J. Math. Anal. Appl. 377 (2011) 161–169

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of blow-up solutions for a non-linear equation with gradient term in $\mathbb{R}^N$

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ARTICLE INFO

Article history:
Received 21 April 2010
Available online 14 October 2010
Submitted by V. Radulescu

Keywords:
Existence
Blow-up
p-Laplacian
Gradient term
Convection term
Radial solution

ABSTRACT

In this paper we study the existence of positive large solutions for the equation $\Delta_p u + \lambda|\nabla u|^{p-1} = \rho(x)f(u)$ in $\mathbb{R}^N$, where f is a non-negative non-decreasing function and ρ is a non-negative continuous function. We show under some hypotheses detailed below the existence of positive solutions which blow up at infinity.

1. Introduction and the main result

There is by now a rich literature on blow-up problems. It is known that the first results on blow-up were obtained by Rademacher and Bieberbach [13,1] for the following problem

$$\begin{cases} \Delta u = \rho(x)f(u) & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } \operatorname{dist}(x, \partial\Omega) \to 0, \end{cases} \tag{1.1}$$

where $\rho = 1$ and f is the exponential function. Later, in [7,12], Keller and Osserman extended the results of [1,13] and proved that

$$\int_1^\infty \frac{1}{\sqrt{F(t)}}\,dt < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds$$

is both necessary and sufficient condition for the existence of blow-up solution. In [4], Ghergu and Rădulescu considered a more general blow-up problem

$$\begin{cases} \Delta u + |\nabla u| = \rho(x)f(u) & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } \operatorname{dist}(x, \partial\Omega) \to 0, \end{cases} \tag{1.2}$$

where f is a non-decreasing function satisfying $f \in C^{0,\nu}[0,\infty)$, $f(0) = 0$, $f > 0$ on $(0,\infty)$ and $\Lambda = \sup_{t\geqslant 1} f(t)/t < \infty$. The authors proved that when Ω is a smooth bounded domain, the problem (1.2) has no solution. When $\Omega = \mathbb{R}^N$, there is a positive solution of (1.2) if and only if

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.008

$$\int_1^\infty e^{-t}t^{1-N}\left(\int_0^t e^s s^{N-1}\min_{|x|=s}\rho(x)\,ds\right)dt=+\infty.$$

Let us announce that several authors have studied extensively the semi-linear case and given various sufficient conditions for existence of blow-up solution under some assumptions on f and ρ. See [3,7,8,12].

Motivated by paper [4], we consider

$$\begin{cases}\Delta_p u+\lambda|\nabla u|^{p-1}=\rho(x)f(u) & \text{for } x\in\Omega,\\ u(x)\to+\infty & \text{as } \operatorname{dist}(x,\partial\Omega)\to 0,\end{cases}\tag{1.3}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p>2$. When $\lambda=0$, the problem (1.3) was investigated by many authors (see [5,10,11]). When $\lambda\neq 0$, under some conditions related to the functions ρ and f, the boundary blow-up problem (1.3) has no positive solution (see [6]). In the present work, we study the problem (1.3) with $\Omega=\mathbb{R}^N$. Namely, we are mainly concerned with existence of solutions $u\in\mathcal{C}^{1,\nu}_{loc}(\mathbb{R}^N)$, with $0<\nu<1$, of the problem

$$\begin{cases}\Delta_p u+\lambda|\nabla u|^{p-1}=\rho(x)f(u) & \text{in } \mathcal{D}'(\mathbb{R}^N),\\ u(x)\to+\infty & \text{as } |x|\to\infty.\end{cases}\tag{1.4}$$

Throughout this paper we will always assume that ρ is non-negative continuous function such that $\alpha:=\inf_{x\in\mathbb{R}^N}\rho(x)>0$ and $\lambda\in\mathbb{R}\backslash\{0\}$. The function f satisfies the following hypotheses.

(H_1) $f\in\mathcal{C}^1[0,\infty)$, $f'\geqslant 0$, $f(0)=0$, $f>0$ on $(0,\infty)$.
(H_2) $\sup_{s>0}\frac{f'(s)}{s^{q-1}}<\infty$, where $1<q<p-1$.
(H_3) $\inf_{s\geqslant 0}(f(s+t)-f(s))>0$ for all $t>0$.

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that* (H_1)–(H_3) *hold. Then problem* (1.4) *has a positive solution if and only if*

$$\int_1^\infty\left(e^{-\lambda t}t^{1-N}\int_0^t e^{\lambda s}s^{N-1}\phi(s)\,ds\right)^{\frac{1}{p-1}}dt=+\infty,\tag{1.5}$$

where $\phi(r):=\inf_{|x|=r}\rho(x)$.

2. Auxiliary results and proof of Theorem 1.1

We need some auxiliary results. We start with the following lemma.

Lemma 2.1. *Suppose that* (H_1)–(H_2) *hold. Then the equation*

$$\Delta_p w+\lambda|\nabla w|^{p-1}=\phi(|x|)f(w)\quad \text{in } \mathbb{R}^N\tag{2.1}$$

has a positive radial solution $w(|x|)$. *If in addition,* (1.5) *is satisfied then* $w(|x|)\to\infty$ *as* $|x|\to\infty$.

Proof. To prove this result, we introduce the following radial problem

$$\begin{cases}\left(|w'|^{p-2}w'\right)'+\dfrac{N-1}{r}|w'|^{p-2}w'+\lambda|w'|^{p-1}=\phi(r)f(w),\\ w(0)=a,\quad w'(0)=0,\end{cases}\tag{2.2}$$

where $a>0$. Firstly, we prove the existence of positive large solution of (2.2). This will be done in two steps.

Step 1. Local existence. The proof is based on the fixed point theorem. By integrating (2.2), we obtain

$$w(r)=a+\int_0^r\mathcal{A}\big(F(w(s))\big)\,ds,\quad r\geqslant 0,$$

where $\mathcal{A}(s)=|s|^{\frac{2-p}{p-1}}s$ and $F(w(s))=s^{1-N}\int_0^s t^{N-1}[-\lambda|w'(t)|^{p-1}+\phi(t)f(w(t))]\,dt$.

Consider the following space

$$E_a=\left\{\varphi\in\mathcal{C}^1\big([0,r_a],\mathbb{R}\big)/\|\varphi\|_a\leqslant c\right\},$$

where c and r_a are positive constants, which will be determined later and

$$\|\varphi\|_a = \max\big(\|\varphi - a\|_\infty, \|\varphi'\|_\infty\big).$$

We define the mapping T on E_a as follows

$$T(\varphi)(r) = a + \int_0^r \mathcal{A}\big(F\big(\varphi(s)\big)\big)\,ds.$$

For $c < a$, we denote

$$\gamma \equiv \gamma(c,a) := \sup_{t\in[a-c,a+c]} f'(t).$$

Thus for all $\varphi \in E_a$ and $s \in (0, r_a]$, we have

$$\big|f\big(\varphi(s)\big) - f(a)\big| \leqslant \gamma\big|\varphi(s) - a\big| \leqslant \gamma c. \tag{2.3}$$

Therefore $\alpha(f(a) - c\gamma) \leqslant \phi(s) f(\varphi(s))$. Hence

$$\frac{\alpha(f(a) - \gamma c)}{N} s \leqslant s^{1-N} \int_0^s t^{N-1}\phi(t) f\big(\varphi(t)\big)\,dt,$$

that is

$$F\big(\varphi(s)\big) \geqslant \begin{cases} \frac{\alpha(f(a)-\gamma c)-\lambda c^{p-1}}{N} s & \text{if } \lambda > 0, \\ \frac{\alpha(f(a)-\gamma c)}{N} s & \text{if } \lambda < 0. \end{cases} \tag{2.4}$$

By (H_2), there exists $M > 0$ such that

$$\begin{aligned} \gamma &\leqslant M \sup_{t\in[a-c,a+c]} t^{q-1} \\ &\leqslant M(a+c)^{q-1} \\ &\leqslant M(2a)^{q-1}. \end{aligned} \tag{2.5}$$

Choose c such that

$$c < \begin{cases} \inf\big(1, a, \frac{\alpha f(a)}{2(M\alpha(2a)^{q-1}+\lambda)}\big) & \text{if } \lambda > 0, \\ \inf\big(1, a, \frac{\alpha f(a)}{2M\alpha(2a)^{q-1}}\big) & \text{if } \lambda < 0. \end{cases}$$

In the case $\lambda > 0$, it follows from (2.5) that $c(\alpha\gamma + \lambda c^{q-1}) \leqslant \frac{\alpha f(a)}{2}$. Therefore

$$\frac{\alpha f(a)}{2} + \lambda\big(c^q - c^{p-1}\big) \leqslant \alpha\big(f(a) - c\gamma\big) - \lambda c^{p-1}.$$

Since $q < p - 1$,

$$\frac{\alpha f(a)}{2N} \leqslant \frac{\alpha(f(a) - c\gamma) - \lambda c^{p-1}}{N}.$$

Also in the case $\lambda < 0$, it is clear that

$$\frac{\alpha f(a)}{2N} \leqslant \frac{\alpha(f(a) - c\gamma)}{N}.$$

According to (2.4), we obtain

$$0 < \Lambda s \leqslant F\big(\varphi(s)\big), \quad \text{for all } 0 < s \leqslant r_a, \tag{2.6}$$

where $\Lambda = \frac{\alpha f(a)}{2N}$.

Claim 1. T maps E_a into itself. Indeed, let $\varphi \in E_a$ and $r \in [0, r_a]$. First, it is easy to see that $T(\varphi) \in \mathcal{C}^1([0, r_a], \mathbb{R})$. On the other hand, we have

$$\begin{aligned}\left|T(\varphi)(r)-a\right| &\leqslant \int_0^r \left|\mathcal{A}\big(F\big(\varphi(s)\big)\big)\right| ds\\ &\leqslant \int_0^r \big(F\big(\varphi(s)\big)\big)^{\frac{2-p}{p-1}} F\big(\varphi(s)\big)\, ds.\end{aligned}$$

Since the function $s \mapsto s^{\frac{2-p}{p-1}}$ is non-increasing in $(0,\infty)$, it follows from (2.6) that

$$\left|T(\varphi)(r)-a\right| \leqslant \int_0^r (\Lambda s)^{\frac{2-p}{p-1}} F\big(\varphi(s)\big)\, ds. \tag{2.7}$$

On account of (2.3), we have

$$f\big(\varphi(s)\big) \leqslant f(a)+c\gamma.$$

Choosing $r_a \leqslant 1$, we get

$$F\big(\varphi(s)\big) \leqslant \frac{|\lambda|c^{p-1}+\beta_1(f(a)+c\gamma)}{N} s,$$

where $\beta_1 = \sup_{t\in[0,1]} \phi(t)$. This and the inequality (2.7) imply that

$$\begin{aligned}\left|T(\varphi)(r)-a\right| &\leqslant \frac{|\lambda|c^{p-1}+\beta_1(f(a)+c\gamma)}{N} \Lambda^{\frac{2-p}{p-1}} \int_0^r s^{\frac{1}{p-1}}\, ds\\ &\leqslant \frac{|\lambda|c^{p-1}+\beta_1(p-1)(f(a)+c\gamma)}{Np} \Lambda^{\frac{2-p}{p-1}} r_a^{\frac{p}{p-1}}.\end{aligned}$$

By choosing

$$r_a \leqslant r_1 := \left[\frac{Npc}{|\lambda|c^{p-1}+\beta_1(p-1)(f(a)+c\gamma)} \Lambda^{\frac{p-2}{p-1}}\right]^{\frac{p-1}{p}}, \tag{2.8}$$

we obtain

$$\left|T(\varphi)(r)-a\right| \leqslant c, \quad \text{for all } r\in[0,r_a]. \tag{2.9}$$

In just the same way, we arrive at

$$\left|T(\varphi)'(r)\right| \leqslant \frac{|\lambda|c^{p-1}+\beta_1(f(a)+c\gamma)}{N} \Lambda^{\frac{2-p}{p-1}} r_a^{\frac{1}{p-1}}.$$

So, choose

$$r_a \leqslant r_2 := \left[\frac{Nc}{|\lambda|c^{p-1}+\beta_1(f(a)+c\gamma)} \Lambda^{\frac{p-2}{p-1}}\right]^{p-1}. \tag{2.10}$$

Therefore

$$\left|T(\varphi)'(r)\right| \leqslant c, \quad \text{for all } r\in[0,r_a].$$

From this last inequality and (2.9), we deduce that $T(\varphi)\in E_a$ and the claim follows.

Claim 2. T is a contraction. In fact, let $\varphi,\psi\in E_a$ and $r\in[0,r_a]$. Then

$$\left|T(\psi)(r)-T(\varphi)(r)\right| \leqslant \int_0^r \left|\mathcal{A}\big(F\big(\psi(s)\big)\big)-\mathcal{A}\big(F\big(\varphi(s)\big)\big)\right| ds.$$

Set $G(s)=\min(F(\varphi(s)),F(\psi(s)))$. Then

$$0<\Lambda s \leqslant G(s), \quad \text{for all } 0<s\leqslant r_a.$$

It is easy to see that

$$\left|\mathcal{A}\big(F\big(\psi(s)\big)\big)-\mathcal{A}\big(F\big(\varphi(s)\big)\big)\right|\leqslant G(s)^{\frac{2-p}{p-1}}\left|F\big(\psi(s)\big)-F\big(\varphi(s)\big)\right|. \tag{2.11}$$

Also by a simple calculation, we get

$$\left|\big(\psi'(s)\big)^{p-1}-\big(\varphi'(s)\big)^{p-1}\right|\leqslant(p-1)c^{p-2}\left\|\psi'-\varphi'\right\|_\infty$$

and

$$\phi(s)\left|f\big(\psi(s)\big)-f\big(\varphi(s)\big)\right|\leqslant\gamma\beta_1\|\psi-\varphi\|_\infty.$$

Therefore

$$\left|F\big(\psi(s)\big)-F\big(\varphi(s)\big)\right|\leqslant\frac{|\lambda|(p-1)c^{p-2}\|\psi'-\varphi'\|_\infty+\gamma\beta_1\|\psi-\varphi\|_\infty}{N}s.$$

Combining this last inequality with (2.11), we obtain

$$\begin{aligned}\left|T(\psi)(r)-T(\varphi)(r)\right|&\leqslant\frac{|\lambda|(p-1)c^{p-2}+\gamma\beta_1}{N}\|\psi-\varphi\|_a\int_0^r G(s)^{\frac{2-p}{p-1}}s\,ds\\&\leqslant\frac{|\lambda|(p-1)c^{p-2}+\gamma\beta_1}{N}\Lambda^{\frac{2-p}{p-1}}\|\psi-\varphi\|_a\int_0^r s^{\frac{1}{p-1}}\,ds\\&\leqslant\frac{(p-1)(|\lambda|(p-1)c^{p-2}+\gamma\beta_1)}{Np}\Lambda^{\frac{2-p}{p-1}}r_a^{\frac{p}{p-1}}\|\psi-\varphi\|_a.\end{aligned}$$

By choosing

$$r_a\leqslant r_3:=\left[\frac{Np}{2(p-1)(|\lambda|(p-1)c^{p-2}+\gamma\beta_1)}\Lambda^{\frac{p-2}{p-1}}\right]^{\frac{p-1}{p}}, \tag{2.12}$$

we get

$$\left|T(\psi)(r)-T(\varphi)(r)\right|\leqslant\frac{1}{2}\|\psi-\varphi\|_a. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}\left|T(\psi)'(r)-T(\varphi)'(r)\right|&\leqslant\left|\mathcal{A}\big(F\big(\psi(r)\big)\big)-\mathcal{A}\big(F\big(\varphi(r)\big)\big)\right|\\&\leqslant G(r)^{\frac{2-p}{p-1}}\left|F\big(\psi(r)\big)-F\big(\varphi(r)\big)\right|\\&\leqslant\frac{|\lambda|(p-1)c^{p-2}+\gamma\beta_1}{N}\|\psi-\varphi\|_aG(r)^{\frac{2-p}{p-1}}r\\&\leqslant\frac{|\lambda|(p-1)c^{p-2}+\gamma\beta_1}{N}\Lambda^{\frac{2-p}{p-1}}r_a^{\frac{1}{p-1}}\|\psi-\varphi\|_a.\end{aligned}$$

Choose

$$r_a\leqslant r_4:=\left[\frac{N}{2(|\lambda|(p-1)c^{p-2}+\gamma\beta_1)}\Lambda^{\frac{p-2}{p-1}}\right]^{p-1}. \tag{2.14}$$

Therefore

$$\left|T(\psi)'(r)-T(\varphi)'(r)\right|\leqslant\frac{1}{2}\|\psi-\varphi\|_a. \tag{2.15}$$

Combining (2.13) with (2.15), we get

$$\left\|T(\psi)-T(\varphi)\right\|_a\leqslant\frac{1}{2}\|\psi-\varphi\|_a.$$

Finally, we choose $r_a\leqslant\inf(1,r_1,r_2,r_3,r_4)$. Consequently, T is a contraction. According to the Banach contraction theorem, the existence of a unique solution of problem (2.2) in $[0,r_a]$ follows.

Step 2. Global existence. Let $w \equiv w(.,a)$ be the maximal solution of (2.2) defined on $[0, r_{max})$. First, note that by continuity of w, there exists $r_0 > 0$ such that $w(r) \geqslant a/2$ for $r \in [0, r_0)$. So, using the fact that f is non-decreasing, we get

$$\phi(r) f\big(w(r)\big) \geqslant \alpha f(a/2) > 0, \quad \forall r \in [0, r_0).$$

Since $w'(0) = 0$, we can find $0 < r_0' \leqslant r_0$ such that

$$-\lambda \big|w'(r)\big|^{p-1} + \phi(r) f\big(w(r)\big) > 0, \quad \forall r \in [0, r_0').$$

Hence, integrating (2.2), we obtain

$$\begin{aligned} r^{N-1}\big|w'\big|^{p-2}w'(r) &= \int_0^r t^{N-1}\big[-\lambda\big|w'(t)\big|^{p-1} + \phi(r) f\big(w(t)\big)\big]\,dt \\ &> 0, \quad \forall r \in \big(0, r_0'\big), \end{aligned}$$

which implies $w' > 0$ in $(0, r_0')$. Particularly, by using the fact that $w'(0) = 0$, we deduce that w is convex in $[0, r_0'')$, $r_0'' < r_0'$. Next, we have $w' \geqslant 0$ in $[0, r_{max})$. In fact, suppose by contradiction that w changes the monotonicity, then there is some $b > r_0''$ such that $w'(b) = 0$ and $(|w'|^{p-2}w')'(b) \leqslant 0$. It follows from (2.2) that $(|w'|^{p-2}w')'(b) = \phi(b) f(w(b)) > 0$, which is impossible and the desired result follows. Finally, suppose again by contradiction that $r_{max} < \infty$. It is clear that $w(r) \to \infty$ as $r \to r_{max}$. Recal that $w' \geqslant 0$ in $[0, r_{max})$. Thus, (2.2) gives

$$\big(e^{\lambda r} r^{N-1}\big(w'\big)^{p-1}\big)' = e^{\lambda r} r^{N-1}\phi(r) f(w).$$

Integrating this equality, we get

$$w(r) = a + \int_0^r \left(e^{-\lambda t} t^{1-N} \int_0^t e^{\lambda s} s^{N-1}\phi(s) f\big(w(s)\big)\,ds\right)^{\frac{1}{p-1}} dt, \quad r \geqslant 0. \tag{2.16}$$

In view of (H_2) and according to w is non-decreasing in $[0, r_{max})$, we have

$$\begin{aligned} w(r) &\leqslant a + C\big[w(r)\big]^{\frac{q}{p-1}} \int_0^r \left(e^{-\lambda t} t^{1-N} \int_0^t e^{\lambda s} s^{N-1}\phi(s)\,ds\right)^{\frac{1}{p-1}} dt \\ &\leqslant a + C\big[w(r)\big]^{\frac{q}{p-1}} \int_0^r \left(e^{-\lambda t} \int_0^t e^{\lambda s}\phi(s)\,ds\right)^{\frac{1}{p-1}} dt, \end{aligned}$$

where $C > 0$. Using the fact that $q < p - 1$ and letting r go to r_{max}, we obtain a contradiction. Consequently $r_{max} = \infty$.

Now, we claim that $\lim_{r\to\infty} w(r) = \infty$. In fact, since $f(w) \geqslant f(a) > 0$, it follows from (2.16) that

$$w(r) \geqslant a + f(a)^{\frac{1}{p-1}} \int_0^r \left(e^{-\lambda t} t^{1-N} \int_0^t e^{\lambda s} s^{N-1}\phi(s)\,ds\right)^{\frac{1}{p-1}} dt, \quad r \geqslant 0.$$

By (1.5), the right side of the last inequality goes to infinity as $r \to \infty$ and therefore $\lim_{r\to\infty} w(r) = \infty$. Consequently $w(|x|)$ is a positive large solution of (2.1). The proof of lemma is now complete. □

We shall use the following weak maximum principle. Its proof is presented in [6].

Theorem 2.1 *(Weak maximum principle). Suppose that* (H_3) *holds. Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $u, v \in W^{1,p}(\Omega)$ *satisfy the following inequality*

$$-\Delta_p u - \lambda|\nabla u|^{p-1} + \rho(x) f(u) \leqslant -\Delta_p v - \lambda|\nabla v|^{p-1} + \rho(x) f(v) \quad \text{in } W^{-1,p'}(\Omega). \tag{2.17}$$

If $|\nabla u|, |\nabla v| \in L^\infty_{loc}(\Omega)$, *then the inequality* $u \leqslant v$ *on* $\partial\Omega$ *implies* $u \leqslant v$ *in* Ω.

The other result that we need is an interior regularity for weak solutions. It is due to DiBenedetto and Tolksdorf [2,14].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose $h(x,t,\eta)$ is a measurable in $x \in \Omega$ and continuous in t and η such that $|h(x,t,\eta)| \leqslant \Gamma(1+|\eta|)^p$ on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of $\Delta_p u = h(x,u,\nabla u)$. Given a sub-domain $\mathcal{O} \subset\subset \Omega$, there is a $\nu > 0$ and a constant C depending on N, p, Γ, $\|u\|_\infty$ and $\mathcal{O}$ such that*

$$\left|\nabla u(x)\right| \leqslant C \quad \text{and} \quad \left|\nabla u(x) - \nabla u(y)\right| \leqslant C|x-y|^\nu, \quad x, y \in \mathcal{O}. \tag{2.18}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $u \in W^{1,p}(\Omega) \cap L^\infty_{loc}(\Omega)$ be a solution of*

$$-\Delta_p u - \lambda|\nabla u|^{p-1} + \rho(x) f(u) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then $|\nabla u| \in L^\infty_{loc}(\Omega)$.

Proof. Let $\mathcal{O}$ be a subset compact and Ω' be a sub-domain of Ω such that $\mathcal{O} \subset \Omega' \subset\subset \Omega$ and define

$$h_M(x,t,\eta) = \begin{cases} -\lambda|\eta|^{p-1} + \rho(x) f(t) & \text{if } t \leqslant M, \\ -\lambda|\eta|^{p-1} + \rho(x) f(M) & \text{if } t > M, \end{cases}$$

where $\|u\|_{\infty,\Omega'} \leqslant M$.

Then, for $x \in \Omega'$,

$$\begin{aligned} \left|h_M(x,t,\eta)\right| &\leqslant |\lambda||\eta|^{p-1} + \rho(x) f(M) \\ &\leqslant \left(|\lambda| + \|\rho\|_{\infty,\Omega'} f(M)\right)\left(1+|\eta|^{p-1}\right) \\ &\leqslant \Gamma\left(1+|\eta|\right)^p. \end{aligned}$$

It is clear that u is a weak solution of $\Delta_p u = h_M(x,u,\nabla u)$ in Ω'. By Theorem 2.2, it follows that $|\nabla u(x)| \leqslant C$, $\forall x \in \mathcal{O}$. Consequently $|\nabla u| \in L^\infty_{loc}(\Omega)$ and the proof of lemma is complete. □

Lemma 2.3. *Suppose that the hypotheses of Theorem 1.1 hold. Then for each $k = 1, 2, \ldots,$ the problem*

$$\left(P^k\right) \quad \begin{cases} L(u) := -\Delta_p u - \lambda|\nabla u|^{p-1} + \rho(x) f(u) = 0 & \text{in } \mathcal{D}'\left(\mathbb{R}^N\right), \\ u(x) \to w(k) & \text{as } |x| \to \infty, \end{cases}$$

admits a positive solution u^k.

Proof. First, let us introduce the following problem

$$\left(P^k_n\right) \quad \begin{cases} L(u) := -\Delta_p u - \lambda|\nabla u|^{p-1} + \rho(x) f(u) = 0 & \text{for } x \in B(0,n), \\ u = w(k) & \text{on } \partial B(0,n), \end{cases}$$

where $n \geqslant k$.

After the transformation $u = v + w(k)$, (P^k_n) becomes

$$\left(P^k_n\right)' \quad \begin{cases} L^k(v) := -\Delta_p v - \lambda|\nabla v|^{p-1} + \rho(x) f\left(v + w(k)\right) = 0 & \text{for } x \in B(0,n), \\ v = 0 & \text{on } \partial B(0,n), \end{cases}$$

therefore

$$\begin{aligned} L^k\left(w\left(|x|\right) - w(k)\right) &= -\Delta_p w\left(|x|\right) - \lambda\left|\nabla w\left(|x|\right)\right|^{p-1} + \rho(x) f\left(w\left(|x|\right)\right) \\ &\geqslant -\Delta_p w\left(|x|\right) - \lambda\left|\nabla w\left(|x|\right)\right|^{p-1} + \phi\left(|x|\right) f\left(w\left(|x|\right)\right). \end{aligned}$$

By using the fact that $w(|x|)$ is solution of (2.1), we get

$$L^k\left(w\left(|x|\right) - w(k)\right) \geqslant 0.$$

Furthermore, $w(|x|) = w(n) \geqslant w(k)$ on $\partial B(0,n)$ and $L^k(-w(k)) = 0$. On the other hand, set

$$h_k(x,s,\eta) = -\lambda|\eta|^{p-1} + \rho(x) f\left(s + w(k)\right),$$

thus

$$\begin{aligned} \left|h_k(x,s,\eta)\right| &\leqslant |\lambda||\eta|^{p-1} + \rho(x) f\left(s + w(k)\right) \\ &\leqslant |\lambda||\eta|^{p-1} + \rho(x) f\left(w\left(|x|\right)\right), \end{aligned}$$

for $-w(k) \leqslant s \leqslant w(|x|) - w(k)$. It is clear that

$$p - 1 < \frac{p}{(p^\star)'} \quad \text{and} \quad \rho(x) f\big(w\big(|x|\big)\big) \in L^\infty\big(B(0,n)\big).$$

Then, we apply Theorem 2.2 in [9] to $(P_k^n)'$ taking $-w(k) \leqslant w(|x|) - w(k)$ as the ordered pair of sub- and super-solution. There exists a solution between $-w(k)$ and $w(|x|) - w(k)$. So, problem (P_n^k) admits a weak solution denoted by u_n^k such that

$$0 \leqslant u_n^k(x) \leqslant w\big(|x|\big), \quad x \in B(0,n). \tag{2.19}$$

According to Lemma 2.2, $|\nabla u_n^k| \in L^\infty_{loc}(B(0,n))$. Since $L(u_n^k) = 0 \leqslant L(w(k))$ in $W^{-1,p'}(B(0,n))$ and $u_n^k = w(k)$ on $\partial B(0,n)$, the maximum principle implies

$$u_n^k(x) \leqslant w(k), \quad x \in B(0,n), \ \text{for all } n \geqslant k. \tag{2.20}$$

This implies $u_{n+1}^k \leqslant w(k) = u_n^k$ on $\partial B(0,n)$. Since $L(u_n^k) = L(u_{n+1}^k) = 0$ in $W^{-1,p'}(B(0,n))$, applying again the maximum principle, we obtain

$$u_{n+1}^k \leqslant u_n^k \quad \text{in } B(0,n), \ \text{for all } n \geqslant k.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Choose $n_0 \geqslant k$ such that $\operatorname{supp}\varphi := \mathcal{K} \subset B(0,n_0/2)$. Then, the sequence $\{u_n^k\}_{n=n_0}^\infty$ is non-increasing and bounded below by 0 and hence converges in $B(0,n_0)$. The remainder of the proof is similar to the proof of Lemma 2.1 in [10]. In view of (2.20), $u_n(x) \leqslant w(k)$, $x \in B(0,n_0)$, for all $n \geqslant n_0$. So, by proceeding as in the proof of Lemma 2.2 with the aid of Theorem 2.2, there is a $\nu > 0$ and $C > 0$ such that for every $n \geqslant n_0$,

$$\big|\nabla u_n^k(x)\big| \leqslant C \quad \text{and} \quad \big|\nabla u_n^k(x) - \nabla u_n^k(y)\big| \leqslant C|x-y|^\nu, \quad x, y \in B(0,n_0/2). \tag{2.21}$$

Therefore the sequences $\{u_n^k\}_{n=n_0}^\infty$ and $\{\nabla u_n^k\}_{n=n_0}^\infty$ are equicontinuous in $B(0,n_0/2)$ and hence, there is a subsequence still denoted by u_n^k such that $u_n^k \to u^k$ and $\nabla u_n^k \to v^k$ uniformly on compact subsets of $B(0,n_0/2)$ for some $u^k \in \mathcal{C}(B(0,n_0/2))$ and $v^k \in \mathcal{C}(B(0,n_0/2))^N$. So, $v^k = \nabla u^k$ in $B(0,n_0/2)$ and $\nabla u^k \in \mathcal{C}^{0,\nu}(B(0,n_0/2))$. By (2.21),

$$\big|\nabla u_n^k\big|^{p-1} |\nabla\varphi| \leqslant C|\nabla\varphi| \quad \text{in } \mathcal{K}.$$

Since $\eta \mapsto |\eta|^{p-2}\eta$ is continuous, it follows that

$$\big|\nabla u_n^k(x)\big|^{p-2} \nabla u_n^k(x) \nabla\varphi(x) \to \big|\nabla u^k(x)\big|^{p-2} \nabla u^k(x) \nabla\varphi(x), \quad x \in \mathcal{K}.$$

According to dominated convergence theorem, we deduce

$$\int \big|\nabla u_n^k\big|^{p-2} \nabla u_n^k \nabla\varphi \to \int \big|\nabla u^k\big|^{p-2} \nabla u^k \nabla\varphi.$$

In the similar way, we get

$$\int \big|\nabla u_n^k\big|^{p-1} \varphi \to \int \big|\nabla u^k\big|^{p-1} \varphi.$$

On the other hand, we have

$$0 \leqslant f\big(u_{n+1}^k\big) \leqslant f\big(u_n^k\big) \quad \text{and} \quad f\big(u_n^k(x)\big) \to f\big(u_n(x)\big), \quad x \in \mathcal{K},$$

thanks to the monotone convergence theorem, we conclude

$$\int \rho f\big(u_n^k\big)\varphi \to \int \rho f\big(u^k\big)\varphi.$$

Consequently,

$$-\int \big|\nabla u^k\big|^{p-2} \nabla u^k \nabla\varphi + \lambda \int \big|\nabla u^k\big|^{p-1} \varphi = \int \rho f\big(u^k\big)\varphi, \quad \forall \varphi \in \mathcal{C}_0^\infty\big(\mathbb{R}^N\big).$$

Finally, we have $u_n^k = w(k)$ on $\partial B(0,n)$. Thus, it follows that $u_n^k(x) \to w(k)$ as $|x| \to \infty$ and u^k is a positive solution of (P^k). The proof of lemma is now complete. □

Proof of Theorem 1.1. `Sufficient condition`. In view of Lemma 2.3, for each $k = 1, 2, \ldots,$

$$\lim_{|x| \to \infty} u^k(x) = w(k).$$

Since $w(k) < w(k+1)$, there exists $R_0 > 0$ such that $u^k(x) \leqslant u^{k+1}(x)$ for $|x| \geqslant R_0$. Thereby,

$$\begin{cases} L(u^k) = L(u^{k+1}) & \text{in } W^{-1,p'}(B(0,R_0)), \\ u^k \leqslant u^{k+1} & \text{on } \partial B(0,R_0). \end{cases}$$

Then, again by the maximum principle, $u^k \leqslant u^{k+1}$ in $B(0,R_0)$. Which implies that $u^k \leqslant u^{k+1}$ in $\mathbb{R}^N$. By (2.19), we deduce $0 \leqslant u^k(x) \leqslant w(|x|)$ for $x \in \mathbb{R}^N$. Then $u^k \to u$ as $k \to \infty$ such that $0 \leqslant u(x) \leqslant w(|x|)$ for $x \in \mathbb{R}^N$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $R > 0$ such that $\operatorname{supp}\varphi := \mathcal{K} \subset B(0,R/2)$. Recalling that w is non-decreasing, thus $u^k(x) \leqslant w(R)$ for $x \in B(0,R)$. So, as in the proof of Lemma 2.3, there is $\nu > 0$ and $C > 0$ such that $u^k \to u \in \mathcal{C}(B(0,R_0/2))$, $\nabla u^k \to \nabla u$ on compact subsets of $B(0,R/2)$ and $|\nabla u^k| \leqslant C$. Moreover $\nabla u \in \mathcal{C}^{0,\nu}(B(0,R/2))$. Similar to the above proof, we obtain

$$-\int |\nabla u|^{p-2}\nabla u \nabla\varphi + \lambda \int |\nabla u|^{p-1}\varphi = \int \rho f(u)\varphi, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

Since $u^k(x) \to w(k)$ as $|x| \to \infty$ and $w(k) \to \infty$ as $k \to \infty$, it follows that $u(x) \to \infty$ as $|x| \to \infty$ and problem (1.1) admits a positive solution $u \in \mathcal{C}^{1,\nu}_{loc}(\mathbb{R}^N)$.

`Necessary condition`. Suppose that

$$\int_1^\infty \left(e^{-\lambda t} t^{1-N} \int_0^t e^{\lambda s} s^{N-1} \phi(s)\, ds \right)^{\frac{1}{p-1}} dt < \infty \tag{2.22}$$

and the problem (1.4) has a positive solution $u \in \mathcal{C}^{1,\nu}_{loc}(\mathbb{R}^N)$. Choose $w(0) = a > u(0)$, with w a solution of (2.2). Then, there is a ball $B(0,R)$ such that

$$w(|x|) > u \quad \text{in } B(0,R). \tag{2.23}$$

In view of (2.16), we have

$$w(r) \leqslant a + C[w(r)]^{\frac{q}{p-1}} \int_0^\infty \left(e^{-\lambda t} t^{1-N} \int_0^t e^{\lambda s} s^{N-1} \phi(s)\, ds \right)^{\frac{1}{p-1}} dt.$$

Using the fact that $q < p-1$, we deduce w is bounded. On the other hand, $u(x) \to \infty$ as $|x| \to \infty$ implies there exists $A > 0$ such that $u(x) \geqslant \sup_{0 \leqslant r} w(r)$ for $|x| = A$. Thus, $L(u) = L(w(|x|)) = 0$ in $B(0,A)$ and $u(x) \geqslant w(|x|)$ for $|x| = A$. The maximum principle gives $u \geqslant w(|x|)$ in $B(0,A)$. Which is contradictory with (2.23). The proof of Theorem 1.1 is now complete. □

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