A Characterization of Buildings of a Spherical Type

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A set of axioms in terms of points and lines is presented which characterize the 'natural' point-line geometries \((C_n,1; D_n,1; E_4,1; E_5,1; E_6,1; E_7,1; E_8,1; F_4,1)\) associated with buildings of spherical type.

1. INTRODUCTION

During the past ten years much work has been carried out to describe the geometries associated with buildings, and in particular those of spherical type, in terms of points and lines. This work has been stimulated by the famous theorem of Buekenhout-Shult [8]. They succeeded in making more beautiful the results of Veldkamp [21] and Tits [19] on polar spaces. Among other things, Cooperstein [14] extended this theorem by characterizing the Grassmannians of projective spaces. In contrast with [18], this characterization appears to allow applications and generalizations to be made in a rather natural way [7], [12], [13], [15]. In particular, Buekenhout [7] provides an axiom system that gives all 'natural' point-line geometries associated with spherical buildings. The aim of this paper is to simplify his list of axioms.

Most of the point-line characterizations of point-line geometries of buildings of spherical type use as a last step the results of Tits [19] and [20], and so do we. For the definitions of building, diagram, associated point-line geometry, etc. we refer to [1], [2], [4], [13], [19] and [20].

2. DEFINITIONS AND MAIN RESULTS

An incidence system \(\Gamma = (\mathcal{P}, \mathcal{L})\) is a set \(\mathcal{P}\) of points together with a family \(\mathcal{L}\) of distinguished subsets of \(\mathcal{P}\) of cardinality at least two, called lines. Two points \(p\) and \(q\) are called collinear if they are together on some line. We denote this by \(p \mathcal{L} q\).

The collinearity graph of \(\Gamma\) is the graph whose vertices are the points of \(\Gamma\) and whose edges consist of the pairs of distinct collinear points. Terms such as connectivity, clique, path, distance will be applied freely to \(\Gamma\) when in fact they are meant for the collinearity graph. For points \(p\) and \(q\), \(d(p, q)\) denotes the distance between \(p\) and \(q\). Further, \(p^\perp\) stands for the set of all points collinear with \(p\). If \(X\) is a set of points, then \(X^\perp = \bigcap_{p \in X} p^\perp\).

A subset \(X\) of \(\mathcal{P}\) is called a subspace of \(\Gamma\) if every line intersecting \(X\) in two distinct points is itself contained in \(X\). A singular subspace is a subspace all of whose points are pairwise collinear. The length \(i\) of a longest chain \(X_1 \subseteq X_2 \subseteq \cdots \subseteq X_i = X\) of nonempty singular subspaces \(X_j\) of a singular subspace \(X\) is called the rank of \(X\). A maximal singular subspace of rank at least three will be called a max space for short.

A geodesic is a path joining \(p\) to \(q\), whose length is equal to the distance \(d(p, q)\). A set of points \(X\) is called geodesically closed if for every pair of points \(p\) and \(q\) of \(X\), every geodesic joining \(p\) to \(q\) is contained in \(X\).

Now it is clear that the intersection of geodesically closed sets is again geodesically closed. Moreover, any intersection of geodesically closed subspaces is also a geodesically closed subspace. This allows us to talk about the geodesic closure of a set of points \(X\), denoted by \((X)\).
The incidence system $\Gamma$ is said to be linear if any two distinct points are on at most one line. If $p$ and $q$ are two collinear, distinct points of a linear incidence system, then $pq$ stands for the unique line through $p$ and $q$.

Finally, we recall from [8] that $\Gamma$ is a polar space of finite rank $r+1$ if:
(a) for every point $p$ and every line $L$, $p$ is collinear with either one or all points of $L$;
(b) no point of $\Gamma$ is collinear with all the others;
(c) every singular subspace has rank at most $r+1$, and there is such a singular subspace of rank $r+1$.

A polar space of rank 2 is also called a generalized quadrangle.

We will say that $\Gamma$ is a polarized space of rank $r+1$ if the following conditions hold:
(1) if $L$ is a line and $p$ is a point collinear with at least two distinct points of $L$, then $p$ is collinear with all points of $L$;
(2) if $p$ and $q$ are non-collinear points with $|p^+ \cap q^+| \geq 2$, then $p^+ \cap q^+$ is a polar space of rank $r$;
(3) (a) the structure is connected but is not complete;
(b) every line contains at least three points;
(c) the set $p^+ \setminus \{p\}$ is connected for every point $p$.

A similar concept has also been introduced by Buekenhout [7] and Cohen [13].

A polarized space of rank $r+1$ is a uniform polarized space of rank $r+1$ if also the following condition is satisfied:
(4) if $H$ and $K$ are maximal polar subspaces of rank $r+1$ of $\Gamma$, having at least one singular subspace of rank $r$ in common, then $H \cap K$ contains a singular subspace of rank $r+1$.

We can now state our main result:

**Theorem 1.** The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the geometry of points and lines of a (weak) thick building of one of the types

\[
C_{n,1} \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n-2 \quad n-1 \quad n
\]

\[
D_{n,1} \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n-2
\]

\[
E_{4,1} \quad 1 \quad 3 \quad 4
\]

\[
E_{5,1} \quad 1 \quad 2 \quad 4 \quad 5
\]

\[
E_{6,1} \quad 1 \quad 2 \quad 3 \quad 5 \quad 6
\]

\[
E_{7,1} \quad 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 7
\]

\[
E_{8,1} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 7 \quad 8
\]

\[
F_{4,1} \quad 1 \quad 2 \quad 3 \quad 4
\]
where the elements of the building whose type is the encircled node represent the points of $\Gamma$, if and only if $\Gamma$ is a uniform polarized space for which the ranks of the maximal singular subspaces are finite and differ by at most one.

**Theorem 2.** The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the geometry of points and lines of a (weak) thick building of one of the types

- $C_{n,1}$
- $D_{n,1}$
- $D_{n,n}$ or a quotient $D_{n,n}/A$ with $A$ a group of automorphisms of $D_{n,n}$ such that for each $a \in A - \{1\}$ the distance between a point and its image under $a$ is at least 5,
- $A_{n,2}$
- $E_{6,1}$
- $E_{7,1}$
- $E_{8,1}$ or a quotient $E_{n,n}/A$ with $A$ a group of automorphisms of $E_{n,n}$ compatible with (1) and (2).
- $F_{4,1}$

where the elements of the building whose type is the encircled node represent the points of $\Gamma$, if and only if $\Gamma$ is a uniform polarized space with singular subspaces of finite rank.

The proof of these theorems depends on [1], [2], [6], [8], [10], [13], [14], [19], [20].

### 3. Some Known Results on Polarized Spaces

In this section we consider a polarized space of rank $r + 1 \geq 3$.

**Proposition 1.** If $p$ and $q$ are distinct collinear points, then there exist points $u$ and $v$ at distance two such that $p, q \in u^- \cap v^+$.

**Proof.** By axiom 3(a) the structure is not complete, so there exist points $a$ and $b$ which are not collinear. In view of the connectivity, we may assume that $d(a, b) = 2$, write $x \in a^- \cap b^+$. By 3(c) there is a path from $a$ to $b$ in $x^+ - \{x\}$. Hence we may assume that $a$ and $b$ are at distance two in $x^+ - \{x\}$. This means that there is at least one point
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\[ y \in a^+ \cap b^+ \cap x^+ \text{ with } y \neq x. \] We conclude that there exist points \( a \) and \( b \) at distance two, such that \( x, y \in a^+ \cap b^+, x \neq y. \)

We claim now that each line \( L \) on \( x \) lies in a polar space of the type \( r^+ \cap s^+ \) described by axiom 2. For suppose the contrary, and choose a point \( p' \) on \( L \) distinct from \( x. \) If in \( x^+ \setminus \{x\} \) there exists a point \( q' \) not collinear with \( p' \) then again by the connectivity of \( x^+ \setminus \{x\} \) we may assume that \( p' \) and \( q' \) are at distance two in \( x^+ \setminus \{x\}. \) Now choose non collinear points \( a' \) and \( b' \) in the polar space \( p'^+ \cap q'^+. \) If in \( p'^+ \cap q'^+ \) there exists a point \( p' \) not collinear with \( a', b' \) then again by the connectivity of \( r^+ \cap s^+ \) we may assume that \( p' \) and \( a' \) and \( b' \) are at distance two such that \( x, p' \in a^+ \cap b^+. \) Here the points \( a, b \) are those constructed in the first part of the proof. Again, \( L \subseteq a \cap b^+ \text{ with } a^+ \cap b^+ \text{ a polar space. This proves the claim.} \)

Next we consider a point \( z \neq x \) collinear with \( x. \) Then by the foregoing claim, any line through \( x \) and \( z \) is contained in a polar space. So there exists a point \( c \in z^+ \setminus x^+. \) The same reasoning as above, with \( z \) playing the role of \( x, \) proves that every line on \( z \) is contained in a polar space of the type \( r^+ \cap s^+ \) described in axiom 2. In view of the connectivity, this property is true for all points. Hence any line on \( p \) and \( q \) is contained in a polar space of the form \( u^+ \cap v^+ \) with \( d(u, v) = 2. \)

**Proposition 2.**

(a) \( \Gamma \) is a linear incidence system and is completely determined by its collinearity graph;
(b) For every pair of points \( p \) and \( q \) satisfying the conditions of axiom 2 the set \( H(p, q) = \{ x \in P | x^+ \cap L \neq \emptyset \text{, for each line } L \text{ contained in } p^+ \cap q^+ \} \) is a geodesically closed subspace isomorphic to a polar space of rank \( r + 1; \)
(c) Every maximal singular subspace is isomorphic to a projective space and contains a line properly;
(d) All lines have the same cardinality.

These results are proved in [14]. Remark that by (b) it can be seen easily that such a set \( H(p, q) \) is a maximal polar subspace. We will call these sets hyperlines.

Let \( p \) be a point of \( \Gamma. \) The residue of \( \Gamma \) at \( p, \) denoted by \( \Gamma_p, \) is an incidence system \( (\mathcal{P}_p, \mathcal{L}_p) \) defined as follows: \( \mathcal{P}_p \) is the set of all lines of \( \Gamma \) containing \( p; \) the set of all lines on \( p \) contained in a plane (singular subspace of rank 3) of \( \Gamma \) is an element of \( \mathcal{L}_p. \) If \( A \) is an incidence system isomorphic to \( \Gamma_p, \) then \( \Gamma \) is said to be locally \( A \) at \( p. \) If \( \Gamma \) is locally \( A \) at every point \( p \) then \( \Gamma \) is locally \( A. \) Remark that axiom 3(c) is equivalent to the connectedness of \( \Gamma_p \) for all \( p. \) We say that \( \Gamma \) is locally connected for every point \( p. \)

Two points \( p \) and \( q \) at distance two satisfying the conditions of axiom 2, form a polar pair \( (p, q). \) In the other case where \( |p^+ \cap q^+| = 1, (p, q) \) is called a special pair. Two lines \( L \) and \( M \) are called coplanar if they are contained in some plane. Then every point of \( L \) is collinear with every point of \( M, \) so we denote this also by \( L \perp M. \) Remark that this notation keeps its right meaning in \( \Gamma_p, \) if we use \( \perp \) as collinearity in \( \Gamma_p \) as well. If \( X_p \) is a subset of \( \Gamma_p, \) then the union \( \cup X_p \) of all its elements (lines of \( \Gamma \)) is clearly a subset of \( \Gamma. \)

**Proposition 3.** The following statements are equivalent:

(a) \( \Gamma \) is a polar space;
(b) \( \Gamma \) is locally a polar space at some point \( p. \)

A proof of this can be found in [13].

**Proposition 4.** If \( \Gamma \) is a uniform polarized space of rank \( r + 1 \geq 4, \) then for any point \( p \) of \( \Gamma \) the residue \( \Gamma_p \) at \( p \) is a uniform polarized space of rank \( r. \) Moreover, in \( \Gamma_p \) there are no special pairs.
**Proposition 5.** If \( \Gamma \) is a uniform polarized space of rank 3, then we have:

(a) two distinct collinear points are contained in exactly one line;
(b) two points at distance two are contained in a geodesically closed subspace isomorphic to a non-degenerate generalized quadrangle (these subspaces will be called quads);
(c) the structure is not complete and is connected;
(d) any two distinct quads on a same point intersect in a line.

**Proof.** Suppose \( \Gamma \) is a uniform polarized space of rank \( r + 1 \geq 4 \); we check the Axioms 1 to 4.

1. Let \( L \) be a line on \( p \) and \( M \) and \( N \) two distinct lines on \( p \) contained in a plane \( \alpha \). Suppose that \( L \perp M \) and \( L \perp N \). Choose any line \( R \) on \( p \) in \( \alpha \). Then we must prove that \( L \perp R \), or that \( R \subset x^1 \) for every \( x \in L \). Since \( \alpha \) is a projective plane, we consider a line \( T \) in \( \alpha \) intersecting \( R, M \) and \( N \) in distinct points \( r, m \) and \( n \). Then \( m, n \in x^1 \), and hence \( T \subset x^1 \) by Axiom 1. In particular \( r \perp x \), and as \( p \perp x \) also, we have indeed that \( rp = R \subset x^1 \) by Axiom 1 again.

2. Let \( L \) and \( M \) be two non coplanar lines on \( p \), such that \( L^1 \cap M^1 \) contains at least one line on \( p \) (this means \( L^1 \cap M^1 \neq \emptyset \) in \( I_p \)). Choose points \( l \) on \( L \setminus \{ p \} \) and \( m \) on \( M \setminus \{ p \} \). Then applying Axiom 2, we obtain a polar space \( S = L^1 \cap M^1 \) of rank \( r \geq 3 \). Call \( S_p \) the set of all lines on \( p \) lying in \( S \), then it is easily checked that \( S_p = L^1 \cap M^1 \) in \( I_p \). Now it is well known that \( S_p \) is itself a polar space of rank \( r - 1 \geq 2 \). Therefore, in \( I_p \) we have that \( L^1 \cap M^1 \) is a polar space of rank \( r - 1 \). Remark that the case that \( |L^1 \cap M^1| = 1 \) in \( I_p \) is impossible. In other words, there are no special pairs in \( I_p \).

3. (a) By Axiom 3(c) the residue \( I_p \) is connected. Suppose that \( I_p \) is complete. Then there cannot be a hyperline on \( p \), because hyperlines are polar spaces. But this contradicts Proposition 1.

(b) In view of Proposition 2(c) and Axiom 3(b) there are at least three lines on a point contained in a plane through that point. This means that in \( I_p \) all lines have cardinality at least three.

(c) Consider a point \( L \) of \( I_p \), and let \( M, N \in L^1 \setminus \{ L \} \) in \( I_p \). If \( M \perp N \) we are already done. So suppose \( M \perp N \). Because \( L \in M^1 \cap N^1 \), by (2) of this proof, \( M^1 \cap N^1 \) is a polar space of rank at least two. Therefore it contains at least one \( R \) collinear with \( L \), different from \( L \). Now \( M, R, N \) is a path joining \( M \) to \( N \) in \( L^1 \setminus \{ L \} \). Hence, \( L^1 \setminus \{ L \} \) is connected in \( I_p \).

4. First, remark that if \( H_p \) is a hyperline of \( I_p \), then \( \langle \cup H_p \rangle = H \) is a hyperline of \( I \) in view of Proposition 2(b). Now suppose that \( H_p \) and \( K_p \) are hyperlines of \( I_p \) intersecting in at least one singular subspace \( M_p \) of rank \( r - 1 \). Then it is a well known property of projective spaces that \( \cup M_p = M \) is a singular subspace of rank \( r \). Both \( H = \langle \cup H_p \rangle \) and \( K = \langle \cup K_p \rangle \) contain \( M \). So axiom 4 applies on \( H \) and \( K \). The hyperlines \( H \) and \( K \) of \( I \) intersect in a singular subspace \( N \) of rank \( r + 1 \). It follows that \( N_p \subset H_p \cap K_p \) with \( N_p \) a singular subspace of rank \( r \) in \( I_p \).

Next suppose \( \Gamma \) is a uniform polarized space of rank 3. Property (a) is an immediate consequence of Proposition 2(a) and 2(c), Property (b) of the same Proposition 2(b). Remark that the reasoning that there exist no special pairs in \( I_p \) can be taken over from the proof of Proposition 4.

Further, (c) follows from Axiom 3(b) and Proposition 1. Finally (d) is proved using Axiom 4 in exactly the same way the last part of Proposition 4 was proved.

**Remark.** It follows from Proposition 2(b) and the theory of polar spaces, that any pair of non-collinear points of \( H(p, q) \) generates \( H(p, q) \). Therefore it is impossible that two distinct hyperlines should have two non-collinear points in common. Hence the
intersection is always a singular subspace (possibly empty). On the other hand, as any hyperline is a polar space of rank \( r + 1 \), the highest rank of a singular subspace contained in it is \( r + 1 \). This gives us a useful restatement of Axiom 4:

(4') two distinct hyperlines having a singular subspace of rank \( r \) in common, intersect in a singular subspace of rank \( r + 1 \).

4. THE SMALLEST CASE: UNIFORM POLARIZED SPACES OF RANK 3

In this case all hyperlines are of rank 3. We reformulate Axiom 4 as follows:

(4) two distinct hyperlines \( H \) and \( K \) having a line in common, intersect in a plane.

Remark that every plane \( \alpha \) is contained in exactly one max space. For suppose there are distinct max spaces \( M \) and \( N \) containing \( \alpha \) (and of course distinct from \( \alpha \)). Choose points \( p \in M - \alpha, q \in N - \alpha \), then \( p \not \perp q \). Indeed, otherwise \( \alpha \subseteq p^\perp \cap q^\perp \) should contradict Axiom 2 for \( r = 2 \). So every point of \( M \) is collinear with every point of \( N \), and hence \( M = N \) as \( M \) and \( N \) are both maximal, a contradiction.

Fix any point \( p \) and consider the residue \( \Gamma_p \). In this new geometry the following properties hold:

**Lemma 1.** If \( Q \) is a quad in \( \Gamma_p \) and \( x \) a point not in \( Q \), then \( x^\perp \cap Q \neq \emptyset \).

**Proof.** In view of the connectedness of \( \Gamma_p \), it suffices to prove that a path \( x, p, q \), with \( q \in Q \) can be shortened. If \( x \perp q \), then we are already done. So assume that \( x \not \perp q \), and consider the quad \( Q' \) on \( x \) and \( q \). Both quads \( Q \) and \( Q' \) contain \( x \), so they intersect in a line \( L \). Because \( Q' \) is a generalized quadrangle, \( x^\perp \cap L \) is not empty, say \( y \in x^\perp \cap L \). In particular \( y \in x^\perp \cap Q \).

**Corollary 1.** In \( \Gamma_p \), the distance from a line \( L \) to a point \( x \) is at most two.

**Proof.** Consider any quad \( Q \) containing \( L \). It is clear from Proposition 5 and the proof of Proposition 1, that such a quad exists. Then by the lemma above, \( x^\perp \cap Q \neq \emptyset \), write again \( y \in x^\perp \cap Q \). Now \( Q \) is a generalized quadrangle, so \( y^\perp \cap L \neq \emptyset \). Therefore, a point \( l \in L \) and a path \( x, y, l \) exist.

**Corollary 2.** The diameter of \( \Gamma_p \) is at most three.

From Proposition 1 and Axiom 2 for \( r = 2 \) it is clear that any line in \( \Gamma \) is contained in at least one plane. Now every plane is contained in exactly one max space. Hence any line is contained in at least one max space in \( \Gamma \). This is not necessarily so for \( \Gamma_p \). Of course, there exist maximal singular subspaces in \( \Gamma_p \). But those spaces do not necessarily contain a line properly. This observation splits up what follows in two cases: in the first one there exists at least one max space in \( \Gamma_p \), in the second there do not exist such spaces. We start with an investigation of the first case.

**Lemma 2.** Every line is contained in at most one max space in \( \Gamma_p \).

**Proof.** This is an immediate corollary of the remark made at the very beginning of this section.

**Lemma 3.** For a max space \( M \) in \( \Gamma_p \) and a point \( x \) of \( \Gamma_p \) not on \( M \), \( x^\perp \cap M \) is a singleton.
Proof. First we show that \( x^+ \cap M \) contains at most one point. Therefore, assume that \( x^+ \cap M \) contains two distinct points \( y \) and \( z \). Then \( y \perp z \) because they are points of a singular subspace \( M \). But now \( x, y, z \) are mutually collinear, and as \( x \notin yz \subset M \), these points span a plane. The line \( yz \) is then contained in at least two max spaces, a contradiction with Lemma 2.

Finally, we establish that \( x^+ \cap M \) is not empty. Choose a line \( A \) in \( M \). Then there is a point \( a \in A \) at distance two of \( x \). Next, choose a line \( B \) in \( M \), \( a \notin B \). Then there is a point \( b \in B \) at distance two of \( x \) again by Corollary 1. Certainly, \( a \) and \( b \) are distinct points.

Now consider the quads \( Q \) on \( x, a \) and \( a' \) on \( x, b \). As \( Q \) and \( Q' \) both contain the point \( x \), they have a line \( L \) in common by Proposition 5(d). Write \( a' \in a^+ \cap L \) and \( b' \in b^+ \cap L \). These points exist because \( Q \) and \( Q' \) are generalized quadrangles. If \( a' = b' \), then \( a, b \in a^+ \cap M \). The same argument of the first part of this proof, tells us that \( a \in M \). In this case the lemma is proved because \( x \perp a' \). Therefore, suppose \( a' \neq b' \). In view of Proposition 5(b), \( Q \) is geodesically closed. Since \( a \) and \( b' \) are non collinear points of \( Q \), both collinear to \( b \), it follows that \( b \) belongs to \( Q \). Again by 5(b), \( b \in Q \), and hence \( ab \subset Q \). Moreover, \( Q \) is a quad, so \( x^+ \cap ab \neq \emptyset \), and a fortiori \( x^+ \cap M \neq \emptyset \).

Lemma 4. Suppose that there exists at least one max space in \( \Gamma_p \). Then there is a max space on every point \( x \) of \( \Gamma_p \).

Proof. First of all, we suppose that the lemma does not hold for a given point \( x \). Let \( M \) be a max space, then we can construct a quad on \( x \) having no common point with \( M \). In view of Lemma 3, we put \( x^+ \cap M = \{ y \} \). Let \( L_1 \) and \( L_2 \) be two distinct lines on \( y \) contained in \( M \). Then there is a quad \( Q_1 \) (resp., \( Q_2 \)) containing \( xy \) and \( L_1 \) (resp., \( L_2 \)). Choose in \( Q_1 \) (resp., \( Q_2 \)) a line \( P_1 \) (resp., \( P_2 \)) on \( x \) different from \( xy \). We claim that \( P_1 \) and \( P_2 \) are distinct lines on \( x \). For suppose \( P_1 = P_2 = P \), and let \( p \in P - \{ x \} \). Then \( p \) and \( y \) are non collinear points of both \( Q_1 \) and \( Q_2 \), whence \( Q_1 = Q_2 \), a contradiction.

Thus we may assume \( P_1 \neq P_2 \). If \( P_1 \) and \( P_2 \) are contained in the same plane, then there is a max space on \( x \). In this case the lemma is proved. So assume moreover that there exists a quad \( Q \) through \( P_1 \) and \( P_2 \). Suppose \( Q \) contains the line \( xy \). Then by Proposition 5(b) \( Q = Q_1 = Q_2 \). In particular, \( L_1, L_2 \subset Q \). This is impossible because \( Q \), as generalized quadrangle, does not contain planes. Thus \( Q \) does not contain \( xy \). We claim that \( Q \) cannot contain a point of \( M \). If this were so, say \( m \in Q \cap M \), then \( m \neq y \), by the reasoning above. Clearly \( x \neq m \), so \( y^+ \cap Q \) contains two distinct points \( x \) and \( m \). Because \( Q \) is geodesically closed, and \( y \notin Q \), \( x \perp m \). But this contradicts Lemma 3. We conclude that \( Q \) is a quad on \( x \) having no common point with \( M \).

Now, choose points \( a \) and \( b \) in \( M \) with \( a \notin yb \). Then the quad \( \langle x, a \rangle \) on \( x \) and \( a \) intersects \( Q \) in a line by Proposition 5(d). Therefore, a point \( a' \in a^+ \cap Q \) exists. Further, the quad \( \langle b, a' \rangle \) intersects \( Q \) also in a line, and gives a point \( b' \in b^+ \cap Q \cap a^{+} \). Finally, the quad \( \langle b', y \rangle \) gives us a point \( x' \in y^+ \cap Q \cap b^{+} \) in the same way. Now \( x' \neq x \), otherwise \( x, a' \) would be on a same line in \( Q \) because they are pairwise collinear and lying in a generalized quadrangle. But then the lines \( ab \) and \( a'b' \) are in a same quad (the quad \( \langle a, b' \rangle \) for example). Hence there is a point \( y' \in ab \cap x^{+} \). As we supposed that \( y \notin ab \), \( y' \neq y \). This means \( x^+ \cap M \) contains two distinct points, which contradicts Lemma 3. Therefore \( x' \neq x' \), but then \( x \perp x' \). Indeed, if \( x \) and \( x' \) would be at distance two in \( Q \), then \( y \in x^+ \cap x^{+} \) would be a point of \( Q \) by the geodesic closure of \( Q \). Finally, as \( Q \) is a subspace, \( y \notin xx' \). We conclude that \( yxx' \) is a plane on \( x \), so there exists at least one max space on \( x \).

Corollary 3. If there exists at least one max space in \( \Gamma_p \), then the diameter of \( \Gamma_p \) is two.
Proof. Clearly the diameter is not one, because in that case \( \Gamma_p \) would be complete. Let \( x \) and \( y \) be two distinct points. By the foregoing lemma a max space \( M \) on \( y \) exists. By Lemma 3, \( x^+ \cap M \) is not empty and hence \( x^+ \cap y^+ \) is not empty.

**Lemma 5.** Suppose there exists at least one max space in \( \Gamma_p \). If \( L \) is a line and \( x \) a point of \( \Gamma_p \) such that \( x^+ \cap L = \emptyset \), then \( x^+ \cap L^+ \) is a singleton.

**Proof.** Choose distinct points \( a \) and \( b \) on \( L \), then by the preceding corollary we have \( d(a, b) = 2 \). So we can consider the quads \( Q_1 \) on \( x \) and \( a \), and \( Q_2 \) on \( x \) and \( b \). It is clear that \( Q_1 \neq Q_2 \) for otherwise \( x^+ \cap L \neq \emptyset \). By Proposition 5(d) both quads intersect in a line \( R \). Put \( r \in a^+ \cap R \) and \( s \in b^+ \cap R \). We claim that \( r = s \). For suppose that \( r 
eq s \), then \( R \) and \( L \) are contained in the same quad \( Q \) on \( r \) and \( b \). But in this quad, \( x^+ \cap L \neq \emptyset \). This contradiction, proves that \( r = s \), and in particular \( r \in x^+ \cap L^+ \).

On the other hand, assume that \( x^+ \cap L^+ \) contains two distinct points \( r \) and \( r' \). First suppose that \( r \neq r' \). The quad on \( r \) and \( r' \) contains both \( x \) and \( L \). But this would imply that \( x^+ \cap L \neq \emptyset \). Consequently \( r \perp r' \). By Lemma 2 there is at most one max space in \( \Gamma_p \) containing the line \( rr' \). Clearly \( x \) and \( L \) are both contained in it, another contradiction.

**Lemma 6.** If \( \Gamma \) is a uniform polarized space of rank 3, having at least one max space of rank greater than 3, then there exists a natural number \( n \geq 4 \), and a skew field \( K \) such that \( \Gamma \equiv A_{n,3}(K) \).

**Proof.** We can reformulate the conclusions of the corollary of Lemma 4 and Lemma 5 in terms of \( \Gamma \). Therefore we show first that if \( \Gamma_p \) contains at least one max space for a given point \( p \), then this is the case for any point \( p \). So take an arbitrary point \( q \) of \( \Gamma \). We must prove that \( \Gamma_q \) contains at least one max space. In view of the connectedness of \( \Gamma \) we may assume that \( p \perp q \). Because \( \Gamma_p \) contains at least one max space, Lemma 4 says that there is a max space on the point \( pq \) of \( \Gamma_p \). In \( \Gamma \), this means that there exists a max space \( M \) containing the line \( pq \) and of rank at least three. But then \( M_q \), the set of all lines on \( q \) lying in \( M \), is a max space in \( \Gamma_p \). Hence, if \( \Gamma \) is a uniform polarized space of rank 3, having at least one max space of rank greater than 3, then every residue \( \Gamma_q \) contains a max space.

By Corollary 3 we know that the diameter of every residue \( \Gamma_x \) is two. Now consider two points \( p \) and \( q \) at distance two. We can choose \( x \in p^+ \cap q^+ \). In \( \Gamma_x \), \( xp \) and \( xq \) are points at distance two also. This yields a line \( X \) on \( x \) with \( X \in (xp)^+ \cap (xq)^+ \) in \( \Gamma_x \) or \( X \in p^+ \cap q^+ \) in \( \Gamma \). Therefore, the case where \( |p^+ \cap q^+| = 1 \) does not occur. In other words, there are no special pairs in \( \Gamma \).

Next, consider a line \( L \) and a point \( p \) in \( \Gamma \) such that \( p^+ \cap L = \emptyset \) and \( p^+ \cap L^+ \neq \emptyset \). We can choose \( x \in p^+ \cap L^+ \). Write \( L_x \) for the set of all lines on \( x \) intersecting \( L \). Then \( xp \) is a point of \( \Gamma_x \) and \( L_x \) a line of \( \Gamma_x \) such that \( (xp)^+ \cap L \) is empty. By Lemma 5, \( (xp)^+ \cap L^+ \) is a singleton \( R \). Here \( R \) is a line on \( x \). In \( \Gamma \), we have that \( p^+ \cap L^+ \supset R \).

Suppose \( y \in p^+ \cap L^+ \). If \( y \perp x \), \( y \neq x \), then \( xy = R \). We claim that \( y \perp x \) is impossible. Indeed, suppose \( d(x, y) = 2 \), then \( p \in x^+ \cap y^+ \) and \( L \subset x^+ \cap y^+ \). But we know that \( x^+ \cap y^+ \) is a generalized quadrangle, so \( p^+ \cap L \neq \emptyset \). This contradiction proves that \( p^+ \cap L^+ = R \).

This suffices to apply the results from [10] (application (a)), proving this lemma.

**Corollary 4.** If we assume moreover that the max spaces of \( \Gamma \) have ranks that differ by at most one, then \( \Gamma_p \equiv A_1 \times A_2 \). So only the case \( A_{4,2} \), or equivalently \( E_{4,1} \), is left.

We now proceed our investigation with the other case, where in the residue \( \Gamma_p \) no max spaces exist. We will show that \( \Gamma_p \) is a classical near hexagon or a generalized quadrangle.
Lemma 7. If there exist no max spaces in $\Gamma_p$, then for any line $L$ and point $x$ not on $L$, there is a unique point of $L$ closest to $x$. In particular, $\Gamma_p$ is a generalized quadrangle if and only if the diameter of $\Gamma_p$ is two.

Proof. If $x^+ \cap L \neq \emptyset$, then $x^+ \cap L$ is a singleton because there are no max spaces in $\Gamma_p$. In view of Corollary 1, we need only to investigate what happens if $d(x, L) = 2$. So assume that $a, b \in L$ with $d(a, x) = d(b, x) = 2$, $a \neq b$.

Two points $a' \in x^+ \cap a^+$ and $b' \in x^+ \cap b^+$ must be distinct because no max spaces exist in $\Gamma_p$. The quads on $x$ and $a$, and on $x$ and $b$ are distinct because $x^+ \cap L = \emptyset$, and hence they intersect in a line $X$ by proposition 5(d). Without loss of generality $a'$ and $b'$ can be assumed to be in $a^+ \cap X$ resp. $b^+ \cap X$. We know that $a' \neq b'$ because no max spaces in $\Gamma_p$ exist. Hence the points $a, a', b$ and $b'$ are contained in a quad $Q'$. This quad contains also the lines $L$ and $X$. Therefore $x^+ \cap L \neq \emptyset$. This contradiction proves that for any point $x$ and line $L$ with $x$ not on $L$, there is a unique point of $L$ closest to $x$ if $d(x, L) = 2$.

Corollary 5. If $\Gamma$ is a uniform polarized space of rank 3 such that for a given point $p$ the residue $\Gamma_p$ contains no max spaces and has diameter two, then $\Gamma$ is a non-degenerate polar space of rank 3 with thick lines.

Proof. This should be clear from Lemma 7 and Proposition 3.

We attack the only case left: the residue $\Gamma_p$ has no max spaces, and has diameter greater than two (and hence three by Corollary 2). First, we introduce some new definitions and results (see also [12]).

If $(\mathcal{P}, \mathcal{L})$ is a connected incidence system, it is called a near hexagon whenever it satisfies the following two axioms:

- (N1) for each point $x$ and line $L$ there is a unique point of $L$ closest to $x$;
- (N2) the diameter is at most three.

It follows from [12] that one can define geodesically closed subspaces isomorphic to a non-degenerate generalized quadrangle. These subspaces are again called quads. Moreover, the near hexagon is called classical if it also satisfies:

- (N3) two points at distance two are contained in a quad;
- (N4) if $Q$ is a quad and $x$ a point not on $Q$, then $x^+ \cap Q \neq \emptyset$. For more comment and results we refer to [12].

Corollary 6. If there exist no max spaces in $\Gamma_p$, the diameter of $\Gamma_p$ is more than two if and only if $\Gamma_p$ is a dual polar space of rank three.

Proof. This follows from Proposition 5, Lemma 1 and Corollaries 1 and 2, combined with the mentioned results.

Corollary 7. If $\Gamma$ is a uniform polarized space of rank 3, such that every residue $\Gamma_p$ contains no max spaces and has diameter greater than two, then $\Gamma$ is the geometry of points and lines of a building of type $F_{4,1}$.

Proof. This follows from Corollary 6, Proposition 2 and the result from [20].

Theorem 3. The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a geometry of points and lines of a (weak) thick building of the types $C_{3,1}(K)$, $A_{n,2}(K)(n \geq 3)$, $F_{4,1}(K)$ if and only if $\Gamma$ is a uniform polarized space of rank 3.
5. Uniform Polarized Spaces of Rank 4

As induction on the rank will be our main tool in the proofs, here are two lemmas that will be very useful in what follows.

**Lemma 8.** If $\Gamma$ is a uniform polarized space of rank $r \geq 3$ such that for a given point $p$ the residue has a diameter greater than two, then $\Gamma$ cannot be itself a residue of a uniform polarized space of rank $r + 1$.

**Proof.** We first show that $\Gamma$ has special pairs. Consider two points $xp$ and $yp$ of the residue $\Gamma_p$, and suppose that $\Gamma$ has only polar pairs. Then $p \in x^+ \cap y^+$ in $\Gamma$, and hence $x^+ \cap y^+$ is a polar space of rank at least 2. In particular a point $qp \in (xp)^+ \cap (yp)^+$ in $\Gamma_p$ exists, so $\Gamma_p$ has diameter two. This contradiction proves that $\Gamma$ contains at least one special pair. By Proposition 4, $\Gamma$ cannot be itself a residue of a uniform polarized space of rank $r + 1$.

**Lemma 9.** If $\Gamma$ is a uniform polarized space of rank $r \geq 4$ such that the ranks of the max spaces differ by at most one unit, then the residue at any point $p$ satisfies this property also.

**Proof.** Easy exercise.

In this paragraph we suppose that all hyperlines have rank 4. Fix a point $p$, then by Proposition 4 and the result of the previous case, we know that $\Gamma_p$ is one of the following:

(a) a polar space of rank 3;
(b) a Grassmannian of lines in an $n$-space $A_{n,2}$;
(c) the geometry of points and lines of a building of type $F_4$.

By Lemma 8 however, case (c) will not occur. Moreover, Proposition 3 handles case (a) separately and provides a polar space of rank 4. So we may assume that for every point $p$, the resulting $\Gamma_p$ is a Grassmannian of lines in an $n$-space. Remark that a priori $n$ depends on the chosen point $p$.

Because in this case $\Gamma_p$ has diameter two, no special pairs exist in $\Gamma$. Hence Axiom 2 can be replaced by:

(2') if $p$ and $q$ are points at distance two, then $p^+ \cap q^+$ is a polar space of rank 3.

On the other hand, if $p$ is a point of $\Gamma$ and $S$ a hyperline of $\Gamma$, then $p^+ \cap S$ is empty, a point or a singular subspace of rank 4. Indeed, suppose that $x \in p^+ \cap S$ and consider $\Gamma_x \cong A_{n,2}$. Then we know in $\Gamma_x$ that $(xp)^+ \cap S_x$ is either empty or a singular subspace of rank 3, from which the assertion follows. It was noted $(F4)_{(-1,0)}$ in [13]. Axiom 2' was called $(P3)_3$ in the same reference.

Both properties $(P3)_3$ and $(F4)_{(-1,0)}$ allow us to apply the result in [2] (for sake of reference we replace the index $n + 1$ by $n$ again): $\Gamma$ is isomorphic to the quotient $D_{n,n}/A$, $n \geq 5$, $A$ being a group of automorphisms of $D_{n,n}$ such that for each $a \in A$, $a \neq 1$, the distance between a point and its image under $a$ in the collinearitygraph of $D_{n,n}$ is at least 5. Remark that in the finite case $A$ is trivial [1], thus we have proved:

**Theorem 4.** The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the geometry of points and lines of a (weak) thick building of the types $C_{n,1}(K)$, $D_{n,1}(K)$, $D_{n,n}(K)$ or a quotient $D_{n,n}(K)/A$ (with $A$ a group of automorphisms of $D_{n,n}(K)$ such that for each $a \in A - \{1\}$ the distance between a point and its image under $a$ is at least 5) $(n > 4)$, if and only if $\Gamma$ is a uniform polarized space of rank 4.
The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a geometry of points and lines of a (weak) thick building of the types $C_{4,1}(K), D_{4,1}(K)$ or $E_{5,1}(K)$ if and only if $\Gamma$ is a uniform polarized space of rank 4 for which the ranks of the maximal singular subspace differ by at most one.

**Proof.** It is a well known property of $D_{n,n}/A$ that the maximal subspaces have rank either 4 or n. This leaves us only the possibilities $4 = n$ and $4 + 1 = n$ or $n = 5$. Thus we have $D_{4,4}/A = D_{4,1}/A$ and $D_{5,5}/A$. But $D_{4,1}$ and $D_{5,5}$ have a diameter two, so an automorphism group $A$ as described in Theorem 4 must be trivial. Now $D_{4,1}$ is already on the list and $D_{5,5}$ can also be regarded as $E_{5,1}$.

### 6. Uniform Polarized Spaces of Rank 5

Next, we take the rank of all hyperlines to be 5. Fixing again a point $p$, and considering $\Gamma_p$, we apply Proposition 4 and the result of Theorem 4 to obtain the following possibilities for $\Gamma_p$:

(a) a non-degenerate polar space of rank 4;
(b) the geometry of points and lines of a building of type $D_{n,n}$, $n \geq 5$, or a quotient of it.

Case (a) is taken care of by Proposition 3: if for any point $p$ the residue $\Gamma_p$ is a non-degenerate polar space of rank 4, then $\Gamma$ is itself a non-degenerate polar space of rank 5.

**Theorem 5.** The geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the geometry of points and lines of a (weak) thick building of the types $C_{5,1}(K)$, $D_{5,1}(K)$ or $E_{6,1}(K)$ if and only if $\Gamma$ is a uniform polarized space of rank 5 for which the ranks of the maximal singular subspaces differ by at most one.

**Proof.** By Lemma 9 we can apply Corollary 8 to conclude that we get only a building of type $E_{5,1}$ for $\Gamma_p$. Now define the following sets:

- $\Gamma_1 = \mathcal{P}$;
- $\Gamma_2 = \mathcal{L}$;
- $\Gamma_3$ is the set of all planes of $\Gamma$;
- $\Gamma_4$ is the set of all max spaces of rank 6;
- $\Gamma_5$ is the set of all max spaces of rank 5;
- $\Gamma_6$ is the set of all hyperlines.

The incidence relation is defined as follows:

1. a max space of rank 5 and a max space of rank 6 are incident if they intersect in a singular subspace of rank 4;
2. a max space of rank 6 and a hyperline are incident if they intersect in a singular subspace of rank 5;
3. inclusion in all the remaining cases.

Now it is easily seen that these sets define a diagram of type $E_{6,1}$:

![Diagram](image)

By the result [20] of Tits, together with [1], $\Gamma$ is the geometry of points and lines of a (weak) building of type $E_{6,1}$.

Both this and the remark made above, prove the theorem.
THEOREM 6. The geometry \( \Gamma = (\mathcal{P}, \mathcal{L}) \) is the geometry of points and lines of a (weak) thick building of the types \( C_{5,1}(K), D_{5,1}(K), E_{6,1}(K), E_{n,n}(K)(n \geq 7) \) or a quotient \( E_{n,n}(K)/A \) with \( A \) a group of automorphisms of \( E_{n,n}(K) \) compatible with (1) and (2), if and only if \( \Gamma \) is a uniform polarized space of rank 5 with singular subspaces of finite rank.

PROOF. We rely on Proposition 4 and Theorem 4 to recall that we only have to investigate the case where \( \Gamma_p \) is isomorphic to \( D_{n,n}/A \) or \( D_{n,n}(n > 4) \).

A priori the index \( n \) and the group \( A \) may depend on the point \( p \). However, we will show that this is not the case for \( n \). In view of the connectedness of the structure, it suffices to prove this if \( p \) and \( q \) are collinear. So suppose that \( \Gamma_p \) isomorphic to \( D_{n,n}(K) \) or \( D_{n,n}(K)/A \) for a given \( n \). Then we know that any point of \( \Gamma_p \), in particular \( pq \), is on at least one max space of rank \( n \). This means in \( \Gamma \) the line \( pq \) is contained in at least one max space of rank \( n + 1 \). Conversely, in \( \Gamma_q \) we have then that the point \( qp \) is contained in at least one max space of rank \( n \). But if \( \Gamma_q \) contains max spaces of that rank, then \( \Gamma_q \) must be isomorphic to \( D_{n,n}(K') \) or \( D_{n,n}(K')/A \) for that same \( n \). Moreover, both fields \( K \) and \( K' \) underly the projective \( n \)-space containing \( pq \), so \( K = K' \). The group \( A \) may still depend on the chosen point \( p \).

First we remark that \( \Gamma \) has exactly two families of max spaces, the one containing all those of rank \( n + 1 \), the other all those of rank 5. This can easily be seen as follows: let \( M \) be a max space of \( \Gamma \), and take a point \( p \) of \( M \). Then \( \Gamma_p \cong D_{n,n}/A \) and the set of all lines on \( p \) contained in \( M \) is a max space of \( D_{n,n}/A \). Such a max space always has rank \( n \) or 4. Hence \( M \) itself has rank \( n + 1 \) or 5. We call \( \Gamma_4 \) the set of all max spaces of rank \( n + 1 \), and \( \Gamma_5 \) the set of all max spaces of rank 5. Moreover, we define \( \Gamma_1 = \mathcal{P}, \Gamma_2 = \mathcal{L}, \) and finally \( \Gamma_3 \) as the set of all planes of \( \Gamma \). In order to apply the results of Tits-Cohen [2], we show that we obtain a truncated diagram of type \( E_{n+1} \) if we define the incidence relation as follows:

(a) a max space of rank \( n + 1 \) is incident with a max space of rank 5 if their intersection is a singular subspace of rank 4;
(b) inclusion in all the remaining cases.

Therefore, we must consider all rank 2 residues of:

\[
\begin{array}{c}
1 & 2 & 3 & 5 \\
\end{array}
\]

i.e. all residues of type \( \{1, 2\}, \{1, 3\}, \ldots, \{4, 5\} \). We know that the residues at a point (an object of type 1), i.e. the set of all lines, planes, max spaces on that point, form a truncation of a building with diagram

\[
\begin{array}{c}
1 & 2 & 3 & 4 \\
\end{array}
\]

so this determines all residues of type \( \{i, j\}, i, j = 2, \ldots, 5 \). Further, the residue of a 4-space, i.e. all points, lines, planes contained in it, and all max spaces of rank \( n + 1 \) incident with it (or, what is the same, all 3-spaces contained in it) constitute the geometry of points, lines, planes and 3-spaces in a 4-space. This corresponds to a diagram

\[
\begin{array}{c}
1 & 2 & 3 & 4 \\
\end{array}
\]
or a truncated diagram

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (2,0);
\draw (0,1) -- (1,1);
\node at (0,0) {1};
\node at (1,0) {2};
\node at (2,0) {3};
\node at (0,1) {4};
\end{tikzpicture}
\end{center}

This argument determines all residues of type \{i, j\}, 5 \neq i \neq j \neq 5. Finally, we verify the residual connectedness over \{1, 2, 3, 4, 5\} (in the sense of [2], which is not equivalent to the connectedness of the residue \(\Gamma_p\) as was defined here). This follows from the fact that for any point \(p\), \(\text{Res}(p) = D_{n,n}/A\) is residual connected, and for any max space of rank 5, the residue of that max space is also residual connected. It follows from Theorem 4 in [2] that \(\Gamma\) is the geometry of points and lines of a (weak) building of type \(E_{n+1,n+1}/B\). Here, the group \(B\) of automorphism of \(E_{n+1,n+1}\) must be compatible with (1) and (2) (and hence with (3) and (4)).

7. Uniform Polarized Spaces of Rank 6

Now we assume that all hyperlines have rank 6.

**Theorem 7.** The geometry \(\Gamma = (\mathcal{P}, \mathcal{L})\) is the geometry of points and lines of a (weak) thick building of the types \(C_{6,1}(K), D_{6,1}(K)\) or \(E_{7,1}(K)\), if and only if \(\Gamma\) is a uniform polarized space of rank 6.

**Proof.** Consider a point \(p\) of \(\Gamma\), and a point \(pq\) of \(\Gamma_p\). Then Lemma 9 tells us that \((\Gamma_p)_{pq}\) has diameter two. In view of Proposition 4 the only possibilities for \((\Gamma_p)_{pq}\) are:

(a) a non-degenerate polar space of rank 4;
(b) the geometry of points and lines of a building of type \(D_{5,5}\).

But now we can use Theorem 5 to say that \(\Gamma_p\) must be one of the following possibilities:

(a) a non-degenerate polar space of rank 5;
(b) the geometry of points and lines of a building of type \(E_{6,1}\).

As always, Proposition 3 tells us what happens if for any point \(p\), the residue \(\Gamma_p\) is a non-degenerate polar space of rank 5: \(\Gamma\) is a non-degenerate polar space of rank 6. So we assume that for all points \(p\), \(\Gamma_p \cong E_{6,1}(K)\). An analogous argument as in the proof of Theorem 6 shows that the field \(K\) does not depend on the point \(p\).

From this we can define a diagram of type \(E_{7,1}\) as can easily be checked (see also [7] and [13].)

By the result already mentioned [19] of Tits, \(\Gamma\) is the geometry of points and lines corresponding to a (weak) building of type \(E_{7,1}\). This settles the theorem.

8. Uniform Polarized Spaces of Rank 7

In the same way as in the second part of the proof of Theorem 7, it can be shown using Proposition 4 and Theorem 7, that the following theorem holds:

**Theorem 8.** The geometry \(\Gamma = (\mathcal{P}, \mathcal{L})\) is the geometry of points and lines corresponding to a (weak) thick building of the types \(C_{7,1}(K), D_{7,1}(K)\) or \(E_{8,1}(K)\), if and only if \(\Gamma\) is a uniform polarized space of rank 7.
9. Uniform Polarized Spaces of Rank Greater Than 7

First we consider the next case, namely rank 8. Again by Proposition 4, Lemma 9 and Theorem 8 we conclude that any residue $I_p$ must be a non-degenerate polar space of rank 7. Hence by Proposition 3, the only possibility left for $I$ is a non-degenerate polar space of rank 8. The same Proposition 3 allows us to apply induction if the rank is greater than 8, to give us:

**Theorem 9.** A uniform polarized space of rank $r+1$ greater than 7 is a non-degenerate polar space of rank $r+1$ and vice versa.

10. Proofs of the Theorems 1 and 2

As before, the 'only if' part can be found in literature ([7], [13], [14] and [12]). The 'if' part follows immediately by putting together the Theorems 3 until 9.

11. Comments

(a) Focusing on the first theorem, we may investigate the mutual independence of the axioms. In view of Theorem 2 it is clear that in a uniform polarized space the ranks of the maximal singular subspaces do not necessarily differ by at most one. Moreover, there are geometries that satisfy all axioms but (4) (for instance $E_{7,4}$), or all axioms but (2). On the other hand examples can be constructed which satisfy all axioms but (3)(a) (for instance a projective space), all but (3)(b) (see for instance Buekenhout–Sprague, polar spaces having a line of cardinality two, *J. Combin. Theory, Ser. A* 223-228 (1982)), all but (3)(c) (a bouquet of polar spaces, see [13]).

This leaves only the question whether or not (1) is independent of all the others, but it seems unlikely that this should not be the case.

(b) The next step should be to characterize all possible point-line geometries (for instance $A_{n,j}$; $C_{n,j}$...). It appears that both axioms (2) and (4) have to be weakened, so classification becomes harder. In [16] an attempt is made in that direction.

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