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A note on approximate Nash equilibria

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ABSTRACT

In view of the intractability of finding a Nash equilibrium, it is important to understand the limits of approximation in this context. A subexponential approximation scheme is known [Richard J. Lipton, Evangelos Markakis, Aranyak Mehta, Playing large games using simple strategies, in: EC, 2003], and no approximation better than $\frac{1}{4}$ is possible by any algorithm that examines equilibria involving fewer than $\log n$ strategies [Ingo Althöfer, On sparse approximations to randomized strategies and convex combinations, Linear Algebra and its Applications (1994) 199]. We give a simple, linear-time algorithm examining just two strategies per player and resulting in a $\frac{1}{2}$-approximate Nash equilibrium in any 2-player game. For the more demanding notion of *approximately well supported Nash equilibrium* due to [Constantinos Daskalakis, Paul W. Goldberg, Christos H. Papadimitriou, The complexity of computing a Nash equilibrium, SIAM Journal on Computing (in press) Preliminary version appeared in STOC (2006)] no nontrivial bound is known; we show that the problem can be reduced to the case of win-lose games (games with all utilities 0 or 1), and that an approximation of $\frac{5}{6}$ is possible, contingent upon a graph-theoretic conjecture.

Subsequent work extends the $\frac{1}{4}$ impossibility result of Ingo Althöfer's paper, as mentioned above, to $\frac{1}{2}$ [Tomás Feder, Hamid Nazerzadeh, Amin Saberi, Approximating nash equilibria using small-support strategies, in: EC, 2007], making our $\frac{1}{2}$-approximate Nash equilibrium algorithm optimal among the algorithms that only consider mixed strategies of sublogarithmic size support. Moreover, techniques similar to our techniques for approximately well supported Nash equilibria are used in [Spyros Kontogiannis, Paul G. Spirakis, Efficient algorithms for constant well supported approximate equilibria in bimatrix games, in: ICALP, 2007] for obtaining an efficient algorithm for 0.658-approximately well supported Nash equilibria, unconditionally.

1. Introduction

Since it was shown that finding a Nash equilibrium is PPAD-complete [8], even for 2-player normal form games [3], the question of approximate Nash equilibrium emerged as the central remaining open problem in the area of equilibrium computation. Since scaling the utilities of a player by any positive factor, and applying any additive constant, results in an equilibrium-equivalent game, it is quite standard to assume that all utilities have been normalized to be between 0 and 1. A set of mixed strategies is then an *ϵ-approximate Nash equilibrium*, where $\epsilon > 0$, if, for each player, all strategies have expected payoff that is at most ϵ more than the expected payoff of the given strategy. Clearly, any mixed strategy combination is a 1-approximate Nash equilibrium, and it is quite straightforward to find a $\frac{3}{4}$-approximate Nash equilibrium

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of a two-player game by examining all pairs of mixed strategies with supports of size two. In fact, [15] provides a scheme that yields, for every $\epsilon > 0$, in time polynomial in the size of the game and $\frac{1}{\epsilon}$, a $\frac{2+\epsilon+\lambda}{4}$-approximate Nash equilibrium, where λ is the minimum, among all Nash equilibria, expected payoff of either player. [18] (building on [1,19]) showed that, for every $\epsilon > 0$, an ϵ-approximate Nash equilibrium can be found in time $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ by examining all supports of size $O\left(\frac{\log n}{\epsilon^2}\right)$. It was pointed out in [1] that, even for zero-sum games, no algorithm that examines supports smaller than about $\log n$ can achieve an approximation better than $\frac{1}{4}$. Can this gap between $\frac{1}{4}$ and $\frac{3}{4}$ be bridged by looking at small supports? And how can the barrier of $\frac{1}{4}$ be broken in polynomial time?

In this note, we concentrate on 2-player games. We point out that a straightforward algorithm, looking at just three strategies in total, achieves a $\frac{1}{2}$-approximate Nash equilibrium. The algorithm is very intuitive: for any strategy i of the row player, let j be the best response of the column player, and let k be the best response of the row player to j. Then the row player plays an equal mixture of i and k, while the column player plays j. The proof of $\frac{1}{2}$-approximation is rather immediate.

We also examine a more sophisticated concept of approximation due to [13,8], called *ϵ-approximately well supported Nash equilibrium*, which does not allow in the support strategies that are suboptimal by more than ϵ. For this concept, no approximation constant better than 1 is known. We show that the problem is reduced — albeit with a loss in the approximation ratio — to the case in which all utilities are either zero or one (this is often called the "win-lose case"). We also prove that, assuming a well-studied and plausible graph-theoretic conjecture, in win-lose games there is a $\frac{2}{3}$-approximately well supported Nash equilibrium with supports of size at most three (and of course it can be found in polynomial time). This yields a $\frac{5}{6}$-approximately well supported Nash equilibrium for any game.

Following the first presentation of our results [9], there have been several improvements. We summarize the subsequent work in Section 5.

2. Definitions

We consider normal form games between two players, the *row player* and the *column player,* each with n strategies at his disposal. The game is defined by two $n \times n$ payoff matrices, R for the row player, and C for the column player. The pure strategies of the row player correspond to the n rows, and the pure strategies of the column player correspond to the n columns. If the row player plays row i and the column player plays column j, then the row player receives a payoff of R_{ij} and the column player gets C_{ij}. Payoffs are extended linearly to pairs of mixed strategies — if the row player plays a probability distribution x over the rows, and the column player plays a distribution y over the columns, then the row player gets a payoff of $x^{\mathrm{T}}Ry$ and the column player gets a payoff of $x^{\mathrm{T}}Cy$.

A *Nash equilibrium*, in this setting, is a pair of mixed strategies, x^* for the row player and y^* for the column player, such that neither player has an incentive to unilaterally defect. Note that, by linearity, the best defection is to a pure strategy. Let e_i denote the vector with a 1 at the ith coordinate and 0 elsewhere. A pair of mixed strategies (x^*, y^*) is a Nash equilibrium if

$$\forall i = 1..n, \quad e_i^{\mathrm{T}} R y^* \le {x^*}^{\mathrm{T}} R y^*$$
$$\forall i = 1..n, \quad {x^*}^{\mathrm{T}} C e_i \le {x^*}^{\mathrm{T}} C y^*.$$

It can be easily shown that every pair of equilibrium strategies of a game remains an equilibrium upon multiplying all the entries of a payoff matrix by a constant, and upon adding the same constant to each entry. We shall therefore assume that the entries of both payoff matrices R and C are between 0 and 1.

For $\epsilon > 0$, we define an *ϵ-approximate Nash equilibrium* to be a pair of mixed strategies x^* for the row player and y^* for the column player, so that the incentive to unilaterally deviate is at most ϵ, that is

$$\forall i = 1..n, \quad e_i^{\mathrm{T}} R y^* \le {x^*}^{\mathrm{T}} R y^* + \epsilon$$
$$\forall i = 1..n, \quad {x^*}^{\mathrm{T}} C e_i \le {x^*}^{\mathrm{T}} C y^* + \epsilon.$$

Given that we have normalized all the payoff entries to [0, 1], the additive notion of approximation that we consider is reasonable, and it is the standard choice in the literature (see, for example, [21]).

A stronger notion of approximation was introduced in [13,8]: for $\epsilon > 0$, an *ϵ-approximately well supported Nash equilibrium*, or *ϵ-Nash equilibrium*, is a pair of mixed strategies, x^* for the row player and y^* for the column player, so that a player plays only approximately best-response pure strategies with non-zero probability, that is

$$\forall i: \ x_i^* > 0 \Rightarrow \ e_i^{\mathrm{T}} R y^* \ge e_j^{\mathrm{T}} R y^* - \epsilon, \ \ \forall j$$
$$\forall i: \ y_i^* > 0 \Rightarrow \ {x^*}^{\mathrm{T}} C e_i \ge {x^*}^{\mathrm{T}} C e_j - \epsilon, \ \ \forall j.$$

If only the first set of inequalities holds, we say that x^* is *ϵ-well supported against* y^*; we also say that every pure strategy in the support of x^* is *ϵ-well supported against* y^*. A similar definition holds for the second set of inequalities, this time for y^* against x^*. The ϵ-Nash equilibrium defines a stronger notion of approximation, in the sense that every ϵ-Nash equilibrium is also an ϵ-approximate Nash equilibrium, but the converse need not be true. Nevertheless, the following lemma from [8] shows that there does exist a tight relationship between the two:

Lemma 2.1 ([8]). *Given an ϵ-approximate Nash equilibrium of a game (R, C) we can compute in polynomial time a $\sqrt{\epsilon}(\sqrt{\epsilon} + 1 + 4U)$-approximately well supported Nash equilibrium, where U is the maximum entry in the payoff matrices R and C.*

In this paper, we show how to find a 1/2-approximate Nash equilibrium. For $\epsilon = 1/2$, the application of the above lemma does not help in getting a non-trivial (i.e. better than 1-approximate) well supported equilibrium. Thus, we also describe algorithms for directly finding non-trivial well supported equilibria.

3. A simple algorithm

We provide, here, a simple way of computing a $\frac{1}{2}$-approximate Nash equilibrium: pick an arbitrary row for the row player, say row i. Let $j \in \arg\max_{j'} C_{ij'}$. Let $k \in \arg\max_{k'} R_{k'j}$. Thus, j is a best-response column for the column player to the row i, and k is a best-response row for the row player to the column j.

The approximate equilibrium is then $x^* = \frac{1}{2}e_i + \frac{1}{2}e_k$ and $y^* = e_j$, i.e., the row player plays row i or row k with probability $\frac{1}{2}$ each, while the column player plays column j with probability 1.

Theorem 3.1. *The strategy pair (x^*, y^*) is a $\frac{1}{2}$-approximate Nash equilibrium.*

Proof. The row player's payoff under (x^*, y^*) is $x^{*\mathrm{T}}Ry^* = \frac{1}{2}R_{ij} + \frac{1}{2}R_{kj}$. By construction, one of his best responses to y^* is to play the pure strategy on row k, which gives a payoff of R_{kj}. Hence, his incentive to defect is equal to the difference:

$$R_{kj} - \left(\frac{1}{2}R_{ij} + \frac{1}{2}R_{kj}\right) = \frac{1}{2}R_{kj} - \frac{1}{2}R_{ij} \le \frac{1}{2}R_{kj} \le \frac{1}{2}.$$

The column player's payoff under (x^*, y^*) is $x^{*\mathrm{T}}Cy^* = \frac{1}{2}C_{ij} + \frac{1}{2}C_{kj}$. Let j' be a best pure strategy response of the column player to x^*: this strategy gives the column player a value of $\frac{1}{2}C_{ij'} + \frac{1}{2}C_{kj'}$, hence her incentive to defect is equal to the difference:

$$\begin{aligned}\left(\frac{1}{2}C_{ij'} + \frac{1}{2}C_{kj'}\right) - \left(\frac{1}{2}C_{ij} + \frac{1}{2}C_{kj}\right) &= \frac{1}{2}(C_{ij'} - C_{ij}) + \frac{1}{2}(C_{kj'} - C_{kj}) \\ &\le 0 + \frac{1}{2}(C_{kj'} - C_{kj}) \\ &\le \frac{1}{2}.\end{aligned}$$

Here the first inequality follows from the fact that column j was a best response to row i, by the first step of the construction. □

4. Well supported Nash equilibria

The algorithm of the previous section yields equilibria that are, in the worst case, as bad as 1-approximately well supported (when $R_{ij} = 0$ and $R_{kj} = 1$). In this section we address the harder computational problem of finding ϵ-approximately well supported equilibria for $\epsilon < 1$.

Our construction has two components. In the first, we transform the given 2-player game into a new game by rearranging and potentially discarding or duplicating some of the columns of the original game. The transformation will be such that well supported equilibria in the new game can be mapped back to well supported equilibria of the original game; moreover, the mapping will result in some sort of *decorrelation of the players*, in the sense that computation of well supported equilibria in the decorrelated game can be carried out by looking at the row player only. The second part of the construction relies in mapping the original game into a win-lose game (a game with 0/1 payoffs) and computing equilibria of the latter. The mapping will guarantee that well supportedness of equilibria is preserved, albeit with some larger approximation.

4.1. Player decorrelation

Let (R, C) be a 2-player game, where the set of strategies of both players is $[n] := \{1, \ldots, n\}$.

Definition 4.1. A mapping $f : [n] \to [n]$ is a *best response mapping for the column player* if, for every $i \in [n]$,

$$C_{if(i)} = \max_j C_{ij}.$$

Definition 4.2 (*Decorrelation Transformation*). The *decorrelated game* (R^f, C^f) corresponding to the best response mapping f is defined as follows

$$\begin{aligned}\forall i, j \in [n]: \quad R^f_{ij} &= R_{if(j)} \\ C^f_{ij} &= C_{if(j)}.\end{aligned}$$

In other words, column j of R^f (resp. C^f) is a copy of column $f(j)$ of R (resp. C). Note that the decorrelation transformation need not be a permutation of the columns of the original game. Some columns of the original game may very well be dropped and others duplicated. So, it is not true, in general, that (exact) Nash equilibria of the decorrelated game can be mapped to Nash equilibria of the original game. However, some specially structured approximately well supported equilibria of the decorrelated game can be mapped to approximately well supported equilibria of the original game, as specified by the following lemmas.

In the following discussion, we assume that we have fixed a best response mapping f for the column player and that the corresponding decorrelated game is (R^f, C^f). Also, if $S \subseteq [n]$, we denote by $\Delta(S)$ the set of probability distributions over the set S. Moreover, if $x \in \Delta(S)$, we denote by $\mathrm{supp}(x) \triangleq \{i \in [n] | x(i) > 0\}$, the support of x.

Lemma 4.3. *In the game (R^f, C^f), for all sets $S \subseteq [n]$, every strategy of the column player in S is $\frac{|S|-1}{|S|}$-well supported against the following strategy x^* of the row player: Define $S' = \left\{i \in S | C^f_{ii} = 0\right\}$.*

- *if $S' \neq \emptyset$, then x^* is the uniform distribution over S'*
- *if $S' = \emptyset$, then*

$$x^*(i) = \begin{cases} \frac{1}{Z}\frac{1}{C^f_{ii}}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

where $Z = \sum_{i \in S} \frac{1}{C^f_{ii}}$ is a normalizing constant.

Proof. Suppose that $S' \neq \emptyset$. By the definition of S', it follows that $\forall i \in S', j \in [n]$, $C^f_{ij} = 0$. Therefore, for all $j \in [n]$,

$$x^{*T} C^f e_j = 0,$$

which proves the claim.

The proof for the $S' = \emptyset$ case is based on the following observations.

- $\forall j \in S$:

$$x^{*T} C^f e_j \geq x^*(j) C^f_{jj} = \frac{1}{Z}\frac{1}{C^f_{jj}} C^f_{jj} = \frac{1}{Z}.$$

- $\forall j \in [n]$:

$$x^{*T} C^f e_j = \frac{1}{Z}\sum_{i \in S}\frac{1}{C^f_{ii}} C^f_{ij} \leq \frac{1}{Z}\sum_{i \in S}\frac{1}{C^f_{ii}} C^f_{ii} = \frac{|S|}{Z}.$$

- $Z = \sum_{i \in S} \frac{1}{C^f_{ii}} \geq |S|$ because every entry of C^f is at most 1.

Therefore, $\forall j_1 \in S, j_2 \in [n]$,

$$x^{*T} C^f e_{j_2} - x^{*T} C^f e_{j_1} \leq \frac{|S|-1}{Z} \leq \frac{|S|-1}{|S|},$$

which completes the claim. □

The following lemma is an immediate corollary of Lemma 4.3.

Lemma 4.4 (*Player Decorrelation*). *In the game (R^f, C^f), if there exists a set $S \subseteq [n]$ and a mixed strategy $y^* \in \Delta(S)$ for the column player such that each of the row player's strategies in S is $\frac{|S|-1}{|S|}$-well supported against the distribution y^*, then there exists a strategy $x^* \in \Delta(S)$ for the row player, of the form specified in the statement of Lemma 4.3, so that the pair (x^*, y^*) is an $\frac{|S|-1}{|S|}$-approximately well supported Nash equilibrium.*

The next lemma describes how well supported equilibria in the games (R, C) and (R^f, C^f) are related.

Lemma 4.5. *$\forall S \subseteq [n]$, if the pair (x^*, y^*), where x^* is defined as in the statement of Lemma 4.3 and y^* is any distribution over S, constitutes an $\frac{|S|-1}{|S|}$-approximately well supported Nash equilibrium for the game (R^f, C^f), then the pair of distributions (x^*, y') is an $\frac{|S|-1}{|S|}$-approximately well supported Nash equilibrium for the game (R, C), where y' is the distribution, defined as follows*

$$y'(i) = \sum_{j \in S} y^*(j)\mathcal{X}_{f(j)=i}, \quad \forall i \in [n],$$

where $\mathcal{X}_{f(j)=i}$ is the indicator function of the condition "$f(j) = i$".

Proof. We have to verify that the pair of distributions (x^*, y') satisfies the conditions of well supportedness for the row and column player in the game (R, C).

Row player: We show that the strategies y^* and y' of the column player give to every pure strategy of the row player the same payoff in the two games. And, since the support of the row player stays the same set S in the two games, the fact that the strategy of the row player is approximately well supported in the game (R^f, C^f) guarantees that the strategy of the row player will be approximately well supported in the game (R, C) as well.

$$\forall i \in [n] : \qquad \begin{aligned} e_i^{\mathrm{T}} R y' &= \sum_{k=1}^{n} R_{ik} \cdot y'(k) \\ &= \sum_{k=1}^{n} R_{ik} \cdot \sum_{j \in S} y^*(j) \mathcal{X}_{f(j)=k} \\ &= \sum_{j \in S} y^*(j) \sum_{k=1}^{n} R_{ik} \cdot \mathcal{X}_{f(j)=k} \\ &= \sum_{j \in S} y^*(j) R_{if(j)} \\ &= \sum_{j \in S} y^*(j) R^f_{ij} = e_i^{\mathrm{T}} R^f y^*. \end{aligned}$$

Column player: As in the proof of Lemma 4.3, the analysis proceeds by distinguishing the cases $S' \neq \emptyset$ and $S' = \emptyset$. The case $S' \neq \emptyset$ is easy, because, in this regime, it must hold that $\forall i \in S', j \in [n], C^f_{ij} = 0$ which implies that, also, $C_{ij} = 0$, $\forall i \in S', j \in [n]$. And since x^* has support S' we see that the column player's strategy is 0-well supported against x^*.

So it is enough to deal with the $S' = \emptyset$ case. The support of y' is clearly the set $S'' = \{j | \exists i \in S \text{ such that } f(i) = j\}$. Moreover, observe the following:

$$\forall j \in S'' : \qquad \begin{aligned} x^{*T} C e_j &= \sum_{i \in S} x^*(i) C_{ij} \\ &\geq \sum_{\substack{i \in S \text{ s.t.} \\ f(i)=j}} x^*(i) C_{if(i)} \\ &= \sum_{\substack{i \in S \text{ s.t.} \\ f(i)=j}} x^*(i) C^f_{ii} \geq \frac{1}{Z}. \end{aligned}$$

The final inequality holds because there is at least one summand, since $j \in S''$. On the other hand,

$$\forall j \notin S'' : \qquad \begin{aligned} x^{*T} C e_j &= \sum_{i \in S} x^*(i) C_{ij} \\ &\leq \sum_{i \in S} x^*(i) C_{if(i)} \\ &= \sum_{i \in S} x^*(i) C^f_{ii} \\ &= \sum_{i \in S} \frac{1}{Z} \frac{1}{C^f_{ii}} C^f_{ii} \\ &= \frac{|S|}{Z}. \end{aligned}$$

Moreover, as we argued in the proof of Lemma 4.3, $\frac{|S|}{Z} - \frac{1}{Z} \leq \frac{|S|-1}{|S|}$. This completes the proof, since the strategy of the column player is, thus, also approximately well supported. □

4.2. Reduction to win-lose games

We now describe a mapping from a general 2-player game to a win-lose game so that approximately well supported equilibria of the win-lose game can be mapped to approximately well supported equilibria of the original game. Let us introduce a bit of notation first. If A is an $n \times n$ matrix with entries in $[0, 1]$, we denote by $round(A)$ the 0/1 matrix defined as follows, for all $i, j \in [n]$,

$$round(A)_{ij} = \begin{cases} 1, & \text{if } A_{ij} \geq \frac{1}{2} \\ 0, & \text{if } A_{ij} < \frac{1}{2}. \end{cases}$$

The following lemma establishes a useful connection between approximately well supported equilibria of the 0/1 game and approximately well supported equilibria of the original game.

Lemma 4.6. *If (x, y) is an ϵ-approximately well supported Nash equilibrium of the game $(round(R), round(C))$, then (x, y) is a $\frac{1+\epsilon}{2}$-approximately well supported Nash equilibrium of the game (R, C).*

Proof. We will show that x is approximately well supported against y in the game (R, C); similar arguments apply to justify that y is approximately well supported. Let us denote $R' = round(R)$ and $C' = round(C)$.

The following claim follows easily from the rounding procedure.

Claim 1. $\forall i, j \in [n] : \frac{R'_{ij}}{2} \leq R_{ij} \leq \frac{1}{2} + \frac{R'_{ij}}{2}$

It follows that for all $i \in [n]$,

$$\frac{1}{2} e_i^{\mathrm{T}} R' y \leq e_i^{\mathrm{T}} R y \leq \frac{1}{2} + \frac{1}{2} e_i^{\mathrm{T}} R' y. \tag{1}$$

We will use (1) to argue that x is approximately well supported. Indeed, $\forall k \in \mathrm{supp}(x)$, and $\forall i \in [n]$

$$e_i^{\mathrm{T}} R y - e_k^{\mathrm{T}} R y \leq \frac{1}{2} + \frac{1}{2} e_i^{\mathrm{T}} R' y - \frac{1}{2} e_k^{\mathrm{T}} R' y \leq \frac{1}{2} + \frac{1}{2} \cdot (e_i^{\mathrm{T}} R' y - e_k^{\mathrm{T}} R' y) \leq \frac{1}{2} + \frac{1}{2} \cdot \epsilon$$

where the last implication follows from the fact that (x, y) is an ϵ-approximately well supported Nash equilibrium of the game (R', C'). □

4.3. Finding well supported equilibria

Lemmas 4.3 through 4.6 suggest the following algorithm, *ALG-WS*, to compute approximately well supported Nash equilibria of a given two player game (R, C):

(1) Map the game (R, C) to the win-lose game $(round(R), round(C))$.
(2) Map game $(round(R), round(C))$ to the game $(round(R)^f, round(C)^f)$, where f is any best response mapping for the column player.
(3) Find a subset $S \subseteq [n]$ and a strategy $y \in \Delta(S)$ for the column player such that all the strategies in S are $\frac{|S|-1}{|S|}$-well supported for the row player in $(round(R)^f, round(C)^f)$ against the strategy y for the column player.
(4) By a successive application of Lemmas 4.4–4.6, we get an $\left(\frac{1}{2} + \frac{1}{2}\frac{|S|-1}{|S|}\right) = \left(1 - \frac{1}{2|S|}\right)$-approximately well supported Nash equilibrium of the original game.

The only non-trivial step of the algorithm is step 3. Let us paraphrase what this task entails:

"Given a 0/1 matrix $round(R)^f$, find a subset of the columns $S \subseteq [n]$ and a distribution $y \in \Delta(S)$, so that all rows in S are $\frac{|S|-1}{|S|}$-well supported against the distribution y over the columns."

It is useful to consider the 0/1 matrix $round(R)^f$ as the adjacency matrix of a directed graph G on n vertices. We shall argue, next, that the task above is easy in two cases: when G has a small cycle, and when G has a small *undominated* set of vertices, that is a set of vertices such that no other vertex has edges to all of them.

(1) Suppose first that G has a cycle of length k, and let S be the set of vertices on the cycle. Then it is easy to see that all the k strategies in S are $\frac{k-1}{k}$-well supported for the row player against y, where y is the uniform strategy for the column player over the set S. The reason is that each strategy in S has an expected payoff of at least $\frac{1}{k}$ against y, and thus no other strategy can dominate it by more than $\frac{k-1}{k}$. This, via the above algorithm, implies a $(1 - \frac{1}{2k})$-approximately well supported Nash equilibrium.
(2) Second, suppose that there is a set S of ℓ undominated vertices. Then every strategy in S is $(1 - \frac{1}{\ell})$-well supported for the row player against the uniform strategy y of the column player over S, simply because there is no row that has payoff better than $1 - \frac{1}{\ell}$ against y. Again, via the algorithm, this implies that we can find a $(1 - \frac{1}{2\ell})$-approximately well supported Nash equilibrium.

This leads us to the following graph theoretic conjecture:

Conjecture 4.7. *There are integers k and ℓ such that every digraph either has a cycle of length at most k or an undominated set of ℓ vertices.*

Now, the next result follows immediately from the preceding discussion:

Theorem 4.8. *If Conjecture 4.7 is true for some values of k and ℓ, then Algorithm* ALG-WS *returns in polynomial time (e.g. by exhaustive search) a* $\max\{1-\frac{1}{2k}, 1-\frac{1}{2\ell}\}$*-approximately well supported Nash equilibrium which has support of size* $\max\{k, \ell\}$.

The statement of the conjecture is false for $k = \ell = 2$, as can be seen by a small construction (with 7 vertices). The statement for $k = 3$, $\ell = 2$ is already non-trivial. In fact, it was stated as a conjecture by Myers [20] in relation to solving a special case of the Caccetta–Häggkvist Conjecture [5]. However, it has recently been proved incorrect in [6] via an involved construction. The case of a constant bound on k for $\ell = 2$ has been left open.

While stronger forms of Conjecture 4.7 seem to be related to well-known and difficult graph theoretic conjectures, we believe that the conjecture itself is true, and even that it holds for some small values of k and ℓ, such as $k = \ell = 3$.

What we can prove is the case of $\ell = \log n$, by showing that every digraph has a set of $\log n$ undominated vertices. This gives a $(1 - \frac{1}{2\log n})$-approximately well supported Nash equilibrium, which does not seem to be easily obtained via other arguments. We can also prove that the statement is true for $k = 3$, $\ell = 1$ in the special case of tournament graphs; this easily follows from the fact that every tournament is either transitive or contains a directed triangle.

5. Subsequent work and open problems

Since the first appearance of our results [9] there have been several improvements, achieving better approximation guarantees for both the approximate Nash equilibrium and the approximately well supported Nash equilibrium. We provide here a summary of the subsequent work, up until the publication of this article.

It was shown in [12] that there exist games in which no pair of strategies with supports of sublogarithmic size can be an ϵ-approximate Nash equilibrium, for $\epsilon < 1/2$; the proof applies the probabilistic method on 1-sum games with uniformly random 0/1 matrices R. This result implies that our $\frac{1}{2}$-approximate Nash equilibrium algorithm is optimal among the algorithms that only consider mixed strategies of sublogarithmic size support.

The first efficient algorithm for computing an ϵ-approximate Nash equilibrium of a two player game, for $\epsilon < 1/2$, was given in [10], which achieved an approximation of $\epsilon = (3-\sqrt{5})/2 \sim 0.38$. The techniques involved random sampling of small support strategies and combining these with the solution to a certain linear program. This result was subsequently improved in [2] to 0.36, while the current best approximation guarantee, given in [22], is 0.34. The latter result, which is based on local search, establishes that any local minimum of a very natural map in the space of pairs of mixed strategies — or its dual point in a certain minimax problem used for finding the local minimum — constitutes a 0.34-approximate Nash equilibrium.

Regarding approximately well supported equilibria, [16] gave a polynomial time algorithm for computing a 0.658-Nash equilibrium, using techniques similar to those presented here. However, while the result here is conditional on a graph theoretic conjecture, the result in [16] is unconditional.

Despite the extensive research on the subject outlined above, it still remains open whether an ϵ-approximate Nash equilibrium can be computed efficiently for arbitrarily small values of ϵ.[1] Nevertheless, there are several positive results for special cases. It is well-known, for example, that two-player zero-sum games (with $R + C = 0$) are solvable exactly in polynomial time using Linear Programming [23,7,14]. In [17] a polynomial-time approximation scheme (PTAS) was provided for a generalization of zero-sum games, called *low-rank games*; these are games in which $R + C$ is a matrix of fixed (constant) rank. Note that, if both R and C have fixed rank, there exists a Nash equilibrium with fixed support [18]. Hence, a Nash equilibrium can be computed exactly in polynomial time by enumerating all fixed size supports and using Linear Programming to check if there is a Nash equilibrium consistent with a certain choice of supports.

In [11], a PTAS was provided for the class of 2-player *large-support games*. These are games with a Nash equilibrium in which both players' mixed strategies spread non-trivially (see [11] for the precise condition) over a linear size subset of the pure strategies. This class of games is known to be PPAD-complete [8,3]. Another class of games which are PPAD-complete to solve exactly, yet have a PTAS, is the class of *bounded-norm games*, in which every player's payoff matrix is the sum of a constant matrix and a matrix with bounded infinity norm. This class of games was shown to be PPAD-complete in [4], and a PTAS was noted in [11].

It turns out that both the PTAS's for large-support games and for bounded-norm games are of a very special kind, called *oblivious* – where the approximate equilibrium is found by sampling a fixed universal distribution over pairs of mixed strategies. *Is there then an oblivious PTAS for the general case?* In [11] it was shown that the answer is no, using a construction similar to [1].

In view of the results described above, it was conjectured in [11] that an important step towards obtaining a polynomial-time approximation scheme for general two-player games is to understand how an approximate Nash equilibrium can be computed in the presence of an exact Nash equilibrium of logarithmic support.

[1] Observe that, in view of Lemma 2.1, a polynomial-time approximation scheme for ϵ-approximate Nash equilibria would imply a polynomial-time approximation scheme for ϵ-approximately well supported Nash equilibria.

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