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The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups

Andrei Agrachev [a], Ugo Boscain [b], Jean-Paul Gauthier [c], Francesco Rossi [a,*]

[a] *SISSA, via Beirut 2-4, 34014 Trieste, Italy*

[b] *CNRS CMAP, École Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France*

[c] *Laboratoire LSIS, Université de Toulon, France*

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Abstract

We present an invariant definition of the hypoelliptic Laplacian on sub-Riemannian structures with constant growth vector using the Popp's volume form introduced by Montgomery. This definition generalizes the one of the Laplace–Beltrami operator in Riemannian geometry. In the case of left-invariant problems on unimodular Lie groups we prove that it coincides with the usual sum of squares.

We then extend a method (first used by Hulanicki on the Heisenberg group) to compute explicitly the kernel of the hypoelliptic heat equation on any unimodular Lie group of type I. The main tool is the non-commutative Fourier transform. We then study some relevant cases: $SU(2)$, $SO(3)$, $SL(2)$ (with the metrics inherited by the Killing form), and the group $SE(2)$ of rototranslations of the plane.

Keywords: Hypoelliptic Laplacian; Generalized Fourier transform; Heat equation

1. Introduction

The study of the properties of the heat kernel in a sub-Riemannian manifold drew an increasing attention since the pioneer work of Hörmander [28].

* Corresponding author.

E-mail addresses: agrachev@sissa.it (A. Agrachev), boscain@sissa.it (U. Boscain), gauthier@univ-tln.fr (J.-P. Gauthier), rossifr@sissa.it (F. Rossi).

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Since then, many estimates and properties of the kernel in terms of the sub-Riemannian distance have been provided (see [7,8,12,13,20,30,35,42,47] and references therein).

In most cases the hypoelliptic Laplacian appearing in the heat equation is the sum of squares of the vector field forming an orthonormal frame for the sub-Riemannian structure. In other cases it is built as the divergence of the horizontal gradient, where the divergence is defined using any $\mathcal{C}^\infty$ volume form on the manifold (see for instance [46]).

The Laplacians obtained in these ways are not intrinsic in the sense that they do not depend only on the sub-Riemannian distance. Indeed, when the Laplacian is built as the sum of squares, it depends on the choice of the orthonormal frame, while when it is defined as divergence of the horizontal gradient, it depends on the choice of the volume form.

The first question we address in this paper is the definition of an invariant hypoelliptic Laplacian. As far as we know, this question has been raised for the first time in a paper by Brockett [11]. Many details can be found in Montgomery's book [38].

To define the intrinsic hypoelliptic Laplacian, we proceed as in Riemannian geometry. In Riemannian geometry the invariant Laplacian (called the Laplace–Beltrami operator) is defined as the divergence of the gradient where the gradient is obtained via the Riemannian metric and the divergence via the Riemannian volume form.

In sub-Riemannian geometry, we define the invariant hypoelliptic Laplacian as the divergence of the horizontal gradient. The horizontal gradient of a function is the natural generalization of the gradient in Riemannian geometry and it is a vector field belonging to the distribution. The divergence is computed with respect to the sub-Riemannian volume form, that can be defined for every sub-Riemannian structure with constant growth vector. This definition depends only on the sub-Riemannian structure. The sub-Riemannian volume form, called the Popp's measure, was first introduced in Montgomery's book [38], where its relation with the Hausdorff measure is also discussed. The definition of the sub-Riemannian volume form is simple in the 3D contact case, and a bit more delicate in general.

We then prove that for the wide class of unimodular Lie groups (i.e. the groups where the right- and left-Haar measures coincide) the hypoelliptic Laplacian is the sum of squares for any choice of a left-invariant orthonormal base. We recall that all compact and all nilpotent Lie groups are unimodular.

In the second part of the paper, we present a method to compute explicitly the kernel of the hypoelliptic heat equation on a wide class of left-invariant sub-Riemannian structures on Lie groups. We then apply this method to the most important 3D Lie groups: $SU(2)$, $SO(3)$, and $SL(2)$ with the metric defined by the Killing form, the Heisenberg group H_2, and the group of rototranslations of the plane $SE(2)$. These groups are unimodular, hence the hypoelliptic Laplacian is the sum of squares. The interest in studying $SU(2)$, $SO(3)$ and $SL(2)$ comes from some recent results of the authors. Indeed, in [9] the complete description of the cut and conjugate loci for these groups was obtained. These results, together with those presented in this paper, open new perspectives for the clarification of the relation between the presence of the cut locus and the properties of the heat kernel, in line with the result of Neel and Stroock [39] in Riemannian geometry. Up to now the only case in which both the cut locus and the heat kernel has been known explicitly was the Heisenberg group [21,22,29].[1]

[1] The Heisenberg group is in a sense a very degenerate example. For instance, in this case the cut locus coincides globally with the conjugate locus (set of points where geodesics lose local optimality) and many properties that one expects to be distinct for more generic situations cannot be distinguished. The application of our method to the Heisenberg group H_2 provides in a few lines the Gaveau–Hulanicki formula [21,29].

The interest in the hypoelliptic heat kernel on $SE(2)$ comes from a model of human vision. It was recognized in [15,41] that the visual cortex V1 solves a nonisotropic diffusion problem on the group $SE(2)$ while reconstructing a partially hidden or corrupted image. The study of the cut locus on $SE(2)$ is a work in progress. Preliminary results can be found in [37].

The method is based upon the generalized (noncommutative) Fourier transform (GFT, for short), that permits to disintegrate[2] a function from a Lie group G to $\mathbb{R}$ on its components on (the class of) non-equivalent unitary irreducible representations of G. This technique permits to transform the hypoelliptic heat equation into an equation in the dual of the group,[3] that is particularly simple since the GFT disintegrate the right-regular representations and the hypoelliptic Laplacian is built with left-invariant vector fields (to which a one parameter group of right-translations is associated).

Unless we are in the abelian case, the dual of a Lie group in general is not a group. In the compact case it is a so called Tannaka category [25,27] and it is a discrete set. In the nilpotent case it has the structure of $\mathbb{R}^n$ for some n. In the general case it can have a quite complicated structure. However, under certain hypotheses (see Section 3), it is a measure space if endowed with the so called Plancherel measure. Roughly speaking, the GFT is an isometry between $L^2(G, \mathbb{C})$ (the set of complex-valued square integrable functions over G, with respect to the Haar measure) and the set of Hilbert–Schmidt operators with respect to the Plancherel measure.

The difficulties of applying our method in specific cases rely mostly on two points:

(i) Computing the tools for the GFT, i.e. the non-equivalent irreducible representations of the group and the Plancherel measure. This is a difficult problem in general: however, for certain classes of Lie groups there are suitable techniques (for instance the Kirillov orbit method for nilpotent Lie groups [33], or methods for semidirect products). For the groups discussed in this paper, the sets of non-equivalent irreducible representations (and hence the GFT) are well known (see for instance [43]).

(ii) Finding the spectrum of an operator (the GFT of the hypoelliptic Laplacian). Depending on the structure of the group and on its dimension, this problem gives rise to a matrix equation, an ODE or a PDE.

Then one can express the kernel of the hypoelliptic heat equation in terms of eigenfunctions of the GFT of the hypoelliptic Laplacian, or in terms of the kernel of the transformed equation. For the cases treated in this paper, see Table 1 (the symbol $\amalg$ means disjoint union).

The idea of using the GFT to compute the hypoelliptic heat kernel is not new: it was already used on the Heisenberg group in [29] at the same time as the Gaveau formula was published in [21], and on all step 2 nilpotent Lie groups in [5,16]. See also the related work [34].

The structure of the paper is the following: in Section 2 we recall some basic definitions from sub-Riemannian geometry and we construct the sub-Riemannian volume form. We then give the definition of the hypoelliptic Laplacian on a regular sub-Riemannian manifold, and we show that the hypothesis of regularity cannot be dropped in general. For this purpose, we show that the invariant hypoelliptic Laplacian defined on the Martinet sub-Riemannian structure is singular. We then move to left-invariant sub-Riemannian structures on Lie groups and we show that a Lie group is unimodular if and only if the invariant hypoelliptic Laplacian is the sum of squares. We

[2] One could also say decompose (possibly continuously).

[3] In this paper, by the dual of the group, we mean the support of the Plancherel measure on the set of non-equivalent unitary irreducible representations of G; we thus ignore the singular representations.

Table 1

Group	Dual of the group	GFT of the hypoelliptic Laplacian	Eigenfunctions of the GFT of the hypoelliptic Laplacian
H_2	$\mathbb{R}$	$\frac{d^2}{dx^2} - \lambda^2 x^2$ (quantum Harmonic oscillator)	Hermite polynomials
$SU(2)$	$\mathbb{N}$	Linear finite dimensional operator related to the quantum angular momentum	Complex homogeneous polynomials in two variables
$SO(3)$	$\mathbb{N}$	Linear finite dimensional operator related to orbital quantum angular momentum	Spherical harmonics
$SL(2)$	$\mathbb{R}^+ \amalg \mathbb{R}^+ \amalg \mathbb{Z}$	Continuous: Linear operator on analytic functions with domain $\{\lvert x\rvert = 1\} \subset \mathbb{C}$ Discrete: Linear operator on analytic functions with domain $\{\lvert x\rvert < 1\} \subset \mathbb{C}$	Complex monomials
$SE(2)$	$\mathbb{R}^+$	$\frac{d^2}{d\theta^2} - \lambda^2 \cos^2(\theta)$ (Mathieu's equation)	2π-periodic Mathieu functions

also provide an example of a 3D non-unimodular Lie group for which the invariant hypoelliptic Laplacian is not the sum of squares. The section ends with the proof that the invariant hypoelliptic Laplacian can be expressed as

$$\Delta_{sr} = -\sum_{i=1}^{m} L_{X_i}^* L_{X_i},$$

where the formal adjoint $L_{X_i}^*$ is built with the sub-Riemannian volume form, providing a connection with existing literature (see e.g. [31]). The invariant hypoelliptic Laplacian is then the sum of squares when L_{X_i} are skew-adjoint.[4]

In Section 3 we recall basic tools of the GFT and we describe our general method to compute the heat kernel of the hypoelliptic Laplacian on unimodular Lie groups of type I. We provide two useful formulas, one in the case where the GFT of the hypoelliptic Laplacian has discrete spectrum, and the other in the case where the GFT of the hypoelliptic heat equation admits a kernel.

In Section 4 we apply our method to compute the kernel on H_2, $SU(2)$, $SO(3)$, $SL(2)$ and $SE(2)$. For the Heisenberg group we use the formula involving the kernel of the transformed equation (the Mehler kernel). For the other groups we use the formula in terms of eigenvalues and eigenvectors of the GFT of the hypoelliptic Laplacian.

The application of our method to higher dimensional sub-Riemannian problems and in particular to the nilpotent Lie groups $(2,3,4)$ (the Engel group) and $(2,3,5)$ is the subject of a forthcoming paper.

[4] This point of view permits to give an alternative proof of the fact that the invariant hypoelliptic Laplacian on left-invariant structures on unimodular Lie groups is the sum of squares. As a matter of fact, left-invariant vector fields are formally skew-adjoint with respect to the right-Haar measure. On Lie groups the invariant volume form is left-invariant, hence is proportional to the left-Haar measure, and is in turn proportional to the right-Haar measure on unimodular groups.

2. The hypoelliptic Laplacian

In this section we give a definition of the hypoelliptic Laplacian Δ_{sr} on a regular sub-Riemannian manifold M.

2.1. Sub-Riemannian manifolds

We start recalling the definition of sub-Riemannian manifold.

Definition 1. A (n,m)-sub-Riemannian manifold is a triple $(M, \blacktriangle, \mathbf{g})$, where

- M is a connected smooth manifold of dimension n;
- $\blacktriangle$ is a smooth distribution of constant rank $m \leqslant n$ satisfying the *Hörmander condition*, i.e. $\blacktriangle$ is a smooth map that associates to $q \in M$ a m-dim subspace $\blacktriangle(q)$ of T_qM and $\forall q \in M$ we have

$$\operatorname{span}\bigl\{\bigl[X_1, \bigl[\ldots [X_{k-1}, X_k]\ldots\bigr]\bigr](q) \bigm| X_i \in \mathrm{Vec}_H(M)\bigr\} = T_qM \tag{1}$$

  where $\mathrm{Vec}_H(M)$ denotes the set of *horizontal smooth vector fields* on M, i.e.

$$\mathrm{Vec}_H(M) = \bigl\{X \in \mathrm{Vec}(M) \bigm| X(p) \in \blacktriangle(p)\ \forall p \in M\bigr\};$$

- $\mathbf{g}_q$ is a Riemannian metric on $\blacktriangle(q)$, that is smooth as function of q.

When M is an orientable manifold, we say that the sub-Riemannian manifold is orientable.

Remark 2. Usually sub-Riemannian manifolds are defined with $m < n$. In our definition we decided to include the Riemannian case $m = n$, since all our results hold in that case. Notice that if $m = n$ then condition (1) is automatically satisfied.

A Lipschitz continuous curve $\gamma : [0, T] \to M$ is said to be *horizontal* if $\dot\gamma(t) \in \blacktriangle(\gamma(t))$ for almost every $t \in [0, T]$.

Given an horizontal curve $\gamma : [0, T] \to M$, the *length of* γ is

$$l(\gamma) = \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}\bigl(\dot\gamma(t), \dot\gamma(t)\bigr)}\, dt. \tag{2}$$

The *distance* induced by the sub-Riemannian structure on M is the function

$$d(q_0, q_1) = \inf\bigl\{l(\gamma) \bigm| \gamma(0) = q_0,\ \gamma(T) = q_1,\ \gamma \text{ horizontal}\bigr\}. \tag{3}$$

The hypothesis of connectedness of M and the Hörmander condition guarantee the finiteness and the continuity of $d(\cdot,\cdot)$ with respect to the topology of M (Chow's Theorem, see for instance [2]). The function $d(\cdot,\cdot)$ is called the Carnot–Carathéodory distance and gives to M the structure of metric space (see [6,23]).

It is a standard fact that $l(\gamma)$ is invariant under reparameterization of the curve γ. Moreover, if an admissible curve γ minimizes the so-called *energy functional*

$$E(\gamma)=\int_0^T \mathbf{g}_{\gamma(t)}\big(\dot\gamma(t),\dot\gamma(t)\big)\,dt$$

with T fixed (and fixed initial and final point), then $v=\sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}$ is constant and γ is also a minimizer of $l(\cdot)$. On the other side, a minimizer γ of $l(\cdot)$ such that v is constant is a minimizer of $E(\cdot)$ with $T=l(\gamma)/v$.

A *geodesic* for the sub-Riemannian manifold is a curve $\gamma:[0,T]\to M$ such that for every sufficiently small interval $[t_1,t_2]\subset[0,T]$, $\gamma_{|_{[t_1,t_2]}}$ is a minimizer of $E(\cdot)$. A geodesic for which $\mathbf{g}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))$ is (constantly) equal to one is said to be parameterized by arclength.

Locally, the pair $(\blacktriangle,\mathbf{g})$ can be given by assigning a set of m smooth vector fields spanning $\blacktriangle$ and that are orthonormal for $\mathbf{g}$, i.e.

$$\blacktriangle(q)=\operatorname{span}\big\{X_1(q),\dots,X_m(q)\big\},\qquad \mathbf{g}_q\big(X_i(q),X_j(q)\big)=\delta_{ij}. \tag{4}$$

In this case, the set $\{X_1,\dots,X_m\}$ is called a local *orthonormal frame* for the sub-Riemannian structure. When $(\blacktriangle,\mathbf{g})$ can be defined as in (4) by m vector fields defined globally, we say that the sub-Riemannian manifold is *trivializable*.

Given a (n,m)-trivializable sub-Riemannian manifold, the problem of finding a curve minimizing the energy between two fixed points $q_0,q_1\in M$ is naturally formulated as the optimal control problem

$$\begin{aligned}&\dot q(t)=\sum_{i=1}^m u_i(t)X_i\big(q(t)\big),\qquad u_i(.)\in L^\infty\big([0,T],\mathbb{R}\big),\qquad \int_0^T\sum_{i=1}^m u_i^2(t)\,dt\to\min,\\ &q(0)=q_0,\qquad q(T)=q_1.\end{aligned} \tag{5}$$

It is a standard fact that this optimal control problem is equivalent to the minimum time problem with controls $u_1,\dots,u_m$ satisfying $u_1(t)^2+\dots+u_m(t)^2\leqslant 1$ in $[0,T]$.

When the manifold is analytic and the orthonormal frame can be assigned through m analytic vector fields, we say that the sub-Riemannian manifold is *analytic*.

We end this section with the definition of the small flag of the distribution $\blacktriangle$:

Definition 3. Let $\blacktriangle$ be a distribution and define through the recursive formula

$$\blacktriangle_1:=\blacktriangle,\qquad \blacktriangle_{n+1}:=\blacktriangle_n+[\blacktriangle_n,\blacktriangle]$$

where $\blacktriangle_{n+1}(q_0):=\blacktriangle_n(q_0)+[\blacktriangle_n(q_0),\blacktriangle(q_0)]=\{X_1(q_0)+[X_2,X_3](q_0)\mid X_1(q),X_2(q)\in\blacktriangle_n(q),\ X_3(q)\in\blacktriangle(q)\ \forall q\in M\}$. The small flag of $\blacktriangle$ is the sequence

$$\blacktriangle_1\subset\blacktriangle_2\subset\cdots\subset\blacktriangle_n\subset\cdots.$$

A sub-Riemannian manifold is said to be *regular* if for each $n=1,2,\dots$ the dimension of $\blacktriangle_n(q_0)=\{f(q_0)\mid f(q)\in\blacktriangle_n(q)\ \forall q\in M\}$ does not depend on the point $q_0\in M$.

A 3D sub-Riemannian manifold is said to be a *3D contact manifold* if $\blacktriangle$ has dimension 2 and $\blacktriangle_2(q_0) = T_{q_0}M$ for any point $q_0 \in M$.

In this paper we always deal with regular sub-Riemannian manifolds.

2.1.1. Left-invariant sub-Riemannian manifolds

In this section we present a natural sub-Riemannian structure that can be defined on Lie groups. All along the paper, we use the notation for Lie groups of matrices. For general Lie groups, by gv with $g \in G$ and $v \in \mathbf{L}$, we mean $(L_g)_*(v)$ where L_g is the left-translation of the group.

Definition 4. Let G be a Lie group with Lie algebra $\mathbf{L}$ and $\mathbf{p} \subseteq \mathbf{L}$ a subspace of $\mathbf{L}$ satisfying the *Lie bracket generating condition*

$$\operatorname{Lie}\mathbf{p} := \operatorname{span}\bigl\{\bigl[p_1, \bigl[p_2, \dots, [p_{n-1}, p_n]\bigr]\bigr] \bigm| p_i \in \mathbf{p}\bigr\} = \mathbf{L}.$$

Endow $\mathbf{p}$ with a positive definite quadratic form $\langle .,. \rangle$. Define a sub-Riemannian structure on G as follows:

- the distribution is the left-invariant distribution $\blacktriangle(g) := g\mathbf{p}$;
- the quadratic form $\mathbf{g}$ on $\blacktriangle$ is given by $\mathbf{g}_g(v_1, v_2) := \langle g^{-1}v_1, g^{-1}v_2 \rangle$.

In this case we say that $(G, \blacktriangle, \mathbf{g})$ is a left-invariant sub-Riemannian manifold.

Remark 5. Observe that all left-invariant manifolds $(G, \blacktriangle, \mathbf{g})$ are regular.

In the following we define a left-invariant sub-Riemannian manifold choosing a set of m vectors $\{p_1, \dots, p_m\}$ being an orthonormal basis for the subspace $\mathbf{p} \subseteq \mathbf{L}$ with respect to the metric defined in Definition 4, i.e. $\mathbf{p} = \operatorname{span}\{p_1, \dots, p_m\}$ and $\langle p_i, p_j \rangle = \delta_{ij}$. We thus have $\blacktriangle(g) = g\mathbf{p} = \operatorname{span}\{gp_1, \dots, gp_m\}$ and $\mathbf{g}_g(gp_i, gp_j) = \delta_{ij}$. Hence every left-invariant sub-Riemannian manifold is trivializable.

The problem of finding the minimal energy between the identity and a point $g_1 \in G$ in fixed time T becomes the left-invariant optimal control problem

$$\begin{aligned} &\dot g(t) = g(t)\biggl(\sum_i u_i(t)p_i\biggr), \qquad u_i(.) \in L^\infty\bigl([0,T], \mathbb{R}\bigr) \qquad \int_0^T \sum_i u_i^2(t)\,dt \to \min, \\ &g(0) = \mathrm{Id}, \qquad g(T) = g_1. \end{aligned} \tag{6}$$

Remark 6. This problem admits a solution, see for instance Chapter 5 of [10].

2.2. Definition of the hypoelliptic Laplacian on a sub-Riemannian manifold

In this section we define the intrinsic hypoelliptic Laplacian on a regular orientable sub-Riemannian manifold $(M, \blacktriangle, \mathbf{g})$. This definition generalizes the one of the Laplace–Beltrami operator on an orientable Riemannian manifold, that is $\Delta\phi := \operatorname{div}\operatorname{grad}\phi$, where grad is a unique operator from $\mathcal{C}^\infty(M)$ to $\mathrm{Vec}(M)$ satisfying $\mathbf{g}_q(\mathrm{grad}\,\phi(q), v) = d\phi_q(v)\ \forall q \in M,\ v \in T_qM$, and the divergence of a vector field X is a unique function satisfying $\operatorname{div} X\mu = L_X\mu$ where μ is the Riemannian volume form.

We first define the sub-Riemannian gradient of a function, that is an horizontal vector field.

Definition 7. Let $(M, \blacktriangle, \mathbf{g})$ be a sub-Riemannian manifold: the *horizontal gradient* is a unique operator grad_{sr} from $\mathcal{C}^\infty(M)$ to $\mathrm{Vec}_H(M)$ satisfying $\mathbf{g}_q(\mathrm{grad}_{sr}\,\phi(q), v) = d\phi_q(v)\ \forall q \in M$, $v \in \blacktriangle(q)$.

One can easily check that if $\{X_1, \dots X_m\}$ is a local orthonormal frame for $(M, \blacktriangle, \mathbf{g})$, then $\mathrm{grad}_{sr}\,\phi = \sum_{i=1}^m (L_{X_i}\phi)X_i$.

The question of defining a sub-Riemannian volume form is more delicate. We start by considering the 3D contact case.

Proposition 8. *Let $(M, \blacktriangle, \mathbf{g})$ be an orientable* 3D *contact sub-Riemannian structure and $\{X_1, X_2\}$ a local orthonormal frame. Let $X_3 = [X_1, X_2]$ and dX_1, dX_2, dX_3 the dual basis, i.e. $dX_i(X_j) = \delta_{ij}$. Then $\mu_{sr} := dX_1 \wedge dX_2 \wedge dX_3$ is an intrinsic volume form, i.e. it is invariant for a orientation preserving change of orthonormal frame.*

Proof. Consider two different orthonormal frames with the same orientation $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$. We have to prove that $dX_1 \wedge dX_2 \wedge dX_3 = dY_1 \wedge dY_2 \wedge dY_3$ with $X_3 = [X_1, X_2]$, $Y_3 = [Y_1, Y_2]$. We have

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos(f(q)) & \sin(f(q)) \\ -\sin(f(q)) & \cos(f(q)) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

for some real-valued smooth function f. A direct computation shows that

$$Y_3 = X_3 + f_1X_1 + f_2X_2 \tag{7}$$

where f_1 and f_2 are two smooth functions depending on f.

We first prove that $dX_1 \wedge dX_2 = dY_1 \wedge dY_2$. Since the change of variables $\{X_1, X_2\} \mapsto \{Y_1, Y_2\}$ is norm-preserving, we have $dX_1 \wedge dX_2(v, w) = dY_1 \wedge dY_2(v, w)$ when $v, w \in \blacktriangle$. Consider now any vector $v = v_1X_1 + v_2X_2 + v_3X_3 = v_1'Y_1 + v_2'Y_2 + v_3'Y_3$: as a consequence of (7), we have $v_3 = v_3'$. Take another vector $w = w_1X_1 + w_2X_2 + w_3X_3 = w_1'Y_1 + w_2'Y_2 + w_3Y_3$ and compute

$$\begin{aligned} dX_1 \wedge dX_2(v, w) &= dX_1 \wedge dX_2(v - v_3X_3, w - w_3X_3) \\ &= dY_1 \wedge dY_2(v - v_3X_3, w - w_3X_3) \\ &= dY_1 \wedge dY_2(v, w), \end{aligned}$$

because the vectors $v - v_3X_3$, $w - w_3X_3$ are horizontal. Hence the two 2-forms coincide.

From (7) we also have $dY_3 = dX_3 + f_1' dX_1 + f_2' dX_2$ for some smooth functions f_1', f_2'. Hence we have $dY_1 \wedge dY_2 \wedge dY_3 = dX_1 \wedge dX_2 \wedge dY_3 = dX_1 \wedge dX_2 \wedge (dX_3 + f_1' dX_1 + f_2' dX_2) = dX_1 \wedge dX_2 \wedge dX_3$, where the last identity is a consequence of skew-symmetry of differential forms. $\square$

Remark 9. Indeed, even if in the 3D contact case there is no scalar product in $T_q M$, it is possible to define a natural volume form, since on $\blacktriangle$ the scalar product is defined by $\mathbf{g}$ and formula (7) guarantees the existence of a natural scalar product in $(\blacktriangle + [\blacktriangle, \blacktriangle])/\blacktriangle$.

The previous result generalizes to any regular orientable sub-Riemannian structure, as presented below.

2.2.1. Definition of the intrinsic volume form

Let $0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$ be a filtration of an n-dimensional vector space E. Let $e_1, \ldots, e_n$ be a basis of E such that $E_i = \operatorname{span}\{e_1, \ldots, e_{n_i}\}$. Obviously, the wedge product $e_1 \wedge \cdots \wedge e_n$ depends only on the residue classes

$$\bar{e}_j = (e_j + E_{i_j}) \in E_{i_j+1}/E_{i_j},$$

where $n_{i_j} < j \leqslant n_{i_j+1}$, $j = 1, \ldots, n$. This property induces a natural (i.e. independent on the choice of the basis) isomorphism of 1-dimensional spaces:

$$\bigwedge^n E \cong \bigwedge^n \left(\bigoplus_{i=1}^{k} (E_i/E_{i-1}) \right).$$

Now consider the filtration

$$0 \subset \blacktriangle_1(q) \subset \cdots \subset \blacktriangle_k(q) = T_q M, \qquad \dim \blacktriangle_i(q) = n_i.$$

Let $X_1, \ldots, X_i$ be smooth sections of $\blacktriangle = \blacktriangle_1$; then the vector

$$\big(\big[X_1, [\ldots, X_i]\ldots\big](q) + \blacktriangle_{i-1}(q)\big) \in \blacktriangle_i(q)/\blacktriangle_{i-1}(q)$$

depends only on $X_1(q) \otimes \cdots \otimes X_i(q)$.

We thus obtain a well-defined surjective linear map

$$\beta_i\colon \quad \begin{array}{rcl} \blacktriangle(q)^{\otimes i} & \to & \blacktriangle_i(q)/\blacktriangle_{i-1}(q), \\ X_1(q) \otimes \cdots \otimes X_i(q) & \mapsto & \big(\big[X_1, [\ldots, X_i]\ldots\big](q) + \blacktriangle_{i-1}(q)\big). \end{array}$$

The Euclidean structure on $\blacktriangle(q)$ induces an Euclidean structure on $\blacktriangle(q)^{\otimes i}$ by the standard formula:

$$\langle \xi_1 \otimes \cdots \otimes \xi_i, \eta_1 \otimes \cdots \otimes \eta_i \rangle = \langle \xi_1, \eta_1 \rangle \ldots \langle \xi_i, \eta_i \rangle, \quad \xi_j, \eta_j \in \blacktriangle(q),\ j = 1, \ldots, i.$$

Then the formula:

$$|v| = \min\{|\bar{\xi}|: \bar{\xi} \in \beta_i^{-1}(v)\}, \quad v \in \blacktriangle^i(q)/\blacktriangle^{i-1}(q)$$

defines an Euclidean norm on $\blacktriangle^i(q)/\blacktriangle^{i-1}(q)$.

Let ν_i be the volume form on $\blacktriangle^i(q)/\blacktriangle^{i-1}(q)$ associated with the Euclidean structure:

$$\langle \nu_i, v_1 \wedge \cdots \wedge v_{m_i} \rangle = \det^{\frac{1}{m_i}} \{\langle v_j, v_{j'} \rangle\}_{j,j'=1}^{m_i},$$

where $m_i = n_i - n_{i-1} = \dim(\blacktriangle^i(q)/\blacktriangle^{i-1}(q))$.

Finally, the intrinsic volume form μ_{sr} on T_qM is the image of $\nu_1 \wedge \cdots \wedge \nu_k$ under the natural isomorphism

$$\bigwedge^n \left(\bigoplus_{i=1}^k \left(\blacktriangle^i(q)/\blacktriangle^{i-1}(q)\right) \right)^* \cong \bigwedge^n (T_qM)^*.$$

Remark 10. To our knowledge, the construction given above was first presented by Brockett [11, pp. 16–17] in the 2-step case, then by Montgomery [38, Section 10.5] in the general case, where the measure μ_{sr} is called Popp's measure. Montgomery also observed that a sub-Riemannian volume form was the only missing ingredient to get an intrinsic definition of hypoelliptic Laplacian.[5]

Once the volume form is defined, the divergence of a vector field X is defined as in Riemannian geometry, i.e. it is the function $\operatorname{div}_{sr} X$ satisfying $\operatorname{div}_{sr} X \mu_{sr} = L_X \mu_{sr}$. We are now ready to define the intrinsic hypoelliptic Laplacian.

Definition 11. Let $(M, \blacktriangle, \mathbf{g})$ be an orientable regular sub-Riemannian manifold. Then the intrinsic hypoelliptic Laplacian is $\Delta_{sr}\phi := \operatorname{div}_{sr} \operatorname{grad}_{sr} \phi$.

Consider now an orientable regular sub-Riemannian structure $(M, \blacktriangle, \mathbf{g})$ and let $\{X_1, \ldots, X_m\}$ be a local orthonormal frame. We want to find an explicit expression for the operator Δ_{sr}. If $n = m$ then Δ_{sr} is the Laplace–Beltrami operator. Otherwise consider $n - m$ vector fields $X_{m+1}, \ldots, X_n$ such that $\{X_1(q), \ldots, X_m(q), X_{m+1}(q), \ldots, X_n(q)\}$ is a basis of T_qM for all q in a certain open set U. The volume form μ_{sr} is $\mu_{sr} = f(q)\, dX_1 \wedge \cdots \wedge dX_n$, with dX_i dual basis of $X_1, \ldots, X_n$: then we can find other $n - m$ vector fields, that we still call $X_{m+1}, \ldots, X_n$, for which we have $\mu_{sr} = dX_1 \wedge \cdots \wedge dX_n$.

Recall that $\Delta_{sr}\phi$ satisfies $(\Delta_{sr}\phi)\mu_{sr} = L_X \mu_{sr}$ with $X = \operatorname{grad}_{sr} \phi$. We have

[5] Montgomery did not use Popp's measure to get the intrinsic definition of the hypoelliptic Laplacian since there are, a priori, two natural measures on a regular sub-Riemannian manifold: the Popp's measure, and the Hausdorff measure (see [36,40]). However, for left-invariant sub-Riemannian manifolds, both of them are proportional to the left-Haar measure. See Remark 16.

$$L_X\mu_{sr} = \sum_{i=1}^{m}(-1)^{i+1}\big\{d\big(\langle d\phi, X_i\rangle\big) \wedge dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n \\ + \langle d\phi, X_i\rangle d(dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n)\big\}.$$

Applying standard results of differential calculus, we have $d(\langle d\phi, X_i\rangle) \wedge dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n = (-1)^{i+1}L^2_{X_i}\phi\mu_{sr}$ and $d(dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n) = (-1)^{i+1}\operatorname{Tr}(\operatorname{ad} X_i)\mu_{sr}$, where the adjoint map is

$$\operatorname{ad} X_i\colon \quad \begin{aligned} \operatorname{Vec}(U) &\to \operatorname{Vec}(U), \\ X &\mapsto [X_i, X] \end{aligned}$$

and by $\operatorname{Tr}(\operatorname{ad} X_i)$ we mean $\sum_{j=1}^{n} dX_j([X_i, X_j])$. Finally, we find the expression

$$\Delta_{sr}\phi = \sum_{i=1}^{m}\big(L^2_{X_i}\phi + L_{X_i}\phi\operatorname{Tr}(\operatorname{ad} X_i)\big). \tag{8}$$

Notice that the formula depends on the choice of the vector fields $X_{m+1}, \ldots, X_n$.

The hypoellipticity of Δ_{sr} (i.e. given $U \subset M$ and $\phi : U \to \mathbb{C}$ such that $\Delta_{sr}\phi \in \mathcal{C}^\infty$, then ϕ is $\mathcal{C}^\infty$) follows from the Hörmander theorem (see [28]):

Theorem 12. *Let L be a differential operator on a manifold M, that locally in a neighborhood U is written as $L = \sum_{i=1}^{m} L^2_{X_i} + L_{X_0}$, where $X_0, X_1, \ldots, X_m$ are $\mathcal{C}^\infty$ vector fields. If $\operatorname{Lie}_q\{X_0, X_1, \ldots, X_m\} = T_qM$ for all $q \in U$, then L is hypoelliptic.*

Indeed, Δ_{sr} is written locally as $\sum_{i=1}^{m} L^2_{X_i} + L_{X_0}$ with the first-order term $L_{X_0} = \sum_{i=1}^{m}\operatorname{Tr}(\operatorname{ad} X_i)L_{X_i}$. Moreover by hypothesis we have that $\operatorname{Lie}_q\{X_1, \ldots, X_m\} = T_qM$, hence the Hörmander theorem applies.

Remark 13. Notice that in the Riemannian case, i.e. for $m = n$, Δ_{sr} coincides with the Laplace–Beltrami operator.

Remark 14. The hypothesis that the sub-Riemannian manifold is regular is crucial for the construction of the invariant volume form. For instance for the Martinet metric on $\mathbb{R}^3$, that is the sub-Riemannian structure for which $L_1 = \partial_x + \frac{y^2}{2}\partial_z$ and $L_2 = \partial_y$ form an orthonormal base, one gets on $\mathbb{R}^3 \setminus \{y = 0\}$

$$\Delta_{sr} = (L_1)^2 + (L_2)^2 - \frac{1}{y}L_2.$$

This is not surprising at all. As a matter of fact, even the Laplace–Beltrami operator is singular in almost-Riemannian geometry (see [3] and references therein). For instance, for the Grushin metric on $\mathbb{R}^2$, that is the singular Riemannian structure for which $L_1 = \partial_x$ and $L_2 = x\partial_y$ form an orthonormal frame, one gets on $\mathbb{R}^2 \setminus \{x = 0\}$

$$\Delta_{\mathrm{LB}} = (L_1)^2 + (L_2)^2 - \frac{1}{x}L_1.$$

2.3. The hypoelliptic Laplacian on Lie groups

In the case of left-invariant sub-Riemannian manifolds, there is an intrinsic global expression of Δ_{sr}.

Corollary 15. *Let $(G, \blacktriangle, \mathbf{g})$ be a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{p_1, \dots, p_m\} \subset \mathbf{L}$. Then the hypoelliptic Laplacian is*

$$\Delta_{sr}\phi = \sum_{i=1}^{m} \big(L_{X_i}^2 \phi + L_{X_i}\phi \operatorname{Tr}(\operatorname{ad} p_i) \big) \tag{9}$$

where L_{X_i} is the Lie derivative w.r.t. the field $X_i = gp_i$.

Proof. If $m \leqslant n$, we can find $n - m$ vectors $\{p_{m+1}, \dots, p_n\}$ such that $\{p_1, \dots, p_n\}$ is a basis for $\mathbf{L}$. Choose the fields $X_i := gp_i$ and follow the computation given above: we find formula (9). In this case the adjoint map is intrinsically defined and the trace does not depend on the choice of $X_{m+1}, \dots, X_n$. $\square$

The formula above reduces to the sum of squares in the wide class of unimodular Lie groups. We recall that on a Lie group of dimension n, there always exist a left-invariant n-form μ_L and a right-invariant n-form μ_R (called respectively left- and right-Haar measures), that are unique up to a multiplicative constant. These forms have the properties that

$$\int_G f(ag)\mu_L(g) = \int_G f(g)\mu_L(g), \qquad \int_G f(ga)\mu_R(g) = \int_G f(g)\mu_R(g),$$
$$\text{for every } f \in L^1(G, \mathbb{R}) \text{ and } a \in G,$$

where L^1 is intended with respect to the left-Haar measure in the first identity and with respect to the right-Haar measure in the second one. The group is called unimodular if μ_L and μ_R are proportional.

Remark 16. Notice that for left-invariant sub-Riemannian manifolds the intrinsic volume form and the Hausdorff measure μ_H are left-invariant, hence they are proportional to the left-Haar measure μ_L. On unimodular Lie groups one can assume $\mu_{sr} = \mu_L = \mu_R = \alpha\mu_H$, where $\alpha > 0$ is a constant that is unknown even for the simplest among the genuine sub-Riemannian structures, i.e. the Heisenberg group.

Proposition 17. *Under the hypotheses of Corollary* 15*, if G is unimodular then $\Delta_{sr}\phi = \sum_{i=1}^{m} L_{X_i}^2 \phi$.*

Proof. Consider the modular function Ψ, that is a unique function such that

$$\int_G f\big(h^{-1}g\big)\mu_R(g) = \Psi(h) \int_G f(g)\mu_R(g)$$

for all f measurable. It is well known that $\Psi(g)=\det(\mathrm{Ad}_g)$ and that $\Psi(g)\equiv 1$ if and only if G is unimodular.

Consider a curve $\gamma(t)$ such that $\dot\gamma$ exists for $t=t_0$: then $\gamma(t)=g_0e^{(t-t_0)\eta+o(t-t_0)}$ with $g_0=\gamma(t_0)$ and for some $\eta\in\mathbf{L}$. We have

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\det(\mathrm{Ad}_{\gamma(t)}) &= \mathrm{Tr}\left((\mathrm{Ad}_{g_0})^{-1}\left[\frac{d}{ds}_{|s=0}\mathrm{Ad}_{g_0e^{s\eta+o(s)}}\right]\right)\\ &= \mathrm{Tr}(\mathrm{Ad}_{g_0^{-1}}\mathrm{Ad}_{g_0}\mathrm{ad}_\eta)=\mathrm{Tr}(\mathrm{ad}_\eta).\end{aligned}\tag{10}$$

Now choose the curve $\gamma(t)=g_0e^{tp_i}$ and observe that $\det(\mathrm{Ad}_{\gamma(t)})\equiv 1$, then $\mathrm{Tr}(\mathrm{ad}_{p_i})=0$. The conclusion follows from (9). □

All the groups treated in this paper (i.e. H_2, $SU(2)$, $SO(3)$, $SL(2)$ and $SE(2)$) are unimodular. Hence the invariant hypoelliptic Laplacian is the sum of squares. A kind of inverse result holds:

Proposition 18. *Let $(G,\blacktriangle,\mathbf{g})$ be a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{p_1,\dots,p_m\}\subset\mathbf{L}$. If the hypoelliptic Laplacian satisfies $\Delta_{sr}\phi=\sum_{i=1}^m L^2_{X_i}\phi$, then G is unimodular.*

Proof. We start observing that $\Delta_{sr}\phi=\sum_{i=1}^m L^2_{X_i}\phi$ if and only if $\mathrm{Tr}(\mathrm{ad}_{p_i})=0$ for all $i=1,\dots,m$.

Fix $g\in G$: due to Lie bracket generating condition, the control system (6) is controllable, then there exists a choice of piecewise constant controls $u_i:[0,T]\to\mathbb{R}$ such that the corresponding solution $\gamma(.)$ is an horizontal curve steering Id to g. Then $\dot\gamma$ is defined for all $t\in[0,T]$ except for a finite set E of switching times.

Consider now the modular function along γ, i.e. $\Psi(\gamma(t))$, that is a continuous function, differentiable for all $t\in[0,T]\backslash E$. We compute its derivative using (10): we have $\frac{d}{dt}_{|t=t_0}\det(\mathrm{Ad}_{\gamma(t)})=\mathrm{Tr}(\mathrm{ad}_\eta)$ with $\eta=\gamma(t_0)^{-1}\dot\gamma(t_0)$. Due to horizontality of γ, we have $\eta=\sum_{i=1}^m a_ip_i$, hence $\mathrm{Tr}(\mathrm{ad}_\eta)=\sum_{i=1}^m a_i\,\mathrm{Tr}(\mathrm{ad}_{p_i})=0$. Then the modular function is piecewise constant along γ. Recalling that it is continuous, we have that it is constant. Varying along all $g\in G$ and recalling that $\Psi(\mathrm{Id})=1$, we have $\Psi\equiv 1$, hence G is unimodular. □

2.3.1. The hypoelliptic Laplacian on a non-unimodular Lie group

In this section we present a non-unimodular Lie group endowed with a left-invariant sub-Riemannian structure. We then compute the explicit expression of the intrinsic hypoelliptic Laplacian: from Proposition 18 we already know that it is the sum of squares plus a first-order term.

Consider the Lie group

$$A^+(\mathbb{R})\oplus\mathbb{R}:=\left\{\begin{pmatrix}a&0&b\\0&1&c\\0&0&1\end{pmatrix}\;\middle|\;a>0,\ b,c\in\mathbb{R}\right\}.$$

It is the direct sum of the group $A^+(\mathbb{R})$ of affine transformations on the real line $x \mapsto ax + b$ with $a > 0$ and the additive group $(\mathbb{R}, +)$. Indeed, observe that

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ d \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ c+d \\ 1 \end{pmatrix}.$$

The group is non-unimodular, indeed a direct computation gives $\mu_L = \frac{1}{a^2}\, da\, db\, dc$ and $\mu_R = \frac{1}{a}\, da\, db\, dc$.

Its Lie algebra $a(\mathbb{R}) \oplus \mathbb{R}$ is generated by

$$p_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which the following commutation rules hold: $[p_1, p_2] = k$, $[p_2, k] = 0$, $[k, p_1] = -k$.

We define a trivializable sub-Riemannian structure on $A^+(\mathbb{R}) \oplus \mathbb{R}$ as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g) = gp_i$ with $i = 1, 2$ and define

$$\blacktriangle(g) = \operatorname{span}\{X_1(g), X_2(g)\}, \qquad \mathbf{g}_g\big(X_i(g), X_j(g)\big) = \delta_{ij}.$$

Using (9), one gets the following expression for the hypoelliptic Laplacian:

$$\Delta_{sr}\phi = L_{X_1}^2\phi + L_{X_2}^2\phi + L_{X_1}\phi.$$

2.4. The intrinsic hypoelliptic Laplacian in terms of the formal adjoints of the vector fields

In the literature another common definition of hypoelliptic Laplacian can be found (see for instance [31]):

$$\Delta^* = -\sum_{i=1}^{m} L_{X_i}^* L_{X_i}, \tag{11}$$

where $\{X_1, \dots, X_m\}$ is a set of vector fields satisfying the Hörmander condition and the formal adjoint $L_{X_i}^*$ is computed with respect to a given volume form. This expression clearly simplifies to the sum of squares when the vector fields are formally skew-adjoint, i.e. $L_{X_i}^* = -L_{X_i}$.

In this section we show that our definition of intrinsic hypoelliptic Laplacian coincides locally with (11), when $\{X_1, \dots, X_m\}$ is an orthonormal frame for the sub-Riemannian manifold and the formal adjoint of the vector fields are computed with respect to the sub-Riemannian volume form.

We then show that left-invariant vector fields on a Lie group G are formally skew-adjoint with respect to the right-Haar measure, providing an alternative proof of the fact that for unimodular Lie groups the intrinsic hypoelliptic Laplacian is the sum of squares.

Proposition 19. *Locally, the intrinsic hypoelliptic Laplacian Δ_{sr} can be written as $-\sum_{i=1}^m L_{X_i}^* L_{X_i}$, where $\{X_1, \dots, X_m\}$ is a local orthonormal frame, and $L_{X_i}^*$ is the formal adjoint of the Lie derivative L_{X_i} of the vector field X_i, i.e.*

$$\big(\phi_1, L^*_{X_i}\phi_2\big) = (\phi_2, L_{X_i}\phi_1), \quad \textit{for every } \phi_1, \phi_2 \in \mathcal{C}^\infty_c(M, \mathbb{R}),\ i = 1, \dots, m, \tag{12}$$

and the scalar product is the one of $L^2(M, \mathbb{R})$ with respect to the invariant volume form, i.e. $(\phi_1, \phi_2) := \int_M \phi_1\phi_2\mu_{sr}$.

Proof. Given a volume form μ on M, a definition of divergence of a smooth vector field X (equivalent to $L_X\mu = \mathrm{div}(X)\mu$) is

$$\int_M \mathrm{div}(X)\phi\mu = -\int_M L_X\phi\mu, \quad \text{for every } \phi \in \mathcal{C}^\infty_c(M, \mathbb{R});$$

see for instance [45]. We are going to prove that

$$\Delta_{sr}\phi = -\sum_{i=1}^m L^*_{X_i} L_{X_i}\phi, \quad \text{for every } \phi \in \mathcal{C}^\infty_c(M, \mathbb{R}); \tag{13}$$

indeed, multiplying the left-hand side of (13) by $\psi \in \mathcal{C}^\infty_c(M)$ and integrating with respect to μ_{sr} we have,

$$\begin{aligned}\int_M (\Delta_{sr}\phi)\,\psi\,\mu_{sr} &= \int_M \big(\mathrm{div}_{sr}(\mathrm{grad}_{sr}\,\phi)\big)\,\psi\,\mu_{sr} = \int_M \mathrm{div}_{sr}\left(\sum_{i=1}^n (L_{X_i}\phi)X_i\right)\psi\,\mu_{sr} \\ &= -\int_M \sum_{i=1}^n (L_{X_i}\phi)(L_{X_i}\psi)\mu_{sr}.\end{aligned}$$

For the right-hand side we get the same expression. Since ψ is arbitrary, the conclusion follows. Then, by density, one concludes that $\Delta_{sr}\phi = -\sum_{i=1}^m L^*_{X_i} L_{X_i}\phi$, for every $\phi \in \mathcal{C}^2(M, \mathbb{R})$. $\square$

Proposition 20. *Let G be a Lie group and X a left-invariant vector field on G. Then L_X is formally skew-adjoint with respect to the right-Haar measure.*

Proof. Let $\phi \in \mathcal{C}^\infty_c(M, \mathbb{R})$ and $X = gp$ $(p \in \mathbf{L},\ g \in G)$. Since X is left-invariant and μ_R is right-invariant, we have

$$\begin{aligned}\int_G (L_X\phi)(g_0)\,\mu_R(g_0) &= \int_G \frac{d}{dt}\bigg|_{t=0} \phi\big(g_0e^{tp}\big)\mu_R(g_0) = \frac{d}{dt}\bigg|_{t=0} \int_G \phi\big(g_0e^{tp}\big)\mu_R(g_0) \\ &= \frac{d}{dt}\bigg|_{t=0} \int_G \phi(g')\mu_R\big(g'e^{-tp}\big) = \frac{d}{dt}\bigg|_{t=0} \int_G \phi(g')\mu_R(g') = 0.\end{aligned}$$

Hence, for every $\phi_1, \phi_2 \in \mathcal{C}^\infty_c(M, \mathbb{R})$ we have

$$0=\int_G L_X(\phi_1\phi_2)\,\mu_R=\int_G L_X(\phi_1)\,\phi_2\,\mu_R+\int_G \phi_1\,(L_X\phi_2)\,\mu_R=(\phi_2,L_X\phi_1)+(\phi_1,L_X\phi_2)$$

and the conclusion follows. □

For unimodular groups we can assume $\mu_{sr}=\mu_L=\mu_R$ (cf. Remark 16) and left-invariant vector fields are formally skew-adjoint with respect to μ_{sr}. This argument provides an alternative proof of the fact that on unimodular Lie groups the hypoelliptic Laplacian is the sum of squares.

3. The generalized Fourier transform on unimodular Lie groups

Let $f\in L^1(\mathbb{R},\mathbb{R})$: its Fourier transform is defined by the formula

$$\hat{f}(\lambda)=\int_{\mathbb{R}} f(x)e^{-ix\lambda}\,dx.$$

If $f\in L^1(\mathbb{R},\mathbb{R})\cap L^2(\mathbb{R},\mathbb{R})$ then $\hat{f}\in L^2(\mathbb{R},\mathbb{R})$ and one has

$$\int_{\mathbb{R}}\left|f(x)\right|^2dx=\int_{\mathbb{R}}\left|\hat{f}(\lambda)\right|^2\frac{d\lambda}{2\pi},$$

called Parseval or Plancherel equation. By density of $L^1(\mathbb{R},\mathbb{R})\cap L^2(\mathbb{R},\mathbb{R})$ in $L^2(\mathbb{R},\mathbb{R})$, this equation expresses the fact that the Fourier transform is an isometry between $L^2(\mathbb{R},\mathbb{R})$ and itself. Moreover, the following inversion formula holds:

$$f(x)=\int_{\mathbb{R}}\hat{f}(\lambda)e^{ix\lambda}\frac{d\lambda}{2\pi},$$

where the equality is intended in the L^2 sense. It has been known from more than 50 years that the Fourier transform generalizes to a wide class of locally compact groups (see for instance [14,19,25,26,32,44]). Next we briefly present this generalization for groups satisfying the following hypothesis:

(H_0) G is a unimodular Lie group of type I.

For the definition of groups of type I see [18]. For our purposes it is sufficient to recall that all groups treated in this paper (i.e. H_2, $SU(2)$, $SO(3)$, $SL(2)$ and $SE(2)$) are of type I. Actually, *both the real connected semisimple and the real connected nilpotent Lie groups are of type* I [17,24] and even though not all solvable groups are of type I, this is the case for the group of the rototranslations of the plane $SE(2)$ [43]. In the following, the L^p spaces $L^p(G,\mathbb{C})$ are intended with respect to the Haar measure $\mu:=\mu_L=\mu_R$.

Let G be a Lie group satisfying (H_0) and $\hat{G}$ be the dual[6] of the group G, i.e. the set of all equivalence classes of unitary irreducible representations of G. Let $\lambda\in\hat{G}$: in the following we

[6] See footnote 3.

indicate by $\mathfrak{X}^\lambda$ a choice of an irreducible representation in the class λ. By definition, $\mathfrak{X}^\lambda$ is a map that to an element of G associates a unitary operator acting on a complex separable Hilbert space $\mathcal{H}^\lambda$:

$$\mathfrak{X}^\lambda\colon \quad \begin{array}{l} G \to U(\mathcal{H}^\lambda), \\ g \mapsto \mathfrak{X}^\lambda(g). \end{array}$$

The index λ for $\mathcal{H}^\lambda$ indicates that in general the Hilbert space can vary with λ.

Definition 21. Let G be a Lie group satisfying (H_0), and $f \in L^1(G,\mathbb{C})$. The generalized (or noncommutative) Fourier transform (GFT) of f is the map (indicated in the following as $\hat{f}$ or $\mathcal{F}(f)$) that to each element of $\hat{G}$ associates the linear operator on $\mathcal{H}^\lambda$:

$$\hat{f}(\lambda) := \mathcal{F}(f) := \int_G f(g)\mathfrak{X}^\lambda\big(g^{-1}\big)\,d\mu. \tag{14}$$

Notice that since f is integrable and $\mathfrak{X}^\lambda$ unitary, then $\hat{f}(\lambda)$ is a bounded operator.

Remark 22. $\hat{f}$ can be seen as an operator from $\int_{\hat{G}}^{\oplus} \mathcal{H}^\lambda$ to itself. We also use the notation $\hat{f} = \int_{\hat{G}}^{\oplus} \hat{f}(\lambda)$.

In general $\hat{G}$ is not a group and its structure can be quite complicated. In the case in which G is abelian then $\hat{G}$ is a group; if G is nilpotent then $\hat{G}$ has the structure of $\mathbb{R}^n$ for some n; if G is compact then it is a Tannaka category (moreover, in this case each $\mathcal{H}^\lambda$ is finite dimensional). Under the hypothesis (H_0) one can define on $\hat{G}$ a positive measure $dP(\lambda)$ (called the Plancherel measure) such that for every $f \in L^1(G,\mathbb{C}) \cap L^2(G,\mathbb{C})$ one has

$$\int_G \big|f(g)\big|^2 \mu(g) = \int_{\hat{G}} \mathrm{Tr}\big(\hat{f}(\lambda) \circ \hat{f}(\lambda)^*\big)\,dP(\lambda).$$

By density of $L^1(G,\mathbb{C}) \cap L^2(G,\mathbb{C})$ in $L^2(G,\mathbb{C})$, this formula expresses the fact that the GFT is an isometry between $L^2(G,\mathbb{C})$ and $\int_{\hat{G}}^{\oplus} \mathbf{HS}^\lambda$, the set of Hilbert–Schmidt operators with respect to the Plancherel measure. Moreover, it is obvious that:

Proposition 23. *Let G be a Lie group satisfying* (H_0) *and $f \in L^1(G,\mathbb{C}) \cap L^2(G,\mathbb{C})$. We have, for each $g \in G$*

$$f(g) = \int_{\hat{G}} \mathrm{Tr}\big(\hat{f}(\lambda) \circ \mathfrak{X}^\lambda(g)\big)\,dP(\lambda) \tag{15}$$

where the equality is intended in the L^2 sense.

It is immediate to verify that, given two functions $f_1, f_2 \in L^1(G,\mathbb{C})$ and defining their convolution as

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2\big(h^{-1}g\big)\, dh, \tag{16}$$

then the GFT maps the convolution into noncommutative product:

$$\mathcal{F}(f_1 * f_2)(\lambda) = \hat{f}_2(\lambda)\hat{f}_1(\lambda). \tag{17}$$

Another important property is that if $\delta_{\mathrm{Id}}(g)$ is the Dirac function at the identity over G, then

$$\hat{\delta}_{\mathrm{Id}}(\lambda) = \mathrm{Id}_{H^\lambda}. \tag{18}$$

In the following, a key role is played by the differential of the representation $\mathfrak{X}^\lambda$, that is the map

$$d\mathfrak{X}^\lambda : X \mapsto d\mathfrak{X}^\lambda(X) := \frac{d}{dt}\bigg|_{t=0} \mathfrak{X}^\lambda\big(e^{tp}\big), \tag{19}$$

where $X = gp$ ($p \in \mathbf{L}$, $g \in G$) is a left-invariant vector field over G. By Stone theorem (see for instance [44, p. 6]) $d\mathfrak{X}^\lambda(X)$ is a (possibly unbounded) skew-adjoint operator on $\mathcal{H}^\lambda$. We have the following:

Proposition 24. *Let G be a Lie group satisfying* (H$_0$) *and X be a left-invariant vector field over G. The GFT of X, i.e. $\hat{X} = \mathcal{F}L_X\mathcal{F}^{-1}$ splits into the Hilbert sum of operators $\hat{X}^\lambda$, each one of which acts on the set $\mathbf{HS}^\lambda$ of Hilbert–Schmidt operators over $\mathcal{H}^\lambda$:*

$$\hat{X} = \int_{\hat{G}}^{\oplus} \hat{X}^\lambda.$$

Moreover,

$$\hat{X}^\lambda \Xi = d\mathfrak{X}^\lambda(X) \circ \Xi, \quad \textit{for every } \Xi \in \mathbf{HS}^\lambda, \tag{20}$$

i.e. the GFT of a left-invariant vector field acts as a left-translation over $\mathbf{HS}^\lambda$.

Proof. Consider the GFT of the operator $R_{e^{tp}}$ of right-translation of a function by e^{tp}, $p \in \mathbf{L}$, i.e.

$$(R_{e^{tp}} f)(g_0) = f\big(g_0 e^{tp}\big),$$

and compute its GFT:

$$\begin{aligned}
\mathcal{F}(R_{e^{tp}} f)(\lambda) = \mathcal{F}\big(f\big(g_0 e^{tp}\big)\big)(\lambda) &= \int_G f\big(g_0 e^{tp}\big)\mathfrak{X}^\lambda\big(g_0^{-1}\big)\mu(g_0) \\
&= \int_G f(g')\mathfrak{X}^\lambda\big(e^{tp}\big)\mathfrak{X}^\lambda\big(g'^{-1}\big)\mu\big(g' e^{-tp}\big) \\
&= \big(\mathfrak{X}^\lambda\big(e^{tp}\big)\big)\hat{f}(\lambda),
\end{aligned}$$

where in the last equality we use the right-invariance of the Haar measure. Hence the GFT acts as a left-translation on $\mathbf{HS}^\lambda$ and it disintegrates the right-regular representations. It follows that

$$\hat{R}_{e^{tp}} = \mathcal{F} R_{e^{tp}} \mathcal{F}^{-1} = \int_{\hat{G}}^{\oplus} \mathfrak{X}^\lambda\big(e^{tp}\big).$$

Passing to the infinitesimal generators, with $X = gp$, the conclusion follows. □

Remark 25. From the fact that the GFT of a left-invariant vector field acts as a left-translation, it follows that $\hat{X}^\lambda$ can be interpreted as an operator over $\mathcal{H}^\lambda$.

3.1. Computation of the kernel of the hypoelliptic heat equation

In this section we provide a general method to compute the kernel of the hypoelliptic heat equation on a left-invariant sub-Riemannian manifold $(G, \blacktriangle, \mathbf{g})$ such that G satisfies the assumption (H_0).

We begin by recalling some existence results (for the semigroup of evolution and for the corresponding kernel) in the case of the sum of squares. We recall that for all the examples treated in this paper, the invariant hypoelliptic Laplacian is the sum of squares.

Let G be a unimodular Lie group and $(G, \blacktriangle, \mathbf{g})$ a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{p_1, \ldots, p_m\}$, and consider the hypoelliptic heat equation

$$\partial_t \phi(t, g) = \Delta_{sr} \phi(t, g). \tag{21}$$

Since G is unimodular, then $\Delta_{sr} = L_{X_1}^2 + \cdots + L_{X_m}^2$, where L_{X_i} is the Lie derivative w.r.t. the vector field $X_i := gp_i$ $(i = 1, \ldots, m)$. Following Varopoulos [47, pp. 20–21, 106], since Δ_{sr} is a sum of squares, then it is a symmetric operator that we identify with its Friedrichs (self-adjoint) extension, that is the infinitesimal generator of a (Markov) semigroup $e^{t\Delta_{sr}}$. Thanks to the left-invariance of X_i (with $i = 1, \ldots, m$), $e^{t\Delta_{sr}}$ admits a right-convolution kernel $p_t(.)$, i.e.

$$e^{t\Delta_{sr}} \phi_0(g) = \phi_0 * p_t(g) = \int_G \phi_0(h) p_t\big(h^{-1} g\big) \mu(h) \tag{22}$$

is the solution for $t > 0$ to (21) with initial condition $\phi(0, g) = \phi_0(g) \in L^1(G, \mathbb{R})$ with respect to the Haar measure.

Since the operator $\partial_t - \Delta_{sr}$ is hypoelliptic, then the kernel is a $\mathcal{C}^\infty$ function of $(t, g) \in \mathbb{R}^+ \times G$. Notice that $p_t(g) = e^{t\Delta_{sr}} \delta_{\mathrm{Id}}(g)$.

The main results of the paper are based on the following key fact.

Theorem 26. *Let G be a Lie group satisfying* (H_0) *and* $(G, \blacktriangle, \mathbf{g})$ *a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{p_1, \ldots, p_m\}$. Let $\Delta_{sr} = L_{X_1}^2 + \cdots + L_{X_m}^2$ be the intrinsic hypoelliptic Laplacian where L_{X_i} is the Lie derivative w.r.t. the vector field $X_i := gp_i$.*

Let $\{\mathfrak{X}^\lambda\}_{\lambda \in \hat{G}}$ be the set of all non-equivalent classes of irreducible representations of the group G, each acting on an Hilbert space $\mathcal{H}^\lambda$, and $dP(\lambda)$ be the Plancherel measure on the dual space $\hat{G}$. We have the following:

(i) *The GFT of* Δ_{sr} *splits into the Hilbert sum of operators* $\hat{\Delta}_{sr}^{\lambda}$, *each one of which leaves* $\mathcal{H}^{\lambda}$ *invariant*:

$$\hat{\Delta}_{sr} = \mathcal{F}\Delta_{sr}\mathcal{F}^{-1} = \int_{\hat{G}}^{\oplus} \hat{\Delta}_{sr}^{\lambda}\, dP(\lambda), \quad \text{where } \hat{\Delta}_{sr}^{\lambda} = \sum_{i=1}^{m} \left(\hat{X}_i^{\lambda}\right)^2. \tag{23}$$

(ii) *The operator* $\hat{\Delta}_{sr}^{\lambda}$ *is self-adjoint and it is the infinitesimal generator of a contraction semi-group* $e^{t\hat{\Delta}_{sr}^{\lambda}}$ *over* $\mathbf{HS}^{\lambda}$, *i.e.* $e^{t\hat{\Delta}_{sr}^{\lambda}}\Xi_0^{\lambda}$ *is the solution for* $t > 0$ *to the operator equation* $\partial_t \Xi^{\lambda}(t) = \hat{\Delta}_{sr}^{\lambda}\Xi^{\lambda}(t)$ *in* $\mathbf{HS}^{\lambda}$, *with initial condition* $\Xi^{\lambda}(0) = \Xi_0^{\lambda}$.

(iii) *The hypoelliptic heat kernel is*

$$p_t(g) = \int_{\hat{G}} \mathrm{Tr}\left(e^{t\hat{\Delta}_{sr}^{\lambda}}\mathfrak{X}^{\lambda}(g)\right) dP(\lambda), \quad t > 0. \tag{24}$$

Proof. Following Varopoulos as above, and using Proposition 24, (i) follows. Item (ii) follows from the split (23) and from the fact that GFT is an isometry between $L^2(G,\mathbb{C})$ (the set of square integrable functions from G to $\mathbb{C}$ with respect to the Haar measure) and the set $\int_{\hat{G}}^{\oplus} \mathbf{HS}^{\lambda}$ of Hilbert–Schmidt operators with respect to the Plancherel measure. Item (iii) is obtained applying the inverse GFT to $e^{t\hat{\Delta}_{sr}^{\lambda}}\Xi_0^{\lambda}$ and the convolution formula (17). The integral is convergent by the existence theorem for p_t, see [47, p. 106]. □

Remark 27. As a consequence of Remark 25, it follows that $\hat{\Delta}_{sr}^{\lambda}$ and $e^{t\hat{\Delta}_{sr}^{\lambda}}$ can be considered as operators on $\mathcal{H}^{\lambda}$.

In the case when each $\hat{\Delta}_{sr}^{\lambda}$ has discrete spectrum, the following corollary gives an explicit formula for the hypoelliptic heat kernel in terms of its eigenvalues and eigenvectors.

Corollary 28. *Under the hypotheses of Theorem* 26, *if in addition we have that for every* λ, $\hat{\Delta}_{sr}^{\lambda}$ *(considered as an operator over* $\mathcal{H}^{\lambda}$*) has discrete spectrum and* $\{\psi_n^{\lambda}\}$ *is a complete set of eigenfunctions of norm one with the corresponding set of eigenvalues* $\{\alpha_n^{\lambda}\}$, *then*

$$p_t(g) = \int_{\hat{G}} \left(\sum_n e^{\alpha_n^{\lambda} t} \left\langle \psi_n^{\lambda}, \mathfrak{X}^{\lambda}(g)\psi_n^{\lambda} \right\rangle \right) dP(\lambda) \tag{25}$$

where $\langle .,. \rangle$ *is the scalar product in* $\mathcal{H}^{\lambda}$.

Proof. Recall that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and that $\mathrm{Tr}(A) = \sum_{i\in I}\langle e_i, Ae_i\rangle$ for any complete set $\{e_i\}_{i\in I}$ of orthonormal vectors. Hence $\mathrm{Tr}(e^{t\hat{\Delta}_{sr}^{\lambda}}\mathfrak{X}^{\lambda}(g)) = \sum_n \langle \psi_n^{\lambda}, \mathfrak{X}^{\lambda}(g)e^{t\hat{\Delta}_{sr}^{\lambda}}\psi_n^{\lambda}\rangle$. Observe that $\partial_t \psi_n^{\lambda} = \hat{\Delta}_{sr}^{\lambda}\psi_n^{\lambda} = \alpha_n^{\lambda}\psi_n^{\lambda}$, hence $e^{t\hat{\Delta}_{sr}^{\lambda}}\psi_n^{\lambda} = e^{\alpha_n^{\lambda}t}\psi_n^{\lambda}$, from which the result follows. □

The following corollary gives a useful formula for the hypoelliptic heat kernel in the case in which for all $\lambda \in \hat{G}$ each operator $e^{t\hat{\Delta}_{sr}^{\lambda}}$ admits a convolution kernel $Q_t^{\lambda}(.,.)$. Here by ψ^{λ}, we intend an element of $\mathcal{H}^{\lambda}$.

Corollary 29. *Under the hypotheses of Theorem* 26*, if for all* $\lambda \in \hat{G}$ *we have* $\mathcal{H}^\lambda = L^2(X^\lambda, d\theta^\lambda)$ *for some measure space* $(X^\lambda, d\theta^\lambda)$ *and*

$$\big[e^{t\hat{\Delta}^\lambda_{sr}}\psi^\lambda\big](\theta) = \int\limits_{X^\lambda} \psi^\lambda(\bar{\theta})\, Q^\lambda_t(\theta,\bar{\theta})\, d\bar{\theta},$$

then

$$p_t(g) = \int\limits_{\hat{G}} \int\limits_{X^\lambda} \mathfrak{X}^\lambda(g)\, Q^\lambda_t(\theta,\bar{\theta})\big|_{\theta=\bar{\theta}}\, d\bar{\theta}\, dP(\lambda),$$

where in the last formula $\mathfrak{X}^\lambda(g)$ *acts on* $Q^\lambda_t(\theta,\bar{\theta})$ *as a function of* θ.

Proof. From (24), we have

$$p_t(g) = \int\limits_{\hat{G}} \mathrm{Tr}\big(e^{t\hat{\Delta}^\lambda_{sr}}\mathfrak{X}^\lambda(g)\big)\, dP(\lambda) = \int\limits_{\hat{G}} \mathrm{Tr}\big(\mathfrak{X}^\lambda(g)e^{t\hat{\Delta}^\lambda_{sr}}\big)\, dP(\lambda).$$

We have to compute the trace of the operator

$$\begin{aligned}\Theta = \mathfrak{X}^\lambda(g)e^{t\hat{\Delta}^\lambda_{sr}} : \psi^\lambda(\theta) \mapsto \mathfrak{X}^\lambda(g)e^{t\hat{\Delta}^\lambda_{sr}}\psi^\lambda(\theta) &= \mathfrak{X}^\lambda(g)\int\limits_{X^\lambda} \psi^\lambda(\bar{\theta})\, Q^\lambda_t(\theta,\bar{\theta})\, d\bar{\theta} \\ &= \int\limits_{X^\lambda} K(\theta,\bar{\theta})\psi^\lambda(\bar{\theta})\, d\bar{\theta} \qquad (26)\end{aligned}$$

where $K(\theta,\bar{\theta}) = \mathfrak{X}^\lambda(g)Q^\lambda_t(\theta,\bar{\theta})$ is a function of $\theta,\bar{\theta}$ and $\mathfrak{X}^\lambda(g)$ acts on $Q^\lambda_t(\theta,\bar{\theta})$ as a function of θ. The trace of Θ is $\int_X K(\bar{\theta},\bar{\theta})\, d\bar{\theta}$ and the conclusion follows. □

4. Explicit expressions on 3D unimodular Lie groups

4.1. The hypoelliptic heat equation on H_2

In this section we apply the method presented above to solve the hypoelliptic heat equation (21) on the Heisenberg group. This kernel, via the GFT, was first obtained by Hulanicki (see [29]). We present it as an application of Corollary 29, since in this case an expression for the kernel of the GFT of this equation is known.

We write the Heisenberg group as the 3D group of matrices

$$H_2 = \left\{ \begin{pmatrix} 1 & x & z+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x,y,z \in \mathbb{R} \right\}$$

endowed with the standard matrix product. H_2 is indeed $\mathbb{R}^3$,

$$(x,y,z)\sim\begin{pmatrix}1 & x & z+\frac{1}{2}xy\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},$$

endowed with the group law

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)=\left(x_1+x_2,y_1+y_2,z_1+z_2+\frac{1}{2}(x_1y_2-x_2y_1)\right).$$

A basis of its Lie algebra is $\{p_1,p_2,k\}$ where

$$p_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\qquad p_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\qquad k=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}. \tag{27}$$

They satisfy the following commutation rules: $[p_1,p_2]=k$, $[p_1,k]=[p_2,k]=0$, hence H_2 is a 2-step nilpotent group. We define a left-invariant sub-Riemannian structure on H_2 as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g)=gp_i$ with $i=1,2$ and define

$$\blacktriangle(g)=\operatorname{span}\{X_1(g),X_2(g)\},\qquad \mathbf{g}_g\big(X_i(g),X_j(g)\big)=\delta_{ij}.$$

Writing the group H_2 in coordinates (x,y,z) on $\mathbb{R}^3$, we have the following expression for the Lie derivatives of X_1 and X_2:

$$L_{X_1}=\partial_x-\frac{y}{2}\partial_z,\qquad L_{X_2}=\partial_y+\frac{x}{2}\partial_z.$$

The Heisenberg group is unimodular, hence the hypoelliptic Laplacian Δ_{sr} is the sum of squares:

$$\Delta_{sr}\phi=\big(L_{X_1}^2+L_{X_2}^2\big)\phi. \tag{28}$$

Remark 30. It is interesting to notice that all left-invariant sub-Riemannian structures that one can define on the Heisenberg group are isometric.

In the next proposition we present the structure of the dual group of H_2. For details and proofs see for instance [33].

Proposition 31. *The dual space of H_2 is $\hat{G}=\{\mathfrak{X}^\lambda\mid\lambda\in\mathbb{R}\}$, where*

$$\mathfrak{X}^\lambda(g):\quad \begin{array}{rcl}\mathcal{H}&\to&\mathcal{H},\\ \psi(\theta)&\mapsto& e^{i\lambda(z-y\theta+\frac{xy}{2})}\psi(\theta-x),\end{array}$$

whose domain is $\mathcal{H}=L^2(\mathbb{R},\mathbb{C})$, endowed with the standard product

$$\langle \psi_1, \psi_2 \rangle := \int_{\mathbb{R}} \psi_1(\theta)\overline{\psi_2}(\theta)\, d\theta$$

where $d\theta$ is the Lebesgue measure.

The Plancherel measure on $\hat{G}$ is $dP(\lambda) = \frac{|\lambda|}{4\pi^2}\, d\lambda$, where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$.

Remark 32. Notice that in this example the domain $\mathcal{H}$ of the representation $\mathfrak{X}^\lambda$ does not depend on λ.

4.1.1. The kernel of the hypoelliptic heat equation

Consider the representation $\mathfrak{X}^\lambda$ of H_2 and let $\hat{X}_i^\lambda$ be the corresponding representations of the differential operators L_{X_i} with $i = 1, 2$. Recall that $\hat{X}_i^\lambda$ are operators on $\mathcal{H}$. From formula (19) we have

$$\big[\hat{X}_1^\lambda \psi\big](\theta) = -\frac{d}{d\theta}\psi(\theta), \qquad \big[\hat{X}_2^\lambda \psi\big](\theta) = -i\lambda\theta\, \psi(\theta), \quad \text{hence}$$
$$\big[\hat{\Delta}_{sr}^\lambda \psi\big](\theta) = \left(\frac{d^2}{d\theta^2} - \lambda^2\theta^2\right)\psi(\theta).$$

The GFT of the hypoelliptic heat equation is thus

$$\partial_t \psi = \left(\frac{d^2}{d\theta^2} - \lambda^2\theta^2\right)\psi.$$

The kernel of this equation is known (see for instance [4]) and it is called the Mehler kernel (its computation is very similar to the computation of the kernel for the harmonic oscillator in quantum mechanics):

$$Q_t^\lambda(\theta, \bar{\theta}) := \sqrt{\frac{\lambda}{2\pi \sinh(2\lambda t)}} \exp\left(-\frac{1}{2}\frac{\lambda \cosh(2\lambda t)}{\sinh(2\lambda t)}\left(\theta^2 + \bar{\theta}^2\right) + \frac{\lambda\theta\bar{\theta}}{\sinh(2\lambda t)}\right).$$

Using Corollary 29 and after straightforward computations, one gets the kernel of the hypoelliptic heat equation on the Heisenberg group:

$$p_t(x, y, z) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}} \frac{2\tau}{\sinh(2\tau)} \exp\left(-\frac{\tau(x^2 + y^2)}{2t \tanh(2\tau)}\right) \cos\left(2\frac{z\tau}{t}\right) d\tau. \tag{29}$$

This formula differs from the one by Gaveau [21] for some numerical factors since he studies the equation

$$\partial_t \phi = \frac{1}{2}\big((\partial_x + 2y\partial_z)^2 + (\partial_y - 2x\partial_z)^2\big)\phi.$$

The Gaveau formula is recovered from (29) with $t \to t/2$ and $z \to z/4$. Moreover, a multiplicative factor $\frac{1}{4}$ should be added, because from the change of variables one gets that the Haar measure is $\frac{1}{4}\, dx\, dy\, dz$ instead of $dx\, dy\, dz$ as used by Gaveau.

4.2. The hypoelliptic heat equation on $SU(2)$

In this section we solve the hypoelliptic heat equation (21) on the Lie group

$$SU(2)=\left\{\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\;\middle|\;\alpha,\beta\in\mathbb{C},\ |\alpha|^2+|\beta|^2=1\right\}.$$

A basis of the Lie algebra $su(2)$ is $\{p_1,p_2,k\}$ where[7]

$$p_1=\frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\qquad p_2=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad k=\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}. \tag{30}$$

We define a trivializable sub-Riemannian structure on $SU(2)$ as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g)=gp_i$ with $i=1,2$ and define

$$\blacktriangle(g)=\operatorname{span}\{X_1(g),X_2(g)\},\qquad \mathbf{g}_g\big(X_i(g),X_j(g)\big)=\delta_{ij}.$$

The group $SU(2)$ is unimodular, hence the hypoelliptic Laplacian Δ_{sr} has the following expression:

$$\Delta_{sr}\psi=\big(L_{X_1}^2+L_{X_2}^2\big)\psi. \tag{31}$$

In the next proposition we present the structure of the dual group of $SU(2)$. For details and proofs see for instance [43].

Proposition 33. *The dual space of $SU(2)$ is $\hat{G}=\{\mathfrak{X}^n \mid n\in\mathbb{N}\}$.*

The domain $\mathcal{H}^n$ of $\mathfrak{X}^n$ is the space of homogeneous polynomials of degree n in two variables (z_1,z_2) with complex coefficients $\mathcal{H}^n:=\{\sum_{k=0}^n a_k z_1^k z_2^{n-k}\mid a_k\in\mathbb{C}\}$, endowed with the scalar product

$$\left\langle\sum_{k=0}^n a_k z_1^k z_2^{n-k},\sum_{k=0}^n b_k z_1^k z_2^{n-k}\right\rangle:=\sum_{k=0}^n k!(n-k)!a_k\bar{b}_k.$$

The representation $\mathfrak{X}^n$ is

$$\mathfrak{X}^n(g):\quad \begin{array}{ccc}\mathcal{H}^n & \to & \mathcal{H}^n,\\ \displaystyle\sum_{k=0}^n a_k z_1^k z_2^{n-k} & \mapsto & \displaystyle\sum_{k=0}^n a_k w_1^k w_2^{n-k}\end{array}$$

with $(w_1,w_2)=(z_1,z_2)g=(\alpha z_1-\bar{\beta}z_2,\beta z_1+\bar{\alpha}z_2)$.

The Plancherel measure on $\hat{G}$ is $dP(n)=(n+1)\,d\mu_\sharp(n)$, where $d\mu_\sharp$ is the counting measure.

Notice that an orthonormal basis of $\mathcal{H}^n$ is $\{\psi_k^n\}_{k=0}^n$ with $\psi_k^n:=\frac{z_1^k z_2^{n-k}}{\sqrt{k!(n-k)!}}$.

[7] See [43, p. 67]: $p_1=X_1$, $p_2=X_2$, $k=X_3$.

4.2.1. The kernel of the hypoelliptic heat equation

Consider the representations $\hat{X}_i^n$ of the differential operators L_{X_i} with $i = 1, 2$: they are operators on $\mathcal{H}^n$, whose action on the basis $\{\psi_k^n\}_{k=0}^n$ of $\mathcal{H}^n$ is (using formula (19))

$$\hat{X}_1^n \psi_k^n = \frac{i}{2}\{k\psi_{k-1}^n + (n-k)\psi_{k+1}^n\}, \qquad \hat{X}_2^n \psi_k^n = \frac{1}{2}\{k\psi_{k-1}^n - (n-k)\psi_{k+1}^n\}$$

hence $\hat{\Delta}_{sr}^n \psi_k^n = (k^2 - kn - \frac{n}{2})\psi_k^n$. Thus, the basis $\{\psi_k^n\}_{k=0}^n$ is a complete set of eigenfunctions of norm one for the operator $\hat{\Delta}_{sr}^n$. We are now able to compute the kernel of the hypoelliptic heat equation using formula (25).

Proposition 34. *The kernel of the hypoelliptic heat equation on* $(SU(2), \blacktriangle, \mathbf{g})$ *is*

$$p_t(g) = \sum_{n=0}^{\infty}(n+1)\sum_{k=0}^{n} e^{(k^2-kn-\frac{n}{2})t} A^{n,k}(g), \tag{32}$$

where

$$A^{n,k}(g) := \langle \psi_k^n, \mathfrak{X}^n(g)\psi_k^n\rangle = \sum_{l=0}^{\min\{k,n-k\}} \binom{k}{k-l}\binom{n-k}{l}\bar{\alpha}^{k-l}\alpha^{n-k-l}\big(|\alpha|^2-1\big)^l$$

with $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}$.

Proof. The formula $p_t(g) = \sum_{n=0}^{\infty}(n+1)\sum_{k=0}^{n} e^{(k^2-kn-\frac{n}{2})t}\langle \psi_k^n, \mathfrak{X}^n(g)\psi_k^n\rangle$ is given by applying formula (25) in the $SU(2)$ case.

We now prove the explicit expression for $\langle \psi_k^n, \mathfrak{X}^n(g)\psi_k^n\rangle$: a direct computation gives

$$\mathfrak{X}^n(g)\psi_k^n = \frac{\sum_{s=0}^n \psi_s^n \sqrt{s!(n-s)!}\big(\sum_{l=\max\{0,s-k\}}^{\min\{s,n-k\}}\binom{k}{s-l}\binom{n-k}{l}\alpha^{s-l}(-\bar{\beta})^{k-s+l}\beta^l\bar{\alpha}^{n-k-l}\big)}{\sqrt{k!(n-k)!}}.$$

Observe that ψ_k^n is an orthonormal frame for the inner product: hence

$$\langle \psi_k^n, \mathfrak{X}^n(g)\psi_k^n\rangle = \left\langle \psi_k^n, \psi_k^n \sum_{l=0}^{\min\{k,n-k\}} \binom{k}{k-l}\binom{n-k}{l}\alpha^{k-l}(-\bar{\beta})^l\beta^l\bar{\alpha}^{n-k-l}\right\rangle.$$

The result easily follows. □

Remark 35. Notice that, as the sub-Riemannian distance (computed in [9]), $p_t(g)$ does not depend on β. This is due to the cylindrical symmetry of the distribution around $e^{\mathbf{k}} = \{e^{ck} \mid c \in \mathbb{R}\}$.

4.3. The hypoelliptic heat equation on $SO(3)$

Let g be an element of $SO(3)=\{A\in \mathrm{Mat}(\mathbb{R},3)\mid AA^T=\mathrm{Id},\ \det(A)=1\}$. A basis of the Lie algebra $so(3)$ is $\{p_1,p_2,k\}$ where[8]

$$p_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\qquad p_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\qquad k=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}. \tag{33}$$

We define a trivializable sub-Riemannian structure on $SO(3)$ as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g)=gp_i$ with $i=1,2$ and define

$$\blacktriangle(g)=\mathrm{span}\{X_1(g),X_2(g)\},\qquad \mathbf{g}_g\big(X_i(g),X_j(g)\big)=\delta_{ij}.$$

The group $SO(3)$ is unimodular, hence the hypoelliptic Laplacian Δ_{sr} has the following expression:

$$\Delta_{sr}\phi=\big(L_{X_1}^2+L_{X_2}^2\big)\phi. \tag{34}$$

We present now the structure of the dual group of $SO(3)$. For details and proofs see [43].

First consider the domain $\mathcal{H}^r$, that is the space of complex-valued polynomials of rth degree in three real variables x,y,z that are homogeneous and harmonic

$$\mathcal{H}^r=\big\{f(x,y,z)\ \big|\ \deg(f)=r,\ f \text{ homogeneous},\ \Delta f=0\big\}.$$

Notice that an homogeneous polynomial $f\in\mathcal{H}^r$ is uniquely determined by its value on $S^2=\{(x,y,z)\mid x^2+y^2+z^2=1\}$, as $f(\rho x,\rho y,\rho z)=\rho^r f(x,y,z)$.

Define $\tilde f(\alpha,\beta):=f(\sin(\alpha)\cos(\beta),\sin(\alpha)\sin(\beta),\cos(\alpha))$. Then endow $\mathcal{H}^r$ with the scalar product

$$\big\langle f_1(x,y,z),f_2(x,y,z)\big\rangle:=\frac{1}{4\pi}\int_{S^2}\tilde f_1(\alpha,\beta)\overline{\tilde f_2(\alpha,\beta)}\sin\alpha\,d\alpha\,d\beta.$$

In the following proposition we present the structure of the dual group.

Proposition 36. *The dual space of $SO(3)$ is $\hat G=\{\mathfrak{X}^r\mid r\in\mathbb{N}\}$.*

Given $g\in SO(3)$, the unitary representation $\mathfrak{X}^r(g)$ is

$$\mathfrak{X}^r(g):\qquad \begin{aligned}\mathcal{H}^r&\to\mathcal{H}^r,\\ f(x,y,z)&\mapsto f(x_1,y_1,z_1)\end{aligned}$$

with $(x_1,y_1,z_1)=(x,y,z)g$.

The Plancherel measure on $SO(3)$ is $dP(r)=(2r+1)\,d\mu_\sharp(r)$, where $d\mu_\sharp$ is the counting measure.

[8] See [43, p. 88]: $p_1=Z_1$, $p_2=Z_2$, $k=Z_3$.

An orthonormal basis for $\mathcal{H}^r$ is given by $\{\phi_s^r\}_{s=-r}^r$ with $\tilde{\phi}_s^r(\alpha,\beta) := e^{is\beta} P_r^{-s}(\cos(\alpha))$, where $P_r^s(x)$ are the Legendre polynomials.[9]

4.3.1. The kernel of the hypoelliptic heat equation

Consider the representations $\hat{X}_i^r$ of the differential operators L_{X_i} with $i=1,2$: using formula (19) we find the following expressions in spherical coordinates[10]

$$\hat{X}_1^r\psi = \sin(\beta)\frac{\partial\psi}{\partial\alpha} + \cot(\alpha)\cos(\beta)\frac{\partial\psi}{\partial\beta}, \qquad \hat{X}_2^r\psi = -\cos(\beta)\frac{\partial\psi}{\partial\alpha} + \cot(\alpha)\sin(\beta)\frac{\partial\psi}{\partial\beta}$$

hence

$$\hat{\Delta}_{sr}^n\psi = \frac{\partial^2\psi}{\partial\alpha^2} + \cot^2(\alpha)\frac{\partial^2\psi}{\partial\beta^2} + \cot(\alpha)\frac{\partial\psi}{\partial\alpha} \tag{35}$$

and its action on the basis $\{\phi_s^r\}_{s=-r}^r$ of $\mathcal{H}^r$ is

$$\hat{\Delta}_{sr}^n\phi_s^r = \big(s^2 - r(r+1)\big)\phi_s^r. \tag{36}$$

Hence the basis $\{\phi_s^r\}_{s=-r}^r$ is a complete set of eigenfunctions of norm one for the operator $\hat{\Delta}_{sr}^n$.

We compute the kernel of the hypoelliptic heat equation, using (25).

Proposition 37. *The kernel of the hypoelliptic heat equation on* $(SO(3), \blacktriangle, \mathbf{g})$ *is*

$$p_t(g) = \sum_{r=0}^{\infty}(2r+1)\sum_{s=-r}^{r} e^{(s^2-r(r+1))t}\big\langle\phi_s^r, \mathfrak{X}^r(g)\phi_s^r\big\rangle. \tag{37}$$

4.3.2. The heat kernel on $SO(3)$ via the heat kernel on $SU(2)$

In this section we verify that the heat kernel on $SO(3)$ given in (37) can be easily retrieved from the one on $SU(2)$ given in (32). In the following, all the objects relative to $SO(3)$ are underlined, e.g. $\underline{g} \in SO(3)$, $\underline{p_i} \in so(3)$, the representations $\underline{\mathfrak{X}^r}$ acting on $\underline{\mathcal{H}^r}$ with basis $\underline{\phi_s^r}$.

Consider the isomorphism of Lie algebras $\mathrm{ad} : su(2) \to so(3)$ defined by $\mathrm{ad}\, p_1 = \underline{p_1}$, $\mathrm{ad}\, p_2 = \underline{p_2}$, $\mathrm{ad}\, k = \underline{k}$: it gives the matrix expression of the adjoint map on $su(2)$ with respect to the basis $\{p_1, p_2, k\}$. There is a corresponding endomorphism of groups $\mathrm{Ad} : SU(2) \to SO(3)$ given by $\mathrm{Ad}(\exp v) = \underline{\exp(\mathrm{ad}(v))}$. It is a covering map of $SO(3)$ by $SU(2)$, such that for each matrix $\underline{g} \in SO(3)$ the preimage is given by two opposite matrices $g, -g \in SU(2)$.

Proposition 38. *The following relation holds between the kernel* $\underline{p_t}$ *on* $SO(3)$ *given in* (37) *and the kernel* p_t *on* $SU(2)$ *given in* (32):

$$\forall \underline{g} \in SO(3),\ g \in \mathrm{Ad}^{-1}(\underline{g}), \quad \underline{p_t}(\underline{g}) = \frac{p_t(g) + p_t(-g)}{2}.$$

---

[9] Recall that $P_r^s(x)$ is defined by $P_r^s(x) := \frac{(1-x^2)^{\frac{s}{2}}}{r!2^r}\frac{d^{r+s}(x^2-1)^r}{dx^{r+s}}$.

[10] I.e. $x = \rho\sin(\alpha)\cos(\beta)$, $y = \rho\sin(\alpha)\sin(\beta)$, $z = \rho\cos(\alpha)$.

Proof. Observe the following key facts (see e.g. [43, II, §7]):

- on $SU(2)$: $\mathfrak{X}^n(-g)\phi = (-1)^n \mathfrak{X}^n(g)\phi$;
- the representation $\underline{\mathfrak{X}^r}$ of $SO(3)$ and the representation $\mathfrak{X}^{2r}$ of $SU(2)$ are unitarily related, i.e. the following relation holds: $\forall g \in SU(2)$, $\underline{\phi_s^r} \in \underline{\mathcal{H}^r}$

$$T^r \underline{\mathfrak{X}^r}(\operatorname{Ad} g)\,\underline{\phi_s^r} = \mathfrak{X}^{2r}(g)\big[T^r\big(\underline{\phi_s^r}\big)\big] \tag{38}$$

where the map

$$T^r\colon \quad \begin{array}{rcl} \underline{\mathcal{H}^r} & \to & \mathcal{H}^{2r}, \\ \underline{\phi_s^r} & \mapsto & \phi_{r+s}^{2r} \end{array}$$

is a unitary isomorphism.

Then we have explicitly

$$\frac{p_t(g) + p_t(-g)}{2} = \frac{\sum_{n=0}^{\infty}(n+1)(1+(-1)^n)\sum_{k=0}^{n} e^{(k^2-kn-\frac{n}{2})t} A^{n,k}(g)}{2}$$
$$= \sum_{r=0}^{\infty}(2r+1)\sum_{s=-r}^{r} e^{(s^2-r(r+1))t}\big\langle \phi_{r+s}^{2r}, \mathfrak{X}^{2r}(g)\phi_{r+s}^{2r}\big\rangle$$

after the substitution $r = 2n$, $s = k - r$. Using Eq. (38), we have $\langle \phi_{r+s}^{2r}, \mathfrak{X}^{2r}(g)\phi_{r+s}^{2r}\rangle = \langle \underline{\phi_s^r}, \underline{\mathfrak{X}^r}(\operatorname{Ad} g)\underline{\phi_s^r}\rangle$, from which the result directly follows. $\square$

4.4. The hypoelliptic heat equation on $SL(2)$

In this section we solve the hypoelliptic heat equation (21) on the Lie group

$$SL(2) = \big\{g \in \operatorname{Mat}(\mathbb{R}, 2)\ \big|\ \det(g) = 1\big\}.$$

A basis of the Lie algebra $sl(2)$ is

$$p_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad p_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad k = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We define a trivializable sub-Riemannian structure on $SL(2)$ as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g) = gp_i$ with $i = 1, 2$ and define

$$\blacktriangle(g) = \operatorname{span}\big\{X_1(g), X_2(g)\big\}, \qquad \mathbf{g}_g\big(X_i(g), X_j(g)\big) = \delta_{ij}.$$

The group $SL(2)$ is unimodular, hence the hypoelliptic Laplacian Δ_{sr} has the following expression:

$$\Delta_{sr}\phi = \big(L_{X_1}^2 + L_{X_2}^2\big)\phi. \tag{39}$$

It is well known that $SL(2)$ and $SU(1,1) = \left\{\left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right) \mid |\alpha|^2 - |\beta|^2 = 1\right\}$ are isomorph Lie groups via the isomorphism

$$\Pi: \quad \begin{array}{rcl} SL(2) & \to & SU(1,1), \\ g & \mapsto & \mathscr{G} = CgC^{-1}, \end{array} \qquad \text{with } C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

This isomorphism also induce an isomorphism of Lie algebras $d\Pi : sl(2) \to su(1,1)$ defined by $d\Pi(p_1) = p_1'$, $d\Pi(p_2) = p_2'$, $d\Pi(k) = k'$ with

$$p_1' = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad p_2' = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad k' = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

This isomorphism induces naturally the definitions of left-invariant sub-Riemannian structure and of the hypoelliptic Laplacian on $SU(1,1)$.

We present here the structure of the dual of the group $SU(1,1)$, observing that the isomorphism of groups induces an isomorphism of representations. For details and proofs, see [43].

The dual space $\hat G$ of $SU(1,1)$ contains two continuous parts and a discrete part: $\hat G = \hat G_C \amalg \hat G_D$ with $\hat G_C = \{\mathfrak{X}^{j,s} \mid j \in \{0, \frac{1}{2}\},\ s = \frac{1}{2} + iv,\ v \in \mathbb{R}^+\}$ and $\hat G_D = \{\mathfrak{X}^n \mid n \in \frac{1}{2}\mathbb{Z},\ |n| \geqslant 1\}$.

We define the domain $\mathcal{H}_C$ of the continuous representation $\mathfrak{X}^{j,s}$: it is the Hilbert space of L^2 complex-valued functions on the unitary circle $S^1 = \{x \in \mathbb{C} \mid |x| = 1\}$ with respect to the normalized Lebesgue measure $\frac{dx}{2\pi}$, endowed with the standard scalar product $\langle f, g\rangle := \int_{S^1} f(x)\overline{g(x)}\,\frac{dx}{2\pi}$. An orthonormal basis is given by the set $\{\psi_m\}_{m\in\mathbb{Z}}$ with $\psi_m(x) := x^{-m}$.

Proposition 39. *The continuous part of the dual space of $SU(1,1)$ is*

$$\hat G_C = \left\{\mathfrak{X}^{j,s} \;\middle|\; j \in \left\{0, \frac{1}{2}\right\},\ s = \frac{1}{2} + iv,\ v \in \mathbb{R}^+\right\}.$$

Given $\mathscr{G} \in SU(1,1)$, the unitary representation $\mathfrak{X}^{j,s}(\mathscr{G})$ is

$$\mathfrak{X}^{j,s}(\mathscr{G}): \quad \begin{array}{rcl} \mathcal{H}_C & \to & \mathcal{H}_C, \\ \psi(x) & \mapsto & |\bar\beta x + \bar\alpha|^{-2s}\left(\dfrac{\bar\beta x + \bar\alpha}{|\bar\beta x + \bar\alpha|}\right)^{2j}\psi\left(\dfrac{\alpha x + \beta}{\bar\beta x + \bar\alpha}\right) \end{array}$$

with $\mathscr{G}^{-1} = \left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right)$.

The Plancherel measure on $\hat G_C$ is

$$dP\left(j, \frac{1}{2} + iv\right) = \begin{cases} \frac{v}{2\pi}\,\mathrm{Tanh}(\pi v)\,dv, & j = 0, \\ \frac{v}{2\pi}\,\mathrm{Cotanh}(\pi v)\,dv, & j = \frac{1}{2}, \end{cases}$$

where dv is the Lebesgue measure on $\mathbb{R}$.

Remark 40. Notice that in this example the domain of the representation $\mathcal{H}_C$ does not depend on j, s.

Now we turn our attention to the description of principal discrete representations.[11]

We first define the domain $\mathcal{H}^n$ of these representations $\mathfrak{X}^n$: consider the space $\mathscr{L}_n$ of L^2 complex-valued functions on the unitary disc $D = \{x \in \mathbb{C} \mid |x| < 1\}$ with respect to the measure $d\mu^*(z) = \frac{2|n|-1}{\pi}(1-|z|^2)^{2n-2}\,dz$ where dz is the Lebesgue measure. $\mathscr{L}_n$ is an Hilbert space if endowed with the scalar product $\langle f, g\rangle := \int_D f(z)\overline{g(z)}\,d\mu^*(z)$. Then define the space $\mathcal{H}^n$ with $n > 0$ as the Hilbert space of holomorphic functions of $\mathscr{L}_n$, while $\mathcal{H}^n$ with $n < 0$ is the Hilbert space of antiholomorphic functions of $\mathscr{L}_{-n}$. An orthonormal basis for $\mathcal{H}^n$ with $n > 0$ is given by $\{\psi^n_m\}_{m\in\mathbb{N}}$ with $\psi^n_m(z) = (\frac{\Gamma(2n+m)}{\Gamma(2n)\Gamma(m+1)})^{\frac{1}{2}} z^m$ where Γ is the Gamma function. An orthonormal basis for $\mathcal{H}^n$ with $n < 0$ is given by $\{\psi^n_m\}_{m\in\mathbb{N}}$ with $\psi^n_m(z) = \overline{\psi^{-n}_m}(z)$.

Proposition 41. *The discrete part of the dual space of $SU(1,1)$ is $\hat{G}_D = \{\mathfrak{X}^n \mid n \in \frac{1}{2}\mathbb{Z},\ |n| \geqslant 1\}$.*

Given $\mathscr{G} \in SU(1,1)$, the unitary representation $\mathfrak{X}^n(\mathscr{G})$ is

$$\mathfrak{X}^n(\mathscr{G}):\quad \begin{aligned} \mathcal{H}^n &\to \mathcal{H}^n,\\ \psi(x) &\mapsto (\bar{\beta}x + \bar{\alpha})^{-2|n|}\psi\left(\frac{\alpha x + \beta}{\bar{\beta}x + \bar{\alpha}}\right), \end{aligned}$$

with $\mathscr{G}^{-1} = \left(\begin{smallmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{smallmatrix}\right)$.

The Plancherel measure on $\hat{G}_D$ is $dP(n) = \frac{2|n|-1}{4\pi}\,d\mu_\sharp(n)$, where $d\mu_\sharp$ is the counting measure.

4.4.1. The kernel of the hypoelliptic heat equation

In this section we compute the representation of differential operators L_{X_i} with $i = 1, 2$ and give the explicit expression of the kernel of the hypoelliptic heat equation.

We first study the continuous representations $\hat{X}^{j,s}_i$, for both the families $j = 0, \frac{1}{2}$. Their actions on the basis $\{\psi_m\}_{m\in\mathbb{Z}}$ of $\mathcal{H}_C$ is

$$\begin{aligned} \hat{X}^{j,s}_1\psi_m &= \frac{s-m-j}{2}\psi_{m-1} + \frac{s+m+j}{2}\psi_{m+1},\\ \hat{X}^{j,s}_2\psi_m &= i\frac{s-m-j}{2}\psi_{m-1} - i\frac{s+m+j}{2}\psi_{m+1}. \end{aligned}$$

Hence

$$\hat{\Delta}^{j,s}_{sr}\psi_m = -\left((m+j)^2 + v^2 + \frac{1}{4}\right)\psi_m.$$

Moreover, the set $\{\psi_m\}_{m\in\mathbb{Z}}$ is a complete set of eigenfunctions of norm one for the operator $\hat{\Delta}^{j,s}_{sr}$.

Remark 42. Notice that the operators $\hat{X}^{j,s}_i$ are only defined on the space of C^∞ vectors, i.e. the vectors $v \in \mathcal{H}_C$ such that the map $g \to [\mathfrak{X}^{j,s}(g)]\,v$ is a C^∞ mapping. This restriction is not crucial for the following treatment.

[11] There exist also the so-called complementary discrete representations, on which the Plancherel measure is vanishing. Hence they do not contribute to the GFT of a function defined on $SU(1,1)$. For details, see for instance [43].

We now turn our attention to the discrete representations in both cases $n > 0$ (holomorphic functions) and $n < 0$ (antiholomorphic functions). Consider the discrete representation $\mathfrak{X}^n$ of $SU(1,1)$ and let $\hat{X}_i^n$ be the representations of the differential operators L_{X_i} with $i = 1, 2$. Their actions on the basis $\{\psi_m^n\}_{m\in\mathbb{N}}$ of $\mathcal{H}^n$ are

$$\hat{X}_1^n \psi_m^n = \frac{\sqrt{(2|n|+m)(m+1)}}{2}\psi_{m+1}^n - \frac{\sqrt{(2|n|+m-1)m}}{2}\psi_{m-1}^n,$$
$$\hat{X}_2^n \psi_m^n = -i\frac{\sqrt{(2|n|+m)(m+1)}}{2}\psi_{m+1}^n - i\frac{\sqrt{(2|n|+m-1)m}}{2}\psi_{m-1}^n.$$

Hence $\hat{\Delta}_{sr}^n \psi_m^n = -(|n| + 2m|n| + m^2)\psi_m^n$, thus the basis $\{\psi_m^n\}_{m\in\mathbb{N}}$ is a complete set of eigenfunctions of norm one for the operator $\hat{\Delta}_{sr}^n$.

We now compute the kernel of the hypoelliptic heat equation using formula (25).

Proposition 43. *The kernel of the hypoelliptic heat equation on* $(SL(2), \blacktriangle, \mathbf{g})$ *is*

$$\begin{aligned} p_t(g) = {} & \int_0^{+\infty} \frac{v}{2\pi}\,\mathrm{Tanh}(\pi v) \sum_{m\in\mathbb{Z}} e^{-t(m^2+v^2+\frac{1}{4})}\big\langle \psi_m, \mathfrak{X}^{0,s}(\mathcal{G})\psi_m\big\rangle\, d\mu(v) \\ & + \int_0^{+\infty} \frac{v}{2\pi}\,\mathrm{Cotanh}(\pi v) \sum_{m\in\mathbb{Z}} e^{-t(m^2+m+v^2+\frac{1}{2})}\big\langle \psi_m, \mathfrak{X}^{\frac{1}{2},s}(\mathcal{G})\psi_m\big\rangle\, d\mu(v) \\ & + \sum_{n\in\frac{1}{2}\mathbb{Z},\ |n|\geqslant 1} \frac{2|n|-1}{4\pi} \sum_{m\in\mathbb{N}} e^{-t(|n|+2m|n|+m^2)}\big\langle \psi_m^n, \mathfrak{X}^n(\mathcal{G})\psi_m^n\big\rangle, \end{aligned} \tag{40}$$

where $\mathcal{G} = \Pi(g^{-1}) \in SU(1,1)$.

4.5. *The hypoelliptic heat kernel on* $SE(2)$

Consider the group of rototranslations of the plane

$$SE(2) = \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & x_1 \\ \sin(\alpha) & \cos(\alpha) & x_2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R}/2\pi,\ x_i \in \mathbb{R} \right\}.$$

In the following we often denote an element of $SE(2)$ as $g = (\alpha, x_1, x_2)$.

A basis of the Lie algebra of $SE(2)$ is $\{p_0, p_1, p_2\}$, with

$$p_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad p_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

We define a trivializable sub-Riemannian structure on $SE(2)$ as presented in Section 2.1.1: consider the two left-invariant vector fields $X_i(g) = gp_i$ with $i = 0, 1$ and define

$$\blacktriangle(g) = \operatorname{span}\{X_0(g), X_1(g)\}, \qquad \mathbf{g}_g\big(X_i(g), X_j(g)\big) = \delta_{ij}.$$

The group $SE(2)$ is unimodular, hence the hypoelliptic Laplacian Δ_{sr} has the following expression:

$$\Delta_{sr}\phi = \big(L_{X_0}^2 + L_{X_1}^2\big)\phi. \tag{42}$$

Remark 44. As for the Heisenberg group, all left-invariant sub-Riemannian structures that one can define on $SE(2)$ are isometric.

In the following proposition we present the structure of the dual of $SE(2)$.

Proposition 45. *The dual space of* $SE(2)$ *is* $\hat{G} = \{\mathfrak{X}^\lambda \mid \lambda \in \mathbb{R}^+\}$.
Given $g = (\alpha, x_1, x_2) \in SE(2)$, *the unitary representation* $\mathfrak{X}^\lambda(g)$ *is*

$$\mathfrak{X}^\lambda(g): \quad \begin{array}{rcl} \mathcal{H} & \to & \mathcal{H}, \\ \psi(\theta) & \mapsto & e^{i\lambda(x\cos(\theta) - y\sin(\theta))}\psi(\theta+\alpha), \end{array}$$

where the domain $\mathcal{H}$ *of the representation* $\mathfrak{X}^\lambda(g)$ *is* $\mathcal{H} = L^2(S^1, \mathbb{C})$, *the Hilbert space of* L^2 *functions on the circle* $S^1 \subset \mathbb{R}^2$ *with respect to the Lebesgue measure* $d\theta$, *endowed with the scalar product* $\langle\psi_1, \psi_2\rangle = \int_{S^1} \psi_1(\theta)\overline{\psi_2(\theta)}\,d\theta$.

The Plancherel measure on $\hat{G}$ *is* $dP(\lambda) = \lambda\,d\lambda$ *where* $d\lambda$ *is the Lebesgue measure on* $\mathbb{R}$.

Remark 46. Notice that in this example the domain of the representation $\mathcal{H}$ does not depend on λ.

4.5.1. The kernel of the hypoelliptic heat equation

Consider the representations $\hat{X}_i^\lambda$ of the differential operators L_{X_i} with $i = 0, 1$: they are operators on $\mathcal{H}$, whose action on $\psi \in \mathcal{H}$ is (using formula (19))

$$\big[\hat{X}_0^\lambda\psi\big](\theta) = \frac{d\psi(\theta)}{d\theta}, \qquad \big[\hat{X}_1^\lambda\psi\big](\theta) = i\lambda\cos(\theta)\psi(\theta),$$

hence

$$\big[\hat{\Delta}_{sr}^n\psi\big](\theta) = \frac{d^2\psi(\theta)}{d\theta^2} - \lambda^2\cos^2(\theta)\psi(\theta).$$

We have to find a complete set of eigenfunctions of norm one for $\hat{\Delta}_{sr}^n$. An eigenfunction ψ with eigenvalue E is a 2π-periodic function satisfying the *Mathieu's equation*

$$\frac{d^2\psi}{dx^2} + \big(a - 2q\cos(2x)\big)\psi = 0 \tag{43}$$

with $a = -\frac{\lambda^2}{2} - E$ and $q = \frac{\lambda^2}{4}$. For details about Mathieu functions see for instance [1, Chapter 20].

Remark 47. Notice that we consider only 2π-periodic solutions of (43) since $\mathcal{H} = L^2(S^1, \mathbb{C})$.

There exists an ordered discrete set $\{a_n(q)\}_{n=0}^{+\infty}$ of distinct real numbers ($a_n < a_{n+1}$) such that the equation $\frac{d^2 f}{dx^2} + (a_n - 2q\cos(2x))f = 0$ admits a unique even 2π-periodic solution of norm one. This function $\mathrm{ce}_n(x, q)$ is called an *even Mathieu function*.

Similarly, there exists an ordered discrete set $\{b_n(q)\}_{n=1}^{+\infty}$ of distinct real numbers ($b_n < b_{n+1}$) such that the equation $\frac{d^2 f}{dx^2} + (b_n - 2q\cos(2x))f = 0$ admits a unique odd 2π-periodic solution of norm 1. This function $\mathrm{se}_n(x, q)$ is called an *odd Mathieu function*.

The set $\mathcal{B}^\lambda := \{\mathrm{ce}_n(x, \frac{\lambda^2}{4})\}_{n=0}^{+\infty} \cup \{\mathrm{se}_n(x, \frac{\lambda^2}{4})\}_{n=1}^{+\infty}$ is a complete set of 2π-periodic eigenfunctions of norm one for the operator $\hat{\Delta}_{sr}^n$. The eigenvalue for $\mathrm{ce}_n(x, \frac{\lambda^2}{4})$ is $a_n^\lambda := -\frac{\lambda^2}{2} - a_n(\frac{\lambda^2}{4})$. The eigenvalue for $\mathrm{se}_n(x, \frac{\lambda^2}{4})$ is $b_n^\lambda := -\frac{\lambda^2}{2} - b_n(\frac{\lambda^2}{4})$.

We can now compute the explicit expression of the hypoelliptic kernel on $SE(2)$.

Proposition 48. *The kernel of the hypoelliptic heat equation on* $(SE(2), \blacktriangle, \mathbf{g})$ *is*

$$p_t(g) = \int_0^{+\infty} \lambda\, d\lambda \left(\sum_{n=0}^{+\infty} e^{a_n^\lambda t} \langle \mathrm{ce}_n(\theta), \mathfrak{X}^\lambda(g)\mathrm{ce}_n(\theta)\rangle + \sum_{n=1}^{+\infty} e^{b_n^\lambda t} \langle \mathrm{se}_n(\theta), \mathfrak{X}^\lambda(g)\mathrm{se}(\theta)\rangle \right). \tag{44}$$

The function (44) *is real for all* $t > 0$.

Proof. The formula (44) is given by writing the formula (25) in the $SE(2)$ case.

We have to prove that $p_t(g)$ is real: we claim that $\langle \mathrm{ce}_n, \mathfrak{X}^\lambda(g)\mathrm{ce}_n\rangle$ is real. In fact, write the scalar product with $g = (\alpha, x, y)$:

$$\langle \mathrm{ce}_n, \mathfrak{X}^\lambda(g)\mathrm{ce}_n\rangle = \int_0^{2\pi} e^{i\lambda(x\cos(\theta) - y\sin(\theta))} \mathrm{ce}_n(\theta)\mathrm{ce}_n(\theta + \alpha)\, d\theta.$$

Its imaginary part is $\int_0^{2\pi} \sin(\lambda(x\cos(\theta) - y\sin(\theta)))\mathrm{ce}_n(\theta)\mathrm{ce}_n(\theta + \alpha)$. Its integrand function assumes opposite values in θ and $\theta + \pi$: indeed

$$\begin{aligned} \sin\big(\lambda\big(x\cos(\theta+\pi) - y\sin(\theta+\pi)\big)\big) &= \sin\big(\lambda\big(-x\cos(\theta) + y\sin(\theta)\big)\big) \\ &= -\sin\big(\lambda\big(+x\cos(\theta) - y\sin(\theta)\big)\big), \end{aligned}$$

while $\mathrm{ce}_n(\theta + \pi) = (-1)^n \mathrm{ce}_n(\theta)$ as a property of Mathieu functions. Thus, the integral over $[0, 2\pi]$ is null. With similar observations it is possible to prove that $\langle \mathrm{se}_n(\theta), \mathfrak{X}^\lambda(g)\mathrm{se}_n(\theta)\rangle$ is real.

Thus, $p_t(g)$ is an integral of a sum of real functions, hence it is real. □

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