A lower bound for the product of eigenvalues of solutions to matrix equations

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ABSTRACT

The following matrix equations:

\[ A^T X B + B^T X^T A = C \quad \text{and} \quad A^T X B + B^T X A = C, \]

are encountered in many systems and control applications, and these matrix equations contain several linear matrix equations as special cases. In the present work, we introduce the inequalities for the determinant of the solutions of these matrix equations, separately. Then using these inequalities, we introduce a lower bound for the product of eigenvalues of the solutions to the matrix equations.

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1. Introduction

In this work, we use \( \mathbb{R}^{n \times n} \) to denote the set of all \( n \times n \) real matrices. \( A^T \) and \( |A| \) stand for the transpose and determinant of a matrix \( A \), respectively. We denote the spectrum of matrix \( A \) with \( \sigma(A) \). Matrix \( A \in \mathbb{R}^{n \times n} \) is said to be a real stability matrix if \( \text{Re}(\lambda_i) < 0 \) for \( i = 1, 2, \ldots, n \), where \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) (\( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of the matrix \( A \)). If \( A \) is a square \( n \)-by-\( n \) matrix with real or complex entries and \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), then \( |A| = \lambda_1 \lambda_2 \cdots \lambda_n \).

Matrix equation is one of the topics of very active research in computational mathematics, and has been widely used in various areas [1]. A large number of papers have studied several matrix equations [2–7]. Ding and Chen [8–12] proposed some iterative methods for solving Sylvester matrix equations. Zhou and Duan [13–15] established the solution of several generalized Sylvester matrix equations. Recently Dehghan and Hajarian proposed some iterative methods for solving several matrix equations [16–18].

So far, various bounds for the eigenvalues of solutions of several matrix equations have been established. In [19], several bounds for the traces of the solutions of the algebraic Riccati and Lyapunov matrix equations were presented in the continuous and discrete domains, respectively. Komaroff [20] introduced simultaneous eigenvalue lower bounds for the solution of the following Lyapunov matrix equation:

\[ A^T P + PA = -Q. \]  (1.1)

For when \( A \) is a stability matrix, a fundamental inequality was established [21] which is satisfied by the extremal eigenvalues of the solutions \( Q \) and \( P \) to Lyapunov matrix equation (1.1). In [22], lower bounds for the sum of the eigenvalues of the solution to the algebraic Riccati equation

\[ A^T P + PA - PRP = -Q, \]  (1.2)

were proposed.

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In this work, two following matrix equations are considered:

\[ A^T XB + B^T X A = C, \]  
(1.3)

and

\[ A^T XB + B^T X A = C, \]  
(1.4)

where \( A, B, C \in \mathbb{R}^{n \times n} \) are known matrices and \( X \in \mathbb{R}^{n \times n} \) is an unknown matrix. In the remainder of this work we first give two inequalities which are satisfied by the determinant of the solutions to the matrix equations (1.3) and (1.4), respectively. Then applying these inequalities, we introduce a lower bound for the product of eigenvalues of the solutions to the matrix equations (1.3) and (1.4).

2. Main results

In this section, we present the main results. Firstly, we give a lemma which is an essential tool for obtaining the main results. The proof of the following lemma can be found in [23].

**Lemma 2.1.** Suppose that \( E = F + G \) and \( F \) is a symmetric positive definite matrix and \( G \) is a skew-symmetric matrix; then \( |E| \geq |F| \).

In the rest of this work, we will suppose that \( C \) is a symmetric positive definite matrix.

**Theorem 2.1.** Assume that the matrix equation (1.3) is consistent; then

\[ |C| \leq 2^n |X||A||B|. \]  
(2.1)

**Proof.** Considering the matrix equation (1.3), we can write

\[ A^T XB = \frac{1}{2} \{ A^T XB + A^T XB + B^T X A - B^T X A \} \]

\[ \quad = \frac{1}{2} (A^T XB + B^T X A) + \frac{1}{2} (A^T XB - B^T X A). \]

Let \( G = \frac{1}{2} (A^T XB - B^T X A) \); therefore

\[ A^T XB = \frac{1}{2} C + G. \]  
(2.2)

It is obvious that \( \frac{1}{2} C \) and \( G \) are a symmetric positive definite matrix and a skew-symmetric matrix, respectively. By Lemma 2.1, we get

\[ |A^T XB| \geq \left| \frac{1}{2} C \right|. \]

\[ |A||X||B| \geq \frac{1}{2^n} |C|. \]  
(2.3)

By (2.3), the proof will be completed. \( \Box \)

Like in the proof of Theorem 2.1, we can prove the following theorem.

**Theorem 2.2.** Assume that the matrix equation (1.4) is consistent over symmetric matrix \( X \); then

\[ |C| \leq 2^n |X||A||B|. \]  
(2.4)

Now by using the above theorems, we propose the following results.

**Theorem 2.3.** Suppose that the matrix equation (1.3) is consistent and \( A, B \) are stability matrices; then

\[ \frac{\Pi_{i=1}^n \gamma_i}{2^n \Pi_{i=1}^n \alpha_i \beta_i} \leq \delta_1 \delta_2 \cdots \delta_n, \]  
(2.5)

where \( \sigma(A) = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \), \( \sigma(B) = \{ \beta_1, \beta_2, \ldots, \beta_n \} \), \( \sigma(C) = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \} \) and \( \sigma(X) = \{ \delta_1, \delta_2, \ldots, \delta_n \} \).

**Proof.** From \( A \) and \( B \) are stability matrices, we have

\[ |A| > 0, \quad |B| > 0 \quad \text{for} \quad n = 2k, \quad \text{and} \quad |A| < 0, \quad |B| < 0 \quad \text{for} \quad n = 2k + 1. \]

Therefore

\[ |A||B| > 0 \quad \text{for} \quad n = 1, 2, 3, \ldots. \]  
(2.6)
Now using (2.1) and (2.6), we can get
\[
\frac{|C|}{2^n|A||B|} \leq |X|.
\] (2.7)

It follows that
\[
\frac{\prod_{i=1}^{n} \gamma_i}{2^n \prod_{i=1}^{n} \alpha_i \beta_i} \leq \delta_1 \delta_2 \cdots \delta_n.
\]

Now the proof is completed. \[\Box\]

By a proof similar to that of the previous theorem we can prove the following theorem.

**Theorem 2.4.** Suppose that the matrix equation (1.4) is consistent over symmetric matrix \(X\) and \(A, B\) are stability matrices; then
\[
\frac{\prod_{i=1}^{n} \gamma_i}{2^n \prod_{i=1}^{n} \alpha_i \beta_i} \leq \delta_1 \delta_2 \cdots \delta_n,
\] (2.8)

where \(\sigma(A) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\), \(\sigma(B) = \{\beta_1, \beta_2, \ldots, \beta_n\}\), \(\sigma(C) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\) and \(\sigma(X) = \{\delta_1, \delta_2, \ldots, \delta_n\}\).

3. Conclusions

In this work, we have proposed the inequalities for the determinant and the product of eigenvalues of the solutions to the matrix equations \(A^T XB + B^T X^T A = C\) and \(A^T XB + B^T XA = C\). We have shown that, knowing \(|A|, |B|\) and \(|C|\), the lower bound for the determinant and the product of eigenvalues of the solutions to these matrix equations can be obtained. It would be interesting to develop the inequalities for some other well known matrix equations. We leave this as a topic for further research.

References