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The equivalence between the convergences of Ishikawa and Mann iterations for an asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive maps

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Abstract

The convergence of modified Mann iteration is equivalent to the convergence of modified Ishikawa iterations, when T is an asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive map.

Keywords: Modified Mann iteration; Modified Ishikawa iteration; Asymptotically nonexpansive in the intermediate sense; Strongly successively pseudocontractive

1. Introduction

Let X be a Banach space and let B be a nonempty subset of X, $u_0, x_0 \in B$ be two arbitrary fixed points and $T : B \to B$ be a map.

Definition 1. The map T is said to be

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(i) asymptotically nonexpansive if there exists a sequence $(k_n)_n$, $k_n \in [1,\infty)$, $\forall n \in \mathbb{N}$, $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leqslant k_n \|x-y\|, \quad \forall x, y \in B,\ \forall n \in \mathbb{N}; \tag{1}$$

(ii) asymptotically nonexpansive in the intermediate sense if T is continuous for some m and

$$\limsup_{n\to\infty} \sup_{x,y\in B} \big(\|T^n x - T^n y\| - \|x-y\|\big) \leqslant 0; \tag{2}$$

(iii) strongly successively pseudocontractive if there exists $k \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\|x-y\| \leqslant \big\|x - y + t\big[(I - T^n - kI)x - (I - T^n - kI)y\big]\big\|, \tag{3}$$

for all $x, y \in B$, $t > 0$ and $n \geqslant n_0$;

(iv) uniformly Lipschitzian if there exists $L > 0$ such that

$$\|T^n x - T^n y\| \leqslant L\|x-y\|, \quad \forall x, y \in B,\ \forall n \in \mathbb{N}. \tag{4}$$

An example of an asymptotically nonexpansive in the intermediate sense which not continuous can be found in [1, Example 1.1, p. 456]. An asymptotically nonexpansive map is uniformly Lipschitzian for some $L \geqslant 1$, i.e., $\exists L \geqslant 1$: $\|T^n x - T^n y\| \leqslant L\|x-y\|$, $\forall x, y \in B$, $\forall n \in \mathbb{N}$. It is clear now that (ii) is weaker then (i).

Remark 2. An asymptotically nonexpansive map is *asymptotically nonexpansive* in the intermediate sense. The converse is not true.

Setting $n = n_0 := 1$ in (3), we get the definition of a strongly pseudocontractive map. In Example 1.2 from [1], there is a map which is not strongly pseudocontractive but which is strongly successively pseudocontractive.

We consider the following iteration, see [3]:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T^n u_n, \quad n = 0, 1, 2, \ldots. \tag{5}$$

This iteration is known as *modified Mann iteration*. We consider the following iteration, known as *modified Ishikawa iteration* (see [2]):

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{6}$$

The sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are such that

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{7}$$

The sequence $\{\alpha_n\}$ remains the same in both iterations. For $\beta_n = 0$, $\forall n \in \mathbb{N}$, from (6) we get (5). We denote by $F(T)$ the set of fixed points of T. Replacing T^n by T in (5) and (6) one obtains ordinary Mann and Ishikawa iteration.

The aim of this note is to prove the equivalence between the convergences of the above two iterations when T is an asymptotically nonexpansive in the intermediate sense or strongly successively pseudocontractive map.

The following lemma is from [7].

Lemma 3 [7]. *Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leqslant (1-\lambda_n)a_n + \delta_n, \tag{8}$$

where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. The case of asymptotically nonexpansive in the intermediate sense

Theorem 4. *Let B be a closed convex bounded subset of an arbitrary Banach space X and $\{x_n\}$ and $\{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying* (7)*. Let $T : B \to B$ be an asymptotically nonexpansive in the intermediate sense and successively strongly pseudocontractive self-map of B. Put*

$$c_n = \max\Bigl(0, \sup_{x,y\in B}\bigl(\|T^n x - T^n y\| - \|x-y\|\bigr)\Bigr) \tag{9}$$

so that

$$\lim_{n\to\infty} c_n = 0. \tag{10}$$

If $u_0 = x_0 \in B$, then the following two assertions are equivalent:

(i) *Modified Mann iteration* (5) *converges to $x^* \in F(T)$.*
(ii) *Modified Ishikawa iteration* (6) *converges to $x^* \in F(T)$.*

Proof. If the modified Ishikawa iteration (6) converges to x^*, then it is clear that this x^* is a fixed point. Setting $\beta_n = 0$, $\forall n \in N$, in (6) we obtain the convergence of modified Mann iteration. Conversely, we shall prove that the convergence of modified Mann iteration implies the convergence of modified Ishikawa iteration. The proof is similar to the proof of Theorem 4 from [5]. From (6) we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n x_{n+1} - \alpha_n T^n x_{n+1} - k\alpha_n x_{n+1} \\ &\quad - 2\alpha_n x_{n+1} + k\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n T^n x_{n+1} - \alpha_n T^n y_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - kI)x_{n+1} - (2-k)\alpha_n x_{n+1} \\ &\quad + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - kI)x_{n+1} - (2-k)\alpha_n\bigl[x_n + \alpha_n(T^n y_n - x_n)\bigr] \\ &\quad + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - kI)x_{n+1} \\ &\quad - (1-k)\alpha_n x_n + (2-k)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n). \end{aligned} \tag{11}$$

Analogously, for (5) we get

$$\begin{aligned} u_n &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T^n - kI)u_{n+1} \\ &\quad - (1-k)\alpha_n u_n + (2-k)\alpha_n^2(u_n - T^n u_n) + \alpha_n(T^n u_{n+1} - T^n u_n). \end{aligned} \tag{12}$$

Compute (11)–(12) to obtain

$$x_n - u_n = (1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n\big[(I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1}\big] - (1-k)\alpha_n(x_n - u_n) + (2-k)\alpha_n^2\big[x_n - u_n - (T^n y_n - T^n u_n)\big] + \alpha_n\big[T^n x_{n+1} - T^n y_n - (T^n u_{n+1} - T^n u_n)\big]. \tag{13}$$

Using the triangular inequality and (3) with $x := x_{n+1}$, $y := u_{n+1}$, $t := \alpha_n/(1+\alpha_n)$,

$$\begin{aligned}\|x_n - u_n\| &\geqslant (1+\alpha_n)\bigg\|(x_{n+1} - u_{n+1}) + \frac{\alpha_n}{1+\alpha_n}\big[(I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1}\big]\bigg\| \\ &\quad - (1-k)\alpha_n\|x_n - u_n\| - (2-k)\alpha_n^2\big\|x_n - u_n - (T^n y_n - T^n u_n)\big\| \\ &\quad - \alpha_n\big\|T^n x_{n+1} - T^n y_n - (T^n u_{n+1} - T^n u_n)\big\| \\ &\geqslant (1+\alpha_n)\|x_{n+1} - u_{n+1}\| - (1-k)\alpha_n\|x_n - u_n\| \\ &\quad - (2-k)\alpha_n^2\big\|x_n - u_n - (T^n y_n - T^n u_n)\big\| \\ &\quad - \alpha_n\big\|T^n x_{n+1} - T^n y_n - (T^n u_{n+1} - T^n u_n)\big\|. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned}&(1+\alpha_n)\|x_{n+1} - u_{n+1}\| \\ &\quad\leqslant \big(1 + (1-k)\alpha_n\big)\|x_n - u_n\| + (2-k)\alpha_n^2\|x_n - u_n - T^n y_n + T^n u_n\| \\ &\qquad + \alpha_n\big\|T^n x_{n+1} - T^n u_{n+1} - (T^n y_n - T^n u_n)\big\| \\ &\quad\leqslant \big(1 + (1-k)\alpha_n\big)\|x_n - u_n\| + (2-k)\alpha_n^2\|x_n - T^n y_n\| + (2-k)\alpha_n^2\|u_n - T^n u_n\| \\ &\qquad + \alpha_n\|T^n u_{n+1} - T^n u_n\| + \alpha_n\|T^n x_{n+1} - T^n y_n\|. \end{aligned} \tag{15}$$

Using the facts that $(1+\alpha_n^2)^{-1} \leqslant 1$ and $(1+\alpha_n^2)^{-1} \leqslant 1 - \alpha_n + \alpha_n^2$ we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leqslant \big(1 + (1-k)\alpha_n\big)\big(1 - \alpha_n + \alpha_n^2\big)\|x_n - u_n\| \\ &\quad + \alpha_n\big\{(2-k)\alpha_n\|x_n - T^n y_n\| + (2-k)\alpha_n\|u_n - T^n u_n\| \\ &\quad + \|T^n u_{n+1} - T^n u_n\| + \|T^n x_{n+1} - T^n y_n\|\big\} \\ &= \big(1 + (1-k)\alpha_n\big)\big(1 - \alpha_n + \alpha_n^2\big)\|x_n - u_n\| + \alpha_n\sigma_n, \end{aligned} \tag{16}$$

where

$$\sigma_n := (2-k)\alpha_n\|x_n - T^n y_n\| + (2-k)\alpha_n\|u_n - T^n u_n\| + \|T^n u_{n+1} - T^n u_n\| + \|T^n x_{n+1} - T^n y_n\|. \tag{17}$$

We have

$$M := \max\big\{\|x_0\|,\ \sup\big\{\|T^n x\|,\ x \in B,\ n \in \mathbb{N}\big\}\big\} < \infty. \tag{18}$$

The sequence $\{\|x_n - T^n y_n\|\}$ is bounded because $\{T^n y_n\}$ is in the bounded set B, and $\{x_n\}$ also is bounded by M. Supposing that $\|x_n\| \leqslant M$, a simple induction leads to

$$\|x_{n+1}\| \leqslant (1-\alpha_n)\|x_n\| + \alpha_n M \leqslant (1-\alpha_n)M + \alpha_n M = M. \tag{19}$$

Modified Mann iteration (5) converges, let x^* be that fixed point. Thus

$$\begin{aligned}0 &\leqslant \|u_n - T^n u_n\| \leqslant \|T^n x^* - T^n u_n\| + \|u_n - x^*\| \\ &= \big(\|T^n x^* - T^n u_n\| - \|u_n - x^*\|\big) + 2\|u_n - x^*\| \\ &\leqslant c_n + 2\|u_n - x^*\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{20}$$

It is clear that $\{\|T^n u_{n+1} - T^n u_n\|\}$ also converges to zero because

$$\begin{aligned}0 &\leqslant \|T^n u_{n+1} - T^n u_n\| \\ &\leqslant \|T^n u_{n+1} - T^n u_n\| - \|u_{n+1} - u_n\| + \|u_{n+1} - u_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{21}$$

From (9) and (10) one obtains

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &= \big[\|T^n y_n - T^n x_{n+1}\| - \|y_n - x_{n+1}\|\big] + \|y_n - x_{n+1}\| \\ &\leqslant c_n + \|y_n - x_{n+1}\| \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{22}$$

since

$$\begin{aligned}\|y_n - x_{n+1}\| &= \|-\beta_n x_n + \beta_n T^n x_n + \alpha_n x_n - \alpha_n T^n y_n\| \\ &\leqslant 2\beta_n M + 2\alpha_n M = 2M(\alpha_n + \beta_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{23}$$

The sequences $\{x_n\}$, $\{T^n x_n\}$ and $\{T^n y_n\}$ are in the bounded set B, and bounded by $M > 0$.

The following inequality is, in fact, inequality (29) from [5]:

$$\begin{aligned}&\big(1 + (1-k)\alpha_n\big)\big(1 - \alpha_n + \alpha_n^2\big) \\ &\quad = 1 - k\alpha_n + k\alpha_n^2 + (1-k)\alpha_n^3 \leqslant 1 - k\alpha_n + k\alpha_n^2 + (1-k)\alpha_n^2 \\ &\quad = 1 - k\alpha_n + \alpha_n^2. \end{aligned} \tag{24}$$

The condition $\lim_{n\to\infty} \alpha_n = 0$ implies the existence of a positive integer N such that for all $n \geqslant N$

$$\alpha_n \leqslant \frac{k}{2}. \tag{25}$$

Substituting inequality (25) into (24) we get

$$\begin{aligned}\big(1 + (1 - k\alpha_n)\big)\big(1 - \alpha_n + \alpha_n^2\big) &\leqslant 1 - k\alpha_n + \alpha_n^2 \leqslant 1 - k\alpha_n + \frac{k}{2}\alpha_n \\ &= 1 - \frac{k}{2}\alpha_n. \end{aligned} \tag{26}$$

Relations (26) and (16) lead to

$$\|x_{n+1} - u_{n+1}\| \leqslant \left(1 - \frac{k}{2}\alpha_n\right)\|x_n - u_n\| + \alpha_n\sigma_n, \tag{27}$$

where $\{\sigma_n\}$is given by (17). From (22) we know that $\lim_{n\to\infty} \sigma_n = 0$. Denote

$$a_n := \|x_n - u_n\|, \qquad \lambda_n := \frac{k}{2}\alpha_n, \qquad \delta_n := \alpha_n\sigma_n = o(\lambda_n). \tag{28}$$

Relations (28) and (27) lead to (8); using Lemma 3 we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \tag{29}$$

to obtain

$$0 \leqslant \|x_n - x^*\| \leqslant \|x_n - u_n\| + \|u_n - x^*\| \to 0 \quad \text{as } n \to \infty. \tag{30}$$

Hence $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

3. The strongly successively pseudocontractive case

3.1. The Lipschitzian case

Theorem 5. *Let B be a closed convex* (*without being necessarily bounded*) *subset of an arbitrary Banach space X and $\{x_n\}$, $\{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying* (7). *Let T be a successively strongly pseudocontractive and uniformly Lipschitzian with $L \geqslant 1$ self-map of B. If $u_0 = x_0 \in B$, then the following two assertions are equivalent*:

(i) *Modified Mann iteration* (5) *converges to $x^* \in F(T)$.*
(ii) *Modified Ishikawa iteration* (6) *converges to $x^* \in F(T)$.*

Proof. Supposing, again, that modified Ishikawa iteration converges, analogously as in the proof of Theorem 4 we obtain the convergence of modified Mann iteration. Conversely, supposing that modified Mann iteration converges, we will prove that modified Ishikawa iteration will converge. For that we need to evaluate $\|x_n - u_n\|$. The map T is successively strongly pseudocontractive. Thus relations (11), (12), (14)–(16), (24) hold:

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leqslant{}& \left(1 - k\alpha_n + \alpha_n^2\right)\|x_n - u_n\| \\ &+ \alpha_n\Big\{(2-k)\alpha_n\|x_n - T^n y_n\| + (2-k)\alpha_n\|u_n - T^n u_n\| \\ &+ \|T^n u_{n+1} - T^n u_n\| + \|T^n x_{n+1} - T^n y_n\|\Big\}.\end{aligned} \tag{31}$$

We have

$$\begin{aligned}\|x_n - T^n y_n\| &\leqslant \|x_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n\| \\ &\leqslant \|x_n - u_n\| + \|u_n - T^n u_n\| + L\|u_n - y_n\|.\end{aligned} \tag{32}$$

$$\begin{aligned}\|u_n - y_n\| &= \big\|(1-\beta_n)(u_n - x_n) + \beta_n(u_n - T^n x_n)\big\| \\ &\leqslant (1-\beta_n)\|x_n - u_n\| + \beta_n\|u_n - T^n x_n\| \\ &\leqslant (1-\beta_n)\|x_n - u_n\| + \beta_n\left(\|T^n u_n - T^n x_n\| + \|u_n - T^n u_n\|\right) \\ &\leqslant (1-\beta_n)\|x_n - u_n\| + \beta_n L\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\| \\ &= (1 - \beta_n + \beta_n L)\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\| \\ &\leqslant L\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\|,\end{aligned} \tag{33}$$

because $1 \leqslant L \Rightarrow 1 - \beta_n + \beta_n L \leqslant L$.

Substituting (33) into (32) we get

$$\begin{aligned}\|x_n - T^n y_n\| &\leqslant \|u_n - x_n\| + \|u_n - T^n u_n\| \\ &\quad + L\big(L\|x_n - u_n\| + \beta_n\|u_n - T^n u_n\|\big) \\ &\leqslant (1 + L^2)\|x_n - u_n\| + (1 + L\beta_n)\|u_n - T^n u_n\|. \end{aligned}\tag{34}$$

Now

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &\leqslant L\|x_{n+1} - y_n\| = L\big\|(1-\alpha_n)x_n + \alpha_n T^n y_n - y_n\big\| \\ &= L\big\|(1-\alpha_n)(x_n - y_n) + \alpha_n(T^n y_n - y_n)\big\| \\ &\leqslant L\big((1-\alpha_n)\|x_n - y_n\| + \alpha_n\|T^n y_n - y_n\|\big). \end{aligned}\tag{35}$$

Using (33),

$$\begin{aligned}\|T^n y_n - y_n\| &\leqslant \|T^n y_n - T^n u_n\| + \|T^n u_n - u_n\| + \|u_n - y_n\| \\ &\leqslant (1+L)\|u_n - y_n\| + \|T^n u_n - u_n\| \\ &\leqslant (1+L)\big(L\|u_n - x_n\| + \beta_n\|T^n u_n - u_n\|\big) + \|T^n u_n - u_n\| \\ &= (1+L)L\|x_n - u_n\| + \big[(1+L)\beta_n + 1\big]\|T^n u_n - u_n\|. \end{aligned}\tag{36}$$

$$\begin{aligned}\|x_n - y_n\| &= \big\|x_n - (1-\beta_n)x_n - \beta_n T^n x_n\big\| = \beta_n\|x_n - T^n x_n\| \\ &\leqslant \beta_n\big[\|x_n - u_n\| + \|T^n u_n - u_n\| + \|T^n x_n - T^n u_n\|\big] \\ &\leqslant \beta_n\big((1+L)\|x_n - u_n\| + \|T^n u_n - u_n\|\big). \end{aligned}\tag{37}$$

Substituting (36) and (37) in (35) one obtains

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &\leqslant L\big[(1-\alpha_n)\|x_n - y_n\| + \alpha_n\|T^n y_n - y_n\|\big] \\ &\leqslant L\Big\{(1-\alpha_n)\big(\beta_n\big((1+L)\|u_n - x_n\| + \|T^n u_n - u_n\|\big)\big) \\ &\quad + \alpha_n\big((1+L)L\|x_n - u_n\| + \big[(1+L)\beta_n + 1\big]\|T^n u_n - u_n\|\big)\Big\} \\ &= (1-\alpha_n)\beta_n(1+L)L\|x_n - u_n\| + L(1-\alpha_n)\beta_n\|T^n u_n - u_n\| \\ &\quad + \alpha_n(1+L)L^2\|x_n - u_n\| \\ &\quad + \alpha_n L\big[(1+L)\beta_n + 1\big]\|T^n u_n - u_n\| \\ &= \big(L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big)\|x_n - u_n\| \\ &\quad + \big(\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big)\|T^n u_n - u_n\|. \end{aligned}\tag{38}$$

Replacing (38) and (32) in (31) we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leqslant \big(1 - k\alpha_n + 2\alpha_n^2\big)\|x_n - u_n\| \\ &\quad + (2-k)\alpha_n^2\big((1+L^2)\|x_n - u_n\| + (1+\beta_n L)\|u_n - T^n u_n\|\big) \\ &\quad + (2-k)\alpha_n^2\|u_n - T^n u_n\| + \alpha_n\|T^n u_{n+1} - T^n u_n\| \\ &\quad + \alpha_n\big(L(1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L^2\big)\|x_n - u_n\| \\ &\quad + \alpha_n\big(\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big)\|u_n - T^n u_n\| \end{aligned}$$

$$
\begin{aligned}
&= \Big\{\big(1 - k\alpha_n + 2\alpha_n^2\big) + (2-k)\alpha_n^2(1+L^2) \\
&\quad + \alpha_n L(1+L)\big((1-\alpha_n)\beta_n + \alpha_n L\big)\Big\}\|x_n - u_n\| \\
&\quad + \Big\{(2-k)\alpha_n^2(2+\beta_n L) + \alpha_n\big[\beta_n L(1-\alpha_n) \\
&\quad + \alpha_n L\big[(1+L)\beta_n + 1\big]\big]\Big\}\|u_n - T^n u_n\| \\
&\quad + \alpha_n\|T^n u_{n+1} - T^n u_n\|. \qquad (39)
\end{aligned}
$$

Formula (30) from [5] with $M = 2 + (2-k)(1+L^2) + L^2(1+L)$ leads us to

$$
\begin{aligned}
&(1 - k\alpha_n) + 2\alpha_n^2 + (2-k)\alpha_n^2(1+L^2) + \alpha_n L(1+L)\big((1-\alpha_n)\beta_n + \alpha_n L\big) \\
&\quad \leqslant 1 - k\alpha_n + \alpha_n\big(2\alpha_n + (2-k)\alpha_n(1+L^2) + L(1+L)\big((1-\alpha_n)\beta_n + \alpha_n L\big)\big) \\
&\quad \leqslant 1 - k\alpha_n + \alpha_n M(\alpha_n + \beta_n) \\
&\quad \leqslant 1 - k\alpha_n + \alpha_n k(1-k) = 1 - k^2\alpha_n, \qquad (40)
\end{aligned}
$$

for all n sufficiently large, since $\lim_{n\to\infty}(\alpha_n + \beta_n) = 0$. Relations (40) and (39) lead to

$$
\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leqslant (1 - k^2\alpha_n)\|x_n - u_n\| \\
&\quad + \alpha_n\Big\{\Big[(2-k)\alpha_n(2+\beta_n L) + \big[\beta_n L(1-\alpha_n) \\
&\quad + \alpha_n L\big[(1+L)\beta_n + 1\big]\big]\Big]\|u_n - T^n u_n\| + \|T^n u_{n+1} - T^n u_n\|\Big\} \\
&= (1 - k^2\alpha_n)\|x_n - u_n\| + \alpha_n\epsilon_n,
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_n &:= \Big[(2-k)\alpha_n(2+\beta_n L) \\
&\quad + \big[\beta_n L(1-\alpha_n) + \alpha_n L\big[(1+L)\beta_n + 1\big]\big]\Big]\|u_n - T^n u_n\| \\
&\quad + \|T^n u_{n+1} - T^n u_n\|. \qquad (41)
\end{aligned}
$$

Supposing that $\lim_{n\to\infty}\|u_n - x^*\| = 0$, with $Tx^* = x^*$, we have $\lim_{n\to\infty}\|u_n - T^n u_n\| = 0$ because

$$
\begin{aligned}
0 &\leqslant \|u_n - T^n u_n\| \leqslant \|u_n - x^*\| + \|T^n x^* - T^n u_n\| \\
&\leqslant \|u_n - x^*\| + L\|u_n - x^*\| = (1+L)\|u_n - x^*\| \to 0 \quad \text{as } n \to \infty. \qquad (42)
\end{aligned}
$$

It is clear that if $\lim_{n\to\infty}\|u_n - T^n u_n\| = 0$, then $\lim_{n\to\infty}\|T^n u_{n+1} - T^n u_n\| = 0$.

Denote by

$$
a_n := \|x_n - u_n\|, \qquad \lambda_n := k^2\alpha_n, \qquad \delta_n := \alpha_n\epsilon_n = o(\lambda_n). \qquad (43)
$$

Supposing that modified Mann iteration converges, i.e., $\lim_{n\to\infty} u_n = x^*$, we get from (42)

$$
\lim_{n\to\infty}\|T^n u_{n+1} - T^n u_n\| = 0
$$

and

$$
\lim_{n\to\infty}\|u_n - T^n u_n\| = 0
$$

because T is uniformly Lipschitzian. Thus $\lim_{n\to\infty}\epsilon_n = 0$, which means that $\delta_n = o(\lambda_n)$. Relations (43) and Lemma 3 lead us to

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{44}$$

The inequality

$$0 \leqslant \|x_n - x^*\| \leqslant \|x_n - u_n\| + \|u_n - x^*\| \to 0 \quad \text{as } n \to \infty \tag{45}$$

leads us to conclusion that $\lim_{n\to\infty} x_n = x^*$. □

3.2. The non-Lipschitzian case

Let X be a real Banach space, B be a nonempty subset of X and $T : B \to B$.

The map $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^*\colon \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}$, $\forall x \in X$, is called *the normalized duality mapping*. The Hahn–Banach theorem assures that $Jx \neq \emptyset$, $\forall x \in X$. It is an easy task to see that $\langle j(x), y\rangle \leqslant \|x\|\|y\|$, $\forall x, y \in X$, $\forall j(x) \in J(x)$.

In [1, Lemma 2.1, p. 459] it is shown that the definition of successively strongly pseudocontractive map is equivalent to the following definition:

Definition 6. T is successively strongly pseudocontractive map if *there exists* $k \in (0, 1)$ and a $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leqslant k\|x - y\|^2, \quad \forall x, y \in B. \tag{46}$$

We need the following lemma from [4].

Lemma 7 [4]. *If X is a real Banach space, then the following relation is true*:

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in X,\ \forall j(x + y) \in J(x + y). \tag{47}$$

We are able now to prove the following result:

Theorem 8. *Let X be a real Banach space with dual uniformly convex and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a successively strongly pseudocontractive operator and $\{x_n\}, \{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying* (7). *Then for $u_0 = x_0 \in B$ the following assertions are equivalent*:

(i) *Modified Mann iteration* (5) *converges to the fixed point of T.*
(ii) *Modified Ishikawa iteration* (6) *converges to the fixed point of T.*

Proof. The proof is similar to the proof of the main result from [6]. If either (5) or (6) converges to a point x^*, then x^* is a fixed point for T. Using (5), (6), (47) with $x := (1 - \alpha_n)(x_n - u_n)$, $y := \alpha_n(T^n y_n - T^n u_n)$ (observe that $x + y = x_{n+1} - u_{n+1}$) and (46) we get

$$\begin{aligned}\|x_{n+1} - u_{n+1}\|^2 &= \big\|(1 - \alpha_n)(x_n - u_n) + \alpha_n(T^n y_n - T^n u_n)\big\|^2 \\ &\leqslant (1 - \alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1})\rangle\end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_n)^2\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\big\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\big\rangle \\
&\quad + 2\alpha_n\big\langle T^n y_n - T^n u_n, J(y_n - u_n)\big\rangle \\
&\leqslant (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n k\|y_n - u_n\|^2 \\
&\quad + 2\alpha_n\big\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\big\rangle \\
&\leqslant (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n k\|y_n - u_n\|^2 \\
&\quad + 2\alpha_n\|T^n y_n - T^n u_n\|\big\|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\big\| \\
&\leqslant (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n k\|y_n - u_n\|^2 \\
&\quad + 2\alpha_n M_1\big\|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\big\|, \qquad (48)
\end{aligned}$$

for some $M_1 > 0$. Observe that $\{\|T^n y_n - T^n u_n\|\}$ is bounded. We prove that

$$J(x_{n+1} - u_{n+1}) - J(y_n - u_n) \to 0 \quad \text{as } n \to \infty. \qquad (49)$$

If the dual is uniformly convex, then J is single map and uniformly continuous on every bounded set. To prove (49) it is sufficient to see that

$$\begin{aligned}
&\big\|(x_{n+1} - u_{n+1}) - (y_n - u_n)\big\| = \big\|(x_{n+1} - y_n) - (u_{n+1} - u_n)\big\| \\
&\quad = \big\|-\alpha_n x_n + \alpha_n T^n y_n + \beta_n x_n - \beta_n T^n x_n + \alpha_n u_n - \alpha_n T^n u_n\big\| \\
&\quad \leqslant \alpha_n\big(\|x_n\| + \|T^n y_n\| + \|u_n\| + \|T^n u_n\|\big) + \beta_n\big(\|x_n\| + \|T^n x_n\|\big) \\
&\quad \leqslant (\alpha_n + \beta_n)M \to 0 \quad \text{as } n \to \infty, \qquad (50)
\end{aligned}$$

where $M = \sup_n((\|x_n\| + \|T^n y_n\| + \|u_n\| + \|T^n u_n\|), (\|x_n\| + \|T^n x_n\|)) < \infty$.

The sequences $\{u_n\}$, $\{x_n\}$, $\{T^n x_n\}$, $\{T^n u_n\}$ and $\{T^n y_n\}$ are bounded, being in the bounded set B. Hence one can see that the M above is finite and (49) holds.

We define

$$\sigma_n := 2\alpha_n M_1\big\|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\big\|. \qquad (51)$$

Again, using (6) and (47) with $x := (1-\beta_n)(x_n - u_n)$, $y := \beta_n(T^n x_n - u_n)$ (observe that $x + y = y_n - u_n$) we get

$$\begin{aligned}
\|y_n - u_n\|^2 &= \big\|(1-\beta_n)(x_n - u_n) + \beta_n(T^n x_n - u_n)\big\|^2 \\
&\leqslant (1-\beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\big\langle T^n x_n - u_n, J(y_n - u_n)\big\rangle \\
&\leqslant \|x_n - u_n\|^2 + \beta_n M_2. \qquad (52)
\end{aligned}$$

The last inequality is true because $\{\langle T^n x_n - u_n, J(y_n - u_n)\rangle\}$ is bounded, with a constant $M_2 > 0$. Replacing (51) and (52) in (48), we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\|^2 &\leqslant (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n k\|x_n - u_n\|^2 + \sigma_n + \alpha_n(2k)\beta_n M_2 \\
&= \big(1 - 2(1-k)\alpha_n + \alpha_n^2\big)\|x_n - u_n\|^2 + o(\alpha_n). \qquad (53)
\end{aligned}$$

The condition $\lim_{n\to\infty}\alpha_n = 0$ implies the existence of an n_0 such that for all $n \geqslant n_0$ we have

$$\alpha_n \leqslant (1-k). \qquad (54)$$

Substituting (54) into (53), we obtain $1-2(1-k)\alpha_n+\alpha_n^2 \leqslant 1-2(1-k)\alpha_n+(1-k)\alpha_n = 1-(1-k)\alpha_n$. Thus, from (53)

$$\|x_{n+1}-u_{n+1}\|^2 \leqslant \bigl(1-(1-k)\alpha_n\bigr)\|x_n-u_n\|^2+o(\alpha_n). \tag{55}$$

Define $a_n := \|x_n - u_n\|^2$, $\lambda_n := (1-k)\alpha_n \in (0,1)$. Then Lemma 3 implies that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n-u_n\|^2 = 0$, i.e.,

$$\lim_{n\to\infty} \|x_n-u_n\| = 0. \tag{56}$$

Suppose that modified Mann iteration converges, i.e., $\lim_{n\to\infty} u_n = x^*$. The inequality

$$0 \leqslant \|x^*-x_n\| \leqslant \|u_n-x^*\|+\|x_n-u_n\| \tag{57}$$

and (56) imply that $\lim_{n\to\infty} x_n = x^*$. Analogously $\lim_{n\to\infty} x_n = x^*$ implies $\lim_{n\to\infty} u_n = x^*$. □

4. The equivalence between T-stability

Let $F(T) := \{x^* \in X\colon x^* = T(x^*)\}$, $x^* \in F(T)$. Consider

$$\varepsilon_n := \bigl\|x_{n+1}-(1-\alpha_n)x_n-\alpha_n T^n y_n\bigr\|, \tag{58}$$

$$\delta_n := \bigl\|u_{n+1}-(1-\alpha_n)u_n-\alpha_n T^n u_n\bigr\|. \tag{59}$$

Definition 9. If $\lim_{n\to\infty} \varepsilon_n = 0$ (respectively $\lim_{n\to\infty} \delta_n = 0$) implies that $\lim_{n\to\infty} x_n = x^*$ (respectively $\lim_{n\to\infty} u_n = x^*$), then (6) (respectively (5)) is said to be T-stable.

It is obvious if we take the limit in (6), respectively (5).

Remark 10. Let X be a normed space with B a nonempty, convex, closed, and bounded subset. Let $T: B \to B$ be a map. If the modified Mann (respectively Ishikawa) iteration converges, then $\lim_{n\to\infty} \delta_n = 0$ (respectively $\lim_{n\to\infty} \varepsilon_n = 0$). The remark holds without the boundeness assumption of B, when the map T is uniformly Lipschitzian.

Proof. Let $\lim_{n\to\infty} u_n = x^*$. Then from (59) we have

$$\begin{aligned} 0 &\leqslant \delta_n \leqslant \|u_{n+1}-u_n\|+\alpha_n\|u_n-T^n u_n\| \\ &\leqslant \|u_{n+1}-x^*\|+\|u_n-x^*\|+\alpha_n\|u_n-x^*\|+\alpha_n\|x^*-T^n u_n\| \\ &\to 0 \quad \text{as } n\to\infty. \quad \square \end{aligned}$$

We are able now to prove the following result:

Theorem 11. *Let B be a closed convex bounded subset of an arbitrary Banach space X and $\{x_n\}$ and $\{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying* (7)*. Let T be an asymptotically nonexpansive in the intermediate sense and successively strongly pseudocontractive self-map of B. Let $\{c_n\}$ be as in* (9) *satisfying* $\lim_{n\to\infty} c_n = 0$. *If $u_0 = x_0 \in B$, then the following two assertions are equivalent*:

(i) *Modified Ishikawa iteration* (6) *is T-stable.*
(ii) *Modified Mann iteration* (5) *is T-stable.*

Proof. From Definition 9 we know that the equivalence (i) $\Leftrightarrow$ (ii) means that $\lim_{n\to\infty}\varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty}\delta_n = 0$. The implication $\lim_{n\to\infty}\varepsilon_n = 0 \Rightarrow \lim_{n\to\infty}\delta_n = 0$ is obvious by setting $\beta_n = 0$ in (6). Conversely, suppose that (5) is T-stable. Using Definition 9, again, we get

$$\lim_{n\to\infty}\delta_n = 0 \quad\Rightarrow\quad \lim_{n\to\infty}u_n = x^*. \tag{60}$$

Theorem 4 assures that $\lim_{n\to\infty}u_n = x^* \Rightarrow \lim_{n\to\infty}x_n = x^*$. Using Remark 10 we have $\lim_{n\to\infty}\varepsilon_n = 0$. Thus we get $\lim_{n\to\infty}\delta_n = 0 \Rightarrow \lim_{n\to\infty}\varepsilon_n = 0$. $\square$

Similarly one can prove the following result.

Theorem 12. *Let B be a closed convex* (*without being necessarily bounded*) *subset of an arbitrary Banach space X and $\{x_n\}, \{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying* (7)*. Let T be a successively strongly pseudocontractive and uniformly Lipschitzian with $L \geqslant 1$ self-map of B. If $u_0 = x_0 \in B$, then the following two assertions are equivalent*:

(i) *Modified Ishikawa iteration* (6) *is T-stable.*
(ii) *Modified Mann iteration* (5) *is T-stable.*

Also the following results holds using Theorem 8:

Theorem 13. *Let X be a real Banach space with dual uniformly convex and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a successively strongly pseudocontractive operator and $\{x_n\}, \{u_n\}$ defined by* (6) *and* (5) *with $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying* (7)*. Then for $u_0 = x_0 \in B$ the following assertions are equivalent*:

(i) *Modified Ishikawa iteration* (6) *is T-stable.*
(ii) *Modified Mann iteration* (5) *is T-stable.*

Our theorems are also true for set-valued mappings, if such maps admit appropriate single-valued selections.

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