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Classification of Normal Operators in Spaces With Indefinite Scalar Product of Rank 2

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ABSTRACT

A finite-dimensional complex space with indefinite scalar product $[\cdot\,,\cdot]$ having $v_- = 2$ negative squares and $v_+ \geq 2$ positive ones is considered. The paper presents a classification of operators that are normal with respect to this product. It relates to the study by Gohberg and Reichstein in which the similar classification was obtained for the case $v = \min\{v_-, v_+\} = 1$.

1. INTRODUCTION

Consider a complex linear space C^n with an indefinite scalar product $[\cdot\,,\cdot]$. By definition, the latter is a nondegenerate sesquilinear Hermitian form. If the ordinary scalar product $(\cdot\,,\cdot)$ is fixed, then there exists a nondegenerate Hermitian operator H such that $[x,y] = (Hx,y)\ \forall x,y \in C^n$. If A is a linear operator ($A\colon C^n \to C^n$), then the *H-adjoint* of A (denoted by $A^{[*]}$) is defined by the identity $[A^{[*]}x,y] = [x,Ay]$ (hence $A^{[*]} = H^{-1}A^*H$). An operator N is called *H-normal* if $NN^{[*]} = N^{[*]}N$, an operator U is called *H-unitary* if $UU^{[*]} = I$, where I is the identity transformation.

Let V be a nontrivial subspace of C^n. V is called *neutral* if $[x,y] = 0$ for all $x,y \in V$. In this case we may write $[V,V] = 0$. V is called *nondegenerate* if from $x \in V$ and $\forall y \in V$ $[x,y] = 0$ it follows that $x = 0$. The subspace $V^{[\perp]}$ is defined as the set of all vectors $x \in C^n$: $[x,y] = 0\ \ \forall y \in V$. If V is nondegenerate, then $V^{[\perp]}$ is also nondegenerate and $V \dot{+} V^{[\perp]} = C^n$.

A linear operator A acting in C^n is called *decomposable* if there exists a nondegenerate subspace $V \subset C^n$ such that both V and $V^{[\perp]}$ are invariant for A. Then A is the *orthogonal sum of* $A_1 = A|_V$ *and* $A_2 = A|_{V^{[\perp]}}$. Since the conditions $AV^{[\perp]} \subseteq V^{[\perp]}$ and $A^{[*]}V \subseteq V$ are equivalent, an operator

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A is decomposable if there exists a nondegenerate subspace V which is invariant both for A and $A^{[*]}$.

Pairs of matrices $\{A_1, H_1\}$ and $\{A_2, H_2\}$, where H_1 and H_2 are Hermitian, are called *unitarily similar* if $A_2 = T^{-1}A_1T$, $H_2 = T^*H_1T$ for some invertible T; in the case when $H_1 = H_2$ they are H_1-*unitarily similar.*

Throughout what follows by a rank of a space we mean $v = \min\{v_-, v_+\}$, where v_- (v_+) is the number of negative (positive) squares of the quadratic form $[x, x]$, or (it is the same) the number of negative (positive) eigenvalues of the operator H. Note that without loss of generality it can be assumed that $v_- \leq v_+$ (otherwise H can be replaced by $-H$; the latter [invertible and Hermitian operator] has opposite eigenvalues).

Our aim is to obtain a complete classification for H-normal operators acting in the space C^n of rank 2, i.e., to find a set of canonical forms such that any H-normal operator could be reduced to one and only one of these forms. This means that for any invertible Hermitian matrix H with $v = 2$ and for any H-normal matrix N we must point out one and only one of the canonical pairs of matrices $\{\widetilde{N}, \widetilde{H}\}$ such that the pair $\{N, H\}$ is unitarily similar to $\{\widetilde{N}, \widetilde{H}\}$.

Since any H-normal operator N: $C^n \to C^n$ is an orthogonal sum of H-normal operators each of which has one or two distinct eigenvalues (Lemma 1 from [1]), it is sufficient to solve our problem only for indecomposable operators having one or two distinct eigenvalues.

Thus, in this paper we consider only indecomposable operators having one or two distinct eigenvalues and assume that $2 = v_- \leq v_+$.

Finally let us introduce some notation. Denote the identity matrix of order $r \times r$ by I_r, the $r \times r$ matrix with 1's on the secondary diagonal and zeros elsewhere by D_r, and a block diagonal matrix with $A, B, \dots, C$ diagonal blocks by $A \oplus B \oplus \cdots \oplus C$:

$$I_r = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \qquad D_r = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$

$$A \oplus B \oplus \cdots \oplus C = \begin{pmatrix} A & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & C \end{pmatrix}.$$

2. SOME PROPERTIES OF INDECOMPOSABLE H-NORMAL OPERATORS

The results of this section hold for any finite-dimensional space with indefinite scalar product.

PROPOSITION 1. *Let an indecomposable H-normal operator N acting in C^n $(n > 1)$ have the only eigenvalue λ; then there exists a decomposition of C^n into a direct sum of subspaces*

$$S_0 = \{x \in C^n : \ (N - \lambda I)x = (N^{[*]} - \overline{\lambda} I)x = 0\}, \tag{1}$$

S, S_1 such that

$$N = \begin{pmatrix} N' = \lambda I & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' = \lambda I \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{2}$$

where N': $S_0 \to S_0$, N_1: $S \to S$, N'': $S_1 \to S_1$, the internal operator N_1 is H_1-normal, and the pair $\{N_1, H_1\}$ is determined up to the unitary similarity.

Proof. Since N and $N^{[*]}$ commute, the subspace S_0 defined by (1) is nontrivial. For N to be indecomposable S_0 must be neutral. Indeed, otherwise $\exists v \in S_0 : Nv = \lambda v$, $N^{[*]} = \overline{\lambda} v$, $[v, v] \neq 0$; therefore, $V = \operatorname{span}\{v\}$ is a nondegenerate subspace that is invariant both for N and $N^{[*]}$, and hence, N is decomposable. Thus, S_0 is neutral. Let us take advantage of the following well-known result: *for any neutral subspace $V_1 \subset C^n$ there exists a subspace V_2 $(V_1 \cap V_2 = \{0\})$ such that*

$$H|_{(V_1 \dot{+} V_2)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3}$$

Therefore, for S_0 there exists a neutral subspace S_1 such that $H|_{(S_0 \dot{+} S_1)}$ has form (3). Since the subspace $(S_0 \dot{+} S_1)$ is nondegenerate, the subspace $S = (S_0 \dot{+} S_1)^{[\perp]}$ is also nondegenerate and $C^n = S_0 \dot{+} S \dot{+} S_1$. As $\forall v \in C^n$ $(N - \lambda I)v \in S_0^{[\perp]}$ and $(N^{[*]} - \overline{\lambda} I)v \in S_0^{[\perp]}$, the matrices N and H has form (2) with respect to the decomposition $C^n = S_0 \dot{+} S \dot{+} S_1$. Since N is H-normal, the internal operator N_1 is H_1-normal.

It is seen that only the subspace S_0 is fixed; S and S_1 may change. However, the pair $\{N_1, H_1\}$ is unique in a sense, namely, it is determined up to the unitary similarity. Indeed, any transformation T such that $TS_0 \subseteq S_0$ has the form

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & T_6 & T_7 \end{pmatrix}.$$

Since

$$\widetilde{H} = \begin{pmatrix} 0 & 0 & I \\ 0 & \widetilde{H}_1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

from condition $\widetilde{H} = T^* H T$ it follows that $T_6 = 0$, $\widetilde{H}_1 = T_4^* H_1 T_4$. As $\widetilde{N} = T^{-1} N T$, $\widetilde{N}_1 = T_4^{-1} N_1 T_4$ so that the pair $\{N_1, H_1\}$ is unitarily similar to $\{\widetilde{N}_1, \widetilde{H}_1\}$. ∎

REMARK. The decomposition $C^n = S_0 \dot{+} S \dot{+} S_1$ was constructed in [1, Section 6] so that the first part of this statement is borrowed from [1].

COROLLARY. *To go over from one decomposition $C^n = S_0 \dot{+} S \dot{+} S_1$ to another by means of a transformation T it is necessary that T would be block triangular with respect to both decompositions.*

THEOREM 1. *If an H-normal operator N acting in a space C^n of rank $k \geq 1$ is indecomposable, then either* (A) *or* (B) *holds:*

(A) *N has two eigenvalues and $n = 2k$;*
(B) *N has one eigenvalue and $2k \leq n \leq 4k$.*

Proof. First show that $n \geq 2k$. Indeed, $n = v_- + v_+ \geq 2\min\{v_-, v_+\} = 2k$. Now prove (A). Let N have two distinct eigenvalues. Then, according to Lemma 1 from [1], C^n is a direct sum of two neutral subspaces of the same dimension m which are invariant for N and $N^{[*]}$. Since in a space with indefinite scalar product no neutral space can be of dimension more than rank of a space, $m \leq k$ and $n \leq 2k$. But it was established before that $n \geq 2k$. Hence, $n = 2k$ and the proof of (A) is completed.

Now prove (B); i.e., show that if N has one eigenvalue, then $n \leq 4k$. For $k = 1$ the proof is given in [1, Theorem 1]. Suppose inductively that for all $i \leq k$ the size of indecomposable operators having one eigenvalue is not more than $4i \times 4i$. Let $v_- = k+1$, $v_+ \geq v_-$, N have the only eigenvalue λ. According to Proposition 1, one can assume that the matrices N and H have form (2). Let $N_1 = N_1^{(1)} \oplus \cdots \oplus N_1^{(p)}$ be a decomposition of the internal operator N_1 into an orthogonal sum of indecomposable operators, and let $H_1 = H_1^{(1)} \oplus \cdots \oplus H_1^{(p)}$, $S = S^{(1)} \oplus \cdots \oplus S^{(p)}$ be the corresponding decompositions of H_1 and S. Let $v_-^{(i)}$ be the number of negative eigenvalues of $H_1^{(i)}$ $(i = 1, \ldots p)$. If $\dim S_0 = s$, then $\sum_{i=1}^p v_-^{(i)} = k + 1 - s$. Let

$$H_1' = \sum_{v_-^{(i)} > 0} H_1^{(i)}, \qquad H_1'' = \sum_{v_-^{(i)} = 0} H_1^{(i)}.$$

Then $H_1 = H_1' \oplus H_1''$, $N_1 = N_1' \oplus N_1''$, where N_1', N_1'' are the corresponding sums of operators $N_1^{(i)}$. Since for any $i = 1, \ldots, p$ rank of the subspace $S_1^{(i)}$ is not more than $v_-^{(i)}$, $v_-^{(i)} \leq k$ (because $k+1-s \leq k$), and the size of an indecomposable operator in a space of rank 0 is equal to 1, by the inductive hypothesis dim $S^{(i)} \leq 4v_-^{(i)}$; hence dim $S' \leq 4(k+1-s)$. Since H_1'' has only positive eigenvalues, N_1'' is a usual normal operator having one eigenvalue λ; therefore, $N_1'' = \lambda I$ so that

$$N = \begin{pmatrix} \lambda I & * & M_1 & * \\ 0 & N_1' & 0 & * \\ 0 & 0 & \lambda I & * \\ 0 & 0 & 0 & \lambda I \end{pmatrix}, \qquad N^{[*]} = \begin{pmatrix} \overline{\lambda} I & * & M_2 & * \\ 0 & N_1'^{[*]} & 0 & * \\ 0 & 0 & \overline{\lambda} I & * \\ 0 & 0 & 0 & \overline{\lambda} I \end{pmatrix}.$$

If dim $S'' = r > 2s$, then the system

$$M_1 X = 0$$
$$M_2 X = 0$$

has a nontrivial solution $X = (x_1, \ldots, x_r)^T$ (where Y^T is Y transposed). Therefore, there exists a nonzero vector $v = \sum_{i=1}^{r} x_i w_i$ (w_i are the basis vectors of S'') that satisfies the condition $(N - \lambda I)v = (N^{[*]} - \overline{\lambda} I)v = 0$, i.e., $v \in S_0$. But $S_0 \cap S = \{0\}$. This contradiction proves that dim $S'' \leq 2s$. Thus, $n = 2$ dim $S_0 +$ dim $S' +$ dim $S'' \leq 2s + 4(k+1-s) + 2s = 4(k+1)$.

Since an indecomposable operator cannot have more than two eigenvalues [1, Lemma 1], either (A) or (B) is true so that the proof of the theorem is completed. ■

3. THE CLASSIFICATION OF INDECOMPOSABLE H-NORMAL OPERATORS

The principal aim of this paper is to prove the following result:

THEOREM 2. *If an indecomposable H-normal operator N (N: $C^n \to C^n$) acts in a space with indefinite scalar product with $v_- = 2$ negative squares and $v_+ \geq 2$ positive ones, then $4 \leq n \leq 8$ and the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs $\{(4), (5)\}$–$\{(31), (32)\}$. The choice of the particular canonical form is determined as follows.*

If N has two distinct eigenvalues λ_1, λ_2, then $\{N, H\}$ is unitarily similar to $\{(4), (5)\}$:

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & x & \lambda_2 \end{pmatrix}, \qquad x \in C,$$

$$\textit{for } x \neq 0 \begin{bmatrix} \operatorname{Im}\{\lambda_1 - \lambda_2\} > 0 & \textit{if } \operatorname{Im}\{\lambda_1 - \lambda_2\} \neq 0, \\ \operatorname{Re}\{\lambda_1 - \lambda_2\} > 0 & \textit{otherwise}, \end{bmatrix} \tag{4}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{5}$$

If N has one eigenvalue λ, $\dim S_0 = 1$, *the internal operator N_1 is indecomposable, and $n = 4$, then $\{N, H\}$ is unitarily similar to* $\{(6), (7)\}$:

$$N = \begin{pmatrix} \lambda & 1 & ir_1 & ir_2 z \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r_1, r_2 \in \Re, \tag{6}$$

$$H = D_4. \tag{7}$$

If N has one eigenvalue λ, $\dim S_0 = 1, N_1$ *is indecomposable, and $n = 5$, then $\{N, H\}$ is unitarily similar to one and only one of pairs* $\{(8), (11)\}$, $\{(9), (11)\}$, $\{(10), (11)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & 1 & ir_1 & -2r_1^2 + ir_2 \\ 0 & 0 & \lambda & 1 & 2ir_1 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re, \tag{8}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & z & r_1 & -2z^2 r_1^2 \operatorname{Im}^2 z + ir_2 z^2 \\ 0 & 0 & \lambda & z & -2ir_1 z^2 \operatorname{Im} z \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{matrix} |z| = 1,\ z \neq i, \\ 0 < \arg z < \pi, \\ r_1, r_2, r_3 \in \Re, \end{matrix} \tag{9}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & r_3 \\ 0 & \lambda & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & \lambda & i & 2ir_1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re, \tag{10}$$

$$H = D_5. \tag{11}$$

If N has one eigenvalue $\lambda, \dim S_0 = 1, N_1$ is decomposable, and $n = 4$, then $\{N, H\}$ is unitarily similar to one and only one of pairs $\{(12), (15)\}$, $\{(13), (15)\}, \{(14), (15)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1, \tag{12}$$

$$N = \begin{pmatrix} \lambda & 1 & 1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & (1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r \in \Re > 0, \tag{13}$$

$$N = \begin{pmatrix} \lambda & 1 & -1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & -(1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r \in \Re > 0, \tag{14}$$

$$H = D_4. \tag{15}$$

If N has one eigenvalue λ, $\dim S_0 = 1, N_1$ is decomposable, and $n = 5$, then $\{N, H\}$ is unitarily similar to $\{(16), (17)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & \lambda & 0 & z & 0 \\ 0 & 0 & \lambda & 0 & r_1 \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r_1, r_2 \in \Re,\ r_1 > 0, \tag{16}$$

$$H = D_5. \tag{17}$$

If N has one eigenvalue $\lambda, \dim S_0 = 1, N_1$ is decomposable, and $n = 6$, then $\{N, H\}$ is unitarily similar to one and only one of pairs $\{(18), (20)\}$, $\{(19), (20)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$r_1, r_2 \in \Re,\ r_2 > 0, \tag{18}$$

$$N = \begin{pmatrix} \lambda & 1 & -2ir_1 \operatorname{Im} z & 0 & 0 & 0 \\ 0 & \lambda & z & r_1 & 0 & (2r_1^2 \operatorname{Im}^2 z - r_2^2/2 + ir_3)z^2 \\ 0 & 0 & \lambda & z & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & z^2 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z| = 1,\ 0 < \arg z < \pi,\ r_1, r_2, r_3 \in \Re,\ r_2 > 0, \tag{19}$$

$$H = \begin{pmatrix} 0 & 0 & 0 & I_1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & I_1 & 0 \\ I_1 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

If N has one eigenvalue $\lambda, \dim S_0 = 2$, and $n = 4$, then $\{N, H\}$ is unitarily similar to one and only one of pairs $\{(21), (23)\}$, $\{(22), (23)\}$:

$$N = \begin{pmatrix} \lambda & 0 & z & re^{-i\pi/3}z \\ 0 & \lambda & 0 & e^{i\pi/3}z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{array}{l} |z| = 1,\ r \in \Re \geq \sqrt{3}, \\ 0 \leq \arg z < \pi \ \text{ if } r > \sqrt{3}, \end{array} \tag{21}$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{22}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{23}$$

If N has one eigenvalue λ, $\dim S_0 = 2$, and $n = 5$, then $\{N, H\}$ is unitarily similar to one and only one of pairs $\{(24),(26)\}, \{(25),(26)\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & \lambda & z & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1, \tag{24}$$

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & r & z \\ 0 & 0 & \lambda & z^2 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r \in \Re > 0, \tag{25}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_1 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{26}$$

If N has one eigenvalue λ, $\dim S_0 = 2$, and $n = 6$, then $\{N, H\}$ is unitarily similar to $\{(27),(28)\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & ir_1 & 0 \\ 0 & \lambda & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & \lambda & 0 & z & 0 \\ 0 & 0 & 0 & \lambda & 0 & z \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{array}{l} |z| = 1,\ z \neq -1, \\ r_1, r_2 \in \Re,\ r_2 > 0, \end{array} \tag{27}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{28}$$

If N has one eigenvalue λ, $\dim S_0 = 2$, and $n = 7$, then $\{N, H\}$ is unitarily similar to $\{(29), (30)\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -z_1\overline{z_2}\cos\alpha & \sin\alpha\cos\beta \\ 0 & 0 & 0 & \lambda & 0 & z_1\sin\alpha & z_2\cos\alpha\cos\beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \quad 0 < \alpha, \beta \leq \pi/2,$$

$$z_1 = 1 \ \ if\ \beta = \pi/2, \quad z_2 = 1 \ \ if\ \alpha = \pi/2, \tag{29}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{30}$$

If N has one eigenvalue λ, $\dim S_0 = 2$, and $n = 8$, then $\{N, H\}$ is unitarily similar to $\{(31), (32)\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -z_1\overline{z_2}\sin\alpha\cos\beta & \cos\alpha\cos\gamma \\ 0 & 0 & 0 & \lambda & 0 & 0 & z_1\cos\alpha\cos\beta & z_2\sin\alpha\cos\gamma \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \sin\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \ 0 \leq \alpha < \pi/2, \ 0 < \beta < \gamma \leq \pi/2,$$

$$z_1 = 1 \ \ if\ \gamma = \pi/2, \quad z_2 = 1 \ \ if\ \alpha = 0 \tag{31}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{32}$$

The following sections contain the proof of this theorem.

4. TWO DISTINCT EIGENVALUES OF N

Suppose an indecomposable H-normal operator N has two distinct eigenvalues. Then [1, Lemma 1] $C^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$, $\dim \mathcal{Q}_1 = \dim \mathcal{Q}_2 = m$, $[\mathcal{Q}_1, \mathcal{Q}_1] = 0$, $[\mathcal{Q}_2, \mathcal{Q}_2] = 0$, $N\mathcal{Q}_1 \subseteq \mathcal{Q}_1$, $N\mathcal{Q}_2 \subseteq \mathcal{Q}_2$, $N_1 = N|_{\mathcal{Q}_1}$ ($N_2 = N|_{\mathcal{Q}_2}$) has only one eigenvalue λ_1 (λ_2). According to Theorem 1, $m = 2$ and $n = 4$. Note that the subspaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are determined up to interchanging. Since N is indecomposable, at least one of the operators N_1, N_2 is not scalar. Consequently, one can assume $N_1 \neq \lambda_1 I$. If both N_1 and N_2 are not scalar, then we can fix $\mathrm{Im}\{\lambda_1 - \lambda_2\} > 0$ if $\mathrm{Im}\{\lambda_1 - \lambda_2\} \neq 0$ and $\mathrm{Re}\{\lambda_1 - \lambda_2\} > 0$ if $\mathrm{Im}\{\lambda_1 - \lambda_2\} = 0$ (let us remember that $\lambda_1 \neq \lambda_2$). Now $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are determined uniquely.

As H is nondegenerate, for any basis in $\mathcal{Q}_1$ there exists a basis in $\mathcal{Q}_2$ such that

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let us fix a basis in $\mathcal{Q}_1$ such that

$$N_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \tag{33}$$

N is H-normal if and only if

$$N_1 N_2{}^* = N_2{}^* N_1. \tag{34}$$

From (34) it follows that $N_2^* = \alpha N_1 + \beta I$. As $N_2 = \overline{\alpha} N_1^* + \overline{\beta} I$ has the only eigenvalue λ_2, we conclude $N_2 = \lambda_2 I + x(N_1^* - \overline{\lambda_1} I)$ $(x \in C)$. Thus, we have reduced N to the form

$$N = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \oplus \begin{pmatrix} \lambda_2 & 0 \\ x & \lambda_2 \end{pmatrix}, \qquad x \in C. \tag{35}$$

Show that forms (35) with different values of x are not H-unitarily similar. To this end suppose that some matrix T satisfies the conditions

$$NT = T\widetilde{N}, \tag{36}$$

$$TT^{[*]} = I, \tag{37}$$

where $N = N_1 \oplus N_2$, $\widetilde{N} = N_1 \oplus \widetilde{N}_2$, N_1 has form (33),

$$N_2 = \begin{pmatrix} \lambda_2 & 0 \\ x & \lambda_2 \end{pmatrix}, \qquad \widetilde{N}_2 = \begin{pmatrix} \lambda_2 & 0 \\ \widetilde{x} & \lambda_2 \end{pmatrix}.$$

From (36) it follows that T is block diagonal with respect to the decomposition $C^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$: $T = T_1 \oplus T_2$, T_1 satisfying the condition $N_1 = T_1^{-1} N_1 T_1$. Taking into account (37), we get $T_2 = T_1^{*-1}$; therefore, $\widetilde{N}_2 = T_2^{-1} N_2 T_2 = N_2$, i.e., $\widetilde{x} = x$.

It can easily be checked that (35) is indecomposable so that we have proved the following lemma:

LEMMA 1. *If an indecomposable H-normal operator acts in a space C^n of rank 2 and has two distinct eigenvalues λ_1, λ_2, then $n = 4$ and the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(4), (5)\}$:*

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & x & \lambda_2 \end{pmatrix}, \qquad x \in C,$$

$$\textit{for } x \neq 0 \begin{bmatrix} \operatorname{Im}\{\lambda_1 - \lambda_2\} > 0 & \quad \textit{if } \operatorname{Im}\{\lambda_1 - \lambda_2\} \neq 0, \\ \operatorname{Re}\{\lambda_1 - \lambda_2\} > 0 & \quad \textit{otherwise,} \end{bmatrix}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where the number x forms a complete and minimal invariant of the pair $\{N, H\}$ under the unitary similarity (in short, we say that x is an H-unitary invariant). In other words, every pair $\{N, H\}$ satisfying the hypothesis of the lemma is unitary similar to pair $\{(4), (5)\}$ and pairs $\{(4), (5)\}$ with different values of x are not H-unitarily similar to each other.

5. ONE EIGENVALUE OF N

Throughout what follows we assume that N has only one eigenvalue λ so that N and H have form (2). Since the neutral subspace S_0 cannot be more than two dimensional, there appear two cases to be considered: dim $S_0 = 1$ and dim $S_0 = 2$. Now let us prove the following proposition which holds for all spaces with indefinite scalar product:

PROPOSITION 2. *An H-normal operator such that* dim $S_0 = 1$ *is indecomposable.*

Proof. Assume the converse. Suppose some nondegenerate subspace V is invariant both for N and for $N^{[*]}$. Let us denote $V_1 = V$, $V_2 = V^{[\perp]}$, $N_1 = N|_{V_1}$, $N_2 = N|_{V_2}$, $H_1 = H|_{V_1}$, $H_2 = H|_{V_2}$. The following conditions must hold: $N_1 N_1^{[*]} = N_1^{[*]} N_1$, $N_2 N_2^{[*]} = N_2^{[*]} N_2$. Here $N_i^{[*]}$ is the H_i-adjoint of N_i $(i = 1, 2)$. Let us define

$$S_0^i = \{x \in V_i : (N_i - \lambda I)x = (N_i^{[*]} - \overline{\lambda} I)x = 0\}, \qquad i = 1, 2.$$

Since the operators N_1 and $N_1^{[*]}$ (N_2 and $N_2^{[*]}$) commute, dim $S_0^i \geq 1$ $(i = 1, 2)$; therefore, $\dim\{S_0 = S_0^1 + S_0^2\} \geq 2$. This contradicts the condition dim $S_0 = 1$. Thus, N is indecomposable. ■

If dim $S_0 = 1$, then rank of S is equal to 1; therefore, to classify the internal operator N_1 we may apply Theorem 1 from [1]. Since the indecomposability (or decomposability) of N_1 is a property that does not change under the unitary similarity of the pair $\{N_1, H_1\}$, we must consider both the case when N_1 is indecomposable and that when N_1 is decomposable.

5.1. dim $S_0 = 1$ *and* N_1 *is Indecomposable*

If N_1 is indecomposable, then, according to Theorem 1, $2 \leq \dim S \leq 4$ (recall that rank of S is equal to 1). Therefore, $4 \leq n \leq 6$. Let us consider the alternatives $n = 4, 5, 6$ one after another.

5.1.1. $n = 4$ According to Theorem 1 of [1], one can assume that N_1 and H_1 are reduced to the form

$$N_1 = \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix}, \qquad |z| = 1, \qquad H_1 = D_2.$$

Hence

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & z & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H = D_4.$$

Throughout what follows only H-unitary transformations are used unless otherwise stipulated. This means that for each case we fix some form of the matrix H and find out to what form it is possible to reduce the matrix N without the change of H.

The condition of the H-normality of N is equivalent to the system

$$a\overline{z} = \overline{e}z \tag{38}$$

$$\operatorname{Re}\{a\overline{b}\} = \operatorname{Re}\{d\overline{e}\}. \tag{39}$$

If $a = 0$, then $e = 0$; therefore, the vector v_2 from S (v_i are the basis vectors) belongs to S_0, which is impossible. Thus, $a \neq 0$. Replace the vector v_1 by av_1 and v_4 by $v_4/\overline{a}$. This transformation reduces $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' & c' \\ 0 & 0 & z & d' \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, apply the transformation

$$T = \begin{pmatrix} 1 & z\overline{d}\,' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\overline{z}d' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to the matrix $N - \lambda I$. We obtain

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b'' & c'' \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows from (39) that $b'' = ir_1$ $(r_1 \in \Re)$. Taking the transformation

$$T = \begin{pmatrix} 1 & 0 & \frac{1}{2}\overline{z}\operatorname{Re}\{c''\overline{z}\} & 0 \\ 0 & 1 & 0 & -\frac{1}{2}z\operatorname{Re}\{c''\overline{z}\} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we reduce $N - \lambda I$ to the final form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & ir_1 & ir_2 z \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1,\ r_1, r_2 \in \Re, \tag{40}$$

where $r_2 = \operatorname{Im}\{c''\overline{z}\}$.

Let us prove that the numbers z, r_1, r_2 are H-unitary invariants. Indeed, let T be an H-unitary transformation of the matrix N to the form $\widetilde{N}$, where

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & i\widetilde{r}_1 & i\widetilde{r}_2\widetilde{z} \\ 0 & 0 & \widetilde{z} & 0 \\ 0 & 0 & 0 & \widetilde{z}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |\widetilde{z}| = 1,\ \widetilde{r}_1, \widetilde{r}_2 \in \Re.$$

This means that T satisfies conditions (36) and (37). From the corollary of Proposition 1 it follows that T is block triangular with respect to the decomposition $C^n = S_0 + S + S_1$. According to Theorem 1 from [1], z is an H_1-unitary invariant of N_1. $T_4 = T|_S$ is an H_1-unitary transformation of N_1 to the form $\widetilde{N}_1$; therefore, z is also an H-unitary invariant of N, i.e., $\widetilde{z} = z$. Applying condition (36), we see that T is uppertriangular and its diagonal terms are equal to each other. From (37) it follows that $|t_{11}| = 1$. Therefore, without loss of generality one can assume that $t_{11} = 1$ (we replace our matrix T by the matrix $T' = \overline{t_{11}}T$; the latter has the same properties (36), (37)).

Thus,

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & t_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For T to be H-unitary it is neccessary and sufficient to have

$$\overline{t_{34}} + t_{12} = 0 \tag{41}$$

$$\overline{t_{24}} + \overline{t_{23}}t_{12} + t_{13} = 0 \tag{42}$$

$$\mathrm{Re}\, t_{14} + \mathrm{Re}\{t_{12}\overline{t_{13}}\} = 0 \tag{43}$$

$$\mathrm{Re}\, t_{23} = 0, \tag{44}$$

for T to reduce N to the form $\widetilde{N}$ it is neccessary and sufficient to have

$$t_{23} + ir_1 = i\widetilde{r}_1 + zt_{12} \tag{45}$$

$$t_{24} + ir_1t_{34} + ir_2z = i\widetilde{r}_2z + z^2t_{13} \tag{46}$$

$$zt_{34} = z^2t_{23}. \tag{47}$$

Express t_{34} in terms of t_{23} from (47) and t_{12} in terms of t_{23} from (45): $t_{34} = zt_{23}$, $t_{12} = \overline{z}(ir_1 - i\widetilde{r}_1) + \overline{z}t_{23}$. Substituting these expressions in (41), we get: $2\,\mathrm{Re}\, t_{23} = i(\widetilde{r}_1 - r_1)$. Since $\mathrm{Re}\, t_{23} = 0$ (condition (44)),

$\widetilde{r}_1 = r_1$. Further, let us express t_{24} in terms of t_{13} and t_{23} (condition (46)): $t_{24} = (i\widetilde{r}_2 - ir_2)z + z^2 t_{13} - ir_1 z t_{23}$. Then condition (42) can be written in the form

$$(ir_2 - i\widetilde{r}_2) + \overline{zt_{13}} + zt_{13} + ir_1\overline{t_{23}} + |t_{23}|^2 = 0.$$

As $\operatorname{Re} t_{23} = 0$, $ir_1\overline{t_{23}} \in \Re$, consequently, $\overline{zt_{13}} + zt_{13} + ir_1\overline{t_{23}} + |t_{23}|^2 \in \Re$. But $i(r_2 - \widetilde{r}_2) \in \Im$. Therefore, $\widetilde{r}_2 = r_2$. Thus, the numbers z, r_1, r_2 are H-unitary invariants.

Due to Proposition 2 matrix (40) is indecomposable so that we have proved the following lemma:

LEMMA 2. *If an indecomposable H-normal operator N (N: $C^4 \to C^4$) has the only eigenvalue λ, $\dim S_0 = 1$, and the internal operator N_1 is indecomposable, then the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(6), (7)\}$:*

$$N = \begin{pmatrix} \lambda & 1 & ir_1 & ir_2 z \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r_1, r_2 \in \Re,$$

$$H = D_4,$$

where z, r_1, r_2 are H-unitary invariants.

5.1.2. $n = 5$ According to [1, Theorem 1], it can be assumed that the pair $\{N_1, H_1\}$ has either form (48) or (49):

$$N_1 = \begin{pmatrix} \lambda & z & r \\ 0 & \lambda & z \\ 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ 0 < \arg z < \pi,\ r \in \Re,\ \ H_1 = D_3, \tag{48}$$

$$N_1 = \begin{pmatrix} \lambda & 1 & ir \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad r \in \Re,\ \ H_1 = D_3. \tag{49}$$

For a while we consider both the cases together, assuming that

$$N_1 = \begin{pmatrix} \lambda & z' & x \\ 0 & \lambda & z' \\ 0 & 0 & \lambda \end{pmatrix}, \qquad |z'| = 1,\ 0 \leq \arg z' < \pi,\ x \in C.$$

Then

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & z' & x & e \\ 0 & 0 & 0 & z' & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the H-normality is equivalent to the system

$$a\overline{z'} = \overline{g}z' \tag{50}$$

$$a\overline{x} + b\overline{z'} = \overline{g}x + \overline{f}z' \tag{51}$$

$$2\ \mathrm{Re}\{a\overline{c}\} + |b|^2 = 2\ \mathrm{Re}\{e\overline{g}\} + |f|^2. \tag{52}$$

As above (see the case when $n = 4$), one can check that $a \neq 0$; hence a can be assumed equal to 1, so $g = z'^2$. Having in mind these equalities, take the (H-unitary) transformation

$$T = \begin{pmatrix} 1 & \overline{z'}b & \overline{z'}(c - x\overline{z'}b) & 0 & -\frac{1}{2}|c - x\overline{z'}b|^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -z'(\overline{c} - \overline{x}z'\overline{b}) \\ 0 & 0 & 0 & 1 & -z'\overline{b} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It reduces $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & d' \\ 0 & 0 & z' & x & e' \\ 0 & 0 & 0 & z' & f' \\ 0 & 0 & 0 & 0 & z'^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now apply either the transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & \mathrm{Re}\, d'/(\mathrm{Re}\, z'^2 + 1) & 0 \\ 0 & 1 & 0 & 0 & -\mathrm{Re}\, d'/(\mathrm{Re}\, z'^2 + 1) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (z' \neq i)$$

or

$$T=\begin{pmatrix}1&0&0&-\frac{1}{2}i\operatorname{Im}d'&0\\0&1&0&0&-\frac{1}{2}i\operatorname{Im}d'\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\qquad(z'=i)$$

to the matrix $N-\lambda I$. We get

$$N-\lambda I=\begin{pmatrix}0&1&0&0&i(\operatorname{Im}d'+\operatorname{Im}\{d'\overline{z'}^2\})/(\operatorname{Re}z'^2+1)\\0&0&z'&x&e'\\0&0&0&z'&f'\\0&0&0&0&z'^2\\0&0&0&0&0\end{pmatrix}\qquad(z'\neq i)$$

or

$$N-\lambda I=\begin{pmatrix}0&1&0&0&\operatorname{Re}d'\\0&0&i&x&e'\\0&0&0&i&f'\\0&0&0&0&-1\\0&0&0&0&0\end{pmatrix}\qquad(z'=i).$$

Now we distinguish cases (48) and (49).

(a) $z'=1,\ \ x=ir_1$ ($r_1\in\Re$). Conditions (51), (52) of the H-normality of N yield: $f'=2ir_1$, $e'=-2r_1^2+ir_2$. Denote $(\operatorname{Im}d'+\operatorname{Im}\{d'\overline{z'}^2\})/(\operatorname{Re}z'^2+1)$ by r_3. We have

$$N-\lambda I=\begin{pmatrix}0&1&0&0&ir_3\\0&0&1&ir_1&-2r_1^2+ir_2\\0&0&0&1&2ir_1\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},\qquad r_1,r_2,r_3\in\Re.\tag{53}$$

There remains to check the H-unitary invariance of the numbers $r_1,\ r_2,\ r_3$. To prove this, let us suppose that some H-unitary matrix T reduces (53) to the form

$$\widetilde{N}-\lambda I=\begin{pmatrix}0&1&0&0&i\widetilde{r}_3\\0&0&1&i\widetilde{r}_1&-2\widetilde{r}_1^2+i\widetilde{r}_2\\0&0&0&1&2i\widetilde{r}_1\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},\qquad \widetilde{r}_1,\widetilde{r}_2,\widetilde{r}_3\in\Re.$$

From condition (36) $NT = T\widetilde{N}$ it follows that T is uppertriangular with diagonal terms which are equal to each other. According to [1, Theorem 1], r_1 is an H_1-unitary invariant for N_1. We already know that in this case r_1 must be an H-unitary invariant (see the previous case $n = 4$), i.e., $\widetilde{r}_1 = r_1$. For T to be H-unitary, i.e., to satisfy (37), $|t_{11}|$ must be equal to 1. Therefore, as in case $n = 4$, one can assume that $t_{11} = 1$. Thus, T has the form

$$T=\begin{pmatrix}1&t_{12}&t_{13}&t_{14}&t_{15}\\0&1&t_{23}&t_{24}&t_{25}\\0&0&1&t_{34}&t_{35}\\0&0&0&1&t_{45}\\0&0&0&0&1\end{pmatrix}. \tag{54}$$

Condition (36) amounts to system (55)–(60), (37) to system (61)–(66):

$$t_{23} = t_{12} \tag{55}$$

$$t_{24} = ir_1t_{12} + t_{13} \tag{56}$$

$$\begin{aligned} t_{25} + ir_3 = {} & i\widetilde{r}_3 + \left(-2r_1^2 + i\widetilde{r}_2\right)t_{12} \\ & + 2ir_1t_{13} + t_{14} \end{aligned} \tag{57}$$

$$t_{34} = t_{23} \tag{58}$$

$$t_{35} + ir_1t_{45} + ir_2 = i\widetilde{r}_2 + 2ir_1t_{23} + t_{24} \tag{59}$$

$$t_{45} = t_{34}, \tag{60}$$

$$\overline{t_{45}} + t_{12} = 0 \tag{61}$$

$$\overline{t_{35}} + \overline{t_{34}}t_{12} + t_{13} = 0 \tag{62}$$

$$\overline{t_{25}} + \overline{t_{24}}t_{12} + \overline{t_{23}}t_{13} + t_{14} = 0 \tag{63}$$

$$2\ \mathrm{Re}\ t_{15} + 2\ \mathrm{Re}\{t_{12}\overline{t_{14}}\} + |t_{13}|^2 = 0 \tag{64}$$

$$\overline{t_{34}} + t_{23} = 0 \tag{65}$$

$$2\ \mathrm{Re}\ t_{24} + |t_{23}|^2 = 0. \tag{66}$$

Express t_{35} in terms of t_{23}, t_{24}, t_{45} from (59) and substitute this expression in (62), taking into account that $t_{12} = t_{23} = t_{34} = t_{45}$ and expressing t_{24} in terms of t_{12} and t_{13} from condition (56). We obtain $ir_2 - i\widetilde{r}_2 = 2ir_1\overline{t_{12}} + 2\operatorname{Re} t_{13} + |t_{12}|^2$. Since $\operatorname{Re} t_{12} = 0$ (Eq. (61)), we have $2ir_1\overline{t_{12}} \in \Re$; hence, the right-hand side of the condition obtained is real and the left one is imaginary. Therefore, $\widetilde{r}_2 = r_2$.

Since $t_{13} = t_{24} - ir_1t_{12}$ (condition (56)), t_{25} can be expressed in terms of t_{12}, t_{24}, and t_{14} in the following way (see condition (57)): $t_{25} = i(\widetilde{r}_3 - r_3) + ir_2t_{12} + 2ir_1t_{24} + t_{14}$. By substituting this expression in (63), we get $ir_3 - i\widetilde{r}_3 = ir_2\overline{t_{12}} + ir_1(2\overline{t_{24}} + |t_{12}|^2) + 2\operatorname{Re}\{t_{12}\overline{t_{24}}\} + 2\operatorname{Re} t_{14}$. Because of condition (66) $ir_1(2\overline{t_{24}} + |t_{12}|^2)$ is real as well as the rest terms of the right-hand side; hence, $\widetilde{r}_3 = r_3$. We have proved the H-unitary invariance of r_1, r_2, r_3.

(b) $z' = z$, $|z| = 1, 0 < \arg z < \pi, x = r_1 \in \Re$. Applying conditions (51), (52) of the H-normality of N, we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & ir_3 \\ 0 & 0 & z & r_1 & -2z^2r_1^2\operatorname{Im}^2 z + ir_2z^2 \\ 0 & 0 & 0 & z & -2ir_1z^2\operatorname{Im} z \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re \ \ (z \neq i)$$

or

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & r_3 \\ 0 & 0 & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & 0 & i & 2ir_1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re \ \ (z = i).$$

We join these cases, assuming that

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & ix \\ 0 & 0 & z & r_1 & -2z^2r_1^2\operatorname{Im}^2 z + ir_2z^2 \\ 0 & 0 & 0 & z & -2ir_1z^2\operatorname{Im} z \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$x = \begin{bmatrix} r_3 \in \Re, & z \neq i \\ -ir_3 \in \Im\ (r_3 \in \Re), & z = i. \end{bmatrix}$$

Let us prove the H-unitary invariance of the numbers z, r_1, r_2, r_3 (or x). Suppose some matrix T realizes the H-unitary transformation of N to the form $\widetilde{N}$, where

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & i\widetilde{x} \\ 0 & 0 & \widetilde{z} & \widetilde{r}_1 & -2\widetilde{z}^2\widetilde{r}_1^2 \operatorname{Im}^2 \widetilde{z} + i\widetilde{r}_2\widetilde{z}^2 \\ 0 & 0 & 0 & \widetilde{z} & -2i\widetilde{r}_1\widetilde{z}^2 \operatorname{Im} \widetilde{z} \\ 0 & 0 & 0 & 0 & \widetilde{z}^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 1 of [1], z and r_1 are H_1-unitary invariants; hence, they are H-unitary invariants, i.e., $\widetilde{z} = z$, $\widetilde{r}_1 = r_1$. Further, from (36) it follows that T is uppertriangular with diagonal terms that are equal to each other. Applying (37), we get that T has form (54). Now condition (37) is equivalent to system (61)–(66), condition (36) to system (67)–(72):

$$t_{23} = zt_{12} \tag{67}$$

$$t_{24} = r_1 t_{12} + zt_{13} \tag{68}$$

$$\begin{aligned} t_{25} + ix = i\widetilde{x} &+ \left(-2z^2 r_1^2 \operatorname{Im}^2 z + i\widetilde{r}_2 z^2\right)t_{12} \\ &- 2ir_1 z^2 \operatorname{Im} z\, t_{13} + z^2 t_{14} \end{aligned} \tag{69}$$

$$t_{34} = t_{23} \tag{70}$$

$$zt_{35} + r_1 t_{45} + ir_2 z^2 = i\widetilde{r}_2 z^2 - 2ir_1 z^2 \operatorname{Im} z\, t_{23} + z^2 t_{24} \tag{71}$$

$$zt_{45} = z^2 t_{34}. \tag{72}$$

Express t_{35} in terms of t_{23}, t_{24}, t_{45} and, taking into account the equalities $t_{12} = \overline{z}t_{23}$ (67), $t_{13} = \overline{z}(t_{24} - r_1 t_{12})$ (68), $t_{34} = t_{23}$ (70), $t_{45} = zt_{23}$ (72), substitute the obtained expression in (62). After multiplying both sides by $\overline{z}$, we have: $(ir_2 - i\widetilde{r}_2) = -2ir_1 \operatorname{Im} z t_{23} + t_{24} + \overline{t_{24}} + |t_{23}|^2 - r_1(\overline{z}t_{23} + z\overline{t_{23}})$. Since $\operatorname{Re} t_{23} = 0$ (65), the right-hand side of this equality is real. Consequently, $\widetilde{r}_2 = r_2$.

Now let us express t_{25} in terms of t_{23}, t_{24}, t_{14} from (69): $t_{25} = i(\widetilde{x} - x) - 2r_1^2 z \operatorname{Im}^2 z\, t_{23} + ir_2 z t_{23} - 2ir_1 z \operatorname{Im} z\, t_{24} + 2ir_1^2 \operatorname{Im} z\, t_{23} + z^2 t_{14}$. Rewrite condition (63) in the form $t_{25} + t_{24}\overline{t_{12}} + t_{23}\overline{t_{13}} + \overline{t_{14}} = 0$, multiply both its sides by $\overline{z}$, and substitute the expression for t_{25} in it. We obtain: $i(x - \widetilde{x})\overline{z} = -2r_1^2 \operatorname{Im}^2 z\, t_{23} + ir_2 t_{23} - 2ir_1 \operatorname{Im} z\, t_{24} + 2ir_1^2 \overline{z} \operatorname{Im} z\, t_{23} + zt_{14} + \overline{z}\overline{t_{14}} + t_{23}\overline{t_{24}} + t_{24}\overline{t_{23}} - zr_1|t_{23}|^2$. Since $-2r_1^2 \operatorname{Im}^2 z + 2ir_1^2 \overline{z} \operatorname{Im} z = ir_1^2 \operatorname{Im} z \operatorname{Re} z$ and $-2ir_1 \operatorname{Im} z\, t_{24} - r_1 z|t_{23}|^2 = r_1(2 \operatorname{Re} z \operatorname{Re} t_{24} + 2 \operatorname{Im} z \operatorname{Im} t_{24})$, the right-hand side is real. Therefore, $\operatorname{Im}[i\overline{z}(x - \widetilde{x})] = 0$. If $z \neq i$, then this condition means $(r_3 - \widetilde{r}_3)\operatorname{Re} z = 0$; hence $\widetilde{r}_3 = r_3$ because $\operatorname{Re} z \neq 0$. If $z = i$, then $\operatorname{Im}[i(\widetilde{r}_3 - r_3)] = 0$; hence we also get $\widetilde{r}_3 = r_3$. This concludes the proof of the H-unitary invariance of z, r_1 r_2, r_3.

Due to Proposition 2 all obtained forms are indecomposable. They are not H-unitarily similar because their internal matrices N_1 are not H_1-unitarily similar due to [1, Theorem 1]. Thus, we have proved the following lemma:

LEMMA 3. *If an indecomposable H-normal operator N (N: $C^5 \to C^5$) has the only eigenvalue λ,* $\dim S_0 = 1$, *and the internal operator N_1 is indecomposable, then the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs $\{(8),(11)\}, \{(9),(11)\}, \{(10),(11)\}$:*

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & 1 & ir_1 & -2r_1^2 + ir_2 \\ 0 & 0 & \lambda & 1 & 2ir_1 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re,$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & z & r_1 & -2z^2 r_1^2 \operatorname{Im}^2 z + ir_2 z^2 \\ 0 & 0 & \lambda & z & -2ir_1 z^2 \operatorname{Im} z \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{array}{c} |z| = 1,\ z \neq i, \\ 0 < \arg z < \pi, \\ r_1, r_2, r_3 \in \Re, \end{array}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & r_3 \\ 0 & \lambda & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & \lambda & i & 2ir_1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad r_1, r_2, r_3 \in \Re,$$

$$H = D_5,$$

where z, r_1, r_2, r_3 are H-unitary invariants.

5.1.3. $n = 6$ In this case, according to [1, Theorem 1], the matrices N_1 and H_1 can be written in the form

$$N_1 = \begin{pmatrix} \lambda & \cos\alpha & \sin\alpha & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad 0 < \alpha \leq \pi/2, \ H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & \cos\alpha & \sin\alpha & 0 & f \\ 0 & 0 & 0 & 0 & 1 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the H-normality of N is equivalent to the following system:

$$a = \overline{p}\cos\alpha \tag{73}$$

$$0 = \overline{p}\sin\alpha \tag{74}$$

$$b\cos\alpha + c\sin\alpha = \overline{g}$$

$$2\,\mathrm{Re}\{a\overline{d}\} + |b|^2 + |c|^2 = 2\,\mathrm{Re}\{f\overline{p}\} + |g|^2 + |h|^2.$$

From (74) and the condition $0 < \alpha \leq \pi/2$ it follows that $p = 0$. Then from (73) we obtain also that $a = 0$. Hence, the vector $v_2 \in S$ belongs to S_0, which is impossible. This contradiction proves that for indecomposable operator $N\colon C^6 \to C^6$ $\dim S_0 \neq 1$.

Recall that if $n > 6$, then the operator N_1 is always decomposable [1, Theorem 1]. Thus, we have obtained the classification for all indecomposable operators N having also indecomposable internal operator N_1.

5.2. $\dim S_0 = 1$ *and* N_1 *is Decomposable*

If the operator N_1 is decomposable, then it can be represented as an orthogonal sum of indecomposable operators $N_1^{(1)}, \ldots, N_1^{(p)}$: $N_1 = N_1^{(1)} \oplus \cdots \oplus N_1^{(p)}$, $H_1 = H_1^{(1)} \oplus \cdots \oplus H_1^{(p)}$. Without loss of generality it can be assumed that $H_1^{(1)}$ has one negative eigenvalue. Denote $H_1^{(1)}$ by H_2, $N_1^{(1)}$ by N_2, $H_1^{(2)} \oplus \cdots \oplus H_1^{(p)}$ by H_3, $N_1^{(2)} \oplus \cdots \oplus N_1^{(p)}$ by N_3. Since H_3 has only positive eigenvalues, one can assume that $H_3 = I$. N_3 is a usual normal operator having the only eigenvalue λ; hence, $N_3 = \lambda I$.

Show that the size of N_3 is equal to 1×1. Indeed, let $\dim V_2 = k$, $\dim V_3 = l > 1$ (V_2 and V_3 are the subspaces of S corresponding to N_2 and N_3, respectively), $V_2 = \mathrm{span}\{w_1^{(2)}, w_2^{(2)}, \ldots, w_k^{(2)}\}$, $V_3 = \mathrm{span}\{w_1^{(3)}, w_2^{(3)}, \ldots, w_l^{(3)}\}$. Then, by the above,

$$N = \begin{pmatrix} \lambda & M_1 & M_2 & * \\ 0 & N_2 & 0 & * \\ 0 & 0 & \lambda I & * \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad N^{[*]} = \begin{pmatrix} \overline{\lambda} & M_3 & M_4 & * \\ 0 & N_2^{[*]} & 0 & * \\ 0 & 0 & \overline{\lambda} I & * \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix},$$

where $M_1 = (a_1, a_2, \dots, a_k)$, $M_2 = (b_1, b_2, \dots, b_l)$, $M_3 = (c_1, c_2, \dots, c_k)$, $M_4 = (d_1, d_2, \dots, d_l)$. Because of the H_2-normality of N_2 dim $S_0^{(2)} \geq 1$ ($S_0^{(2)} = \{x \in V_2 \colon (N_2 - \lambda I)x = (N_2^{[*]} - \overline{\lambda} I)x = 0\}$); hence, without loss of generality it can be assumed that $w_1^{(2)} \in S_0^{(2)}$. Since $l > 1$, $\exists \{\alpha_i\}_1^{n+1}$ ($\sum_1^{n+1} |\alpha_i| \neq 0$):

$$\sum_1^n \alpha_i b_i + \alpha_{n+1} a_1 = 0 \tag{75}$$

$$\sum_1^n \alpha_i d_i + \alpha_{n+1} c_1 = 0. \tag{76}$$

Therefore, $\exists v = \sum_1^n \alpha_i w_i^{(3)} + \alpha_{n+1} w_1^{(2)} \neq 0$: $(N - \lambda I)v = (N^{[*]} - \overline{\lambda} I)v = 0$; i.e., some nonzero vector from S belongs to S_0. This is impossible, so dim $V_3 = 1$.

As N_2 is indecomposable and rank of V_2 is less than or equal to 1, dim $V_2 \leq 4$ in accordance with Theorem 1. Thus, $1 \leq$ dim $V_2 \leq 4$, dim $V_3 = 1$ so that $4 \leq n \leq 7$. Consider the cases $n = 4, 5, 6, 7$ one after another.

5.2.1. $n = 4$ Then dim $V_2 = 1$, dim $V_3 = 1$,

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $H_1 = -1 \oplus 1$ is congruent to D_2, we assume that $H_1 = D_2$ so that $H = D_4$. Having fixed $H = D_4$, we apply, as is customary, only H-unitary transformations.

The condition of the H-normality of N is now equivalent to the following:

$$\operatorname{Re}\{a\overline{b}\} = \operatorname{Re}\{d\overline{e}\}. \tag{77}$$

Since the assumption $a = b = 0$ contradicts the condition $S \cap S_0 = \{0\}$ (because then either v_2 or v_3 belongs to S_0), one can assume that $a \neq 0$ and, therefore, $a = 1$ (see the paragraph after (39)). Keeping in mind that $a = 1$, reduce $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' = \operatorname{sgn} \operatorname{Re} b & c' \\ 0 & 0 & 0 & d' \\ 0 & 0 & 0 & e' \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

having applied either the transformation

$$T = \begin{pmatrix} \sqrt{|\operatorname{Re} b|} & 0 & 0 & 0 \\ 0 & \sqrt{|\operatorname{Re} b|} & -i \operatorname{Im} b/\sqrt{|\operatorname{Re} b|} & 0 \\ 0 & 0 & 1/\sqrt{|\operatorname{Re} b|} & 0 \\ 0 & 0 & 0 & 1/\sqrt{|\operatorname{Re} b|} \end{pmatrix} \qquad (\operatorname{Re} b \neq 0)$$

or

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\operatorname{Re} b = 0).$$

Now consider the three cases ($\operatorname{Re} b' = 0, 1$ or -1) separately.

(a) $b' = 0$. Since $\operatorname{Re}\{d'\overline{e'}\} = 0$ (condition (77) of the H-normality of N) and $d' \neq 0$ (otherwise $v_3 \in S_0$), the representation $d' = \varrho_1 z$, $e' = i\varrho_2 z$ ($|z| = 1$, $\varrho_1, \varrho_2 \in \Re$, $\varrho_1 > 0$) is valid. Therefore, taking

$$T = \begin{pmatrix} \sqrt{\varrho_1} & 0 & 0 & 0 \\ 0 & \sqrt{\varrho_1} & 0 & 0 \\ 0 & i\varrho_2/\sqrt{\varrho_1} & 1/\sqrt{\varrho_1} & 0 \\ 0 & 0 & 0 & 1/\sqrt{\varrho_1} \end{pmatrix},$$

we reduce $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c'' \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can assume that $c'' = 0$. To achieve this it is sufficient to apply the transformation

$$T = \begin{pmatrix} 1 & 0 & \overline{c''} & 0 \\ 0 & 1 & 0 & -c'' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There remains to prove that z is an H-unitary invariant. Indeed, any matrix T satisfying condition (36) $(N-\lambda I)T = T(\widetilde{N}-\lambda I)$ for the matrices

$$N-\lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{N}-\lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{z} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = |\widetilde{z}| = 1$$

and condition (37) $TT^{[*]} = I$ has the form

$$T = t_{11} \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad |t_{11}| = 1.$$

This follows the desired equality $z = \widetilde{z}$.

(b) $b' = 1$. As $\mathrm{Re}\{d'\overline{e'}\} = 1$ (condition (77)), $d' = \varrho z$, $e' = (1/\varrho + ir)z$ ($|z| = 1$, $\varrho, r \in \Re$, $\varrho > 0$). Consider the transformation

$$T = I_1 \oplus \begin{pmatrix} -it/(1-it) & 1/(1-it) \\ 1/(1-it) & -it/(1-it) \end{pmatrix} \oplus I_1, \qquad t \in \Re, \tag{78}$$

where t is a root of the equation $1 + t^2 = 1/\varrho^2 + (t\varrho + r)^2$. Its discriminant $\mathcal{D}/4 = 1/\varrho^2 + \varrho^2 + r^2 - 2$ is nonnegative so that t is in fact real. Subjecting to (78), the matrix $N - \lambda I$ becomes the following:

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 1 & c'' \\ 0 & 0 & 0 & z' \\ 0 & 0 & 0 & (1+ir')z' \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z'| = 1, \qquad r' \in \Re.$$

Note that if $r' = 0$, then there exists a nonzero vector $v = \alpha v_2 + \beta v_3 \in S_0$, which is impossible. Applying (78) with $t = -\frac{1}{2}r'$, we can replace r' by $-r'$. Thus, we can assume $r' > 0$. Finally, to get $c'' = 0$ it is sufficient to take

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & -\mathrm{Re}\{t_{12}\overline{t_{13}}\} \\ 0 & 1 & 0 & -\overline{t_{13}} \\ 0 & 0 & 1 & -\overline{t_{12}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $t_{12} = e^{-i\varphi/2}(rc_1'' - 2c_2'')/(2r)$, $t_{13} = e^{-i\varphi/2}c_2''/r$ (we mean that $z' = e^{i\varphi}$, $c_1'' = \mathrm{Re}\{c''e^{-i\varphi/2}\}$, $c_2'' = \mathrm{Im}\{c''e^{-i\varphi/2}\}$).

Thus, we have reduced the matrix $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & (1+ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1, \qquad r \in \Re > 0.$$

Now there remains to show that the numbers z and r are H-unitary invariants.

First note that for a block triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix} \tag{79}$$

to reduce $N - \lambda I$ to the form $\widetilde{N} - \lambda I$, where

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & N_3 & N_4 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N}_1 & \widetilde{N}_2 \\ 0 & \widetilde{N}_3 & \widetilde{N}_4 \\ 0 & 0 & 0 \end{pmatrix},$$

it is necessary and sufficient to have

$$N_1 T_4 = T_1 \widetilde{N}_1 + T_2 \widetilde{N}_3 \tag{80}$$

$$N_1 T_5 + N_2 T_6 = T_1 \widetilde{N}_2 + T_2 \widetilde{N}_4 \tag{81}$$

$$N_3 T_4 = T_4 \widetilde{N}_3 \tag{82}$$

$$N_3 T_5 + N_4 T_6 = T_4 \widetilde{N}_4. \tag{83}$$

If

$$H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

then for (79) to be H-unitary it is necessary and sufficient to have

$$T_1 T_6^* = I \tag{84}$$

$$T_4 H_1 T_2^* + T_5 T_1^* = 0 \tag{85}$$

$$T_1 T_3^* + T_2 H_1 T_2^* + T_3 T_1^* = 0 \tag{86}$$

$$T_4 H_1 T_4^* H_1 = I. \tag{87}$$

Since any H-unitary transformation T, such that

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & (1+ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix} T = T \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \widetilde{z} \\ 0 & 0 & 0 & (1+i\widetilde{r})\widetilde{z} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$|z| = |\widetilde{z}| = 1$, r, $\widetilde{r} \in \Re > 0$, must be block triangular (by the corollary of Proposition 1), systems (80)–(83), (84)–(87) are applicable. Combining (80) and (87), we get $|t_{11}| = 1$; hence (condition (84)) $t_{44} = t_{11}$. Now from (80) and (83) it follows that $(2+ir)z = (2+i\widetilde{r})\widetilde{z}$; hence $\widetilde{z} = z$, $\widetilde{r} = r$.

(c) $b' = -1$. The matrix $N - \lambda I$ can be carried into the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & -(1+ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1, \qquad r \in \Re > 0,$$

where z and r are H-unitary invariants. The proof is analogous to the case (b) above.

Thus, we have obtained the canonical form for each case considered. By using conditions (80)–(87) one can easily check that these forms are not H-unitarily similar to each other. They are indecomposable due to Proposition 2. Thus, we have proved the following lemma:

LEMMA 4. *If an indecomposable H-normal operator N (N: $C^4 \to C^4$) has the only eigenvalue λ, $\dim S_0 = 1$, and the internal operator N_1 is decomposable, then the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs* $\{(12), (15)\}, \{(13), (15)\}, \{(14), (15)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,$$

$$N = \begin{pmatrix} \lambda & 1 & 1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & (1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1, \qquad r \in \Re > 0,$$

$$N = \begin{pmatrix} \lambda & 1 & -1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & -(1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1, \qquad r \in \Re > 0,$$

$$H = D_4,$$

where z, r are H-unitary invariants.

5.2.2. $n = 5$ Then $\dim V_2 = 2$, $\dim V_3 = 1$ and, according to [1, Theorem 1], after interchanging the third and fourth rows and columns, we get

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & 0 & z & e \\ 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1, \; H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the H-normality of N is equivalent to the system

$$a\bar{z} = \bar{g}z \tag{88}$$

$$2\,\mathrm{Re}\{a\bar{c}\} + |b|^2 = 2\,\mathrm{Re}\{e\bar{g}\} + |f|^2. \tag{89}$$

It is readily seen that $a \neq 0$; consequently, it can be assumed that $a = 1$ and $g = z^2$ (see the paragraph after (39)). Further, take the (H-unitary) transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -b & -\frac{1}{2}|b|^2 & 0 \\ 0 & 0 & 1 & \bar{b} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and reduce $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c' & d' \\ 0 & 0 & 0 & z & e' \\ 0 & 0 & 0 & 0 & f' \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applying now the transformation

$$T = I_1 \oplus \begin{pmatrix} 1 & 0 & i\,\mathrm{Im}\{e'\overline{z}^2\} \\ 0 & e^{i\arg f'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_1,$$

we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c'' & d'' \\ 0 & 0 & 0 & z & r_1 z^2 \\ 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_1, r_2 \in \Re, \qquad r_2 \geq 0.$$

We can assume that $r_2 > 0$ because otherwise $v_3 \in S_0$, which is impossible. From condition (89) of the H-normality of N it follows that $c'' = r_1 + \frac{1}{2}r_2^2 + ir_3$ ($r_3 \in \Re$). Keeping in mind these conditions, apply the transformation

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & 0 & -\frac{1}{2}|t_{13}|^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\overline{t_{13}} \\ 0 & 0 & 0 & 1 & -\overline{t_{12}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $t_{12} = r_1\overline{z}$, $t_{13} = (d'' - r_1 z(r_1 + \frac{1}{2}r_2^2 + ir_3))/r_2$, to the matrix $N - \lambda I$. Then $c''' = \frac{1}{2}r_2^2 + ir_3$, $d''' = 0$, and the remaining terms of $N - \lambda I$ do not change. Renaming r_2 and r_3, write out the final form of $N - \lambda I$:

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_1, r_2 \in \Re,\ r_1 > 0,\ |z| = 1.$$

To prove the H-unitary invariance of z, r_1, r_2 assume that

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2}\widetilde{r}_1^2 + i\widetilde{r}_2 & 0 \\ 0 & 0 & 0 & \widetilde{z} & 0 \\ 0 & 0 & 0 & 0 & \widetilde{r}_1 \\ 0 & 0 & 0 & 0 & \widetilde{z}^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{r}_1, \widetilde{r}_2 \in \Re,\ \widetilde{r}_1 > 0,\ |\widetilde{z}| = 1,$$

and there exists a matrix T such that $NT = T\widetilde{N}$ (condition (36)) and $TT^{[*]} = I$ (condition (37)). Recall that T has block form (79) so that conditions (80)–(87) hold. From (82) it follows that $t_{23} = 0$ and $zt_{44} = \widetilde{z}t_{22}$. Since $t_{22}\overline{t_{44}} = 1$ (87), $z|t_{44}|^2 = \widetilde{z}$; i.e., $\widetilde{z} = z$, $|t_{44}| = 1$. Therefore, one can assume that

$$T_4 = \begin{pmatrix} 1 & 0 & it \\ 0 & t_{33} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad |t_{33}| = 1,\ t \in \Re$$

because it is allowed to divide T by its term $t_{22} = t_{44}$ of modulus 1. Now from (83) it follows that $t_{45} = itz$, $\widetilde{r}_1 t_{33} = r_1$. As r_1, $\widetilde{r}_1 > 0$, $t_{33} = 1$ and $\widetilde{r}_1 = r_1$. Since $t_{12} = -\overline{t_{45}}$ (condition (85)) and $t_{24} + (\frac{1}{2}r_1^2 + ir_2)t_{44} = (\frac{1}{2}\widetilde{r}_1^2 + i\widetilde{r}_2)t_{11} + \widetilde{z}t_{12}$ (condition (80)), $\widetilde{r}_2 = r_2$. This completes the proof of the H-unitary invariance of z, r_1, r_2.

Due to Proposition 2 the obtained form is indecomposable. Thus, we have proved the following lemma:

LEMMA 5. *If an indecomposable H-normal operator N (N: $C^5 \to C^5$) has the only eigenvalue λ, $\dim S_0 = 1$, and the internal operator N_1 is decomposable, then the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(16), (17)\}$:*

$$N = \begin{pmatrix} \lambda & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & \lambda & 0 & z & 0 \\ 0 & 0 & \lambda & 0 & r_1 \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r_1, r_2 \in \Re,\ r_1 > 0,$$

$$H = D_5,$$

where r_1, r_2, z are H-unitary invariants.

5.2.3. $n = 6$ In this case $\dim V_2 = 3$, $\dim V_3 = 1$. The matrices $N - \lambda I$ and H, according to [1, Theorem 1], have the form

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & z & r & 0 & f \\ 0 & 0 & 0 & z & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1,\ r \in \Re \tag{90}$$

or

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & 1 & ir & 0 & f \\ 0 & 0 & 0 & 1 & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r \in \Re, \tag{91}$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For a while we consider these two cases together, assuming that

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & z & x & 0 & f \\ 0 & 0 & 0 & z & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1,\ x \in C.$$

Then the condition of the H-normality of N is equivalent to the system

$$a\overline{z} = z\overline{h} \tag{92}$$

$$a\overline{x} + b\overline{z} = x\overline{h} + z\overline{g} \tag{93}$$

$$2\operatorname{Re}\{a\overline{c}\} + |b|^2 + |d|^2 = 2\operatorname{Re}\{f\overline{h}\} + |g|^2 + |p|^2. \tag{94}$$

As is customary, we can assume that $a = 1$, $h = z^2$. Let us use the (H-unitary) transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2}|d|^2 & -d & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It reduces $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' & c' & 0 & e' \\ 0 & 0 & z & x & 0 & f' \\ 0 & 0 & 0 & z & 0 & g' \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & p' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, take the transformation

$$T = \begin{pmatrix} 1 & z\overline{g'} & \overline{z}c' - x\overline{g'} & 0 & 0 & -\frac{1}{2}|\overline{z}c' - x\overline{g'}|^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -z\overline{c'} + \overline{x}g' \\ 0 & 0 & 0 & 1 & 0 & -\overline{z}g' \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and carry the matrix $N - \lambda I$ into the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b'' & 0 & 0 & e'' \\ 0 & 0 & z & x & 0 & f'' \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & p'' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now note that $p'' \neq 0$ because otherwise $v_5 \in S_0$. Since the rotation of the vector v_5 about any angle does not change the matrix H, we can assume that $p'' = r_2 \in \Re > 0$ (we put $\widetilde{v_5} = e^{i \arg p''} v_5$). The transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & e''/r_2 & -\frac{1}{2}|e''/r_2|^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\overline{e}''/r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

reduces the matrix $N - \lambda I$ to the final form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b''' & 0 & 0 & 0 \\ 0 & 0 & z & x & 0 & f''' \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we distinguish the cases (90) and (91).

(a) $z = 1, x \in \Im$. According to conditions (93) and (94) of the H-normality of N,

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{array}{l} r_1,\ r_2,\ r_3 \in \Re, \\ r_2 > 0. \end{array}$$

Let us show that r_1, r_2, r_3 are H-unitary invariants. Indeed, suppose some matrix T satisfies conditions (37) $TT^{[*]} = I$ and (36) $(N - \lambda I)T = T(\widetilde{N} - \lambda I)$, where

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 2i\widetilde{r}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\widetilde{r}_1 & 0 & 2\widetilde{r}_1^2 - \widetilde{r}_2^2/2 + i\widetilde{r}_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{r}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{array}{l} \widetilde{r}_1,\ \widetilde{r}_2,\ \widetilde{r}_3 \in \Re, \\ \widetilde{r}_2 > 0. \end{array}$$

From (36) it follows that

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ 0 & t_{11} & t_{23} & t_{24} & 0 & t_{26} \\ 0 & 0 & t_{11} & t_{34} & 0 & t_{36} \\ 0 & 0 & 0 & t_{11} & 0 & t_{46} \\ 0 & 0 & 0 & t_{54} & t_{55} & t_{56} \\ 0 & 0 & 0 & 0 & 0 & t_{11} \end{pmatrix}.$$

Using (87), we get: $t_{54} = 0$, $|t_{11}| = 1$. As above (see the argument before Lemma 5), we can assume that $t_{11} = 1$. Then $t_{34} = -\overline{t_{23}}$ (condition (87)) and $i(\widetilde{r}_1 - r_1) = t_{34} - t_{23}$ (condition (82)); hence, $\widetilde{r}_1 = r_1$ and $\operatorname{Re} t_{23} = 0$. Further, from (83) it follows that $r_2 = \widetilde{r}_2 t_{55}$ and from (87) that $|t_{55}| = 1$. As $r_2, \widetilde{r}_2 > 0$, $\widetilde{r}_2 = r_2$ and $t_{55} = 1$. Thus,

$$T = \begin{pmatrix} 1 & it & t_{24} & 0 \\ 0 & 1 & it & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad t \in \Re,\ 2\operatorname{Re} t_{24} + t^2 = 0.$$

Substituting T_4 in (80), we get $t_{12} = it$, $t_{13} = t_{24} - r_1 t$; replacing T_5 by $-T_4 H_1 T_2^*$ in (83), we have $i\widetilde{r}_3 = ir_3 - 2\operatorname{Re} t_{24} - t^2$; hence $\widetilde{r}_3 = r_3$. This completes the proof of the H-unitary invariance of r_1, r_2, r_3.

(b) $0 < \arg z < \pi, x \in \Re$. Applying the condition of the H-normality, we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & -2ir_1 \operatorname{Im} z & 0 & 0 & 0 \\ 0 & 0 & z & r_1 & 0 & (2r_1^2 \operatorname{Im}^2 z - r_2^2/2 + ir_3)z^2 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where r_1, r_2, $r_3 \in \Re$, $r_2 > 0$. That the numbers z, r_1, r_2, r_3 are H-unitary invariants can be checked as in (a) above. That the forms obtained are not H-unitary similar can also be checked by the reader by using formulas (80)–(87).

Because of Proposition 2 the forms obtained are indecomposable so that we have proved the following lemma:

LEMMA 6. *If an indecomposable H-normal operator N (N: $C^6 \to C^6$) has the only eigenvalue λ, $\dim S_0 = 1$, and the internal operator N_1 is decomposable, then the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs $\{(18), (20)\}, \{(19), (20)\}$:*

$$N = \begin{pmatrix} \lambda & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad r_1, r_2 \in \Re,\ r_2 > 0,$$

$$N = \begin{pmatrix} \lambda & 1 & -2ir_1 \operatorname{Im} z & 0 & 0 & 0 \\ 0 & \lambda & z & r_1 & 0 & \left(2r_1^2 \operatorname{Im}^2 z - r_2^2/2 + ir_3\right) z^2 \\ 0 & 0 & \lambda & z & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & z^2 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$|z| = 1,\ 0 < \arg z < \pi,\ r_1, r_2, r_3 \in \Re,\ r_2 > 0,$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where z, r_1, r_2, r_3 are H-unitary invariants.

5.2.4. $n = 7$ We show that this alternative is impossible. Indeed, if $\dim V_2 = 4$, $\dim V_3 = 1$, then, in accordance with [1, Theorem 1],

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e & f \\ 0 & 0 & \cos\alpha & \sin\alpha & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 1 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad 0 < \alpha \leq \pi/2,$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the conditions of the H-normality of N are as follows:

$$a = \overline{q} \cos\alpha$$

$$0 = \overline{q} \sin\alpha$$

$$b\ \cos\alpha + c\ \sin\alpha = \overline{h}$$
$$2\,\mathrm{Re}\{a\overline{d}\} + |b|^2 + |c|^2 + |e|^2 = 2\,\mathrm{Re}\{g\overline{q}\} + |h|^2 + |p|^2 + |r|^2.$$

Since $\sin\alpha \neq 0$, $q = 0$; hence $a = 0$. Thus, $(N - \lambda I)v_2 = (N^{[*]} - \overline{\lambda} I)v_2 = 0$, which contradicts the condition $S_0 \cap S = \{0\}$.

Thus, we have classified all indecomposable operators with one-dimensional subspace S_0. Now let us consider the case when $\dim S_0 = 2$.

5.3. $\dim S_0 = 2$

Let S_0 be two-dimensional. Since the operator $H_1 = H|_S$ has only positive eigenvalues, one can assume that $H_1 = I$. N_1 is a usual normal operator having the only eigenvalue λ; hence, $N_1 = \lambda I$. As a result, we have

$$N = \begin{pmatrix} \lambda I & N_1 & N_2 \\ 0 & \lambda I & N_3 \\ 0 & 0 & \lambda I \end{pmatrix}, \tag{95}$$

$$H = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}. \tag{96}$$

Below we do not stipulate that the pair $\{N, H\}$ has form $\{(95), (96)\}$.

For N to be H-normal it is necessary and sufficient to have

$$N_1 N_1^* = N_3^* N_3. \tag{97}$$

According to Theorem 1, for indecomposable operators, $n \leq 8$. Let us consider the cases $n = 4,\ 5,\ 6,\ 7,\ 8$ one after another.

5.3.1. $n = 4$ In this case $C^4 = S_0 \dotplus S_1$,

$$N - \lambda I = \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Condition (97) of the H-normality of N does not restrict the submatrix N_2 (its terms a, b, c, d). If $N_2 = 0$, the operator N is decomposable because the nondegenerate subspace $V = \mathrm{span}\{v_1, v_3\}$ is invariant for N and $N^{[*]}$. Thus, N_2 can be either of rank 1 or of rank 2 ($rg\, N_2 = 1$ or 2).

(a) $rg\, N_2 = 2$. Suppose an H-unitary transformation T

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

reduces $N - \lambda I$ to the form $\widetilde{N} - \lambda I$:

$$N - \lambda I = \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix}, \qquad \widetilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N}_2 \\ 0 & 0 \end{pmatrix}.$$

Then conditions (98)–(100) must be satisfied:

$$N_2 T_3 = 0 \tag{98}$$
$$N_2 T_4 = T_1 \widetilde{N}_2 \tag{99}$$
$$0 = T_3 \widetilde{N}_2. \tag{100}$$

Since N_2 is invertible, (98) holds only if $T_3 = 0$. Hence, T is H-unitary iff

$$T_1 T_4^* = I \tag{101}$$
$$T_1 T_2^* + T_2 T_1^* = 0. \tag{102}$$

From system (101)–(102) it follows that without loss of generality we can consider only block diagonal transformations of the form $T = T_1 \oplus T_1^{*-1}$ because T_2 does not figure in Eqs. (98)–(100).

Thus, only condition (99) $N_2 = T_1 \widetilde{N}_2 T_1^*$ must be satisfied. Applying Proposition 3 from the Appendix, we obtain that the submatrix N_2 can be reduced to one of the canonical forms

$$N_2 = \begin{pmatrix} z & \varrho e^{-i\pi/3} z \\ 0 & e^{i\pi/3} z \end{pmatrix}, \qquad N_2 = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where z, z_1, z_2, ϱ ($|z| = 1$, $\varrho \in \Re \geq \sqrt{3}$, $0 \leq \arg z < \pi$ if $\varrho > \sqrt{3}$, $|z_1| = |z_2| = 1$, $\arg z_1 \leq \arg z_2$) are invariants. For the latter form the operator N is decomposable because the nondegenerate subspace $V = \operatorname{span}\{v_1, v_3\}$ is invariant both for N and $N^{[*]}$. For the former we obtain the following canonical form:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & z & re^{-i\pi/3} z \\ 0 & 0 & 0 & e^{i\pi/3} z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{array}{l} |z| = 1,\ r \in \Re \geq \sqrt{3}, \\ 0 \ \leq \arg z < \pi \ \text{ if } r > \sqrt{3}. \end{array}$$

(b) $rg\, N_2 = 1$. Then

$$N_2 = \begin{pmatrix} ka & kb \\ la & lb \end{pmatrix}, \qquad |a| + |b| \neq 0, \qquad |k| + |l| \neq 0.$$

If $l\bar{a} = k\bar{b}$, then $v = bv_3 - av_4 \neq 0$ belongs both to S_0 and S_1, which is impossible ($S_0 \cap S_1 = \{0\}$). Thus, we can assume that $l\bar{a} \neq k\bar{b}$. Taking the transformation $T = T_1 \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} \bar{a} & k \\ \bar{b} & l \end{pmatrix},$$

we obtain one more canonical form:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

LEMMA 7. *If an indecomposable H-normal operator N (N: $C^4 \to C^4$) has the only eigenvalue λ and* $\dim S_0 = 2$, *then the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs $\{(21), (23)\}, \{(22), (23)\}$:*

$$N = \begin{pmatrix} \lambda & 0 & z & re^{-i\pi/3}z \\ 0 & \lambda & 0 & e^{i\pi/3}z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{array}{l} |z| = 1,\ r \in \Re \geq \sqrt{3}, \\ 0 \ \leq \arg z < \pi \text{ if } r > \sqrt{3}, \end{array}$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where r, z are H-unitary invariants.

Proof. The possibility of reducing N to one of forms (21), (22) is proved before the lemma. The argument in (a) above shows that these forms are not similar; hence, they are not H-unitarily similar. Thus, we must prove only the indecomposability of N.

Show that the first canonical form is indecomposable. Assume the converse. Let some nondegenerate subspace V be invariant for N and $N^{[*]}$. Then there exists a nonzero vector $w_1 \in V\colon w_1 \in S_0$. Therefore, $\exists w_2 = av_3 + bv_4 + v \in V$ $(v \in S_0,\ |a| + |b| \neq 0)$:

$$(N - \lambda I)w_2 = azv_1 + b(re^{-i\pi/3}zv_1 + e^{i\pi/3}zv_2),$$
$$(N^{[*]} - \overline{\lambda} I)w_2 = a(\overline{z}v_1 + re^{i\pi/3}\overline{z}v_2) + be^{-i\pi/3}\overline{z}v_2.$$

Since $\min\{\dim V, \dim V^{[\perp]}\} \leq 2$, it can be assumed that $\dim V \leq 2$. As the vectors w_1 and w_2 are linearly independent, we get $\dim V = 2$. Therefore, the vectors $(N - \lambda I)w_2$ and $(N^{[*]} - \overline{\lambda} I)w_2$ must be linearly dependent; i.e., the following condition must be satisfied:

$$(a + bre^{-i\pi/3})(are^{i\pi/3} + be^{-i\pi/3}) = abe^{i\pi/3}. \tag{103}$$

Since (103) breaks if either a or b is equal to zero, we can rewrite (103) as

$$\left(\frac{a}{b}\right)^2 re^{i\pi/3} + \left(\frac{a}{b}\right)(e^{-i\pi/3} - e^{i\pi/3} + r^2) + re^{-2i\pi/3} = 0. \tag{104}$$

Discriminant of (104) is equal to $r^4 - 2r^2 - 3$. Since $r^2 \geq 3$, it is nonnegative. Therefore,

$$\frac{a}{b} = \frac{i\sqrt{3} - r^2 \pm \sqrt{r^4 - 2r^2 - 3}}{r(1 + i\sqrt{3})}.$$

Consequently, $|a/b|^2 = \frac{1}{2}(r^2 - 1 \mp \sqrt{r^4 - 2r^2 - 3})$; therefore, $[w_2, (N - \lambda I)w_2] = z|b|^2(|a/b|^2 + (a/b)re^{i\pi/3} + e^{-i\pi/3}) = 0$. Thus, the subspace V is degenerate; i.e., the operator N is indecomposable.

For the second matrix N we see that the vectors $(N - \lambda I)w_2$ and $(N^{[*]} - \overline{\lambda} I)w_2$ $(w_2 = av_3 + bv_4 + v,\ v \in S_0)$ can be linearly dependent only if $a = b = 0$. Therefore, N is also indecomposable. This concludes the proof of the lemma. ∎

5.3.2. $n = 5$ The matrix $N - \lambda I$ has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & c & d \\ 0 & 0 & b & e & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that condition (97) of the H-normality of N amounts to the system

$$\begin{aligned} |a| &= |g| \\ a\bar{b} &= \bar{g}h \\ |b| &= |h|. \end{aligned}$$

The latter means that $g = \bar{a}z$, $h = \bar{b}z$ ($|z| = 1$). Note that a and b are not equal to zero simultaneously because otherwise $v_3 \in S_0$, which is impossible.

Take the transformation $T = T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} a & t_{12} \\ b & t_{22} \end{pmatrix}, \qquad at_{22} \neq bt_{12},$$

and reduce $N - \lambda I$ to the form

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & c' & d' \\ 0 & 0 & 0 & e' & f' \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad |z| = 1.$$

Now we fix the form of the submatrices N_1 and N_3 so that the following transformations change only the submatrix N_2. At first, apply the transformation

$$T = \begin{pmatrix} I & T_2 & -\frac{1}{2}T_2T_2^* \\ 0 & I & -T_2^* \\ 0 & 0 & I \end{pmatrix}, \tag{105}$$

where $T_2^* = (0 \;\; d')$, and reduce N_2 to the form

$$N_2 = \begin{pmatrix} c'' & 0 \\ e'' & f'' \end{pmatrix}.$$

Now let us consider two cases: $f'' = 0$ and $f'' \neq 0$.

(a) $f'' = 0$. Then $e'' \neq 0$ because otherwise $v_5 \in S_0$. Subjecting $N - \lambda I$ to the transformation $T = T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} 1 & c'' \\ 0 & e'' \end{pmatrix},$$

we get

$$N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(b) $f'' \neq 0$. Then one can assume that $|f''| = 1$ (to this end it is sufficient to put $\widetilde{v_2} = \sqrt{|f''|}v_2$, $\widetilde{v_5} = v_5/\sqrt{|f''|}$). Thus, $f'' = z_1$, $|z_1| = 1$.

If $z_1^2 \neq z$, then N is decomposable. Indeed, applying

$$T = \begin{pmatrix} T_1 & -T_1T_5^* & -\frac{1}{2}T_1T_5^*T_5 \\ 0 & I & T_5 \\ 0 & 0 & T_1^{*-1} \end{pmatrix}, \tag{106}$$

where

$$T_1 = \begin{pmatrix} 1 & z_1\overline{e''}/(1-\overline{z}z_1^2) \\ 0 & 1 \end{pmatrix}, \qquad T_5 = \begin{pmatrix} 0 & z_1^2\overline{e''}/(1-\overline{z}z_1^2) \end{pmatrix},$$

we reduce N_2 to the diagonal form $N_2 = c''' \oplus z_1$. Now the nondegenerate subspace $V = \operatorname{span}\{v_2, v_5\}$ is invariant for N and $N^{[*]}$; hence, N is decomposable.

Let $z_1^2 = z$. Note that if $e'' = 0$, then N is decomposable ($V = \operatorname{span}\{v_2, v_5\}$ is nondegenerate, $NV \subseteq V$, $N^{[*]}V \subseteq V$). Thus, $e'' \neq 0$. Taking transformation (106) with

$$T_1 = \begin{pmatrix} 1 & iz_1c_2''/|e''| \\ 0 & e^{i\arg e''} \end{pmatrix}, \qquad T_5 = \begin{pmatrix} -z_1(c_1'' + c_2''^2/|e''|^2)/2 & iz_1^2c_2''/|e''| \end{pmatrix},$$

where $c_1'' = \operatorname{Re}\{c''\overline{z_1}\}$, $c_2'' = \operatorname{Im}\{c''\overline{z_1}\}$, we reduce N_2 to the final form

$$N_2 = \begin{pmatrix} 0 & 0 \\ r & z_1 \end{pmatrix}, \qquad r = |e''| > 0.$$

LEMMA 8. *If an indecomposable H-normal operator N (N: $C^5 \to C^5$) has the only eigenvalue λ, $\dim S_0 = 2$, then the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs* $\{(24), (26)\}, \{(25), (26)\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & \lambda & z & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,$$

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & r & z \\ 0 & 0 & \lambda & z^2 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad |z| = 1,\ r \in \Re > 0,$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_1 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where z, r are H-unitary invariants.

Proof. The possibility of reducing N to one of forms (24), (25) is proved before the lemma. Hence, it is necessary to show that these forms are indecomposable and are not H-unitarily similar to each other and their terms z, r are H-unitary invariants. These statements may be proved as follows.

For the block triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix} \tag{107}$$

to satisfy condition (36) $NT = T\widetilde{N}$, where

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N}_1 & \widetilde{N}_2 \\ 0 & 0 & \widetilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

it is necessary and sufficient to have

$$N_1 T_4 = T_1 \widetilde{N}_1 \tag{108}$$
$$N_1 T_5 + N_2 T_6 = T_1 \widetilde{N}_2 + T_2 \widetilde{N}_3 \tag{109}$$
$$N_3 T_6 = T_4 \widetilde{N}_3. \tag{110}$$

If H has form (96), then for (107) to be H-unitary it is necessary and sufficient to have

$$T_1 T_6^* = I \tag{111}$$
$$T_1 T_5^* + T_2 T_4^* = 0 \tag{112}$$
$$T_1 T_3^* + T_2 T_2^* + T_3 T_1^* = 0 \tag{113}$$
$$T_4 T_4^* = I. \tag{114}$$

If an H-unitary transformation T reduces matrix (25) (the second) to form (24) (the first), then from the corollary of Proposition 1 it follows that T has block form (107) and, according to (36),

$$T_1 = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}. \tag{115}$$

Apply condition (109), replacing T_6 by T_1^{*-1} (111) and T_2 by $-T_1T_5^*T_4$ (112). Then we get: $z/\overline{t_{22}} = 0$. This contradiction proves that the canonical forms are not H-unitarily similar.

If

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r & z \\ 0 & 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} T = T \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{r} & \widetilde{z} \\ 0 & 0 & 0 & \widetilde{z}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$|z| = |\widetilde{z}| = 1$, and $r, \widetilde{r} \in \Re > 0$, then T has form (107), the submatrix T_1 having form (115) and $t_{11} = t_{33}$. Since $|t_{33}| = 1$ (condition (114)), we can assume that $t_{11} = t_{33} = 1$. Replace T_6 by T_1^{*-1} and apply (110); we have $\widetilde{z}^2 = z^2$. Now substitute T_1^{*-1} for T_6 and $-T_1T_5^*$ for T_2 in (109). We obtain

$$t_{35} = \widetilde{z}t_{12} \tag{116}$$

$$r - z\overline{t_{12}}/\overline{t_{22}} = \widetilde{r}t_{22} - z^2t_{22}\overline{t_{35}} \tag{117}$$

$$z/\overline{t_{22}} = \widetilde{z}t_{22}. \tag{118}$$

From (118) it follows that $|t_{22}| = 1$, $\widetilde{z} = z$. Hence, $1/\overline{t_{22}} = t_{22}$, $t_{35} = zt_{12}$, and $r = \widetilde{r}t_{22}$. Therefore, $r = \widetilde{r}|t_{22}|$, i.e., $\widetilde{r} = r$. Thus, the numbers z, r are H-unitary invariants of canonical form (25). That z is an H-unitary invariant of (24) can be checked in the similar way.

There remains to prove that matrices (24) and (25) are indecomposable. The proof is by reductio ad absurdum. Suppose some nondegenerate subspace V is invariant for N and $N^{[*]}$ (N has form (24)). As $\min\{\dim V, \dim V^{[\perp]}\} \le 2$, we can assume that $\dim V \le 2$. Since there exists a vector $w_1 \ne 0 \in S_0 : w_1 \in V$, there exists also a vector $w_2 = av_3 + bv_4 + cv_5 + v \in V$ ($v \in S_0$, $|b| + |c| \ne 0$). As the vectors $(N - \lambda I)w_2 = av_1 + b(v_2 + zv_3)$ and $(N^{[*]} - \overline{\lambda} I)w_2 = a\overline{z}v_1 + bv_3 + cv_1$ must be linearly dependent, we obtain $b = 0$. But in this case the subspace V is degenerate because $[(N - \lambda I)w_2, w_2] = 0$. This contradiction proves the indecomposability of (24). Now let us check the indecomposability of (25). Suppose a nondegenerate subspace V is invariant both for N and $N^{[*]}$. Then, as before, $\exists w_1 \neq 0 \in S_0 : w_1 \in V$ and $\exists w_2 = av_3 + bv_4 + cv_5 + v \in V$ ($v \in S_0$, $|b| + |c| \neq 0$). Therefore, the vectors $(N - \lambda I)w_2 - z^2(N^{[*]} - \overline{\lambda} I)w_2 = brv_2 - crz^2 v_1$ and $(N - \lambda I)w_2 = av_1 + brv_2 + bz^2 v_3 + czv_2$ must be linearly dependent. Hence, $b = 0 \Rightarrow c = 0$. The contradiction obtained proves that (25) is also indecomposable. The proof of the lemma is completed. ■

5.3.3. $n = 6$ The matrix $N - \lambda I$ has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } N_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The submatrix N_1 is not equal to zero because then condition (97) of the H-normality of N implies $N_3 = 0$ so that $v_3, v_4 \in S_0$, which is impossible. Thus, we must consider two alternatives: $rg\, N_1 = 2$ and $rg\, N_1 = 1$.

(a) $rg\, N_1 = 2$. At first apply the transformation $T = N_1 \oplus I \oplus N_1^{*-1}$; it takes N_1 to I. Since N_1 has become equal to I, N_3, according to (97), has become unitary. Recall that any unitary matrix is unitarily similar to some diagonal one with nonzero terms of modulus 1; moreover, this representation is unique to within order of diagonal terms. Thus, $\exists U\ (UU^* = I) : \ \widetilde{N}_3 = U^* N_3 U$, where

$$\widetilde{N}_3 = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, \qquad |z_1| = |z_2| = 1, \qquad \arg z_1 \leq \arg z_2. \tag{119}$$

If we subject $N - \lambda I$ to the transformation $T = U \oplus U \oplus U$, then N_3 maps to (119) and $N_1 = I$ does not change.

Note that if $z_1 \neq z_2$, N is decomposable. To check this it is sufficient to reduce

$$N_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{120}$$

to the diagonal form by means of transformation (105) with the submatrix

$$T_2 = \begin{pmatrix} 0 & (\overline{g} - \overline{z_1} f)/(1 - \overline{z_1} z_2) \\ (\overline{f} - \overline{z_2} g)/(1 - z_1 \overline{z_2}) & 0 \end{pmatrix}$$

(this transformation does not change N_1 and N_3). Now the nondegenerate subspace $V = \operatorname{span}\{v_1, v_3, v_5\}$ is invariant for N and $N^{[*]}$; hence, N is decomposable.

Thus, for N to be indecomposable N_3 must be equal to zI. Show that in the case when $z = -1$, N is also decomposable. Indeed, apply the transformation

$$T = \begin{pmatrix} U & -\frac{1}{2}N_2U & -\frac{1}{8}N_2N_2^*U \\ 0 & U & \frac{1}{2}N_2^*U \\ 0 & 0 & U \end{pmatrix},$$

where U is a unitary matrix reducing $N_2 + N_2^*$ to the diagonal form (U is known to exist). Then N_2 becomes diagonal; we already know that in this case N is decomposable.

Thus, $N = zI$, $z \neq -1$. Now we apply only transformations preserving the submatrices N_1 and N_3. First let us take (105) with

$$T_2 = \begin{pmatrix} 0 & 0 \\ \overline{f} & 0 \end{pmatrix}$$

and carry submatrix (120) to the form

$$N_2 = \begin{pmatrix} e' & 0 \\ g' & h' \end{pmatrix}.$$

Further, apply transformation (105) with

$$T_2 = \begin{pmatrix} t_{13} & 0 \\ 0 & t_{24} \end{pmatrix},$$

where $\operatorname{Re}\{\overline{t_{13}} + zt_{13}\} = \operatorname{Re} e'$, $\operatorname{Re}\{\overline{t_{24}} + zt_{24}\} = \operatorname{Re} h'$ (since $z \neq -1$, these equations are solvable for any e' and h'). After this transformation

$$N_2 = \begin{pmatrix} ir_1 & 0 \\ g' & ir_2 \end{pmatrix}.$$

One can assume that $g' = r_3 \in \Re \geq 0$. To this end it is sufficient to put $\widetilde{v_2} = e^{i \arg g'} v_2$, $\widetilde{v_4} = e^{i \arg g'} v_4$, $\widetilde{v_6} = e^{i \arg g'} v_6$. Now apply the transformation

$$T = \begin{pmatrix} T_1 & T_1T_2 & -\frac{1}{2}T_1T_2T_2^* \\ 0 & T_1 & -T_1T_2^* \\ 0 & 0 & T_1 \end{pmatrix},$$

where

$$T_1 = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ -\overline{(z+1)}/|z+1| & \overline{(z+1)}/|z+1| \end{pmatrix},$$

$$T_2 = \tfrac{1}{2}\begin{pmatrix} -r_3/|z+1| & 0 \\ (ir_2 - ir_1) - r_3\overline{(z+1)}/|z+1| & r_3/|z+1| \end{pmatrix}.$$

We get

$$N_2 = \begin{pmatrix} ir_1' & 0 \\ g'' & ir_1' \end{pmatrix}, \qquad r_1' = \frac{1}{2}(r_1 + r_2).$$

As above, we can assume that $g'' \in R \geq 0$. For N to be indecomposable g'' must be nonzero so that $g'' > 0$. This is the final form of the matrix $N - \lambda I$:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & ir_1 & 0 \\ 0 & 0 & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{matrix} |z| = 1,\ z \neq -1, \\ r_1, r_2 \in \Re,\ r_2 > 0. \end{matrix} \tag{121}$$

Let us show that z, r_1, r_2 are H-unitary invariants. To this end suppose that an H-unitary matrix T reduces (121) to the form

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & I & \widetilde{N}_2 \\ 0 & 0 & \widetilde{z} I \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{N}_2 = \begin{pmatrix} i\widetilde{r}_1 & 0 \\ \widetilde{r}_2 & i\widetilde{r}_1 \end{pmatrix}, \qquad \begin{matrix} |\widetilde{z}| = 1,\ \widetilde{z} \neq -1, \\ \widetilde{r}_1, \widetilde{r}_2 \in \Re,\ \widetilde{r}_2 > 0. \end{matrix}$$

By the corollary of Proposition 1, T must have block triangular form (107); therefore, systems (108)–(110) and (111)–(114) must hold. From (108), (114), and (111) it follows that $T_1 = T_4 = T_6 = T_6^{*-1}$. Now from (110) it follows that $\widetilde{z} = z$. Combining (112) and (109), we get $N_2 = T_1 \widetilde{N}_2 T_1^* + z T_2 T_1^* + T_1 T_2^*$. If we denote

$$T_2' = T_2 T_1^* = \begin{pmatrix} t_{11}' & t_{12}' \\ t_{21}' & t_{22}' \end{pmatrix}$$

and write out the general form for 2×2 unitary matrix

$$T_1 = \begin{pmatrix} \varrho s_1 & \sqrt{1-\varrho^2} s_2 \\ \sqrt{1-\varrho^2} s_3 & -\varrho \overline{s_1} s_2 s_3 \end{pmatrix}, \qquad \varrho \in [0,1],\ |s_1| = |s_2| = |s_3| = 1, \tag{122}$$

then we obtain

$$\begin{aligned} ir_1 &= i\widetilde{r}_1 + \varrho\sqrt{1-\varrho^2}\,\overline{s_1} s_2 \widetilde{r}_2 + zt'_{11} + \overline{t'_{11}} \\ ir_1 &= i\widetilde{r}_1 - \varrho\sqrt{1-\varrho^2}\,\overline{s_1} s_2 \widetilde{r}_2 + zt'_{22} + \overline{t'_{22}}. \end{aligned}$$

Summing these equalities, we get

$$2ir_1 = 2i\widetilde{r}_1 + zt'_{11} + \overline{t'_{11}} + zt'_{22} + \overline{t'_{22}}.$$

It is easy to check that if $\operatorname{Re}\{zt + \overline{t}\} = 0$ $(z \neq -1)$, then $\operatorname{Im}\{zt + \overline{t}\} = 0$. In our case $t'_{11} + t'_{22}$ plays the role of t; therefore, we have $zt'_{11} + \overline{t'_{11}} + zt'_{22} + \overline{t'_{22}} = 0$. Hence $\widetilde{r}_1 = r_1$. Let us check that from the obtained equality $\widetilde{r}_1 = r_1$ it follows that $\widetilde{r}_2 = r_2$. Indeed, $zN_2^* - N_2 = T_1(z\widetilde{N}_2^* - \widetilde{N}_2)T_1^*$,

$$zN_2^* - N_2 = \begin{pmatrix} -ir_1(z+1) & zr_2 \\ -r_2 & -ir_1(z+1) \end{pmatrix};$$

the determinant of $zN_2^* - N_2$, which does not change under the similarity, is equal to $-r_1^2(z+1)^2 + zr_2^2$; hence $r_2^2 = \widetilde{r}_2^2$. Since the sign of r_2 coincides with that of $\widetilde{r}_2$, $\widetilde{r}_2 = r_2$. The proof of the H-unitary invariance of the numbers r_1, r_2 is completed.

(b) $rg\, N_1 = 1$. Let us show that in this case N is decomposable. In fact,

$$N_1 = \begin{pmatrix} ka & kb \\ la & lb \end{pmatrix}, \qquad |a| + |b| \neq 0, \qquad |k| + |l| \neq 0.$$

Taking $T = T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} t_{11} & k \\ t_{21} & l \end{pmatrix}, \qquad lt_{11} \neq kt_{21},$$

we reduce N_1 to the form

$$N_1 = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

Without loss of generality one can assume that $a \neq 0$ and, therefore, that $a = 1$ (this may be achieved by putting $\widetilde{v}_2 = av_2$, $\widetilde{v}_6 = v_6/\overline{a}$). If $b \neq 0$, apply the transformation $T_1 \oplus T_4 \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{|b|^2+1} \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 1/\sqrt{|b|^2+1} & |b|/\sqrt{|b|^2+1} \\ \bar{b}/\sqrt{|b|^2+1} & -e^{-i\arg b}/\sqrt{|b|^2+1} \end{pmatrix},$$

to the matrix $N - \lambda I$ (we mean that $a = 1$). Then we obtain

$$N_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

According to (97),

$$N_3 = \begin{pmatrix} 0 & z_1 \cos\alpha \\ 0 & z_2 \sin\alpha \end{pmatrix}, \qquad |z_1| = |z_2| = 1, \qquad 0 \le \alpha \le \pi/2.$$

Since $v_4 \bar{\in} S_0$, $\sin\alpha \neq 0$. Therefore, we can apply the transformation T of form (105), where

$$T_2 = \begin{pmatrix} \bar{g} & (f - z_1\bar{g}\cos\alpha)/(z_2\sin\alpha) \\ 0 & 0 \end{pmatrix}$$

(N_2 has form (120)). Under the action of T the submatrices N_1 and N_3 do not change but the submatrix N_2 becomes diagonal. Now the nondegenerate subspace $V = \operatorname{span}\{v_1, v_5\}$ is invariant for N and $N^{[*]}$; hence, N is decomposable.

LEMMA 9. *If an indecomposable H-normal operator N ($N\colon C^6 \to C^6$) has the only eigenvalue λ, $\dim S_0 = 2$, then the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(27), (28)\}$:*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & ir_1 & 0 \\ 0 & \lambda & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & \lambda & 0 & z & 0 \\ 0 & 0 & 0 & \lambda & 0 & z \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \begin{matrix} |z| = 1,\ z \neq -1, \\ r_1, r_2 \in \Re,\ r_2 > 0, \end{matrix}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where z, r_1, r_2 are H-unitary invariants.

Proof. It is necessary to prove only the indecomposability of the canonical form because the rest was proved before the lemma. Suppose that a nondegenerate subspace V satisfies the conditions $NV \subseteq V$, $N^{[*]}V \subseteq V$. As above, we can assume that $\dim V \leq 3$ (see the proofs of the previous lemmas). Since $\exists w_1 \neq 0 \in S_0 : w_1 \in V$, $\exists w_2 = av_5 + bv_6 + v \in V$ $(v \in (S_0 + S)$, $|a| + |b| \neq 0)$. The vectors $(N - \lambda I)(N^{[*]} - \overline{\lambda} I)w_2 = av_1 + bv_2$ and $(N - \lambda I - z(N^{[*]} - \overline{\lambda} I))w_2 = air_1(1 + z)v_1 - br_2 z v_1 + bir_1(1 + z)v_2 + ar_2 v_2$ must be linearly dependent because otherwise $S_0 \subset V$ and $\dim V \geq 4$. Therefore, $-b^2 r_2 z = a^2 r_2$. Since $z \neq -1$, $a = b = 0$. This contradiction proves that N is indecomposable. The proof of the lemma is completed. ∎

5.3.4. $n = 7$ The matrix $N - \lambda I$ has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } N_1 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

As in the case when $n = 6$, one can check that $N_1 \neq 0$; therefore, we must consider the cases $rg\, N_1 = 1$ and $rg\, N_1 = 2$. Show that the former alternative is also impossible. Indeed, if $rg\, N_1 = 1$, then

$$N_1 = \begin{pmatrix} ka & kb & kc \\ la & lb & lc \end{pmatrix}, \qquad |a| + |b| + |c| \neq 0, \qquad |k| + |l| \neq 0.$$

Applying the transformation $T = T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} t_{11} & k \\ t_{21} & l \end{pmatrix}, \qquad lt_{11} \neq kt_{21},$$

we reduce N_1 to the form

$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \end{pmatrix}.$$

Then from condition (97) of the H-normality of N it follows that

$$N_3 = \begin{pmatrix} 0 & s \\ 0 & u \\ 0 & w \end{pmatrix}.$$

Since there exists a nontrivial solution $\{\alpha_i\}_1^3$ of the system

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$$
$$\overline{s}\alpha_1 + \overline{u}\alpha_2 + \overline{w}\alpha_3 = 0,$$

the nonzero vector $v = \alpha_1 v_3 + \alpha_2 v_4 + \alpha_3 v_5$ belongs to S_0, which contradicts the condition $S_0 \cap S = \{0\}$.

Thus, $rg\, N_1 = 2$. Then without loss of generality it can be assumed that

$$\det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0.$$

Take the block diagonal transformation $T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} a & b \\ d & e \end{pmatrix}.$$

It reduces N_1 to the form

$$N_1 = \begin{pmatrix} 1 & 0 & c' \\ 0 & 1 & f' \end{pmatrix}.$$

Further, apply the transformation $T_1 \oplus T_2 \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1+|f'|^2} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{1+|f'|^2} & -f'/\sqrt{1+|f'|^2} \\ 0 & \overline{f'}/\sqrt{1+|f'|^2} & 1/\sqrt{1+|f'|^2} \end{pmatrix}.$$

Then we get

$$N_1 = \begin{pmatrix} 1 & b'' & c'' \\ 0 & 1 & 0 \end{pmatrix}.$$

Now take $T = T_1 \oplus T_2 \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} \sqrt{1+|c''|^2} & b'' \\ 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1/\sqrt{1+|c''|^2} & 0 & -c''/\sqrt{1+|c''|^2} \\ 0 & 1 & 0 \\ \overline{c''}/\sqrt{1+|c''|^2} & 0 & 1/\sqrt{1+|c''|^2} \end{pmatrix},$$

and get the final form of the submatrix N_1:

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now consider the submatrix

$$N_3 = \begin{pmatrix} r & s \\ t & u \\ v & w \end{pmatrix}.$$

If v and w are both equal to zero, then $v_5 \in S_0$. Therefore, we can assume that $|v|^2 + |w|^2 \neq 0$ and can apply the transformation $T = T_1 \oplus T_1 \oplus I \oplus T_1$, where

$$T_1 = \begin{pmatrix} w/\sqrt{|v|^2+|w|^2} & \overline{v}/\sqrt{|v|^2+|w|^2} \\ -v/\sqrt{|v|^2+|w|^2} & \overline{w}/\sqrt{|v|^2+|w|^2} \end{pmatrix}.$$

Then

$$N_3 = \begin{pmatrix} r' & s' \\ t' & u' \\ 0 & w' \end{pmatrix}, \qquad w' = \sqrt{|v|^2+|w|^2} > 0.$$

If $s' \neq 0$, replace s' by $|s'|$ by putting $\widetilde{v_1} = e^{i \arg s'} v_1$, $\widetilde{v_3} = e^{i \arg s'} v_3$, $\widetilde{v_6} = e^{i \arg s'} v_6$. If $s' = 0$, then apply the transformation $\widetilde{v_1} = e^{-i \arg t'} v_1$, $\widetilde{v_3} = e^{-i \arg t'} v_3$, $\widetilde{v_6} = e^{-i \arg t'} v_6$ and replace t' by $|t'|$. Now we can assume that $s' \in \Re \geq 0$ and if $s' = 0$, then $t' \in \Re \geq 0$.

Now let us apply condition (97) of the H-normality of N. We obtain

$$N_3 = \begin{pmatrix} -z_1 \overline{z_2} \cos\alpha & \sin\alpha \cos\beta \\ z_1 \sin\alpha & z_2 \cos\alpha \cos\beta \\ 0 & \sin\beta \end{pmatrix},$$

$|z_1| = |z_2| = 1$, $0 \leq \alpha, \beta \leq \pi/2$, $\beta \neq 0$, $z_1 = 1$ if $\sin\alpha\cos\beta = 0$, $z_2 = 1$ if $\alpha = \pi/2$. Let us show that in the case when $\alpha = 0$, N is decomposable. Indeed, under the action of (105), where

$$T_2 = \begin{pmatrix} 0 & \overline{p} & (h - \overline{p} z_2 \cos\alpha \cos\beta)/\sin\beta \\ 0 & 0 & 0 \end{pmatrix},$$

the submatrix

$$N_2 = \begin{pmatrix} g & h \\ p & q \end{pmatrix}$$

becomes diagonal. The nondegenerate subspace $V = \operatorname{span}\{v_1, v_3, v_6\}$ is now invariant for N and $N^{[*]}$; hence, N is decomposable.

Thus, $\alpha \neq 0$. Applying transformation (105) with

$$T_2 = \begin{pmatrix} 0 & t_{14} & t_{15} \\ 0 & t_{24} & t_{25} \end{pmatrix},$$

where

$$\begin{aligned} t_{14} &= g/(z_1 \sin\alpha) \\ t_{15} &= (h - t_{14} z_2 \cos\alpha\cos\beta)/\sin\beta \\ t_{24} &= (p - \overline{t_{14}})/(z_1 \sin\alpha) \\ t_{25} &= (q - \overline{t_{24}} - t_{24} z_2 \cos\alpha\cos\beta)/\sin\beta, \end{aligned}$$

we reduce N_2 to zero without changing N_1 and N_3. This is the final form of the matrix $N - \lambda I$:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z_1\overline{z_2}\cos\alpha & \sin\alpha\cos\beta \\ 0 & 0 & 0 & 0 & 0 & z_1 \sin\alpha & z_2\cos\alpha\cos\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$|z_1| = |z_2| = 1,\ 0 < \alpha, \beta \leq \pi/2, z_1 = 1 \text{ if } \beta = \pi/2, z_2 = 1 \text{ if } \alpha = \pi/2.$

Show that z_1, z_2, α, β are H-unitary invariants. Suppose an H-unitary matrix T reduces $N - \lambda I$ to the form

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & \widetilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \widetilde{N}_3 = \begin{pmatrix} -\widetilde{z}_1\overline{\widetilde{z}_2}\cos\widetilde{\alpha} & \sin\widetilde{\alpha}\cos\widetilde{\beta} \\ \widetilde{z}_1 \sin\widetilde{\alpha} & \widetilde{z}_2\cos\widetilde{\alpha}\cos\widetilde{\beta} \\ 0 & \sin\widetilde{\beta} \end{pmatrix},$$

$|\widetilde{z}_1| = |\widetilde{z}_2| = 1$, $0 < \widetilde{\alpha}, \widetilde{\beta} \leq \pi/2$, $\widetilde{z}_1 = 1$ if $\widetilde{\beta} = \pi/2$, $\widetilde{z}_2 = 1$ if $\widetilde{\alpha} = \pi/2$. Therefore, T has block triangular form (107) and conditions (108)–(114) hold. Combining (108), (114), and (111), we get: $T_4 = T_1 \oplus t_{55}$ ($|t_{55}| = 1$),

$T_1 = T_6 = T_6^{*-1}$. Now from (110) it follows that $T_4 = t_{11} \oplus t_{22}$ ($|t_{11}| = |t_{22}| = 1$),

$$\begin{aligned} t_{22} \sin\alpha \cos\beta &= t_{11} \sin\widetilde{\alpha} \cos\widetilde{\beta} \\ t_{11} z_1 \sin\alpha &= t_{22} \widetilde{z}_1 \sin\widetilde{\alpha} \\ t_{22} \sin\beta &= t_{55} \sin\widetilde{\beta}. \end{aligned}$$

Hence $t_{11} = t_{22} = t_{55}$, and hence $N_3 = \widetilde{N}_3$; i.e., $\widetilde{\alpha} = \alpha$, $\widetilde{\beta} = \beta$, $\widetilde{z}_1 = z_1$, $\widetilde{z}_2 = z_2$. Thus, α, β, z_1, z_2 are H-unitary invariants.

LEMMA 10. *If an indecomposable H-normal operator N ($N: C^7 \to C^7$) has the only eigenvalue λ, $\dim S_0 = 2$, then the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(29), (30)\}$:*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -z_1\overline{z_2}\cos\alpha & \sin\alpha\cos\beta \\ 0 & 0 & 0 & \lambda & 0 & z_1\sin\alpha & z_2\cos\alpha\cos\beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1,\ 0 < \alpha, \beta \le \pi/2,\ z_1 = 1 \text{ if } \beta = \pi/2,\ z_2 = 1 \text{ if } \alpha = \pi/2.$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where z_1, z_2, r, α, β are H-unitary invariants.

Proof. We have to prove only the indecomposability of the canonical form because the rest was proved above. The proof, as is customary, is by inductio ad absurdum. Suppose a nondegenerate subspace V is invariant for N and $N^{[*]}$; then we can assume (see the proofs of the previous lemmas) that $\dim V \le 3$ and $\exists w_2 = av_6 + bv_7 + v \in V$ ($v \in (S_0 + S)$, $|a| + |b| \neq 0$). Then some nontrivial linear combination of the vectors $(N^{[*]} - \overline{\lambda} I)w_2 = av_3 + bv_4 + v'$ ($v' \in S_0$) and $(N - \lambda I)w_2 = a(-z_1\overline{z_2}\cos\alpha v_3 + z_1 \sin\alpha v_4) + b(\sin\alpha\cos\beta v_3 + z_2\cos\alpha\cos\beta v_4 + \sin\beta v_5) + v''$ ($v'' \in S_0$) must belong to S_0. This implies $b = 0 \Rightarrow a = 0$. The contradiction obtained proves that N is indecomposable. The proof is completed. ∎

5.3.5. $n = 8$ In this case

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } N_1 = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}.$$

As in the case when $n = 7$, one can check that for the condition $S \cap S_0 = \{0\}$ to hold the rank N_1 must be equal to 2. Without loss of generality it can be assumed that

$$\det\begin{pmatrix} a & b \\ e & f \end{pmatrix} \neq 0.$$

As before (in the case when $n = 7$), taking the block diagonal transformation $T = T_1 \oplus I \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} a & b \\ e & f \end{pmatrix},$$

we reduce N_1 to the form

$$N_1 = \begin{pmatrix} 1 & 0 & c' & d' \\ 0 & 1 & g' & h' \end{pmatrix}.$$

The results for the previous case $n = 7$ let the submatrix N_1 reduce to the form $(I \;\; 0)$. Indeed, there exists a transformation

$$T = T_1 \oplus T_2 \oplus T_1^{*-1}, \qquad \text{where } T_2 = T_2^{*-1} = \begin{pmatrix} t_{33} & t_{34} & t_{35} & 0 \\ t_{43} & t_{44} & t_{45} & 0 \\ t_{53} & t_{54} & t_{55} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that reduces the submatrix N_1 to the form

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & d' \\ 0 & 1 & 0 & h' \end{pmatrix}$$

and there exists a transformation

$$T = T_1 \oplus T_2 \oplus T_1^{*-1}, \qquad \text{where } T_2 = T_2^{*-1} = \begin{pmatrix} t_{33} & t_{34} & 0 & t_{36} \\ t_{43} & t_{44} & 0 & t_{46} \\ 0 & 0 & 1 & 0 \\ t_{63} & t_{64} & 0 & t_{66} \end{pmatrix},$$

that reduces the obtained submatrix N_1 to the desired form

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{123}$$

Now consider the submatrix N_3 and its submatrices N_3' and N_3'':

$$N_3 = \begin{pmatrix} N_3' \\ N_3'' \end{pmatrix}, \qquad N_3' = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad N_3'' = \begin{pmatrix} t & u \\ v & w \end{pmatrix}.$$

Note that N_3'' must be nondegenerate because otherwise the system

$$\begin{aligned} \bar{t}\alpha_1 + \bar{v}\alpha_2 &= 0 \\ \bar{u}\alpha_1 + \bar{w}\alpha_2 &= 0 \end{aligned}$$

has a nontrivial solution $\{\alpha_i\}_1^2$; hence, the nonzero vector $v = \alpha_1 v_5 + \alpha_2 v_6$ belongs to S_0.

Thus, N_3'' is nondegenerate. Recall that any nondegenerate matrix is a product of some self-adjoint positive definite matrix and some unitary one. Consequently, $N_3'' = RU$, where R is self-adjoint positive definite and U is unitary. Let U_1 be a unitary matrix reducing R to the real positive diagonal form. Taking $T = U^*U_1 \oplus U^*U_1 \oplus U_1 \oplus U^*U_1$, we carry N_3'' into the form

$$N_3'' = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad r_1, r_2 \in \Re,\ 0 < r_1 \le r_2$$

without changing the submatrix N_1. Now we have

$$N_3 = \begin{pmatrix} N_3' \\ N_3'' \end{pmatrix} = \begin{pmatrix} p' & q' \\ r' & s' \\ r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

Further, apply transformation (105) with

$$T_2 = \begin{pmatrix} 0 & \overline{m} & (k - r'\overline{m})/r_1 & (l - s'\overline{m})/r_2 \\ 0 & 0 & 0 & n/r_2 \end{pmatrix}$$

and reduce the submatrix

$$N_2 = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

to zero. Finally apply condition (97) of the H-normality of N. We get: $r_2 \leq 1$. Show that if $r_1 = r_2$, then N is decomposable. In fact, if $r_1 = r_2 = 1$, then from (97) it follows that $N_3' = 0$; hence, the nondegenerate subspace $V = \operatorname{span}\{v_1, v_3, v_5, v_7\}$ is invariant for N and $N^{[*]}$, and hence, N is decomposable. If $r_1 = r_2 < 1$, then the matrix $N_3'/\sqrt{1-r_1^2}$ is unitary; therefore, there exists a unitary matrix U that reduces N_3' to the diagonal form. Then the transformation $T = U \oplus U \oplus U \oplus U$ does not change the submatrices $N_1 = (I \;\; 0)$, $N_2 = 0$, $N_3'' = r_1 I$ and reduces N_3' to the diagonal form. Now it is seen that N is decomposable ($V = \operatorname{span}\{v_1, v_3, v_5, v_7\}$ is nondegenerate, $NV \subseteq V$, $N^{[*]}V \subseteq V$). Thus, in either case N is decomposable.

There remains to consider the case when $r_1 < r_2$. If $q' \neq 0$, let us replace q' by $|q'|$ by means of the transformation $\widetilde{v_1} = e^{i \arg q'} v_1$, $\widetilde{v_3} = e^{i \arg q'} v_3$, $\widetilde{v_5} = e^{i \arg q'} v_5$, $\widetilde{v_7} = e^{i \arg q'} v_7$. If $q' = 0$, let us put $\widetilde{v_1} = e^{-i \arg r'} v_1$, $\widetilde{v_3} = e^{-i \arg r'} v_3$, $\widetilde{v_5} = e^{-i \arg r'} v_5$, $\widetilde{v_7} = e^{-i \arg r'} v_7$. Then r' will be replaced by $|r'|$. Thus, one can assume that $q' \in \Re \geq 0$ and if $q' = 0$, then $r' \in \Re \geq 0$. Applying (97) and renaming the terms of N_3, we get

$$N_3 = \begin{pmatrix} -z_1\overline{z_2}\sin\alpha\cos\beta & \cos\alpha\cos\gamma \\ z_1\cos\alpha\cos\beta & z_2\sin\alpha\cos\gamma \\ \sin\beta & 0 \\ 0 & \sin\gamma \end{pmatrix}, \tag{124}$$

$|z_1| = |z_2| = 1$, $0 < \beta < \gamma \leq \pi/2$, $0 \leq \alpha \leq \pi/2$, $z_1 = 1$ if $\cos\alpha\cos\gamma = 0$, $z_2 = 1$ if $\alpha = 0$. We already know that if N_3' is diagonal, N is decomposable. Therefore, $\alpha \neq \pi/2$. As a result, we have

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{125}$$

and N_3 has form (124),

$$\begin{gathered} |z_1| = |z_2| = 1, \qquad 0 < \beta < \gamma \leq \pi/2, \qquad 0 \leq \alpha < \pi/2, \\ z_1 = 1 \quad \text{if} \quad \gamma = \pi/2, \qquad z_2 = 1 \quad \text{if} \quad \alpha = 0. \end{gathered} \tag{126}$$

Check the H-unitary invariance of the numbers α, β, γ, z_1, and z_2. To this end suppose that an H-unitary matrix T reduces $N - \lambda I$ to the form $\widetilde{N} - \lambda I$, where $N - \lambda I$ has form (124), (125), (126),

$$\widetilde{N} - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & \widetilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

N_1 has form (123), and N_3 has form (124),

$$\widetilde{N}_3 = \begin{pmatrix} -\widetilde{z}_1\overline{\widetilde{z}_2}\sin\widetilde{\alpha}\cos\widetilde{\beta} & \cos\widetilde{\alpha}\cos\widetilde{\gamma} \\ \widetilde{z}_1\cos\widetilde{\alpha}\cos\widetilde{\beta} & \widetilde{z}_2\sin\widetilde{\alpha}\cos\widetilde{\gamma} \\ \sin\widetilde{\beta} & 0 \\ 0 & \sin\widetilde{\gamma} \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \qquad 0 < \widetilde{\beta} < \widetilde{\gamma} \le \pi/2, \qquad 0 \le \widetilde{\alpha} < \pi/2,$$

$$\widetilde{z}_1 = 1 \text{ if } \widetilde{\gamma} = \pi/2, \qquad \widetilde{z}_2 = 1 \text{ if } \widetilde{\alpha} = 0.$$

Then T has form (107) and conditions (108)–(114) hold. From (108), (114), and (111) it follows that $T_4 = T_1 \oplus T_4'$, $T_4'T_4'^* = I$, $T_1 = T_6 = T_6^{*-1}$. From (110) it follows that $N_3''T_1 = T_4'\widetilde{N}_3''$. Taking into account the general form (122) of a 2×2 unitary matrix, we can check that this equality implies $T_4' = T_1 = t_{11} \oplus t_{22}$ ($|t_{11}| = |t_{22}| = 1$), $\widetilde{\beta} = \beta, \widetilde{\gamma} = \gamma$. Applying (110) again, we get

$$t_{22}\cos\alpha\cos\gamma = t_{11}\cos\widetilde{\alpha}\cos\widetilde{\gamma}$$
$$t_{11}z_1\cos\alpha\cos\beta = t_{22}\widetilde{z}_1\cos\widetilde{\alpha}\cos\widetilde{\beta}.$$

Hence $t_{11} = t_{22}$, and hence $\widetilde{N}_3 = N_3$; i.e., $\widetilde{\alpha} = \alpha$, $\widetilde{z}_1 = z_1$, $\widetilde{z}_2 = z_2$.

LEMMA 11. *If an indecomposable H-normal operator N ($N\colon C^8 \to C^8$) has the only eigenvalue $\lambda, \dim S_0 = 2$, then the pair $\{N, H\}$ is unitarily similar to canonical pair $\{(31), (32)\}$:*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -z_1\overline{z_2}\sin\alpha\cos\beta & \cos\alpha\cos\gamma \\ 0 & 0 & 0 & \lambda & 0 & 0 & z_1\cos\alpha\cos\beta & z_2\sin\alpha\cos\gamma \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \sin\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \qquad 0 \le \alpha < \pi/2, \qquad 0 < \beta < \gamma \le \pi/2,$$

$$z_1 = 1 \text{ if } \gamma = \pi/2, \qquad z_2 = 0 \text{ if } \alpha = 0.$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where $z_1, z_2, \alpha, \beta, \gamma$ are H-unitary invariants.

Proof. We must prove only the indecomposability of the canonical form. Assume the converse. Then (see the proofs of the previous lemmas) we can assume that $\dim V \geq 4$, $w_2 = av_7 + bv_8 + v \in V$ ($v \in (S_0 + S)$, $|a| + |b| \neq 0$). The vectors $(N - \lambda I)(N^{[*]} - \overline{\lambda} I)w_2 = av_1 + bv_2$, $(N^{[*]} - \overline{\lambda} I)^2 w_2 = a(-\overline{z_1} z_2 \sin\alpha\cos\beta v_1 + \cos\alpha\cos\gamma v_2) + b(\overline{z_1}\cos\alpha\cos\beta v_1 + \overline{z_2}\sin\alpha\cos\gamma v_2)$ and $(N - \lambda I)^2 w_2 = a(-z_1\overline{z_2}\sin\alpha\cos\beta v_1 + z_1\cos\alpha\cos\beta v_2) + b(\cos\alpha\cos\gamma v_1 + z_2\sin\alpha\cos\gamma v_2)$ must be collinear because otherwise we get $S_0 \subset V$, but since the condition $NS_1 \subset (S_1 + S_0)$ does not hold, we obtain $\dim V > 4$. Thus, let us write the conditions of the linear dependence (if a or b is equal to zero, the vectors are not collinear):

$$-\overline{z_1} z_2 \sin\alpha\cos\beta + \overline{z_1}\cos\alpha\cos\beta\frac{b}{a} = \cos\alpha\cos\gamma\frac{a}{b} + \overline{z_2}\sin\alpha\cos\gamma$$
$$-z_1\overline{z_2}\sin\alpha\cos\beta + \cos\alpha\cos\gamma\frac{b}{a} = z_1\cos\alpha\cos\beta\frac{a}{b} + z_2\sin\alpha\cos\gamma.$$

If we replace the last condition by its complex conjugate and subtract it from the first, we obtain

$$\overline{z_1}\cos\alpha\cos\beta\frac{b}{a} - \cos\alpha\cos\gamma\left(\frac{\overline{b}}{\overline{a}}\right) = \cos\alpha\cos\gamma\frac{a}{b} - \overline{z_1}\cos\alpha\cos\beta\left(\frac{\overline{a}}{\overline{b}}\right)$$

or

$$\overline{z_1}\cos\alpha\cos\beta\frac{|a|^2 + |b|^2}{a\overline{b}} = \cos\alpha\cos\gamma\frac{|a|^2 + |b|^2}{\overline{a}b}.$$

Modulus of the left-hand side must be equal to that of the right-hand side, i.e., $\cos\alpha\cos\beta = \cos\alpha\cos\gamma$. Since $\cos\alpha \neq 0$, $\cos\beta = \cos\gamma$; hence, $\beta = \gamma$. But for our canonical form $\beta < \gamma$. This contradiction proves the indecomposability of the operator N.

We have considered all alternatives for an indecomposable operator N and have obtained canonical forms for each case. Thus, we have proved Theorem 2. ∎

APPENDIX: CANONICAL FORMS FOR 2 × 2 MATRICES UNDER CONGRUENCE

PROPOSITION 3. *Any invertible matrix A of order 2×2 is congruent to one and only one of the following canonical forms*:

$$A = \begin{pmatrix} z & \varrho e^{-i\pi/3} z \\ 0 & e^{i\pi/3} z \end{pmatrix}, \quad |z| = 1,\ \varrho \in \Re \geq \sqrt{3},\ 0 \leq \arg z < \pi \text{ if } \varrho > \sqrt{3}, \tag{127}$$

$$A = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, \qquad |z_1| = 1,\ |z_2| = 1,\ \arg z_1 \le \arg z_2, \tag{128}$$

where z, z_1, z_2, ϱ form a complete and minimal set of invariants.

Proof. Consider the matrix $A' = AA^{*-1}$. If $\widetilde{A} = TAT^*$, then $\widetilde{A}' = TA'T^{-1}$ so that spectral properties of A' do not change under congruence of A. Reduce A' to the Jordan normal form. Since $|\det A'| = 1$, there exist three such forms:

$$A' = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \qquad x_1 \ne x_2,\ |x_1 x_2| = 1,\ |x_1| \le 1, \tag{129}$$

$$A' = xI, \qquad |x| = 1, \tag{130}$$

$$A' = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \qquad |x| = 1. \tag{131}$$

(a) A' is reduced to form (129). Since $A = A'A^*$, we have

$$A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a}x_1 & \bar{c}x_1 \\ \bar{b}x_2 & \bar{d}x_2 \end{pmatrix} = A'A^*. \tag{132}$$

It is seen that either $b = c = 0$ or $\arg x_1 = \arg x_2$.

If $|x_1| < 1$, then from (132) it follows that $a = d = 0$; since A is invertible, b and c are nonzero; therefore, $\arg x_1 = \arg x_2$. Now let us consider the function $f(\varrho) = \frac{1}{2}(1 - \varrho^2 - \sqrt{(\varrho^2+1)(\varrho^2-3)})$ of the real variable ϱ. It monotonically decreases on the interval $(\sqrt{3}, +\infty)$; $f(\sqrt{3}) = -1$, and $\lim_{\varrho\to+\infty} f(\varrho) = -\infty$; therefore, the equation $f(\varrho) = s$ has a root $\varrho > \sqrt{3}$ for all $s < -1$. Let ϱ be a root of the equation $f(\varrho) = -|x_2|$ and let $e^{i\arg x_2} = -e^{i\pi/3}z^2$, where $|z| = 1$, $0 \le \arg z < \pi$. Then $x_1 = \frac{1}{2}e^{i\pi/3}z^2(1 - \varrho^2 + \sqrt{(\varrho^2+1)(\varrho^2-3)})$, $x_2 = \frac{1}{2}e^{i\pi/3}z^2(1 - \varrho^2 - \sqrt{(\varrho^2+1)(\varrho^2-3)})$, and from (132) it follows that

$$A = \begin{pmatrix} 0 & b \\ e^{i\pi/3}z^2 f(\varrho)\bar{b} & 0 \end{pmatrix}, \qquad b \ne 0.$$

Now the transformation

$$T = \begin{pmatrix} 1 & \bar{z}(e^{-i\pi/3}f(\varrho) - 1)/(\bar{b}(f(\varrho)^2 - 1)) \\ e^{2i\pi/3}\varrho f(\varrho)/(e^{i\pi/3}f(\varrho) - 1) & -e^{i\pi/3}\bar{z}\varrho/(\bar{b}(f(\varrho)^2 - 1)) \end{pmatrix}$$

reduces A to form (127) with $\varrho > \sqrt{3}$. The numbers ϱ and z cannot be changed under congruence because the eigenvalues of A' are invariants and from the condition $e^{i\pi/3}z^2f(\varrho) = e^{i\pi/3}\widetilde{z}^2f(\widetilde{\varrho})$ ($|z|=|\widetilde{z}|=1$, $0 \le \arg z$, $\arg\widetilde{z} < \pi$, $\varrho, \widetilde{\varrho} \in \Re > \sqrt{3}$) it follows that $\widetilde{z} = z$, $\widetilde{\varrho} = \varrho$.

If $|x_1| = 1$, then from the condition $x_1 \neq x_2$ it follows that $\arg x_1 \neq \arg x_2$; hence $b = c = 0$. By taking $T = D_2$ one can interchange the terms a and d of the matrix A. Hence, we can assume that $\arg a \le \arg d$. Applying the transformation

$$T = \begin{pmatrix} 1/\sqrt{|a|} & 0 \\ 0 & 1/\sqrt{|d|} \end{pmatrix},$$

we reduce A to form (128) with $z_1 = e^{i\arg a}$, $z_2 = e^{i\arg d}$.

To prove the invariance of z_1 and z_2 suppose that $\widetilde{A} = TAT^*$, where $A = z_1 \oplus z_2$, $\widetilde{A} = \widetilde{z}_1 \oplus \widetilde{z}_2$, $|z_1| = |z_2| = |\widetilde{z}_1| = |\widetilde{z}_2| = 1$, $\arg z_1 \le \arg z_2$, $\arg\widetilde{z}_1 \le \arg\widetilde{z}_2$. Then

$$z_1|t_{11}|^2 + z_2|t_{12}|^2 = \widetilde{z}_1 \tag{133}$$

$$z_1t_{11}\overline{t_{21}} + z_2t_{12}\overline{t_{22}} = 0 \tag{134}$$

$$z_1\overline{t_{11}}t_{21} + z_2\overline{t_{12}}t_{22} = 0 \tag{135}$$

$$z_1|t_{21}|^2 + z_2|t_{22}|^2 = \widetilde{z}_2. \tag{136}$$

Since $t_{11}\overline{t_{21}} = -\overline{z_1}z_2t_{12}\overline{t_{22}}$ (condition (134)), (135) holds only if $(z_2^2 - z_1^2)\overline{t_{12}}t_{22} = 0$. If $z_1^2 \neq z_2^2$, then t_{12} must be zero because if $t_{22} = 0$, then $t_{11} = 0$ and, therefore, $\widetilde{z}_1 = z_2$, $\widetilde{z}_2 = z_1$, which contradicts the condition $\arg\widetilde{z}_1 \le \arg\widetilde{z}_2$. Thus, $t_{12} = 0$; hence, $t_{21} = 0$, $\widetilde{z}_1 = z_1$, $\widetilde{z}_2 = z_2$. If $z_1 = z_2$, then, according to (133)–(136), $\widetilde{z}_1 = z_1(|t_{11}|^2 + |t_{12}|^2)$, $\widetilde{z}_2 = z_1(|t_{21}|^2 + |t_{22}|^2)$; hence $\widetilde{z}_1 = \widetilde{z}_2 = z_1 = z_2$. If $z_2 = -z_1$ and $\overline{t_{12}}t_{22} \neq 0$, then $t_{11}\overline{t_{21}} \neq 0$ and $\widetilde{z}_1 = z_1(|t_{11}|^2 - |t_{12}|^2)$. Since $|t_{21}|/|t_{22}| = |t_{12}|/|t_{11}|$, $\widetilde{z}_2 = z_1(|t_{21}|^2 - |t_{22}|^2) = -\widetilde{z}_1|t_{22}|^2/|t_{11}|^2$. As $\arg\widetilde{z}_1 \le \arg\widetilde{z}_2$, we get $\widetilde{z}_1 = z_1$, $\widetilde{z}_2 = z_2$. The case when $z_2 = -z_1$ and $\overline{t_{12}}t_{22} = 0$ can be considered as before. Thus, we have proved the invariance of the numbers z_1 and z_2.

(b) A' is reduced to form (130). Then $A = xA^*$, $|x| = 1$, this property being invariant with respect to congruence. Since A is invertible, $A = RU$, where R is self-adjoint positive definite matrix and U is unitary. Let T be a unitary matrix reducing U to the diagonal form Λ. After the application of T we have: $A = \widetilde{R}\Lambda$, where $\widetilde{R} = TRT^*$ is also self-adjoint positive definite. Now let T be a lowertriangular matrix such that $T\widetilde{R}T^* = I$. Then we reduce A to the uppertriangular form $T^{*-1}\Lambda T^*$. Since the term c of A is now equal to zero, from the condition $A = xA^*$ it follows that b is also equal to zero; i.e., A is diagonal. We already know that a diagonal matrix is congruent to (128) (see case (a) above). Thus, A can be reduced to form (128).

(c) A' is reduced to form (131). Let $x = -e^{i\pi/3}z^2$ ($|z| = 1$). Then the application of the condition $A = A'A^*$ yields

$$A = \begin{pmatrix} a & b \\ -e^{i\pi/3}z^2\overline{b} & 0 \end{pmatrix}, \qquad b = \overline{a} + e^{-i\pi/3}\overline{z}^2 a.$$

For A to be invertible b must be nonzero. Since $|b| = |a + e^{i\pi/3}z^2\overline{a}| = |a\overline{z} + e^{i\pi/3}\overline{a}z| = |a\overline{z} - e^{-2i\pi/3}\overline{a}z| = |e^{i\pi/3}a\overline{z} - e^{-i\pi/3}\overline{a}z| = 2|\operatorname{Im}\{e^{i\pi/3}a\overline{z}\}|$, we see that $\operatorname{Im}\{e^{i\pi/3}a\overline{z}\} \neq 0$. Let us choose z so that $\operatorname{Im}\{e^{i\pi/3}a\overline{z}\} > 0$. Applying the transformation

$$T = \frac{\sqrt[4]{3}}{\sqrt{|b|}^3} \begin{pmatrix} |b| & \frac{2}{3}i\overline{z}\operatorname{Im}\{a\overline{z}\}|b|/\overline{b} \\ e^{i\pi/3}\overline{z}\overline{b} & \overline{z}^2\left(-\frac{2}{3}i\operatorname{Im}\{a\overline{z}\} + a\overline{z}\right) \end{pmatrix},$$

we reduce A to form (127) with $\varrho = \sqrt{3}$. It is clear that matrix (127) with $\varrho = \sqrt{3}$ is not congruent to that with $\varrho > 3$ because in the former case A' has the diagonal Jordan normal form in contrast to the latter. Therefore, we must prove only the invariance of z. Note that if $\widetilde{A} = TAT^*$, where

$$A = \begin{pmatrix} z & \sqrt{3}e^{-i\pi/3}z \\ 0 & e^{i\pi/3}z \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} \widetilde{z} & \sqrt{3}e^{-i\pi/3}\widetilde{z} \\ 0 & e^{i\pi/3}\widetilde{z} \end{pmatrix}, \qquad |z| = |\widetilde{z}| = 1,$$

then $\widetilde{z}^2 = z^2$ because the eigenvalue $x = -e^{i\pi/3}z^2$ of A' does not change under congruence of A. Therefore,

$$A' = z^2 \begin{pmatrix} 1 - 3e^{i\pi/3} & \sqrt{3} \\ \sqrt{3} & e^{2i\pi/3} \end{pmatrix} = \widetilde{A}'.$$

For T to satisfy the condition $A'T = TA'$ the matrix T must have the form

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{11} + it_{12} \end{pmatrix}.$$

Now from the condition $\widetilde{A} = TAT^*$ it follows that

$$z|t_{11}|^2 + \sqrt{3}e^{-i\pi/3}zt_{11}\overline{t_{12}} + e^{i\pi/3}z|t_{12}|^2 = \widetilde{z} \tag{137}$$

$$z\overline{t_{11}}t_{12} + \sqrt{3}e^{-i\pi/3}z|t_{12}|^2 + e^{i\pi/3}z\left(t_{11}\overline{t_{12}} + i|t_{12}|^2\right) = 0. \tag{138}$$

If $t_{12} \neq 0$, from (138) it follows that

$$e^{-i\pi/6}\frac{\overline{t_{11}}}{\overline{t_{12}}} + \sqrt{3}e^{-i\pi/2} + e^{i\pi/6}\frac{t_{11}}{t_{12}} + e^{2i\pi/3} = 0,$$

which is impossible because the imaginary part of the left-hand side is equal to $\mathrm{Im}\{\sqrt{3}e^{-i\pi/2} + e^{2i\pi/3}\} = -\sqrt{3}/2$. Therefore, $t_{12} = 0$, and hence (condition (137)) $\widetilde{z} = z$; i.e., z is an invariant. This concludes the proof of the proposition. ■

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