Incomplete self-orthogonal latin squares ISOLS(6m + 6, 2m) exist for all m

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Abstract
An incomplete self-orthogonal latin square of order v with an empty subarray of order n, an ISOLS(v, n) can exist only if v ≥ 3n + 1. We show that an ISOLS(6m + 6, 2m) exists for all values of m and thus only the existence of an ISOLS(6m + 2, 2m), m ≠ 2, remains in doubt.

1. Introduction

A self-orthogonal latin square of order v, an SOLS(v), is a latin square of order v which is orthogonal to its transpose. It is known [5] that an SOLVS(v) exists for all values of v, v ≠ 2, 3 or 6.

An incomplete self-orthogonal latin square of order v is a v × v latin array A = (a_{ij}) with row and column indices and entries taken from the set \( \mathbb{Z}_v \times X \), \( X = \{x_1, x_2, \ldots, x_n\} \), and with an empty subarray of order n so that
\[
(Z_v \times Z_v \setminus X) \cup (X \times Z_v \setminus X)
= \{(a_{ij}, a_{ji}): (i, j) \in (Z_v \times Z_v \setminus X) \cup (X \times Z_v \setminus X) \cup (X \times X)\}.
\]

We denote such an array by ISOLS(v, n) and it is the question of the existence of

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these arrays that is studied in this paper. There has been considerable work done on the problem and we begin by summarizing the current state of affairs.

Simple counting shows that a necessary condition for the existence of an ISOLS\((v, n)\) is that \(v \geq 3n + 1\). As already mentioned, Brayton, Coppersmith and Hoffman [5] showed that an ISOLS\((v, 1)\) (equivalently an SOLS\((v)\)) exists for all \(v \geq 4\) except for \(v = 6\). Parker [14] constructed many ISOLS\((3n + 1, n)\) (notably an ISOLS\((10, 3)\) which yielded the first pair of orthogonal latin squares of order 10) as did Hedayat [9]. Heinrich [10] constructed an ISOLS\((3n + 1, n)\) for all values of \(n\) and ISOLS\((3n + 2, n)\) for all odd \(n\). Just prior to this Crampin and Hilton [7] had shown that given \(n\) there exists an integer \(v(n)\) so that for all \(v > v(n)\) an ISOLS\((v, n)\) can be constructed. This was greatly improved by Drake and Lenz [8] who showed the existence of an ISOLS\((v, n)\) when \(n \geq 304\) and \(v \geq 4n + 3\). Most recently, Heinrich and Zhu [13] almost completed the problem when they showed that ISOLS\((v, n)\) exist for all \(v \geq 3n + 1\), \(v \neq 6\), and \((v, n) \neq (8, 2)\), except possibly for ISOLS\((6m + 2, 2m)\), \(m \geq 2\), and ISOLS\((6m + 6, 2m)\). In this paper we will construct ISOLS\((6m + 6, 2m)\) for all values of \(m\). However, the question of the existence of an ISOLS\((6m + 2, 2m)\), \(m \geq 2\), remains in doubt and at this point no such array is known to exist.

Before continuing we note that the transpose of a latin square is one of the six conjugates of the square and is in fact the \((2, 1, 3)\)-conjugate. The \((i, j, k)\)-conjugate of \(A\) is obtained by applying the permutation \(\sigma\) where \(\sigma(1) = i\), \(\sigma(2) = j\) and \(\sigma(3) = k\), to the three rows of the corresponding \(3 \times v^2\) orthogonal array. Analogous to the definition of SOLS\((v)\) and ISOLS\((v, n)\) we can define an \((i, j, k)\)-conjugate orthogonal (idempotent) latin square and an incomplete \((i, j, k)\)-conjugate orthogonal (idempotent) latin square. The existence of these other conjugate arrays has been studied in some detail and the reader is referred to the papers of Bennett, Wu and Zhu [1–2] and the references therein. We now return to the construction of ISOLS\((v, n)\).

In constructing ISOLS\((v, n)\) two techniques have been employed. The so called ‘starter-adder’ technique is used in many of the small cases and then recursive techniques based on the ‘Wilson-type’ constructions for group-divisible designs are used (see the generalizations of Brouwer and van Rees [6] and of Stinson [15], as well as the book of Beth, Jungnickel and Lenz [3]). In this paper we will rely heavily on these constructions (as they relate to ISOLS\((v, n)\)) which are described in detail in [13]; the lemmas we require from that paper we will restate (usually in the simpler form required here) but without proof. To further assist the reader, the recursive constructions we use will each be presented with a diagram. For small cases we rely on the starter-adder constructions of Wu [17].

2. The small cases

We begin with the construction of several ISOLS\((6m + 6, 2m)\) for small values of \(m\). These were all previously constructed by Wu [17] using the starter-added method which we now describe.
The starter-adder method. Let \( e = (a_{00}, a_{01}, a_{02}, \ldots, a_{(v-n-1)}) \) be a vector of length \( v-n \) with entries in \( \mathbb{Z}_{v-n} \cup X \), where \( X = \{x_1, x_2, \ldots, x_n\} \). Let \( f = (a_{0(v-n)}, a_{0(v-n+1)}, \ldots, a_{0(v-1)}) \) and \( g = (a_{(v-n)}, a_{(v-n+1)}, \ldots, a_{(v-1)}) \) be vectors of length \( n \) with entries in \( \mathbb{Z}_{v-n} \). These vectors are used to construct an array \( A = (a_{ij}) \) of order \( v \) with an empty subarray of order \( n \) with row and column indices, and entries in \( \mathbb{Z}_{v-n} \cup X \). The procedure is as follows. (Note that all arithmetic calculations are made in \( \mathbb{Z}_{v-n} \).)

(i) If \( a_{ij} \in \mathbb{Z}_{v-n}, 0 \leq i, j \leq v - n - 1 \), then \( a_{(i+1)(j+1)} = a_{ij} + 1 \).

(ii) If \( a_{ij} \in \{x_1, x_2, \ldots, x_n\}, 0 \leq i, j \leq v - n - 1 \), then \( a_{(i+1)(j+1)} = a_{ij} \).

(iii) If \( 0 \leq i \leq v - n - 1 \) and \( j \in X \), then \( a_{(i+1)j} = a_{ij} + 1 \).

(iv) If \( 0 \leq j \leq v - n - 1 \) and \( i \in X \), then \( a_{ij(j+1)} = a_{ij} + 1 \).

Conditions can be described for the vectors \( e, f \) and \( g \) so that the array as constructed is an ISOLS(\( v, n \)). However, we will simply give the vectors and the reader can check for herself that they do indeed yield the desired ISOLS(\( v, n \)). (We remark that this method cannot yield an ISOLS(14, 4).)

Theorem 2.1. There is an ISOLS(\( 6m + 6, 2m \)) for \( m \in \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13\} \).

Proof. An ISOLS(12, 2) and an ISOLS(18, 4) appear in [13] (the ISOLS(18, 4) being first constructed by Wang [16]). For the ISOLS(12, 2) put \( e = (1, 4, x_1, 8, x_2, 5, 3, 6, 0, 7) \), \( f = (2, 9) \) and \( g = (4, 6) \) and for the ISOLS(18, 4) put \( e = (0, 4, x_1, x_2, x_3, x_4, 1, 8, 10, 5, 7, 9, 11, 6) \), \( f = (2, 3, 12, 13) \) and \( g = (6, 5, 4, 8) \). Although an ISOLS(24, 6) is given in [13] (it is not constructed by the starter-adder technique) we will also give a construction here. The starter-adder method devised by Wu [17] to construct an ISOLS(\( 6m + 6, 2m \)) is to put

\[
e = (0, a, a_2, a_3, \ldots, a_{2m}, a + 1, a + 2m + 3, a + 2, a + 2m + 6, a + 2m + 5, x_1, x_2, \ldots, x_{2m}),
\]

\[
f = (a - 1, b_2, b_3, \ldots, b_{2m}) \quad \text{and} \quad g = (a + 2m + 3, a_2, a_3, \ldots, a_{2m}),
\]

where \( a, a \), and \( b \) are chosen to satisfy

1. \( a_i - b_i \equiv i \pmod{4m + 6}, 2 \leq i \leq 2m, \text{and} \)
2. \( \mathbb{Z}_{4m+6} = \{0, a - 1, a, a + 1, a + 2, a + 2m + 3, a + 2m + 5, a + 2m + 6, a_2, a_3, \ldots, a_{2m}, b_2, b_3, \ldots, b_{2m}\} \).

Straightforward calculation verifies that such a choice results in an ISOLS(\( 6m + 6, 2m \)). In Table 1 we give the vectors \( a = (a_2, a_3, \ldots, a_{2m}) \) and \( b = (b_2, b_3, \ldots, b_{2m}) \), and the value of \( a \), for \( m \in \{3, 4, 5, 6, 7, 8, 10, 12, 13\} \).

For the remaining values of \( m \) we use recursive constructions. These recursive constructions rely on the existence of other orthogonal arrays and on information regarding the location of transversals in certain latin squares. To this end we need more notation. A POLS(\( v \)) is a pair of orthogonal latin squares of order \( v \) and an IPOLS(\( v, n \)), an incomplete pair of orthogonal latin squares, is a pair of
orthogonal latin squares of order $v$ each with a common empty subarray of order $n$ (that is, with an empty subarray of order $n$ positioned at the same location in each square). More generally, $\text{IPOLS}(v, n_1, n_2, \ldots, n_k)$ denotes a pair of orthogonal latin squares of order $v$ with $k$ common disjoint empty subarrays of orders $n_1, n_2, \ldots, n_k$ on the main diagonal. Similarly we define an $\text{ISOLS}(v, n_1, n_2, \ldots)$. It is useful to note that if $v > n_1 + n_2 + \cdots + n_k$, then an $\text{ISOLS}(v, n_1, n_2, \ldots, n_k)$ exists if and only if an $\text{ISOLS}(v, n_1, n_2, \ldots, n_k)$ exists. If any $n_i$ is zero we will simply ignore it; so in particular an $\text{ISOLS}(v, 0)$ is an $\text{SOLS}(v)$.

Let $A = (a_{ij})$ be a latin square. We call two transversals disjoint if they have no cell in common. A transversal $T$ is symmetric if $(i, j) \in T$ if and only if $(i, i) \in T$. A pair of transversals $T$ and $S$ are symmetric if $(i, j) \in T$ if and only if $(j, i) \in S$. Finally, a pair of symmetric transversals will be called $(0, 0)$-intersecting if the only element they have in common is $(0, 0)$.

The following theorems provide the 'ingredients' for the application of the recursive constructions given in Lemmas 2.6–2.12. Although reference [12] is given for Theorem 2.3 we point out that this paper is merely the last in a series of papers (by many authors) on the problem. For a complete proof the reader is referred to the survey paper [11].

**Theorem 2.2.** [4]. There exists a $\text{POLS}(v)$ for all values of $v$, $v \neq 2, 6$.

**Theorem 2.3** [12]. There exists an $\text{IPOLS}(v, n)$ for all values of $v$ and $n$ satisfying $v \geq 3n$ except that an $\text{IPOLS}(6, 1)$ does not exist.

**Theorem 2.4** [13]. There exists an $\text{ISOLS}(v, n)$ for all values of $v$ and $n$ satisfying $v \geq 3n + 1$, $v \neq 6$, except possibly for $n = 2m$ and $v = 6m + 2$ or $v = 6m + 6$. 

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**Table 1**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$a = (a_1, a_2, \ldots, a_{2m})$</th>
<th>$b = (b_1, b_2, \ldots, b_{2m})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$(10, 9, 17, 16, 7)$</td>
<td>$(8, 6, 13, 11, 1)$</td>
</tr>
<tr>
<td>4</td>
<td>$(8, 20, 11, 14, 5, 19, 18)$</td>
<td>$(6, 17, 7, 9, 21, 12, 10)$</td>
</tr>
<tr>
<td>5</td>
<td>$(17, 25, 13, 2, 16, 8, 19, 12, 24)$</td>
<td>$(15, 22, 9, 23, 10, 1, 11, 3, 14)$</td>
</tr>
<tr>
<td>6</td>
<td>$(29, 16, 26, 17, 1, 18, 10, 28, 14, 20, 15)$</td>
<td>$(27, 13, 22, 12, 25, 11, 2, 19, 4, 9, 3)$</td>
</tr>
<tr>
<td>7</td>
<td>$(31, 27, 16, 14, 25, 18, 6, 1, 17, 21, 8, 28, 13)$</td>
<td>$(29, 24, 12, 9, 19, 11, 32, 26, 7, 10, 30, 15, 33)$</td>
</tr>
<tr>
<td>8</td>
<td>$(32, 14, 33, 15, 31, 16, 35, 17, 36, 18, 34, 19, 13, 20, 28)$</td>
<td>$(30, 11, 29, 10, 25, 9, 27, 8, 26, 7, 22, 6, 37, 5, 12)$</td>
</tr>
<tr>
<td>10</td>
<td>$(12, 36, 11, 37, 15, 38, 16, 39, 13, 40, 18, 41, 19, 42, 20, 43, 35, 44, 34)$</td>
<td>$(10, 33, 7, 32, 9, 31, 8, 30, 3, 29, 6, 28, 5, 27, 4, 26, 17, 25, 14)$</td>
</tr>
<tr>
<td>12</td>
<td>$(14, 42, 19, 43, 17, 44, 18, 45, 13, 46, 20, 47, 21, 48, 22, 49, 23, 50, 24, 51, 9, 52, 40)$</td>
<td>$(12, 39, 15, 38, 11, 37, 10, 36, 3, 35, 8, 34, 7, 33, 6, 32, 5, 31, 4, 30, 41, 29, 16)$</td>
</tr>
<tr>
<td>13</td>
<td>$(48, 20, 49, 21, 50, 22, 51, 23, 52, 24, 38, 25, 54, 56, 55, 27, 36, 28, 57, 29, 11, 30, 19, 31, 1)$</td>
<td>$(46, 17, 45, 16, 44, 15, 43, 14, 42, 13, 26, 12, 40, 41, 39, 10, 18, 9, 37, 8, 47, 7, 53, 6, 33)$</td>
</tr>
</tbody>
</table>
Theorem 2.5. (a) If $q$ is an odd prime power, $q \geq 5$, then there exists a self-orthogonal Latin square of order $q$ with $q - 1$ disjoint transversals, each disjoint from the main diagonal and occurring as $(q - 1)/2$ pairs of symmetric transversals.

(b) If $q$ is an even prime power, $q \geq 4$, then there exists a self-orthogonal Latin square of order $q$ with $q - 1$ disjoint symmetric transversals.

(c) If $q$ is a prime power, $q \geq 7$, then there exists a self-orthogonal Latin square of order $q$ with a pair of $(0, 0)$-intersecting transversals.

Proof. All are easily derived from the set of $q - 1$ pairwise orthogonal Latin squares $L_1, L_2, \ldots, L_{q-1}$ of order $q$ defined over the field $\mathbb{GF}(q) = \{0, -1 = a_1, a_2, \ldots, a_{q-1}\}$ by $L_i(r, s) = (1 + a_i)^{-1}(a_r + a_s a_i)$, $2 \leq i \leq q - 1$ and $L_i(r, s) = a_r - a_s$. □

Observe that when $q \geq 7$ conditions (a) and (c) hold simultaneously, as do conditions (b) and (c) when $q \geq 8$.

Lemma 2.6. If there exists an SOLS$(d)$ and an ISOLS$(a, e, f)$, then there exists an ISOLS$(ad, ed, fd)$

Lemma 2.7. If there exists an SOLS$(d)$ with $k$ pairs of symmetric transversals, all $2k$ transversals being pairwise disjoint, a POLS$(a)$, an IPOLS$(a + b_1, b_i)$, $i = 1, 2 \ldots, k$ and an ISOLS$(a + b, b)$, then there exists an ISOLS$(ad + b + 2(b_1 + b_2 + \cdots + b_k), b + 2(b_1 + b_2 + \cdots + b_k))$.

Lemma 2.8. If there exists an SOLS$(d)$ with $k$ pairs of symmetric transversals, all $2k$ transversals being pairwise disjoint, a POLS$(a, e)$, an IPOLS$(a + b_1, e, b_i)$, $i = 1, 2 \ldots, k$, an ISOLS$(a + b, e, b)$, and an SOLS$(b + 2(b_1 + b_2 + \cdots + b_k))$, then there exists an ISOLS$(ad + b + 2(b_1 + b_2 + \cdots + b_k), de)$.

Lemma 2.9. If there exists an SOLS$(d)$ with $k$ pairs of symmetric transversals (including the main diagonal—and thus implying that $d$ is even) and a pair of $(0, 0)$-intersecting transversals, an IPOLS$(a, e)$, an IPOLS$(a + b_1, e, b_i)$, $i = 2 \ldots, k$, an ISOLS$(a + b_1, e, b_i)$, an IPOLS$(a + c + b_1, e, c, b_i)$, $i = 2 \ldots, k$, an ISOLS$(b_1 + b_2 + \cdots + b_k, b_1)$, and an ISOLS$(a + 2c + b_1, e)$, then there exists an ISOLS$(ad + 2c + b_1 + b_2 + \cdots + b_k, de)$.

Lemma 2.10. If there exists an SOLS$(d)$ with $k$ pairs of symmetric transversals (so that all $2k$ transversals are pairwise disjoint—and therefore cannot contain any cell of the main diagonal) and a pair of $(0, 0)$-intersecting transversals, a POLS$(a)$, an ISOLS$(a + b, b)$, an IPOLS$(a + c, c)$, an ISOLS$(a + b + 2c, b)$ and both IPOLS$(a + b_1, b_i)$, and IPOLS$(a + b_1 + c, b_i, c)$, $i = 1, 2 \ldots, k$, then there is an ISOLS$(ad + b + 2(b_1 + b_2 + \cdots + b_k) + 2c, b + 2(b_1 + b_2 + \cdots + b_k))$. 
Lemma 2.11. If there exists an SOLS(d) with a pair of (0, 0)-intersecting transversals, an IPOLS(a, e), an ISOLS(a + b, e, b) an IPOLS(a + c, e, c), and an ISOLS(a + b + 2c, e), then there is an ISOLS(ad + b + 2c, de).

Lemma 2.12. If there exists an SOLS(d) with a symmetric transversal which intersects the main diagonal in exactly one cell, an IPOLS(a, e), an ISOLS(a + b, e, b), an IPOLS(a + c, e, c), and an ISOLS(a + b + c, e), then there is an ISOLS(ad + b + c, de).

It is also important to observe that if we 'fill in' the empty subarray of an ISOLS(v, n) we can then delete any order n' symmetrically positioned subsquare and so obtain an ISOLS(v, n').

We are now ready to begin the constructions. The diagrams are to be used only as a guide to the constructions. Black areas of the diagrams denote the empty subarray and other shaded areas denote certain ISOLS or IPOLS as indicated.

Theorem 2.13. There is an ISOLS(6m + 6, 2m) for m \in \{20, 24, 30, 40, 60, 120\}.

Proof. Each case will be dealt with in turn.

\textbf{ISOLS(126, 40)}

This is an application of Lemma 2.10 with d = 7, k = 3, a = 12, b = 4, c = 1 and b_1 = b_2 = b_3 = 6.
This is an application of Lemma 2.11 with $d = 8$, $a = 18$, $e = 6$, $b = 4$ and $c = 1$. The application requires an ISOLS(22, 6, 4) which is constructed as follows. Proceed as in Lemma 2.12 using $d = 5$, $a = 4$, $e = 0$, $b = 0$ and $c = 2$. Since we have no ISOLS(6, 0) we leave this subarray empty and obtain an ISOLS(22, 6) which also has a subsquare of order 4 positioned symmetrically about the main diagonal. Thus we have an ISOLS(22, 6, 4).

This is Lemma 2.7 with $d = 7$, $k = 3$, $a = 18$, $b = 6$ and $b_1 = b_2 = b_3 = 9$. 
The construction is an application of Lemma 2.12 with \(d = 5, a = 48, e = 16, b = 4\) and \(c = 2\). This requires an ISOLS(52, 16, 4) which will be constructed in Lemma 3.1. Also required is an ISOLS(50, 16, 2) which is constructed using Lemma 2.9 but with \(c = 0\) which eliminates the need for a pair of intersecting transversals and then putting \(d = 16, k = 2, a = 3, e = 1, b_1 = b_2 = 1\). Since there is no ISOLS(2, 1) we obtain an ISOLS(50, 16, 2).

**ISOLS(366, 120)**

This construction is Lemma 2.9 with \(d = 8, k = 4, a = 45, e = 15, b_1 = b_2 = b_3 = b_4 = c = 1\).

**ISOLS(726, 240)** This array is constructed using Lemma 2.7 with \(d = 9, k = 4, a = 54, b = 24\) and \(b_1 = b_2 = b_3 = b_4 = 27\). \(\Box\)
3. The general construction

We have now dealt with all cases which will not be covered by the main recursive constructions (of which there are two). However, before presenting these we need another lemma.

**Lemma 3.1.** There exists an ISOLS$(12r + 4, 4r, 4)$ and an IPOLS*(12r + 4, 4r + 1, 4), where the * indicates that the subarrays of orders 4r + 1 and 4 have exactly one cell in common, for all values of r.

**Proof.** The first is easily constructed from Lemma 2.6 with \( d = 4, a = 3r + 1, e = r \) and \( f = 1 \). To construct the second begin with Lemma 2.8 putting \( d = 4r + 1, k = 0, a = 3, e = 1 \) and \( b = 1 \) to obtain an ISOLS$(12r + 4, 4r + 1)$ with a subsquare of order 4 (intersecting the empty array of order \( 4r + 1 \)) which we delete to obtain the result. 0

**Theorem 3.2.** There exists an ISOLS$(12t + 12, 4t + 2)$ for all values of t.

**Proof.** Let \( t = pr \), where \( p \) is a prime power, \( p \neq 4 \). This is possible for all values of \( t \) except \( t \in \{1, 2, 3, 6\} \), but in these four cases the ISOLS$(12t + 12, 4t + 2)$ are constructed in Theorem 2.1. Begin with Lemma 2.8, putting \( d = p, k = 1, a = 12r, e = 4r \) and \( b = b_1 = 4 \). Now use the IPOLS*(12r + 4, 4r + 1, 4) instead of IPOLS$(12r + 4, 4r, 4)$ (making sure, however, that the 4r + 1 array covers the cells the 4r array would have covered) and instead of an SOLS(12) use an ISOLS(12, 2) making sure that it is 'lined up' as indicated in the diagram. The result is an ISOLS$(12t + 12, 4t + 2)$. 0
Theorem 3.3. There exists an ISOLS(12t + 6, 4t) for all values of t.

Proof. Let \( t = pr \), where \( p \) is a prime power, \( p \geq 7 \). This is possible for all values of \( t \) except \( t \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} \); but in these cases the ISOLS(12t + 6, 4t) are constructed in Theorems 2.1 and 2.13. Suppose that there exists an ISOLS(12r + 6, 4r). We apply Lemma 2.11 with \( d = p \), \( a = 12r \), \( e = 4r \), \( b = 4 \) and \( c = 1 \), and use the arrays described in Lemma 3.1. (This is the same construction as used for the ISOLS(150,48) in Lemma 2.13 and the diagram given there adequately describes the situation.)

By the induction hypothesis the proof is complete. \( \Box \)

We are now in a position to make the following claim.

Theorem 3.4. There exists an ISOLS(v, n) for all values of \( v \) and \( n \) satisfying \( v \geq 3n + 1 \), except for \( v = 6 \) and \( (v, n) = (8, 2) \) and perhaps excepting \( (v, n) = (6m + 2, 2m) \), \( m \geq 2 \).

References

[10] K. Heinrich, Self-orthogonal latin squares with self-orthogonal subsquares, Ars Combin. 3 (1977) 251–266.
[17] Lisheng Wu, Some incomplete self-orthogonal latin squares ISOLS(6m + 6, 2m), unpublished manuscript.