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# THE NOTION OF INDEPENDENCE IN CATEGORIES OF ALGEBRAIC STRUCTURES, PART I: BASIC PROPERTIES*

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We define a formula $\phi(x; t)$ in a first-order language $L$, to be an equation in a category of $L$-structures $\mathbf{K}$ if for any $H$ in $\mathbf{K}$, and set

$$p = \{\phi(x; a_i); i \in I, a_i \in H\}$$

there is a finite set $I_0 \subset I$ such that for any $f: H \to F$ in $\mathbf{K}$,

$$\bigcap_{i \in I_0} \phi(F; fa_i) = \bigcap_{i \in I} \phi(F; fa_i).$$

We say that an elementary first-order theory $T$ which has the amalgamation property over substructures is equational if every quantifier-free formula is equivalent in $T$ to a boolean combination of (quantifer-free) equations in Mod($T$), the category of models of $T$ with embeddings for morphisms.

Thus, we develop a theory of independence with respect to equations in general categories of structures, which is similar to the one introduced in stability (and actually identical to it in the case of equational theories) but which, in our context, has an algebraic character.

## Introduction

The motivation for this work comes largely from the work of Shelah on stable theories. In his investigation of a stable theory $T$, Shelah showed the existence of an independence relation between the subsets of a model of $T$ which satisfy some natural properties (see axioms 1–5 below). This notion was crucial for developing a dimension theory on structures and the proof of structure theorems.

However, the notion of a stable theory, and consequently the independence relation introduced by Shelah does not take into consideration the 'algebraic' character of many of the familiar examples of algebra. In particular, it does not take into consideration the existence in such theories of distinguished formulas which play the same role that algebraic equations play in the theory of fields or linear equations in the theory of vector spaces. Our aim in this paper is to show that there is a good notion of a distinguished formula, that we call 'equation',

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which is a natural generalization of standard equations in algebra (e.g., algebraic equations, linear equations, differential equations, etc.). And also that we can define (say, in a given class of structures) an independence relation with respect to a set of 'equations' in very much the same manner algebraic independence is defined in the class of fields, or linear independence in the class of vector spaces.

But let us start from the beginning and assume only a naive understanding of what an independence relation means. The following are some examples.

(a) In the category of fields $\mathbf{F}$ with field homomorphisms, we have the notion of algebraic independence. More precisely, given fields $F_0 \subset F_1$, $F_2 \subset F$, we say that $F_1$ is algebraically independent from $F_2$ over $F_0$ if whenever $A$ is a finite set of elements in $F_1$ which is algebraically independent over $F_0$, then $A$ remains algebraically independent over $F_2$. For $A$, $B$, $C$ subsets of a field $F$, write $B \downarrow_A C$ if $\langle A \cup B \rangle$ (the field generated by $A \cup B$) is algebraically independent from $\langle A \cup C \rangle$ over $\langle A \rangle$. Let us also say that $B$ is isomorphic to $C$ over $A$ if there is a field isomorphism from $\langle A \cup B \rangle$ onto $\langle A \cup C \rangle$ keeping $A$ fixed. Then the relation $\downarrow$ satisfies the following properties.

1. (Existence). There exists $L \in \mathbf{F}$ extending $F$ which contains an isomorphic copy $B'$ of $B$ over $A$ such that $B' \downarrow_A C$.
2. (Monotonicity-transitivity). Given $D \subset C$, $B \downarrow_A C$ iff $B \downarrow_D C$ and $B \downarrow_A D$.
3. (Local character). $B \downarrow_A C$ iff $B \downarrow_A C_0$ for every finite subset $C_0$ of $C$.
4. (Symmetry). $B \downarrow_A C$ iff $C \downarrow_A B$.
5. (Stability). (i) There exists $C_0 \subset C$ such that $B \downarrow_{C_0} C$ and $|C_0| \leq |B| + \aleph_0$.

(ii) Assuming $B$ finite, there are at most $2^{\aleph_0}$ (actually here at most finitely many) isomorphic copies $B'$ of $B$ over $A$ which are non-isomorphic over $C$ and such that $B' \downarrow_A C$.

(b) In the category $\mathbf{M}_R$ of modules over a fixed ring, we have the notion of direct sum over a module. For $A$, $B$, $C$ subsets of a module $M$, write $B \downarrow_A C$ if $\langle A \cup B \rangle \cap \langle A \cup C \rangle = \langle A \rangle$ ($\langle A \rangle$ the module generated by $A$), i.e., if the sum $\langle A \cup B \rangle + \langle A \cup C \rangle$ is direct over $\langle A \rangle$. Again, one verifies that the relation $\downarrow$ satisfies the properties 1–5 described above.

(c) More generally, in an arbitrary category $\mathbf{K}$ of $L$-structures ($L$ a first-order language) one has the notion of free amalgamation, which is defined as follows: let $h_i : H_0 \to H_i$ and $f_i : H_i \to H$, $i = 1, 2$, be morphisms in $\mathbf{K}$ with $f_1 h_1 = f_2 h_2$. Then $H$ (in principle, we should say $(H, f_1, f_2)$) is a free amalgam of $H_1$ and $H_2$ if for any $g_i : H_i \to G$ ($i = 1, 2$) with $g_1 h_1 = g_2 h_2$, there exists a homomorphism (i.e., a map preserving the atomic formulas) $h : H \to G$ such that $h f_i = g_i$, $i = 1, 2$.

Write $(H_1, f_1) \downarrow_{H_0} (H_2, f_2)$ if there exists $g_i : H_i \to G$, $i = 1, 2$, and $g : G \to H$ such that $g_1 h_1 = g_2 h_2$, $g g_i = f_i$ and $G$ is a free amalgam of $H_1$ and $H_2$ over $H_0$.

In general, the relation $\downarrow$ defined above does not satisfy properties 1, 2, 3, 5. However, if $\mathbf{K}$ is an elementary class with embeddings for morphisms, which has the amalgamation property and which is closed under substructures and products, then $\downarrow$ satisfies 1, 2, 3, 4.

(d) Let **K** be the category described in (c). By substituting the function symbols in **L** by corresponding relation symbols, we can assume **L** contains only relation symbols. Given $H$ in **K**, let $H^*$ be the $L$-structure which has the same underlying set as $H$, but where a relation symbol in **L** is interpreted by its negation in **H**. Let $\mathbf{K}^*$ be the category with objects the structures $H^*$ and with morphisms the original morphisms of **K**. Obviously, the same relation $\downarrow$ that we defined in **K** can be defined in $\mathbf{K}^*$, and clearly it satisfies the properties 1–5 according to their satisfaction in **K**. Note however that $\mathbf{K}^*$ is not preserved under products.

[To analyze the independence relation in these examples, we resort to the notion of type: Let $L$ be a first-order language and **K** a class of $L$-structures where a certain set $\Delta$ of formulas in $L$ with $\Delta = \text{cl}(\Delta)$, has been fixed. We call such a class a $\Delta$-category. (For instance, it is natural to consider the class of fields **F** as a $\Delta$-category with $\Delta$ the set of quantifier-free formulas.) We will define a notion of type generalizing the notion in model theory. Since we have not assumed that **K** is the class of models of a complete quantifier eliminable theory, whenever we speak of the truth of a sentence we must specify an ambient structure.

If $A, B \subset H \in \mathbf{K}$, the type of $B$ over $A$ in $H$ is defined as

$$\text{tp}(B, A; H) = \{\phi(\bar{x}_{\bar{b}}; \bar{a}) : \bar{a} \in A, \bar{b} \in B, \phi \in \Delta, H \models \phi(\bar{b}; \bar{a})\}.$$

A collection $p$ of $\Delta$-formulas $\phi(\bar{x}; \bar{a})$ with parameters from $A \subset H$ is *realized over* $H$ if for some $H_1 \in \mathbf{K}$, $H \subset H_1$, and $\bar{b}$ in $H_1$, $p \subset \text{tp}(\bar{b}, A; H_1)$. The collection $p$ is *consistent over* $H$, and if so we call $p$ a *type over* $A$, if every finite subset of $p$ is realized over $H$. If $p$ is a type over $A \subset H \in \mathbf{K}$ and $\phi$ is a formula with parameters in $H$, we write $p \vdash_H \phi$ if for some finite $p_0 \subset p$, for every $F \in \mathbf{K}$ with $H \subset F$, $F \models \bigwedge p_0 \rightarrow \phi$. For $A \subset B \subset H$ and $p$ a type over $A$ in $H$ we define the consequences of $p$ on $B$

$$p_B = \{\phi(\bar{x}; \bar{b}) : \bar{b} \in B, \phi \in \Delta, p \vdash_H \phi(\bar{x}; \bar{b})\}.$$

We sometimes restrict $p_B$ to the formulas from a subset $S$ of $\Delta$ and write $p_B^S$.

Note that the preceding discussion takes place with respect to a category of structures **K** where the morphisms are embeddings. However, modulo a slight change of notation we could work with any class of morphisms.]

Now, in the examples above, the relation $\downarrow$ has been defined in terms of a particular set of formulas $S$: in example (a), $S$ is the set of algebraic equations; in (b), $S$ is the set of linear equations; in (c), $S$ is the set of atomic formulas; and in (d), $S$ is the set of negated atomic formulas. (Example (d) shows that the syntax of the formulas $S$ with respect to which $\downarrow$ is defined is not essential. If the formulas in $S$ are atomic formulas, then we can easily transform them into negated atomic formulas without essentially changing the class of structures.) Let us naively call the formulas in $S$ 'equations'. Then, informally speaking, for $A, B, C \subset F \in \mathbf{K}$ we had $B \downarrow_A C$ if the 'equations' that elements of $B$ satisfy over $C$ are induced from the relations that these elements satisfy over $A$. More precisely, in examples (b), (c) and (d) one can check that $B \downarrow_A C$ if $\operatorname{tp}(B, A; F) \vdash_F \operatorname{tp}^S_C(B, C; F)$—here we usually say that $B \downarrow_A C$ if $B$ does not satisfy non-trivial equations over $C$ which it did not already satisfy over $A$—and the same holds in example (a) in case $\langle A \rangle$ is algebraically closed.

At that point we can ask whether, for arbitrary categories $\mathbf{K}$, there is a general property of formulas which makes them behave like equations. To begin with, such a property should be such that if $S$ is a set of 'equations' in $\mathbf{K}$, then we should be able to define in $\mathbf{K}$ an independence relation $\downarrow$ with respect to $S$ which is similar to those described in the examples; and conversely if $\mathbf{K}$ is a category with an independence relation defined with respect to a particular set of formulas, then we should be able to consider these formulas as 'equations'. In the following paragraphs we present a generalization of the notion of independence described in the examples, to an arbitrary category. Subsequently we define the notion of an equation and state the theorem which shows that the two notions correspond to each other in the sense given above.

Given a $\Delta$-category $\mathbf{K}$, $A \subset C \subset H \in \mathbf{K}$, $S \subset \Delta$, $p$ a type over $A$, $q$ a complete type over $C$, $q \supset p$, $q$ is an '$S_H$-minimal extension' of $p$ if for every complete type $r$ over $C$, $r \supset p$, $q^S_H \supset r^S_H \Rightarrow q^S_H = r^S_H$. (For instance, in case $\mathbf{K} = \mathbf{F}$, one verifies that $q$ is an $S_H$-minimal extension of $p$ if the algebraic equations $\bigwedge q^S_H$ is an irreducible component of $\bigwedge p^S_H$ over $H$.)

The notion of $S_H$-minimal extension of a type is best appreciated if it is compared to the following: Given $A \subset B, C \subset H$, let us say that $B$ and $C$ are $S$-freely amalgamated over $A$ if whenever $B'$, $C'$ are subsets of $H$ such that $B'$, $C' \supset A$, $\operatorname{tp}^S_A(B; A) \subset \operatorname{tp}^S_A(B'; A)$, and $\operatorname{tp}^S_A(C; A) \subset \operatorname{tp}^S_A(C'; A)$, then for every tuples $\bar{b}$, $\bar{c}$ in $B$, $C$ respectively,

$$\{\phi(\bar{x}; \bar{y}) \in S; H \models \phi(\bar{b}; \bar{c})\} \subset \{\phi(\bar{x}; \bar{y}) \in S; H \models \phi(\bar{b}'; \bar{c}')\}.$$

We will see that under some conditions—as for instance the case of a complete theory with $A$ an elementary submodel of $H$ and $S$ a set of equations (see below)—we have that $\operatorname{tp}(B, C; H)$ is an $S_H$-minimal extension of $\operatorname{tp}(B, A; H)$ iff $B$ and $C$ are $S$-freely amalgamated over $A$.

Later on we write $B \downarrow_A C$ (for $A, B, C \subset H \in \mathbf{K}$) if $\operatorname{tp}(B, C; H)$ is an $S_H$-minimal extension of $\operatorname{tp}(B, A; H)$. In that way, properties 1–5 above translate into properties of $S_H$-minimal extensions of types. For instance, property 1 becomes:

(Existence). Given $A \subset C \subset H \in \mathbf{K}$ and $p$ a type over $A$, there exists an $S_H$-minimal extension of $p$ to $C$.

**Definition.** A set of formulas $R$ is *equational* if for any $H$ in $\mathbf{K}$ and set $p = \{\phi_i(x; a_i); i \in I, \phi_i \in R, a_i \in H\}$, there is a finite subset $p_0$ of $p$ such that $p_0 \vdash_H p$; a formula $\phi(x; t)$ is an *equation* (in $\mathbf{K}$) if $\{\phi(x; t)\}$ is equational.

(Algebraic equations in $\mathbf{F}$ and linear equations in $\mathbf{M}_R$ are equations in the sense given above.)

We show:

**Theorem.** (1) *If* $\mathbf{K}$ *is the* $\Delta$*-category of models of an elementary first-order theory, with embeddings for morphisms,* $\Delta$ *the set of quantifier-free formulas, such that the atomic formulas are equations and* $\mathbf{K}$ *has the amalgamation property over substructures, then the relation* $\downarrow$ *indeed defines an independence relation in existentially closed structures satisfying properties* 1–5. ($\mathbf{K}$ has the amalgamation property over substructures if whenever $A \subset H_1$, $H_2 \in \mathbf{K}$ and $(H_1, A) \equiv_\Delta (H_2; A)$, then there exist $h_i : H_i \to H$, $i = 1, 2$, such that $h_1 | A = h_2 | A$.)

(2) *Conversely, if* $\mathbf{K}$ *is a category such as in* (1) *above* (*but without the assumptions on the atomic formulas being equations*), *which satisfy the conclusion of* (1), *then the atomic formulas are equations.*

A category $\mathbf{K}$ satisfying the hypotheses of the theorem, we have called equational.

As mentioned above the motivation for this work comes largely from the work of Shelah on stable theories. For our purpose, we shall say that a $\Delta$-category $\mathbf{K}$ of models of an elementary first-order theory with embeddings for morphisms, $\Delta$ the set of quantifier-free formulas, which has the amalgamation property over substructures, is stable if every atomic formulas does not have the order property; a formula $\phi(x; t)$ has the order property in $\mathbf{K}$ if there is $H$ in $\mathbf{K}$, $(a_i)_{i<\omega}$, $(b_i)_{i<\omega}$ in $H$ such that $H \models \phi(a_j, b_i)$ iff $j < i$.

Shelah defined in such categories the notion of a 'non-forking' extension of a type which corresponds to our notion of $S_H$-minimal extensions (but is more subtle); consequently he defined an independence relation $\downarrow$ (e.g., $B \downarrow_A C$ if $\mathrm{tp}(B; C; H)$ is a non-forking extension of $\mathrm{tp}(B, A; H)$) and showed that it satisfies in existentially closed structures the properties 1–5 described above (cf. [8] and [9]). (To be precise, the results in [8] are not presented in exactly the same manner we stated them.)

It is easy to see that a formula which is an equation in $\mathbf{K}$ does not have the order property, so that an equational category is in fact stable and Shelah's results apply to our context. However, we are interested here in 'algebraic' structures, and the assumption of stability alone does not convey the algebraic character of many of the familiar examples of algebra.

Indeed, the non-order property alone of a formula $\phi$ does not establish any distinction between $\phi$ and $\neg\phi$ since, by compactness, it is easily seen that $\phi$ has the order property if $\neg\phi$ has, while a crucial aspect of 'algebraic' theories is that some formulas are distinguished and considered 'positive' and their negations are considered 'negative'; also most of the basic notions defined in these theories, among which, notions of independence, mainly depend on these formulas. For instance, in the theory of fields, the fundamental formulas are the algebraic equations; definitions or results in this theory are usually stated in terms of algebraic equations. Actually, in this case, we note that an important distinction between the algebraic equations and the inequations is the fact that the varieties of some fixed dimension $n$ in a given field $F$ are the basic closed sets of a noetherian compact topology on $F$.

It is our aim to show that the assumption of equationality (and chain condition properties in general) translates well this idea of positiveness. For instance, if $R$ is an equational set of formulas in $\mathbf{K}$, then the $R$-definable subsets of a structure $M$ in $\mathbf{K}$ constitute the basic closed sets of a compact noetherian topology on $M$, which, in the case of fields with $R$ the set of algebraic equations, is identified to the Zariski topology. We do not pretend that the assumptions of equationality are all that make one set of formulas more interesting than another. For instance, in the case of fields, we work with the set of algebraic equations rather than with the set of algebraic equations with in addition the formula $x \neq 1$, although the latter set satisfy all the chain condition properties that we can think of. But the notion of equation gives a natural context in which these considerations can be investigated.

In general, we have attempted to show that the notion of an equation relates in a natural way to algebra and that it is possible to attach to it many of the properties and definitions that are usually attached to the standard equations in algebra (e.g., algebraic equations in $\mathbf{F}$, linear equations in $\mathbf{M}_R$).

Finally, we ought to say a few words on how our notion of an equational category relates to the standard notion (or to varieties) in universal algebra. Clearly, if we stick to the definitions, the two notions do not compare. However, a comprehensive definition — based on the existence of an independence relation with some properties, (excluding 5(i)) — that will include both notions is possible. We have not done so at least for practical reasons. In any case, the way we presented our material, it will be clear in the sequel, if not explicitly stated, whether a certain result will hold true for varieties; while on the other hand, certain concepts of universal algebra will be seen to extend to our context.

For the convenience of the reader and of publication, we have divided the paper in four parts: Basic properties (Sections 0–2), $S$-minimal extensions (Sections 3–5), Equational categories (Sections 6–7), and the case of a complete theory (Section 8). These will be published separately. However they will have to be read sequentially.

In Section 0, we fix some notation, make precise the setting in which we want to work and define the notion of an equational set of formulas.

In Section 1, we compare the notion of equationality to some natural variants as for instance a formula having finite height; then we investigate the basic properties of equational formulas.

In Section 2 we introduce some terminology on types. Then we give two useful criteria for equationality using complete types. As an application we prove that the set of differential equations in the category of differential fields of characteristic 0 is equational; and that every quantifier-free formula in the category of radical differential fields of characteristic $p$ ($p \neq 0$) is a boolean combination of equations.

In Section 3, we investigate $S_H$-minimal extensions of a type, show that such extensions always exist (cf. II.2), that they satisfy the monotonicity-transitivity property when considering types over $S_H$-closed subsets of $H$ and that, for $S$ equational, a type $p$ over $A \subset B \subset H$ has, up to $S_H$-equivalence, finitely many $S_H$-minimal extensions to $B$.

In Section 4, we define what $S$-irreducible and $S$-full types are, as well as $S$-irreducible and $S$-full structures. We also define what an $S_H$-component of a type is and investigate their existence.

In Section 5, we observe that the theory of Sections 3 and 4 goes through in a very general abstract context; we then describe such a context.

In Section 6 we study the case of a $\Delta$-category $\mathbf{K}$ of models of a first-order theory $T$ with the $\Gamma$-elementary embeddings for morphisms, where we assume $\Delta$ is the set of all formulas in $L$, $\Gamma$ is a boolean-closed set of formulas, $S \subset \Gamma$, $S = \mathrm{cl}^+(S)$ and $\mathbf{K}$ reflects $S$. We show then, for $S$ a set of equations, that any $\Sigma_1(\Gamma)$-closed structure $H$ in $\mathbf{K}$ is $S$-full and that any subset of such a structure is $S_H$-closed in $H$; furthermore we prove the local-character property for $S_H$-minimal extensions of type over subsets of $H$.

In Section 7, we introduce the notion of $S$-minimal amalgam and relate it to $S_H$-minimal extensions. We also define the notion of an equational category and prove the symmetry property in such categories.

In Section 8, we consider the case of the category of models of a complete first-order theory $T$ with the elementary embeddings for morphisms. We consider the case $T$ stable and show that if $p$ is a complete type over $A \subset B$ and $q$ is a non-forking extension of $p$ to $B$ then $q$ is an $S_H$-minimal extension of $p$ to $B$ whenever $S$ is a set of equations in $T$. We define what an equational theory is. We then classify equational theories in terms of equational theories having the d.c.c. and equational theories having the d.c.c. on irreducible types. We show that equational theories with the d.c.c. (resp. with the d.c.c. on irreducible types) are totally-transcendental (resp. superstable) and give (in both cases) criteria for the two notions to be equivalent. We also characterize the fundamental order in equational theories in terms of equations.

It is possible to infer some of our results from stability theory (cf. [9] or [7]). However, as we believe that the notions of equations and equational categories are natural concepts that deserve to be studied for their own sake, we preferred to give a direct approach which is particular to these notions. We point out here to the fact that we do not have an example of a stable theory which is not equational. Also, we attempted to begin our study of $S$-minimal extensions of types from scratch, starting with arbitrary categories of $L$-structures and arbitrary sets of formulas $S$. We like to believe that this attempt has not been pursued purely for the sake of generalization, but that such a presentation describes a general setting which is applicable to particular situations. (For instance, given a ring $R$ and a polynomial $P(\bar{x})$ over $R$, the ring $R[\bar{x}]/P(\bar{x})$ can be construed as the ring over $R$ generated by a tuple $\bar{x}$ which realizes the $S_R$-minimal extension of the type $\{(P(\bar{x})=0)\}$, for $S$ the set of algebraic equations in the category of rings. Similarly, free-amalgamation of groups can be described using the notion of $S$-minimal extensions of types, for $S$ the set of atomic formulas in the category of groups. Note that the two preceding examples fall within the context of example (c) above, but, as we have seen in (d), a simple alternation will give us less trivial cases.) On the whole we tried to understand what conditions are necessary and sufficient to define a reasonable independence relation in a given category of algebraic structures.

Thus, the technical work in the paper is self-contained. And, for the sake of clarity as well as of completeness, although the similarities with stability theory are evident throughout the paper, we made almost no reference to this theory until Section 8 where in any case we state the facts that are needed. Section 8 is in fact devoted to comparisons between the concepts introduced here and the concepts of stability theory. Familiarity with stability theory is therefore helpful but not necessary. (Useful introductory references to stability theory are [4], [5], [3], [1].)

The arguments used in the proofs are fairly simple — that actually was one of our original motivations for writing the paper. However, the new terminology and consequently the notation that we have had to introduce might sometimes induce a statement to appear more complicated than it really is. We realize that we have not been very imaginative in such cases, but we still hope that the reader will recognize this tediousness as indeed just a matter of notation which therefore has to be dealt with carefully.

Unless expressly stated to the contrary, all the results and concepts in this paper are due to the author. The notions of equation and equational theory have first been defined in [11].

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## 0. Preliminaries

In this section we fix the setting in which we want to work and define the notions of equational and strongly equational sets of formulas. We also give some examples.

(a) We fix once and for all a first-order language $L$.

$\phi, \psi, \chi, \ldots$ denote formulas in $L$; $H, H', M, \ldots$ denote $L$-structures; $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$ denote finite tuples of elements in given $L$-structures.

We do not distinguish between $L$-structures and their underlying sets.

We divide all variables in $L$ in two classes $X$ and $T$, and call the variables in $X$ types variables, the variables in $T$ parameter variables.

Unless stated otherwise, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_1, \ldots$ denote finite tuples of type variables;

$\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{t}_1, \ldots$ denote finite tuples of parameter variables.

A formula in $\boldsymbol{x}$ is a formula of the form $\phi(\boldsymbol{x}; \boldsymbol{t})$.

If $S$ is a set of formulas in $L$, we let $S^{\boldsymbol{x}}$ denote the set of formulas in $\boldsymbol{x}$ which are in $S$. We frequently write $\phi$ or $\phi(\boldsymbol{x})$ for $\phi(\boldsymbol{x}; \boldsymbol{t})$ or $\phi(\boldsymbol{x}; \boldsymbol{a})$, $\boldsymbol{a}$ a tuple in some given structure, that is, when the context makes it clear which one is meant.

A set of formulas $p$ in $\boldsymbol{x}$ with parameters in $H$ is realized in $H$ if there is a finite tuple $\boldsymbol{a}$ of elements in $H$ such that $H \models \phi(\boldsymbol{a})$ whenever $\phi(\boldsymbol{x})$ belongs to $p$.

(b) We fix a category $\mathbf{K}$: the objects of $\mathbf{K}$ are $L$-structures and the morphisms of $\mathbf{K}$ are maps between the underlying sets of objects in $\mathbf{K}$; composition of morphisms is then the composition of maps, and, for $H$ in $\mathbf{K}$, the identity morphism on $H$ is the identity map from $H$ into $H$.

Later on we shall consider additional assumptions on $\mathbf{K}$ as for instance that $\mathbf{K}$ is the category of models of a first-order theory with embeddings or elementary embeddings for morphisms.

In fact, most of the time, for the sake of clarity, we will assume that the morphisms in $\mathbf{K}$ are inclusions (or at least one-to-one maps), the arbitrary case being an immediate generalization. The notation that follows below will then become just the standard notation and will be used indiscriminately as such.

Thus, although a precise formalization of the notation is necessary, at least so as to make the generalization from inclusions to arbitrary maps possible, once taken into consideration it can be readily confused with the standard one.

To simplify the presentation we extend $\mathbf{K}$ to the category $\hat{\mathbf{K}}$ which includes the subsets of structures in $\mathbf{K}$ as objects and the inclusion maps between subsets of a structure in $\mathbf{K}$ as morphisms. Formally speaking:

– Object$(\hat{\mathbf{K}}) = \{(A, H); H \in \mathbf{K}$ and $A \subset H\}$.

– A morphism $f:(A, H) \to (B, F)$ in $\hat{\mathbf{K}}$ is defined as a morphism, denoted again $f$, $f: H \to F$ in $\mathbf{K}$ such that range$(f \mid A) \subset B$. [We mean by that, that $f:(A, H) \to (B, F)$ is identified with the triple $\langle f: H \to F; (A, H); (B, F)\rangle$.]

Thus, the identity morphism id$:(A, H) \to (A, H)$ is formally defined as the identity morphism on $H$.

If $f:(A,H)\to(B,F)$ and $g:(B,F)\to(C,G)$ are in $\hat{\mathbf{K}}$ so that $f:H\to F$ and $g:F\to G$ are in $\mathbf{K}$, then $g\cdot f:(A,H)\to(C,G)$ is formally defined as the morphism $g\cdot f:H\to G$.

For $f:(A,H)\to(B,F)$ we let $f[(A,H)]=(f(A),F)$. So if $f:H\to H$ is the identity on $H$ and $A\subset B\subset H$, then $f:(A,H)\to(B,H)$ is a morphism in $\hat{\mathbf{K}}$ and $f[(A,H)]=(A,H)$; in that case, we refer to $f$ as an inclusion map.

Note that two morphisms $f,g:(A,H)\to(B,F)$ in $\hat{\mathbf{K}}$ are identified if the morphisms $f,g:H\to F$ are equal and not just if $f$ and $g$ take the same values on $A$.

When there is no ambiguity, we write $A$ instead of $(A,H)$ and $f;A\to B$ instead of $f:(A,H)\to(B,F)$.

If $\phi=\phi(x,a)$ is a formula in $x$ with parameters in $A$ ($A\in\hat{K}$) and $f:A\to B$ is a morphism in $\hat{\mathbf{K}}$, we let $f\phi=\phi(x,fa)$ (where $fa=\langle fa_1,\ldots,fa_n\rangle$ when $a=\langle a_1,\ldots,a_n\rangle$). Of course $f\phi$ is a formula in $x$ with parameters in $B$.

If $p$ is a set of formulas in $x$ with parameters in $A$, we let $fp=\{f\phi;\phi\in p\}$.

We say $p$ is realized (in $\mathbf{K}$) over $(A,H)$ if there is a morphism $f:(A,H)\to F$ such that $fp$ is realized in $F$; $p$ is consistent over $(A,H)$ if every finite subset of $p$ is realized over $(A;H)$; $p$ is inconsistent if $p$ is not consistent.

For $A=(A,H)$ in $\hat{\mathbf{K}}$, $\phi=\phi(x)$ and $\psi=\psi(x)$ formulas in $x$ with parameters in $A$, we write $\phi\vdash_A\psi$ if for any morphism $f:H\to F$, $(f\phi)(F)\subset(f\psi)(F)$. Write $\phi\sim_A\psi$ iff $\phi\vdash_A\psi$ and $\psi\vdash_A\phi$.

If $p$ and $q$ are sets of formulas in $x$ with parameters in $A$, $\phi$ as above, we write $p\vdash_A\phi(x)$ if there is a finite subset $p_0$ of $p$ such that $p_0\vdash_A\phi$; $p\vdash_A q$ if $p\vdash_A e$ for every $e\in q$; $p\sim_A q$ if $p\vdash_A q$ and $q\vdash_A p$.

If $p$ and $q$ are sets of formulas in $x$ with parameters in $A$, or just single formulas, and $p\sim_A q$ we say that $p$ is equivalent to $q$ over $A$.

**Remark.** For $p$ a set of formulas in $x$ with parameters in $A$ ($A\subset H$), $p$ is inconsistent over $A$ iff some finite subset of $p$ is inconsistent over $A$ iff some finite subset of $p$ is not realized over $A$ iff there is a finite set $\{\phi_i;i\in I\}$ ($I$ finite) of formulas in $p$ such that $(x=x)\vdash_A\bigvee_{i\in I}\neg\phi_i$.

**Example.** Let $\mathbf{K}$ be a category of $L$-structures with embeddings for morphisms, e.g., the category of fields.

Suppose that $\mathbf{K}$ has the amalgamation property and is closed under unions of increasing chains of structures. Let $F$ be a structure in $\mathbf{K}$, $\phi(x)$ and $\psi(x)$ quantifier-free formulas with parameters in $F$.

Then one easily checks that $\phi(x)\vdash_F\psi(x)$ (resp. $\phi(x)$ is consistent over $F$) iff for some existentially-closed structure $E$ in $\mathbf{K}$ containing $F$ we have $\phi(E)\subset\psi(E)$ (resp. $\phi(x)$ is realized in $E$).

**0.1. Definition.** Let $\Delta$ be a set of formulas in $L$, $f:H\to F$ a morphism in $\mathbf{K}$.

Then

(i) $f$ is *$\Delta$-elementary* if for any formula $\phi(x)$ in $\Delta$ with parameters in $H$ and $a$ a tuple of elements in $H$, $H \models \phi(a) \Leftrightarrow F \models f\phi(fa)$.

(ii) $f$ *reflects* $\Delta$ if for $\phi(x)$ and $\psi(x)$ in $\Delta \cup \{(x = x), (x \neq x)\}$ with parameters in $H$, $f\phi \vdash_F f\psi \Rightarrow \phi \vdash_H \psi$.

(iii) $\mathbf{K}$ is *$\Delta$-elementary* (resp. *reflects* $\Delta$) if every morphism in $\mathbf{K}$ is $\Delta$-elementary (resp. reflects $\Delta$).

**0.2. Remark.** A standard application of the method of diagrams shows that if $\mathbf{K}$ is the category of models of a first-order theory $T$ with embeddings for morphisms $\Delta$ is the set of quantifier-free formulas, then $\mathbf{K}$ reflects $\Delta$ iff $T$ has the amalgamation property. (Note that, in general, if $\mathbf{K}$ has the amalgamation property and $\Delta$ is a set of formulas in $L$ such that $\mathbf{K}$ is $\Delta$-elementary, then $\mathbf{K}$ reflects $\Delta$.)

**0.3. Definition.** Let $S$ be a set of formulas in $L$, $n$ a natural number.

(i) $S$ is *equational* (resp. *n-strongly-equational*) if for any $H$, $x$ and any set $p$ of formulas in $S^x$ with parameters in $H$, there is $p_0 \subset p$, $p_0$ finite (resp. $\mathrm{card}(p_0) \leq n$) and $p_0 \vdash_H p$.

$S$ is *strongly* equational if there is a natural number $m$ such that $S$ is $m$-strongly equational.

(ii) $\phi(x; t)$ is an *equation* (resp. *n-strong equation, strong equation*) if $\{\phi(x; t)\}$ is equational (resp. $n$-strongly equational, strongly equational).

**0.4. Examples.** (i) Let $\mathbf{F}$ be the category of fields with field embeddings for morphisms; $L = \{+, \cdot, 0, 1\}$. Let $S$ be the set of atomic formulas in $L$.

**Claim.** *$S$ is equational.*

**Proof.** Suppose $H \in \mathbf{K}$ and $p = \{\phi_i(x; a_i); i \in I, \phi_i \in S, a_i \in H\}$. Each $\phi_i(x; a_i)$ is equivalent in $\mathbf{K}$ to an algebraic equation $(P_i(x; a_i) = 0)$; where $P_i$ is a polynomial in the variables $x$ with coefficients in $H$.

Since the ring of polynomials $H[x]$ is noetherian, there is a finite set $J \subset I$ such that for any $i \in I$, $P_i$ is a linear combination of $P_j$'s for $j \in J$. It follows that for any morphism $f$ in $\mathbf{K}$, $fP_i$ is a linear combination of $fP_j$'s for $j \in J$. Clearly then, if $p_0 = \{\phi_j; j \in J\}$, $p_0 \vdash_H (P_i(x; a_i) = 0)$ for every $i \in I$, i.e., $p_0 \vdash_H p$ which is what we wanted.

(Note that the same as above holds with the category of noetherian rings with ring homomorphisms instead of $\mathbf{F}$.)

(ii) Let $R$ be a fixed ring and let $L$ be the standard language of $R$-modules. A homomorphism of modules $f : H \to F$, is pure if for any positive primitive formula (p.p.f. in short) $\phi(x)$ and $a \in H$,

$$H \models \phi(a) \Leftrightarrow F \models \phi(fa).$$

In other words $f$ is pure if $f$ is $\Delta$-elementary with $\Delta$ the set of p.p.f.

Let $\mathbf{M}_{p,R}$ be the category of modules with pure embeddings for morphisms. The following claim is now part of the folklore.

**Claim.** *Every positive primitive formula $\phi(x;t)$ is a strong equation in $\mathbf{M}_{p,R}$.*

**Proof.** for $H$ an $R$-module, $\phi(H;0)$ $(0=\langle 0,\ldots,0\rangle)$ is an additive subgroup of $H^n$ ($n$ = length $x$) and $\phi(H;a)$ $(a\in H)$, if not empty, is a coset of $\phi(H;0)$ in $H^n$. It follows that for $a,b\in H$, $\phi(x;a)$ and $\phi(x;b)$ are either equivalent in $H$ or contradictory in $H$.

Since the morphisms in $\mathbf{K}$ are pure, in fact, either $\phi(x;a)$ and $\phi(x;b)$ are equivalent in $\mathbf{K}$ or $\{\phi(x;a),\phi(x;b)\}$ is inconsistent in $\mathbf{K}$. The claim follows immediately.

Similarly, one shows that in $\mathbf{M}_R$, the category of modules with embeddings for morphisms, the atomic formulas are strong equations.

(iii) If $\mathbf{K}$ is the category of models of a first-order theory $T$ with elementary embeddings for morphisms and $E(x;t)$ is a formula which defines an equivalence relation in models of $T$, then $E(x;t)$ is a 2-strong-equation.

**0.5. Remark.** By definition, if $S$ is equational and $p$ is a set of formulas in $S^x$ over $H$, then $p\sim_H p_0$ for some finite subset $p_0$ of $p$. So if $p$ is consistent over $H$, i.e., if every finite subset of $p$ is realized over $H$, then $p$ itself is realized. This is a compactness property. Now we could define a weaker notion as follows:

$S$ is 'equational' if for every $H$ in $\mathbf{K}$ and $p$, a set of formulas in $S^x$ over $H$ which is *realized over* $H$, there is a finite subset $p_0$ of $p$ such that $p\sim_H p_0$.

This notion is in general too weak for what follows, but it is worthwhile bearing it in mind and checking at different stages what additional conditions on $\mathbf{K}$ make it sufficient to obtain analogous results.

**Example.** Consider the ring of integers $\mathbb{Z}$. Let $\mathbf{K}$ be the category with single object $\mathbb{Z}$ and single morphism the identity on $\mathbb{Z}$; $L=\{+,\cdot,0,1\}$.

Let $\phi(x,t)=\exists s\,(x=s\cdot t)$. $\phi$ says '$x$ is a multiple of $t$'.

$\phi$ is not an equation: take $p=\{\phi(x;k);k\in\mathbb{Z}\}$; $p$ says '$x$ is a multiple of all integers', and clearly there is no finite set $\mathbb{Z}_0$ of integers such that '$x$ is a multiple of all integers' iff '$x$ is a multiple of all integers in $\mathbb{Z}_0$'.

However, if $p=\{\phi(x;a);i\in I,a_i\in\mathbb{Z}\}$ is realized in $\mathbb{Z}$, then $p$ is equivalent to a finite subset: for let $k$ realize $p$. Then $a_i$ divides $k$ for any $i\in I$. Since $k$ has finitely many divisors it follows that there are at most finitely many distinct $a_i$'s. The assertion has now become obvious.

Note here that the formula $\phi(t;x)=\exists s\,(t=sx)$ is an equation.

**0.6. Question.** Let $G$ be a free non-abelian group and let $\mathbf{K}$ be the category with single object $G$ and single morphism the identity on $G$; $L=\{\cdot,{}',e\}$. Let $\phi(x;t)$

(resp. $\phi(\boldsymbol{x};\boldsymbol{t})$) be an atomic formula in a single variable $x$ (resp. in a tuple of variables $\boldsymbol{x}$).

Is $\phi(x;\boldsymbol{t})$ (resp. $\phi(\boldsymbol{x};\boldsymbol{t})$) an equation in $\mathbf{K}$? (Question 0.6 is of course related to the open question of whether a free group is stable.)

## 1. Basic combinatorial properties

We consider some variations on the notion of an equation which come immediately to mind and then work out their basic properties. Many of the assertions below will be presented as matters of fact and their proofs will be left to the reader. This section could presently be skipped and returned to later on.

Throughout this section, $S$ denotes a set of formulas in $L$ closed under substitution of parameter variables.

If $\Delta$ is a set of formulas in $L$ we let $\mathrm{cl}^+(\Delta)$ denote the closure of $\Delta$ under finite conjunctions, finite disjunctions and substitution of parameter variables; $\mathrm{cl}(\Delta)$ denotes the closure of $\Delta$ under boolean combinations and substitution of parameter variables.

For every $H$ in $\mathbf{K}$ we let $\mathbf{K}_H$ denote the category consisting of the single object $H$ and the identity morphism on $H$.

**1.0. Definition.** Let $n$ be a positive integer.

(i) *$S$ has height less than $n$* if there is no structure $H$ in $\mathbf{K}$, $\boldsymbol{x}$ and sequence $(\phi_i)_{i<n}$ of formulas in $S^{\boldsymbol{x}}$ with parameters in $H$ such that the formulas $\bigwedge_{i<k}(\phi_i \wedge \neg\phi_k)$ for $0<k<n$ as well as $\bigwedge_{i<n}\phi_i$ are consistent over $H$.

(ii) (The letter $l$ italicized will stand for 'local'.) $S$ is *$l$-equational* (resp. is *$l$-$n$-strongly equational, has $l$-height less than $n$*) *in* $\mathbf{K}$ iff for every $H\in\mathbf{K}$, $S$ is equational (resp. $n$-strongly equational, has height less than $n$) in $\mathbf{K}_H$.

$S$ has *finite* $l$-height (resp. height) if there is a natural number $m$ such that $S$ has $l$-height (resp. height) less than $m$; $S$ has infinite $l$-height (resp. height) otherwise.

Thus, $S$ has $l$-height less than 1 if for any $H$ in $\mathbf{K}$ and $\phi$ in $S$ with parameters in $H$, $\phi$ is inconsistent in $H$; $\phi(\boldsymbol{x},\boldsymbol{t})$ has height 1 if any two instances of $\phi$ (in some $H$) are either equivalent or contradictory over $H$. The latter type of formulas have been called normal by Pillay (cf. [6]).

**1.1. Lemma.** (i) *$S$ is not $l$-equational (resp. equational) in $K$ iff there is a structure $H$ in $\mathbf{K}$, and a countable sequence $(\phi_n)_{n<\omega}$ of formulas in $S^{\boldsymbol{x}}$ with parameters in $H$ such that for any $k<\omega$, $\bigwedge_{i\leq k}\phi_i \wedge \neg\phi_{k+1}$ is consistent in $H$ (resp. in $\mathbf{K}$ over $H$).*

(ii) *$S$ is not $l$-$n$-strongly equational (resp. $n$-strongly equational) iff either $S$ is not $l$-equational (resp. equational) or there is a structure $H$ in $\mathbf{K}$, $\boldsymbol{x}$ and a finite set $q$ of formulas in $S^{\boldsymbol{x}}$ with parameters in $H$ such that $q$ has cardinality $n+1$ and $q$ is not logically equivalent in $H$ (resp. in $\mathbf{K}$ over $H$) to any proper subset.*

**1.2. Corollary.** *We have the following (best possible) diagram of implications:*

$$\begin{array}{ccccc} S \text{ has finite height} & \to & S \text{ is strongly equational} & \to & S \text{ is equational} \\ \downarrow & & \downarrow & & \downarrow \\ S \text{ has finite } l\text{-height} & \to & S \text{ is } l\text{-strongly-equational} & \to & S \text{ is } l\text{-equational.} \end{array}$$

**1.3. Corollary.** *If the objects of* $\mathbf{K}$ *are the models of a first-order theory* $T$, *then* $\phi(x; t)$ *is l-equational iff* $\phi(x; t)$ *has finite l-height.*

Lemmas 1.4 and 1.5 below should read in view of Proposition 1.6.

**1.4. Lemma.** *Suppose* $S$ *is closed under finite conjunctions and disjunctions, and* $\mathbf{K}$ *reflects* $S$. *Then, for any morphism* $f: H \to F$ *in* $\mathbf{K}$ *and* $p$, *a set of formulas in* $\mathrm{cl}(S^\pi)$ *with parameters in* $H$, $p$ *is consistent over* $H$ *iff* $fp$ *is consistent over* $F$.

Let us say that $\mathbf{K}$ is *ω-conservative*, if for any sequence of morphisms $(f_\beta : H_\beta \to H_{\beta+1})_{\beta<\omega}$ in $\mathbf{K}$ there is a structure $H$ and morphisms $g_\beta : H_\beta \to H$ $(\beta < \omega)$ such that $g_{\beta+1} \cdot f_\beta = g_\beta$ for any $\beta < \omega$.

$\omega$-conservativeness is similar to closure under unions of countable chains, but here we do not require $H$ to be a limit to the chain.

(Similarly, one defines $\alpha$-conservativeness for arbitrary ordinals $\alpha$, and conservativeness as $\alpha$-conservativeness for every $\alpha$.)

**1.5. Lemma.** *Suppose in addition to the assumptions in Lemma 1.4, that* $\mathbf{K}$ *is S-elementary. If* $f_{-1}: H \to H_0$ *is a morphism in* $\mathbf{K}$ *and for* $i < \alpha$, $\alpha$ *a finite ordinal,* $p_i$ *is a finite set of formulas in* $\mathrm{cl}(S^\pi)$ *with parameters in* $H$ *which is realized over* $H$, *then there exists a morphism* $g: H_0 \to G$ *such that for every* $i < \alpha$, $g \cdot f_{-1} p_i$ *is realized in* $G$.

*If in fact* $\mathbf{K}$ *is ω-conservative, then the claim above holds with* $\alpha = \omega$ *instead of a finite ordinal.*

**1.6. Proposition.** *Assume* $S$ *closed under finite conjunctions,* $\mathbf{K}$ *reflects* $S$ *and* $\mathbf{K}$ *is S-elementary. Then,*

(i) $l$-height$(S) = m$ *iff* height$(S) = m$.

(ii) *If in addition* $K$ *is ω-conservative, then*

(a) *$S$ is l-equational iff $S$ is equational.*

(b) *$S$ is l-m-strongly equational iff $S$ is m-strongly equational.*

Let $\bigwedge S$ denote the closure of $S$ under finite conjunctions. For $n$ a positive integer, let

$$\mathrm{cl}_n(S) = \left\{ \psi = \bigvee_{i<n} \phi_i;\ \phi_i \in S \right\}.$$

**1.7. Proposition.** (i) height$(S^x)$ = height$(\bigwedge(S^x))$.

(ii) *If S has height less than* $m-1$, *then, for* $n \geq 2$, $\mathrm{cl}_n(S^x)$ *has height less than* $n^m$ *(for any given* $x$*).*

(iii) *S is equational iff* $\mathrm{cl}^+(S)$ *is equational.*

(**1.7′. Proposition.** *The analogue of Proposition* 1.7 *with the notions of l-height and l-equationality.*)

**Proof of Proposition 1.7.** *For* $\phi$ and $\psi$, formulas with parameters in $H$, write $\phi \subset \psi$ if $\phi \vdash_H \psi$ and $\psi \nvdash_H \phi$.

(i) Clearly, if height$(S^x) \geq m$, then height$(\bigwedge(S^x)) \geq m$. To show the converse we need first the following:

(∗) for $i<m$, let $\psi_i = \bigwedge_{j\in J_i} \phi_j$ where $J_i$ is a finite set and $\phi_j$ ($j \in J_i$) is a formula in $S^x$ with parameters in $H$; and suppose $\bigwedge_{i\leq k+1} \psi_i \subset \bigwedge_{i\leq k} \psi_i$ for any $k<m-1$. Then, we can find a sequence $(j_i)_{i<m}$ such that $j_i \in J_i$ and for any $k<m-1$,

$$\bigwedge_{i\leq k+1} \phi_{j_i} \subset \bigwedge_{i\leq k} \phi_{j_i}.$$

Moreover $j_0$ can be arbitrarily chosen.

**Proof of (∗).** We choose $j_i$ by induction on $i<m$. Take $j_0$ to be any element of $J_0$ and suppose $j_0, \ldots, j_i$ have been chosen. We have

$$\bigwedge_{k\leq i+1} \psi_k \subset \bigwedge_{k\leq i} \psi_k \vdash_H \bigwedge_{k\leq i} \phi_{j_k};$$

hence $\bigwedge_{k\leq i}(\phi_{j_i} \wedge \neg\psi_{i+1})$ is consistent in $K$. It follows that $\bigwedge_{k\leq i}(\phi_{j_k} \wedge \neg\phi_j)$ is consistent in **K** for some $j \in J_{i+1}$; let then $j_{i+1} = j$. Obviously $\bigwedge_{k\leq i+1} \phi_{j_k} \subset \bigwedge_{k\leq i} \phi_{j_k}$. That proves (∗).

Now, if height$(\bigwedge(S^x)) \geq m$, then by definition, a sequence $(\psi_i)_{i<m}$ as above does exist with in addition the property that $\bigwedge_{i\leq m} \psi_i$ is consistent.

But then the sequence $(\phi_{j_i})_{i<m}$ given by (∗) is such that $\bigwedge_{i\leq k+1} \phi_{j_i} \subset \bigwedge_{i\leq k} \phi_{j_i}$ and furthermore $\bigwedge_{i<m} \phi_{j_i}$ is consistent in **K**, since $\bigwedge_{i<m} \psi_i \vdash_H \bigwedge_{i<m} \phi_{j_i}$. That implies height$(S^x) \geq m$.

(ii) We can assume $S = S^x$ and by (i) we can assume $\bigwedge S = S$. We first show the following.

(+) For $i<n^m$, let $\psi_i = \bigvee_{j\in J_i} \phi_{j_i}$ where card $J_i = n$ and $\phi_j$ is a formula in $S^x$ with parameters in $H$. Suppose that

$$\bigwedge_{i\leq k+1} \psi_i \subset \bigwedge_{i\leq k} \psi_i \quad \text{for any } k<n^m-1.$$

then there is a sequence $(j_i)_{i<m}$, such that $j_i \in J_i$ and

$$\bigwedge_{i\leq k+1} \phi_{j_i} \subset \bigwedge_{i\leq k} \phi_{j_i} \quad \text{for any } k<m-1.$$

**Proof of (+).** By induction on $m$.

For $m=0$ the assertion is trivial.

Suppose the assertion holds for $m-1$, and $(\psi_i)_{i<n}$ is given as above.

For any $k$, $0<k<n^m$, there is $j_k \in J_0$ such that

$$\phi_{j_k} \wedge \bigwedge_{i\leq k} \psi_i \subset \phi_{j_k} \wedge \bigwedge_{i\leq k-1} \psi_i,$$

for if $\phi_j \wedge \bigwedge_{i\leq k} \psi_i \sim_H \phi_j \wedge \bigwedge_{i\leq k-1} \psi_i$ for any $j \in J_0$, then

$$\left(\bigvee_{j\in J_0} \phi_j\right) \wedge \bigwedge_{i\leq k} \psi_k \sim_H \left(\bigvee_{j\in J_0} \phi_j\right) \wedge \bigwedge_{i\leq k-1} \psi_i,$$

i.e., $\psi_0 \wedge \bigwedge_{i<k} \psi_i \sim_H \psi_0 \wedge \bigwedge_{i\leq k-1} \psi_i$; which implies $\bigwedge_{i\leq k} \psi_{i_m} \sim_H \bigwedge_{i\leq k-1} \psi_i$.

For $0<i$, $k<n^m$ write $i \equiv k$ if $j_i = j_k$. We partition in this way the set $\{i, 0<i<n^m\}$ into $n$ subsets. Since $n \geq 2$, one such subset $I$ must have cardinality at least $n^{m-1}$. Let $j_0$ denote the common value of the $j_i$'s for $i \in I$.

If $i \in I$ and $k<i$, then $\phi_{j_0} \wedge \bigwedge_{l\leq i} \psi_l \subset \phi_{j_0} \wedge \bigwedge_{l\leq k} \psi_l$; for

$$\phi_{j_0} \wedge \bigwedge_{l\leq i} \psi_l \subset \phi_{j_0} \wedge \bigwedge_{l\leq i-1} \psi_l \vdash_H \phi_{j_0} \wedge \bigwedge_{l\leq k} \psi_l.$$

In particular, $\phi_{j_0} \wedge \bigwedge_{l\leq i} \psi_l \subset \phi_{j_0}$ (since $\phi_{j_0} \wedge \psi_0 \sim_H \phi_{j_0}$). Let $(i_k)_{0<k\leq n^{m-1}}$ be an increasing sequence of elements in $I$; let $\chi_0 = \phi_{j_0}$ and, for $k<n^{m-1}$, $\chi_{k+1} \cdot{=}\cdot \bigwedge_{i_k<l\leq i_{k+1}} (\phi_{j_0} \wedge \psi_l)$ (put $i_0=0$). We have

$$\bigwedge_{l\leq k+1} \chi_l \subset \bigwedge_{l\leq k} \chi_k \quad \text{for any } k<n^{m-1}.$$

By (*) (see the proof of (i)) it follows there is a sequence $(h_k)_{k\leq n^{m-1}}$, $i_k < h_{k+1} \leq i_{k+1}$ and $h_0=0$ such that

$$\bigwedge_{l\leq k+1} (\phi_{j_0} \wedge \psi_{h_l}) \subset \bigwedge_{l\leq k} (\phi_{j_0} \wedge \psi_{h_l})$$

for any $k<n^{m-1}$. Clearly now, the formulas $\phi_{j_0} \wedge \psi_{h_l}$, $l \leq n^{m-1}$, can be considered as formulas in $\mathrm{cl}_n(S)$; thus the induction hypothesis applies to the sequence $(\phi_{j_0} \wedge \psi_{h_l})_{0<l<n^{m-1}}$. In other words there is a sequence $(j_l)_{0<l<m}$ such that $j_l \in J_{h_l}$ and

$$\bigwedge_{i\leq k+1} (\phi_{j_0} \wedge \phi_{j_i}) \subset \bigwedge_{i\leq k} (\phi_{j_0} \wedge \phi_{j_i}) \quad \text{for } 0<k<m-1.$$

Upon observing that

$$\phi_{j_0} \wedge \phi_{j_i} \vdash \phi_{j_0} \wedge \psi_{h_i} \subset \phi_{j_0} \wedge \psi_{h_0} = \phi_{j_0},$$

we conclude that

$$\bigwedge_{i\leq k+1} \phi_{j_i} \subset \bigwedge_{i\leq k} \phi_{j_i}$$

for any $k<m$. That finishes the proof of (+).

Now if $\mathrm{cl}_n(S^x)$ has height greater or equal to $n^m$, then a sequence $(\psi_i)_{i<n^m}$ such as given in (+) does exist; let $\phi_j$ be the sequence of formulas in $S^x$ with parameters in $H$ given by (+).

We have that $\bigwedge_{i\leq k}(\phi_{j_i}\wedge\neg\phi_{j_{k+1}})$ is consistent for any $k<m-1$; in particular, $\bigwedge_{i\leq k}(\phi_{j_i}\wedge\neg\phi_{j_{k+1}})$ is consistent for any $k<m-2$ and $\bigwedge_{i<m-1}\phi_{j_i}$ is consistent.

But that implies height$(S)\geq m-1$, which is what we wanted.

(iii) See Corollary 2.3. (One could prove (iii) by using a direct pigeon-hole argument. Cf. [7] or [11].) □

**Remark.** Although $\mathrm{cl}_n(-)$ preserves equationality, it does not preserve strong-equationality. More precisely there are strongly equational sets $S$ with $\mathrm{cl}_2(S)$ not strongly equational. Indeed, let $L=\{R\}$, $R$ a 2-ary relation. Let $|H|$ be an infinite set and $\langle P_n\rangle_{n<\omega}$, $\langle Q_n\rangle_{n<\omega}$ two sequences of chains of subsets of $|H|$ such that:

(i) For $n<\omega$, $P_n=(C_i^n)_{i<n}$, $Q_n=(D_i^n)_{i<n}$ with $C_i^n, D_i^n\subset|H|$,

$$C_i^n\subsetneq C_{i+1}^n,\qquad D_i^n\subsetneq D_{i+1}^n\quad(i<n-1),\qquad C_0^n, D_0^n\neq\emptyset.$$

(ii) If $n\neq m$, then $C_i^n\cap C_j^m=\emptyset$, $D_i^n\cap D_j^m=\emptyset$, for any $i<n$, $j<m$.

Choose an interpretation of $R$ in $|H|$ such that the $C_i^n$'s and $D_i^n$'s, $n<\omega$, $i<n$, are the only interpretations in $H$ of instances of $R$. (An instance of $R$ is here meant to be a formula of the form $R(x;h)$, $h\in H$.) Let $H$ be the structure thus obtained, $\mathbf{K}$ the category with single object $H$ and single morphism the identity on $H$. It is easy to check that $R(x;t)$ is 2-strongly-equational in $\mathbf{K}$.

Fix $n<\omega$. Consider the sequence $(E_i)_{i<m}$ in $\mathrm{cl}_2(R(x;t))_H$ where $E_i=C_i^n\cup D_{n-i-1}^n$ (for simplicity we shall write $C_i$ and $D_i$ for $C_i^n$ and $D_i^n$ respectively). We shall construct the $C_i$'s and $D_i$'s in such a way that $\bigcap_{i<n}E_i\neq\emptyset$ and is not equal to any proper subintersection. Moreover it will be immediate that this can be done for every $n<\omega$, preserving the conditions (i) anJ (ii) above. This then clearly implies that $\mathrm{cl}_2(R(x;r))$ is not strongly equational.

Construction of $C_i$ and $D_i$: let $(B_{ij})$, $0\leq i,\ j\leq n+1$, be pairwise disjoint, non-empty subsets of $H$. Present the $B_{ij}$'s in a $(n+1)\times(n+1)$-matrix,

$$\begin{array}{lll}
B_{1,1} & B_{1,2} \text{------} & B_{1,n+1}\\
B_{2,1} & B_{2,2} \text{------} & B_{2,n+1}\\
\vdots & \vdots & \vdots\\
B_{n+1,1} & B_{n+1,2} & B_{n+1,n+1}
\end{array}$$

For $i<n$, let $C_i$ be the union of the $i+1$ first lines of the matrix, while $D_i$ is the union of the $i+1$ first columns of the matrix.

$$C_i=\bigcup\{B_{kl}:1\leq k\leq i+1,\ 1\leq l\leq v+1\},$$

$$D_i=\{B_{kl}:1\leq k<n+1,\ 1\leq l\leq i+1\}$$

Note that $B_{i+2,n+1-i}$ $(i<n)$ belongs to $E_j = C_j \cup D_{n-j-1}$ if $j \neq i$ (since $B_{i+2,n+1-i}$ is the meet of the $(i+2)$th line with the $(n-i+1)$th column).

Thus $\bigcap_{i<n} E_i \neq \bigcap_{i\in I} E_i$ once $I \subsetneq n$, for if $i \in n\setminus I$, then $B_{i+2,n+1-i} \subset \bigcap_{i\in I} E_i$ while $B_{i+2,n+1-i} \not\subset \bigcap_{i<N} E_i$.

Moreover, $\bigcap_{i<n} E_i \neq 0$ since $B_{1,1} \subset \bigcap_{i<n} E_i$. □

Let $\phi = \phi(x;t)$ and let $u$ be a subtuple of $x \frown t$. Clearly we can consider $\phi$ as a formula in $u$ instead of a formula in $x$; In other words $u$ is now the tuple of type variables and the rest of the variables become parameter variables. We sometimes write $\phi^u$ instead of $\phi$ to underline the fact that we consider $\phi$ as a formula in $u$.

**1.8. Proposition.** *Let* $\phi = \phi(x;t)$.

(i) $l$-height$(\phi^x) = m$ *iff* $l$-height$(\phi^t) = m$.

(ii) *Assuming* $\phi^t$ *is l-equational, if* $\phi^x$ *is l-m-strongly equational then* $\phi^t$ *is l-m-strongly-equational.*

**Proof.** We prove (ii); the argument for (i) is similar and is left to the reader (see 1.? for a different proof).

Suppose $\phi^t$ is $l$-equational but not $l$-$m$-strongly equational. By Lemma 1.1(ii), there is $H$, $a_0, \ldots, a_m$ in $H$ such that for any $k \leq m$, the formula

$$\psi_k(t) = \bigwedge_{i\in m+1-\{k\}} (\phi(a_i;t) \wedge \neg\phi(a_k;t))$$

is consistent in $H$.

Let $b_k$ realize $\psi_k$ in $H$. We have the following diagram of true statements in $H$:

$$\begin{array}{llll}
\neg\phi(a_0;b_0) & \phi(a_1;b_0) & \text{------------} & \phi(a_m;b_0) \\
\phi(a_0;b_1) & \neg\phi(a_1;b_1) & \text{------------} & \phi(a_m;b_1) \\
\vdots & & & \vdots \\
\phi(a_0;b_{m-1}) & \phi(a_1;b_{m-1}) & \text{---}\ \neg\phi(a_{m-1};b_{m-1}) & \phi(a_m;b_{m-1}) \\
\phi(a_0;b_m) & \phi(a_1;b_m) & \text{---------}\ \phi(a_{m-1};b_m) & \neg\phi(a_m;b_m)
\end{array}$$

Considering the columns of this diagram we obtain that for any $l$, $0 \leq l \leq m$, the formula

$$\bigwedge_{i\in m+1-\{l\}} (\phi(x;b_i) \wedge \neg\phi(x;b_l))$$

is realized in $H$ by $a_l$. That easily implies $\phi^x$ is not $l$-$m$-strongly equational. □

**Example.** Let $R$ be a noetherian ring with a unit, $L$ the language of rings, $\mathbb{K} = \mathbb{K}_R$. Consider the formula $\phi(x;t) = \exists s\,(x = st)$; for $a$ in $R$, $\phi(a;t)$ defines the set of divisors of $a$, while $\phi(x;a)$ defines the set of multiples of $a$. $\phi$ is an equation in $t$: for if $(a_i)_{i<\omega}$ is a sequence of elements in $R$, the ideal $\langle a_i; i<\omega\rangle$ is finitely generated, hence equals $\langle a_i; i<n\rangle$ for some $n$. It follows that $t$ divides $a_i$ for $i<\omega$ iff $t$ divides $a_i$ for $i<n$.

However, if $R$ contains an infinite sequence $(a_i)_{i<\omega}$ such that $a_i$ strictly divides $a_{i+1}$, take for instance $R=\mathbb{Z}\times\mathbb{Z}$, and $a_i=(2^i;1)$, then $\phi$ is not equational in $x$.

(Note that in $\mathbb{Z}\times\mathbb{Z}$ there is an element $a_\omega\neq 0$ such that $a_i$ divides $a_\omega$ for every $i<\omega$, namely $a_\omega=(0,1)$, (in other words $\{\phi(x;a_i); i<\omega\}$ is realized by $a_\omega$ in $R$).

**1.9.** Let $\phi=\phi(x;t)$, $m=\text{length}\, x$, $n=\text{length}\, t$ and $H\in\mathbf{K}$. Let $L_x$ (resp. $L_t$) be the (obvious) semi-lattice whose underlying set is the class of subsets of $H^m$ (resp. $H^n$) which are definable by conjuncts of instances of $\phi^x$ (resp. $\phi^t$) (an instance of $\phi^t$ is a formula of the kind $\phi(a;t)$, $a$ in $H$).

Assume $\phi$ is $l$-equational in $t$. We define a map

$$*_t = *_{H,\phi,t}: L_x \to L_t$$
$$X \to \bigcap\{\phi(b_i;H); b_i\in X\}$$
$$(\phi(b_i;H)=\{a\in H^n; H\models\phi(b_i;a)\}).$$

The intersection mentioned in the definition of $*_t$ above is finite because of $l$-equationality.

Similarly, if $\phi$ is $l$-equational in $x$, we define the map $*_x: L_t\to L_x$. We have the following properties:

(a) For $X, Y\in L_x$, $X\subset Y \Rightarrow *_t(X)\supset *_t(Y)$.

(b) If $X=\bigcap_{i<n}\phi(H;a_i)$, then $a_i\in *_t(X)$ for every $i<n$.

(c) Suppose $Y=\bigcap_{j<k}\phi(H;a_j)$ and $*_t(X)=\bigcap_{i<n}\phi(b_i;H)$, $b_i\in X$. Then, $X\subset Y$ iff $b_i\in Y$ for every $i<n$ iff $a_j\in *_t(X)$ for every $j<k$.

(d) For $X, Y\in L_x$, $X\subset Y \Leftrightarrow *_t(X)\supset *_t(Y)$.

**Proof of (d).** We already have one direction (a). Suppose $Y=\bigcap_{j<k}\phi(H;a_j)$ and $*_t(X)\supset *_t(Y)$. In (b) we have noted that $a_j\in *_t(Y)$ for $j<k$. Hence, $a_j\in *_t(X)$, $j<k$. By (c), we conclude that $X\subset Y$.

As a corollary to (d) one gets another proof of Lemma 1.8(i), namely that $l$-height $(\phi^x)=l$-height$(\phi^t)$. For it follows from (d) that any chain of elements in $L_x$ of length $m$ gives rise to a chain of elements in $L_t$ of length $m$.

**1.10. Proposition.** *If $\phi(x;t)$ is $l$-equational in $x$ and $t$, then $*_t$ is a dual isomorphism from $L_x$ onto $L_t$ with inverse $*_x$.* (By dual we just mean the property stated in 1.9(d).)

**Proof.** We already know from 1.9(d) above that $*_t$ is a dual isomorphism from $L_x$ into $L_t$. It remains to show that $*_x$ is the inverse of $*_t$.

We show $*_x \cdot *_t$ is the identity on $L_t$. Let $X = \bigcap_{i<n} \phi(H; a_i)$. By definition,

$$*_x \cdot *_t(X) = \bigcap \{\phi(H; c_i); c_i \in *_t(X)\}.$$

From 1.9(b) $a_i \in *_t(X)$ for $i<n$; hence $*_x \cdot *_t(X) \subset X$.

On the other hand, by definition,

$$c \in *_t(X) \quad \text{iff} \quad H \models \phi(b; c) \quad \text{for all } b \in X.$$

Thus, $c \in *_t(X) \Rightarrow X \subset \phi(H; c)$; hence $*_x \cdot *_t(X) \supset X$.

We conclude $*_x \cdot *_t(X) = X$. □

## 2. Types

We give two criteria for equationality using complete types. We then apply them to prove that in the category of differential fields of characteristic 0, the set of differential equations is equational; and that in the category of radical differentially closed fields of characteristic $p$ (in the language of differential fields with one additional unary symbol r(−)) every atomic formula is equivalent to a boolean combination of equations.

Throughout this section, $S$ denotes a set of formulas in $L$ closed under substitution of parameter variables.

**2.1. Terminology.** Let $A \subset B \subset H \in \mathbf{K}$.

Given a set $\Delta$ of formulas in $L$, a set $p$ of formulas in $\mathrm{cl}(\Delta^x)$ with parameters in $A$ is called a *$\Delta$-type* in $x$ over $A$ if $p$ is consistent in $K$ over $H$;

A $\Delta$-type over $A$ is a $\Delta$-type in some tuple of variables $x$ over $A$.

If $p$ is a type over $B$, we let

$$p \restriction A = \{\phi(x; a); a \in A, \phi(x; a) \in p\}.$$

$$p \restriction \Delta = \{\phi(x; b); \phi(x; t) \in \mathrm{cl}(\Delta), \text{ and } \phi(x, b) \in p\}.$$

A $\Delta$-type $p$ in $x$ over $A$ is *$\Delta$-complete* if $p$ is maximal with respect to inclusion among $\Delta$-types in $x$ over $A$. If $C \subset H$, we let

$$\mathrm{tp}(C, A; H) \cdot = \cdot \{\phi(x_c; a), a \in A, c \in C, H \models \phi(c, a)\}.$$

**2.2. Lemma.** $A \subset B \subset H \in \mathbf{K}$.

(a) *A $\Delta$-type $p$ in $x$ over $A$ is $\Delta$-complete iff for any formula $\phi$ in* $\mathrm{cl}(\Delta^x)$ *over $A$, either $\phi$ or $\neg\phi$ belongs to $p$.*

(b) *If $p$ is an $L$-complete $L$-type over $B$, then $p \restriction A$ (resp. $p \restriction \Delta$) is $L$-complete (resp. $\Delta$-complete).*

(c) *An $L$-type over $A$ can always be extended to an $L$-complete $L$-type over $A$.*

**Proof.** Clear. □

Until stated otherwise, by type or complete type we mean an $L$-type or $L$-complete $L$-type.

For $p$ a complete type over $H$ let

$$p^S = \{\phi(x; a); a \in H, \phi \in S, p \vdash_H \phi\}.$$

**2.3. Proposition.** *The following assertions are equivalent:*

(i) *$S$ is equational.*

(ii) *For any structure $H$ in $\mathbf{K}$, $x$, and complete type $q$ in $x$ over $H$, there is a finite subset $q_0$ of $q$ such that $q_0 \vdash_H q^S$.*

**Proof.** (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i). Assume (ii) holds. Let $H \in K$, $p$ a set of formulas in $S^x$ with parameters in $H$. We want to show the existence of $p_0 \subset p$, $p_0$ finite such that $p_0 \vdash_H p$.

If $p$ is inconsistent in $\mathbf{K}$, then clearly such a $p_0$ exists. Suppose $p$ is consistent. Let $P$ be the set of complete types in $x$ over $H$ containing $p$; by Lemma 2.2(c), $P$ is not empty.

Moreover, by assumption, for any $q \in P$ we can find $q_0 \subset q$, $q_0$ finite and $q_0 \vdash_H q^S$. Let $p' = p \cup \{\neg(\bigwedge q_0); q \in P\}$.

$p'$ is inconsistent in $\mathbf{K}$: For if $p'$ is consistent, we can extend it to a complete type $q$ over $H$ so that $q \in P$. But then, $\neg(\bigwedge q_0) \in p' \subset q$ and $q_0 \subset q$.

Hence, there are $q^1, \ldots, q^n$ in $P$ and $p_0$ a finite subset of $p$ such that $p_0 \vdash_H \bigvee_{i=1}^n (\bigwedge q_0^i)$. On the other hand, for any $i$, $1 < i < n$, $q_0^i \vdash_H q^S \vdash_H p$; hence $\bigvee_{i=1}^n (\bigwedge q_0^i) \vdash_H p$. We conclude $p_0 \vdash_H p$, which is what we wanted. □

**Corollary.** *$S$ is equational iff $R = \mathrm{cl}^+(S)$ is equational.*

**Proof.** Indeed, let $H \in K$ and $q$ a complete type in $x$ over $H$. Then clearly $q^R \sim_H q^S$. Since $S$ is equational, there is a finite subset $q_0$ of $q$ such that $q^R \sim_H q^S \sim_H q_0$. We conclude by Proposition 2.3 that $R$ is equational. □

The following application of Proposition 2.3 is somewhat typical. It relies on the existence of a 'division rule'. A similar argument could be applied to the category of fields $\mathbf{F}$ to prove directly without the use of Hilbert's theorem (see 0.4(i)) that the set of algebraic equations in $n$ variables is equational.

Let $\mathbf{DF}_p$ be the category of differential fields of characteristic $p$ with differential field embeddings. $L$ is then the language of fields plus a unary operation symbol d(–) representing the derivation function.

**Application.** *With $\mathbf{K} = \mathbf{DF}_0$ and $S$ the set of atomic formulas in $x = \langle x_0, \ldots, x_n \rangle$, $S$ is equational.*

**Proof.** Let $H$ be a differential field. Recall that a differential polynomial $P$ in $x$ with coefficients in $H$ is a polynomial in a sequence of variables $X$ with coefficients in $H$ where $x$ is of the form:

$$X = \langle x_0, \ldots, x_n, \mathrm{d}x_0, \ldots, \mathrm{d}x_n, \ldots, \mathrm{d}^m x_0, \ldots, \mathrm{d}^m x_n \rangle,$$

for some $m < \omega$. Let $\operatorname{ord} P$ (order of $P$ in $x_n$) be the highest number $m$ such that $\mathrm{d}^m x_n$ occurs non-trivially in $P$; let $u_P = \mathrm{d}^m x_n$ for $m = \operatorname{ord} P$.

Thus, we can write the formal equality of polynomials: $P = \sum_{i=0}^{r} I_i u_P^i$ where $I_i$, for $0 \leq i \leq r$, is a polynomial in the sequence of variables

$$\langle x_0, \ldots, x_n, \mathrm{d}x_0, \ldots, \mathrm{d}x_n, \ldots, \mathrm{d}^{m-1}x_0, \ldots, \mathrm{d}^{m-1}x_n \rangle,$$

$m = \operatorname{ord} P$; let $I_P = I_r$ and $S_P = \sum_{i=1}^{r} iI_i u_P^{i-1}$.

Note first that, since a differential equation over $H$ can be considered as an algebraic equation over $H$ ($H$ as a field) and since algebraic equations are equational in the category of fields it follows immediately that a differential equation is equational in the category of differential fields (of any characteristic).

However, this does not entail that the set $S$ of all differential equations is equational. To show that $S$ is equational we need a division rule on differential equations. Such a rule is given by Lemma 5 of Chapter I.8 in [2] which we reproduce below:

**Lemma** (cf. [2, I.8]). *For any differential polynomial $P$ and $m$, $0 < m < \omega$, $\mathrm{d}^m P - S_P \mathrm{d}^m u_P$ has lower order than $\mathrm{d}^m u_P$.*

**Proof of the lemma.** Write $P = \sum_{i=0}^{r} I_i u_P^i$. Then,

$$\mathrm{d}P = S_P \,\mathrm{d}u_P + \sum_{i=0}^{r} \mathrm{d}(I_i) u_P^i.$$

Since every derivative of $x_n$ present in $I_i$, is strictly lower than $u_P$ (i.e., $I_i$ has a lower order than $u_P$) and $u_P$ has a lower order than $\mathrm{d}u_P$, we find that $\mathrm{d}P - S_P \,\mathrm{d}u_P$ has lower order than $\mathrm{d}u_P$. This proves the lemma for $m = 1$. The lemma for arbitrary $m$ follows quickly by induction on $m$.

**Back to our proof.** Let $H \in \mathbf{DF}_0$. Expanding polynomial expressions and using the properties of the derivation, every atomic formula $\phi(x; a)$ with coefficients in $H$, can be written in a natural way in the form:

$$\phi(x; a) \sim_H (P_\phi(x) = 0)$$

where $P_\phi(x)$ is a differential polynomial in $x$ with coefficients in $H$. ($P_\phi$ is uniquely determined up to formal equality of polynomials.) Let $\operatorname{ord} \phi$, $I_\phi$, $S_\phi$, $u_\phi$ denote respectively $\operatorname{ord} P_\phi$, $I_{P_\phi}$, $S_{P_\phi}$, and $u_{P_\phi}$.

Let $p$ be a complete type in $x$ over $H$; let

$$p^S = \{\phi(x; a);\ \phi \in S,\ a \in H,\ p \vdash_H \phi\},$$
$$q = \{\phi \in p^S;\ (I_\phi = 0) \notin p,\ u_\phi \neq 0\},$$
$$p_1 = \{\phi \in p^S;\ u_\phi = 0\}.$$

$p_1$ is the set of atomic formulas in $p$ which do not mention (non-trivially) $x_n$. By induction hypothesis, we can assume $p_1$ equivalent in $\mathbf{K}$ to a finite subset. Let $m = \min\{\operatorname{ord}\phi;\ \phi \in q\}$; let $\phi$ be an element of $q$ with $\operatorname{ord}\phi = m$ and such that $P_\phi$ has lowest possible degree, say $r$, in $d^m x_n$. Let $P = P_\phi$.

**Claim.** $(S_\phi = 0) \notin p^s$.

For, either $r > 1$, in which case $\operatorname{ord} S_\phi = m$, the degree of $S_\phi$ in $d^m x_n$ is strictly less than $r$, $I_{S_\phi} = r \cdot I_\phi$, whence $(I_{S_\phi} = 0) \notin p$ (since $(I_\phi = 0) \notin p$), and therefore $S_\phi$ cannot belong to $p$ by the minimal choice of $\phi$; or $r = 1$ in which case $S_\phi = I_\phi$ and $(S_\phi = 0) \notin p$ since $\phi \in q$.

Consider now an element $\psi$ of $p^s \backslash p_1$. Write

$$P_\psi = Q = \sum_{i=0}^{k} I_i u_Q^i, \qquad u_Q = d^l x_n.$$

*Case* 1: $l > m$. Then, let

$$\begin{aligned} R_0 &= S_P \cdot Q - I_Q \cdot d^{l-m}P \cdot (d^{l-m}u_P)^{k-1} \\ &= \sum_{i=0}^{k-1} S_P I_i u_Q^i + S_P I_Q u_Q^k - I_Q \cdot d^{l-m}P \cdot u_Q^{k-1} \\ &= \sum_{i=0}^{k-1} I_i S_P \cdot u_Q^i + I_Q \cdot u_Q^{k-1} \cdot (S_P \cdot u_Q - d^{l-m}P). \end{aligned}$$

By the lemma above we see that either $\operatorname{ord} R_0 < l$ or the degree of $R_0$ in $u_Q$ is strictly less than $k$ (recall $k$ is the degree of $Q$ in $u_Q$).

Moreover, since $(Q = 0) \wedge (P = 0) \vdash_H (R_0 = 0)$, $(R_0 = 0)$ belongs to $p^S$. On the other hand

$$(R_0 = 0) \wedge (P = 0) \wedge (S_P \neq 0) \vdash_H (Q = 0).$$

If $\operatorname{ord} R_0 > m$, we repeat the same process with $R_0$ instead of $Q$ and obtain $R_1$. Ultimately we find $R_0, R_1, \ldots, R_j$, $j < \omega$, $R_j \in p^S$,

$$\begin{aligned} &(R_j = 0) \wedge (P = 0) \wedge (S_P \neq 0) \vdash_H \\ &\quad (R_{j-1} = 0) \wedge (P = 0) \wedge (S_P \neq 0) \vdash_H \cdots \vdash_H (Q = 0), \quad \text{and} \quad \operatorname{ord} R_j \leq m. \end{aligned}$$

Thus we have come down to the case of $l \leq m$.

*Case* 2: $l = m$ *and* $k \geq r$. Clearly then, if $Q_0 = I_P Q - I_Q P \cdot u_P^{k-r}$, the degree of $Q_0$ in $u_P$ is strictly less than $k$. Moreover $(Q_0 = 0) \in p^S$ and

$$(Q_0 = 0) \wedge (P = 0) \wedge (I_P \neq 0) \vdash_H (Q = 0).$$

If the degree of $Q_0$ is greater or equal to $r$ and $\operatorname{ord} Q_0 = m$ we repeat the same process with $Q_0$ instead of $Q$ to obtain $Q_1$. Ultimately we find, $d < \omega$, $(Q_d = 0) \in p^S$,

$$(Q_d = 0) \wedge (P = 0) \wedge (I_P \neq 0) \vdash_H (Q = 0),$$

and, either $\operatorname{ord} Q_d < m$ or [$\operatorname{ord} Q_d = m$ and the degree of $Q_d$ in $u_p$ is strictly less than $r$].

*Case* 3: $l < m$ *or* [$l = m$ *and* $k < r$].

**Claim.** $p_1 \vdash_H (Q = 0)$.

Indeed, by the minimal choice of $P$, $(Q = 0)$ cannot belong to $q$. Hence, either $u_Q = 0$, in which case $(Q = 0) \in p_1$, or $(I_Q = 0) \in p^S$, in which case

$$(I_Q = 0) \wedge (Q_0 = 0) \vdash_H (Q = 0)$$

where $Q = Q_0 + I_Q u_Q^k$ ($(Q_0 = 0) \in p^S$); by induction on the order of $Q$ and the degree of $Q$ in $u_Q$ we can assume $p_1 \vdash_H (Q_0 = 0)$ and since $p_1 \vdash_H (I_Q = 0)$, we conclude $p_1 \vdash_H (Q = 0)$.

Combining the three cases above we deduce that

$$p_1 \wedge (P = 0) \wedge (S_P \neq 0) \wedge (I_P \neq 0) \vdash_H (Q = 0).$$

Since $p_1$ is equivalent to a finite subset,

$$(P = 0) \in p, \qquad (S_P \neq 0) \in p \quad \text{and} \quad (I_P \neq 0) \in p,$$

it follows that there is a finite subset $p_0$ of $p$ such that $p_0 \vdash_H p^S$. By Proposition 2.3 we conclude that $S$ is equational. □

Let us say temporarily that a category $\mathbb{K}$ is elementary if

(i) Obj($\mathbb{K}$) is the class of models of a first-order theory $T$ in the language $L$.

(ii) Mor($\mathbb{K}$) is the class of models of a first-order theory $R$ in the 2-sorted language $L' = L \cup \{f\}$ (say with sorts $Y$ and $Z$), where $L$ is interpreted in both $Y$ and $Z$ and $f$ is a 1-ary function symbols from $Y$ into $Z$. (In fact, closure of Mor($\mathbb{K}$) under ultraproducts could be seen to be sufficient for our purposes.)

(iii) Mor($\mathbb{K}$) includes all elementary embeddings between objects in $\mathbb{K}$.

For instance if Obj($\mathbb{K}$) = Mod($T$) and $\mathbb{K}$ has homomorphisms for morphisms, then $\mathbb{K}$ is elementary.

**2.4. Proposition.** (i) *If* $\phi(\bar{x}; \bar{y})$ *is an equation, then the formula* $\psi(\bar{y}; \bar{t}) = \forall \bar{x}\, (\theta(\bar{x}; \bar{y}) \rightarrow \phi(\bar{x}; \bar{t}))$ *is an equation in* $\bar{y}$, *for any* $\theta$ *in* $L$.

(ii) *Assume* $\mathbb{K}$ *is elementary in the sense given above. Then* $\phi(\bar{x}, x; \bar{y})$ *is an equation in* $\bar{x} \frown x$ *iff there is* $n < \omega$ *such that for every* $H$ *in* $\mathbb{K}$, $b$ *and* $(\bar{b}_i)_{i \in I}$ *in* $H$ *there exists* $J \subset I$, $|J| \leqslant n$, *and* $H \models \forall \bar{x}\, (\bigwedge_{j \in J} \phi(\bar{x}, b, \bar{b}_j) \rightarrow \phi(\bar{x}, b, \bar{b}_i))$ *for every* $i \in I$, *and for every* $H$ *in* $\mathbb{K}$, $a = \langle a_1, \ldots, a_n \rangle$ *and* $(\bar{c}_i)_{i < \omega}$ *in* $H$ *there exists* $m < \omega$ *such that* $\bigwedge_{i < m} \psi(x, a, \bar{c}_i) \vdash_H \psi(x, a, \bar{c}_j)$ *for every* $j < \omega$, *where* $\psi(x, y, \bar{z}) \cdot = \cdot$

$\forall\bar{x}\,(\bigwedge_{i<n}\phi(\bar{x};x,\bar{y}_i)\rightarrow\phi(\bar{x};x,\bar{z}))$, $y=\langle\bar{y}_1,\ldots,\bar{y}_n\rangle$. (Note that the conditions above are met if $\phi$ is an equation in $\bar{x}$ and $\psi$ is an equation in $x$ or in $x\frown y$.)

**Proof.** (i) Let $H$ be in $\mathbb{K}$ and $(\bar{a}_i)_{i<\omega}$ a sequence of elements in $H$. Since $\phi$ is an equation, there is $n<\omega$ such that $\bigwedge_{i<n}\phi(\bar{x};\bar{a}_i)\vdash_H\phi(\bar{x};\bar{a}_j)$ for every $j<\omega$. It is easy then to check that $\bigwedge_{i<n}\psi(\bar{y};\bar{a}_i)\vdash_H\psi(\bar{y};a_j)$ for every $j<\omega$. We conclude that $\psi$ is an equation in $\bar{y}$.

(ii) One direction is similar to (i). Suppose the right-hand side of the equivalence holds.

We apply Proposition 2.3. Let $H$ be in $\mathbb{K}$ and $p$ an $L$-complete type in $\bar{x}\frown x$ over $H$. Let $\bar{a}\frown b$ realize $p$ in some extension $F$ of $H$ and let

$$C=\{\bar{c}\in H;F\models\phi(\bar{a},b,\bar{c})\}.$$

By assumption there is a finite subset $E$ of $C$, $|E|<n$, such that $F\models\forall\bar{x}\,(\bigwedge_{\bar{e}\in E}\phi(\bar{x},b,\bar{e})\rightarrow\phi(\bar{x},b,\bar{c}))$ for every $\bar{c}$ in $C$. Thus we have $\psi(x,e,\bar{c})\in p$ for every $\bar{c}$ in $C$. By the assumption on $\psi$ there exists a finite subset $U$ of $C$ such that $\bigwedge_{\bar{u}\in U}\psi(x,e,\bar{u})\vdash_H\phi(x,e,\bar{c})$ for every $\bar{c}\in C$. Let

$$p_0=\{\psi(x,e,\bar{u});\bar{u}\in U\}\cup\{\phi(\bar{x},x,\bar{e}),\bar{e}\in E\}.$$

Clearly $p_0\vdash_H\phi(\bar{x},x,\bar{c})$ for every $\bar{c}$ in $C$. By Proposition 2.3, we conclude that $\phi(\bar{x},x;\bar{y})$ is an equation in $\bar{x}\frown x$. □

**Remark.** (a) Proposition 2.4 can be easily generalized to an arbitrary set of formulas $S$ instead of just one formula $\phi$. Similar results can also be obtained if one works with any of the notions we have defined ($l$-equations, etc.) instead of equations.

(b) Given that $\phi(\bar{x},x;y)$ is equational in $\bar{x}\frown x$, the passage from $\phi$ to a formula like $\psi(x,y,z)$ in Proposition 2.4(ii) above can be seen as a process of 'elimination' of the variable $\bar{x}$. For instance, over a field $F$, if $\phi(x_0,\ldots,x_{n-1},\bar{y})=(\sum_{i<n}y_ix_i=0)$, then

$$\forall x_0\,(\phi(x_0,\ldots,x_{n-1};\bar{y})\rightarrow\phi(x_0,\ldots,x_{n-1};\bar{z}))\leftrightarrow\Big[\Big(\sum_{0<i<n}(y_0z_i-z_0y_i)x_i=0\Big)\vee\Big(y_0=0\wedge z_0=0\wedge\sum_{0<i<n}(y_i-z_i)x_i=0\Big)\Big].$$

Thus, it is interesting to consider the cases where formulas like $\psi(x;y,\bar{z})$ are equational in $x$.

(c) It might happen in some instances that in addition to $\phi(\bar{x};\bar{y})$ being an equation in $\bar{x}$ we can in fact, whenever given $M$ and $(\bar{b}_i)_{i<\omega}$ in $H$, find $J\subset\omega$, $J$ finite, such that for every $i<\omega$, $\bigwedge_{j\in J}\phi(\bar{x};\bar{b}_j)\vdash_H\phi(\bar{x};\bar{b}_i)$, with the latter implication of a special kind. For instance, in the case of fields with $\phi$ an algebraic equation, we can choose $J$ such that the $\phi(\bar{x};\bar{b}_j)$'s generate the ideal $\langle\phi(\bar{x};\bar{b}_i):i<\omega\rangle$, a fact which a priori is stronger than the mere implication given by equationality. Thus, in the case of $\phi(\bar{x}; x, \bar{y})$ in Proposition 2.4(ii) let us assume the existence of an equational set $R$ in $x$ of formulas of the form $\theta(x; y, \bar{z})$ such that whenever $M \in \mathbf{K}$, $a$ and $(\bar{b}_i)_{i\in I}$ are in $M$, there exists a family $(\theta_i)_{i\in I}$ of formulas in $R$, and $J \subset I$, $|J| = n$, such that $H \models \theta_i(a, b; \bar{b}_i)$, $b = \langle \bar{b}_j : j \in J\rangle$, for every $i \in I$, and $\theta_i \vdash_H \psi$ ($\psi$ as in 2.4(ii)). Then, using an argument similar to 2.4(ii), one shows that $\phi$ is equational in $\bar{x} \frown x$ once it has height $n$ in $\bar{x}$.

Note however that once we know $\phi$ is equational in $\bar{x} \frown x$, then we know that $\psi$ is equational in $x \frown y$ so that in principle we could have applied Proposition 2.4(ii) directly.

In the case where $\mathbf{K}$ has elementary embeddings for morphisms, Theorem 2.5 below is readily proved via the use of indiscernibles as has been suggested by J. Baldwin; the argument given below is an adaption of that proof to the general case. Our original proof using ultraproducts was more tedious but need only the closure of Mor($\mathbf{K}$) under ultraproducts.

**2.5. Theorem.** *Assume* $\mathbf{K}$ *is elementary in the sense given above and* $\phi(x; t)$ *is a formula in* $\Delta$. *The following assertions are equivalent*:

(a) $\phi$ *is an equation.*

(b) *For any regular cardinal* $\lambda \geq |L| + \aleph_0$, $H \in \mathbf{K}$ *and* $q$ *a complete type in* $x$ *over* $H$, *there is* $A \subset H$, card $A < \lambda$, *such that* $p^{\Phi} \vdash_H q^{\Phi}$, *where* $p = q \mid A$ *and* $\Phi = \mathrm{cl}^+(\phi)$.

(c) *There is a regular cardinal* $\lambda$ *such that*: *for every* $H \in \mathbf{K}$ *with* card $H = \lambda$, *and* $q$ *a complete type in* $x$ *over* $H$, *there is* $A \subset H$, card $A < \lambda$, *such that* $p \vdash_H q^{\Phi}$ *where* $p = q \mid A$ *and* $\Phi = \mathrm{cl}^+(\phi)$.

**Proof.** (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is immediate.

For simplicity let us assume the morphisms in $\mathbf{K}$ are inclusions. Suppose (c) holds but not (a). Then there is a sequence $\langle f_n : G \to G_n; n < \omega\rangle$ of morphisms in $\mathbf{K}$, a sequence $\langle a_n : n < \omega\rangle$ of elements in $G$ and a sequence $\langle b_n : n < \omega\rangle$ such that $b_n \in G_n$ and $G_n \models \bigwedge_{i<n} \phi(b_n; a_i) \wedge \neg\phi(b_n; a_n)$ for every $n < \omega$ (we write $a_n$ instead of $f_n a_n$). (Cf. 1.1.)

Consider the language $L' = L \cup \{U, U_\beta, a_\beta, b_\beta : \beta \leq \lambda\} \cup F$, where $U$, $U_\beta$ are new unary predicates, $a_\beta$, $b_\beta$ are new individual constants and $F$ is a set of function symbols.

Let $T'$ be the first-order theory in $L$ which says:

(i) $U$ and $U_\beta$ "are models of $T$", "$U \subset U_\beta$" and this inclusion is a morphism in $\mathbf{K}$ ($\beta \leq \lambda$).

(ii) $a_\beta \in U$, $b_\beta \in U_\beta$ and "$U_\beta \models \bigwedge_{\gamma<\beta} \phi(b_\beta, a_\gamma) \wedge \neg\phi(b_\beta, a_\beta)$" ($\beta \leq \lambda$).

(iii) $F$ is a set of Skolem functions of $U$.

(iv) for every $\theta$ in $L'$ and sequence $\gamma_0, \ldots, \gamma_n < \alpha$, $\beta < \lambda$,

$$\text{"}U_\alpha \models \theta(b_\alpha, a_{\gamma_0}, \ldots, a_{\gamma_n})\text{"} \iff \text{"}U_\beta \models \theta(b_\beta, a_{\gamma_0}, \ldots, a_{\gamma_n})\text{"}.$$

An immediate application of Ramsey's theorem to the sequence $\langle G_n, a_n, b_n : n < \omega\rangle$ above (here given $\theta \in L$, $\langle a_1 \frown b_1, \ldots, a_n \frown b_n\rangle$ is coloured white if, say, $G_n \models \theta(b_n, a_{n-1}, \ldots, a_1)$) where $G$ is assumed Skolemized, shows that $T'$ is consistent.

Let $F$ be a model of $T$; and let $H$, $H_\beta$, $a_\beta$, $b_\beta$ be the interpretations of $U$, $U_\beta$, $a_\beta$, $b_\beta$ respectively in $F$.

Taking $L$ to be the Skolem hull of $\{a_\beta : \beta < \lambda\}$ in $H$ and $p = \text{tp}(b_\lambda; L)$ (taken in $H_\lambda$). We get a contradiction to (c), since by (iv) for any $\beta < \lambda$, $\text{tp}(b_\lambda; \{a_\alpha : \alpha < \beta\})$ (taken $H_\lambda$) equals $\text{tp}(b_\gamma : \{a_\beta : \alpha < \beta\})$ (taken in $H_\beta$) but $H_\beta \models \neg\phi(b_\beta; a_\beta)$. □

**Remark.** Let $\mathbf{K}$ be the category of linearly ordered sets, with embeddings for morphisms ($L = \{>\}$); let $\phi(x; t) = (x > t)$. Then for any singular cardinal $\lambda$, $H \in \mathbf{K}$, $\text{card}\, H = \lambda$ and type $p$ over $H$, there is $A \subset H$, $\text{card}\, A < \lambda$ such that $p \mid A \vdash_H p^\Phi$, but obviously $\phi$ is not an equation.

**Application 1.** Suppose $\mathbf{K}$ is universal with embeddings (or homomorphisms) for morphisms and which admits free amalgams in the following sense: given arbitrary morphisms $f_i : H_0 \to H_i$, $i = 1, 2$, in $\mathbf{K}$, which can be amalgated in $\mathbf{K}$, there exists morphisms $h_i : H_i \to H$ such that $h_1 \cdot f_1 = h_2 \cdot f_2$ and whenever $g_i : H_i \to G$ are morphisms such that $g_1 \cdot f_1 = g_2 \cdot f_2$, then there is a homomorphism $h : H \to G$ with $h \cdot h_i = g_i$, $i = 1, 2$. Suppose furthermore that for some regular cardinal $\lambda$ the following property holds in $\mathbf{K}$; whenever $F$ is a structure generated by $H_1$ and $H_2$, $\text{card}\, H_2 = \lambda$, $\text{card}\, H_1 = \aleph_0$ (or just 'small' with respect to $\lambda$), then there exists $H_0 \subset H_2$, $\text{card}\, H_0 < \lambda$ such that $F$ is the free amalgam of $H'$ and $H_2$ over $H_0$, where $H'$ is the substructure of $F$ generated by $H_0 \cup H_1$. *Then*, by Theorem 2.5, every atomic formula in $L$ is an equation.

**Application 2.** Let $L$ be the language of differential fields with one additional unary function symbol r(–), $L = \{+, \cdot, 0, 1, \mathrm{d}, r\}$; let $T_0$ be the theory of radical differential fields of characteristic $p$ ($p \neq 0$), i.e., the theory of differential fields + the axiom:

$$(*) \qquad [(\mathrm{d}x = 0) \to (\mathrm{r}(x)^p = x)] \wedge [(\mathrm{d}x \neq 0) \to (\mathrm{r}(x) = 0)].$$

It is known (cf. [13]) that $T_0$ has the A.P. and has a model completion $T$ which has elimination of quantifiers. Given an atomic formula $\phi$, let $\hat{\phi}$ be the formula obtained from $\phi$ by adding as a conjunct the formula $(\mathrm{d}t = 0)$ ($t$ a term in $L$) whenever the term $\mathrm{r}(t)$ occurs in $\phi$. Let $S$ be the set of formulas $\hat{\phi}$ obtained in that manner. Using axiom $(*)$ it is easy to check that every atomic formula in $L$ (and hence every formula in $L$) is equivalent in $T$ to a boolean combination of formulas in $S$.

**Proposition.** *Every formula in S is an equation.*

We will need the following result which, stated slightly differently, is due to Shelah (cf. [10, Theorem 9]).

**Fact.** *Given $F_0 \subset F_1$, $F_2 \subset F_3$, models of $T$, with $F_1$ and $F_2$ linearly independent over $F_0$, the ring $F = F_1 \cdot F_2$ generated by $F_1$ and $F_2$ is closed under $r$.* (Shelah showed that if $c$ belongs to $F$ and $\mathrm{d}c = 0$, then $c$ is of the form $\sum_{i,j} \gamma_{ij} d_i e_j$ for some $\gamma_{ij}$ in $F_0^p$, $d_i$ in $F_1^p$ and $e_j$ in $F_2^p$.)

**Claim 1.** *With $F_0$, $F_1$, $F_2$, $F_3$ as above,* $\mathrm{tp}(F_1 \cup F_2, F_0; F_3)$ *is the S-free amalgam of* $\mathrm{tp}(F_1, F_0; F_3)$ *and* $\mathrm{tp}(F_2, F_0; F_3)$. *I.e., if $F_0 \subset G_1$, $G_2 \subset G$, $G$ a model of $T$, and* $\mathrm{tp}^S(F_i, F_0; F_3) \subset \mathrm{tp}^S(G_i, F_0; G)$, *then* $\mathrm{tp}^S(F_1 \cup F_2, F_0, F_3) \subset \mathrm{tp}^S(G_1 \cup G_2, F_0; G)$.

**Proof.** We note first that if $G$ is a model of $T$ and $f_i : F_i \to G$, $i = 1, 2$, are ($L$-) embeddings in $G$, say $G_i = f_i(F_i)$, with $f_1 \mid F_0 = f_2 \mid F_0$, then there is a ring embedding $f : F \to G$ extending $f_1$ and $f_2$: this is due to the fact that $F_1$ and $F_2$ are linearly independent over $F_0$. But then $f$ preserves the operation d(–): since, if $c = \sum_i a_i b_i$, $a_i \in F_1$, $b_i \in F_2$, then $\mathrm{d}c = \sum_i a_i \,\mathrm{d}b_i + \sum_i b_i \,\mathrm{d}a_i$; hence

$$f(\mathrm{d}c) = \sum_i f_1 a_i \cdot f_2(\mathrm{d}b_i) + \sum_i f_2 b_i \cdot f_1(\mathrm{d}a_i)$$

$$= \sum_i f_1 a_i \cdot \mathrm{d}(f_2 b_i) + \sum_i f_2 b_i \cdot \mathrm{d}(f_1 a_i) = \mathrm{d}(fc).$$

Moreover, by the fact above, $f$ clearly preserves the relation $(r(x) = y) \wedge (\mathrm{d}x = 0)$. Thus if $\phi(x, t) \in \mathrm{tp}(F_1 \cup F_2, F_0; F_3) \cap S$; then $\phi(x; t) \in \mathrm{tp}(G_1 \cup G_2, F_0; G) \cap S$, which proves the claim.

**Claim 2.** *given a model $F_2$ of $T$ and a complete type $p$ over $F_2$, there is a countable subset $A$ of $F_2$ such that $p \mid A \vdash p^S_{F_2}$.*

**Proof.** Let $a$ realize $p$ in an elementary extension $F_3$ of $F_2$. It is easy to construct $F_1$ in such a way that $a \in F_1$, $F_1$ is countable, and $F_1$ linearly independent from $F_2$ over $F_0$, where $F_0 = F_1 \cap F_2$. The claim now follows from Claim 1. By Theorem 2.5 we conclude that every formula in $S$ is an equation. □

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