}(1 + n^2\pi^2)\beta_n^2 = \frac{1}{2}\,\|v\|_{H^1}^2,$$

where

$$v(x) = \sum_{n=1}^{\infty} \beta_n \sqrt{2} \sin n\pi x$$

is the Fourier expansion of $v \in H_0^1(I)$. It is easy to see that B is even computable and by Proposition 5.10 it follows that V_B is a computable Hilbert space which is computably isomorphic to V. Using the computable Theorem of Fréchet–Riesz 4.3 one can show that the weak solution operator W is computable, analogously to the discussion in the previous propositions. □

As a side result of the proof above we note that the Hilbert space $H_0^1(I)$ can be equipped with a computably equivalent norm given by B.

Corollary 7.4. *The space* $H_0^1(I)$ *endowed with the inner product given by*

$$\langle u, v\rangle = \int_I u'v' \, \mathrm{d}x$$

and the fundamental sequence $(e_n)_{n\in\mathbb{N}}$ *as given in Example* 2.3 *is a computable Hilbert space that is computably isomorphic to* $H_0^1(I)$ *as a subspace of* $H^1(I)$.

8. Conclusions

In this paper we have studied some simple one-dimensional elliptic boundary value problems and their computability properties. In order to establish computable solutions for these problems we had to provide computable versions of the Fréchet–Riesz Theorem and the Lax–Milgram Theorem. It turned out that compactness properties of certain embeddings even play a more important role in the computable setting than classically. Compactness and computability are both finiteness conditions of different type, but our results confirm that there is a close interaction between these properties. In the context of the results presented here compactness turns out to be important since the functionals $f : H \to \mathbb{F}$ to which we apply the Fréchet–Riesz Theorem have to come with the additional information of their norm $\|f\|$, an information that comes for free classically but has to be provided explicitly in the computable case. The computable compactness properties are used to take care of this fact.

The elliptic boundary value problems discussed in this paper are the most elementary ones in the one-dimensional case. However, these examples are sufficient to demonstrate that the variational tools can be used in the computable case very much along the same lines as classically. In order to study more advanced elliptic boundary value problems a further development of the theory of computable Sobolev spaces seems to be necessary. In particular, it would be desirable to study computable versions of trace and embedding theorems as well as related computability conditions of domains in the higher dimensional case.

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