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Commuting Toeplitz operators on the Segal–Bargmann space

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Abstract

Consider two Toeplitz operators T_g, T_f on the Segal–Bargmann space over the complex plane. Let us assume that g is a radial function and both operators commute. Under certain growth condition at infinity of f and g we show that f must be radial, as well. We give a counterexample of this fact in case of bounded Toeplitz operators but a fast growing radial symbol g. In this case the vanishing commutator $[T_g, T_f] = 0$ does not imply the radial dependence of f. Finally, we consider Toeplitz operators on the Segal–Bargmann space over $\mathbb{C}^n$ and $n > 1$, where the commuting property of Toeplitz operators can be realized more easily.

Keywords: Toeplitz operator; Mellin transform; Reproducing kernel Hilbert space; Radial symbol

1. Introduction

The study of commuting Toeplitz operators on the Bergman and Hardy spaces over various domains and related operator algebras has a long lasting history; cf. [1,6,8,10] and recently [7,11,13, 14,16,17]. The problem of characterizing commuting Toeplitz operators with arbitrary bounded

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symbols seems quite challenging and is not fully understood until now. However, methods for an analysis are available if one restricts attention to certain sub-classes of symbols.

In the present paper we study classes of commuting Toeplitz operators acting on the Segal–Bargmann space $H^2(\mathbb{C}^n, d\mu)$ of all Gaussian square integrable entire functions on $\mathbb{C}^n$. Here the case $n = 1$ is of particular interest. Moreover, we admit the (not so frequently studied) situation of unbounded operator symbols of a certain type since their growth behavior near infinity essentially influences our results; cf. Theorem 4.17 and Example 5.8. As a consequence we have to deal with a space of i.g. unbounded, densely defined Toeplitz operators. We use the construction in [3] in order to check that they can be embedded into an operator algebra and therefore commutators are well defined.

An easy calculation shows that a Toeplitz operator with a radial symbol f is diagonal with respect to the standard orthonormal basis of $H^2(\mathbb{C}^n, d\mu)$ and therefore such type of operators commute. Here a function f is called radial if $f(z) = f(|z|)$. Conversely, one can ask whether for a non-trivial radial symbol f and an arbitrary symbol g (both in our symbol class) the commutator condition $[T_f, T_g] = 0$ implies that g is radial. The analog question in the case of Toeplitz operators with bounded symbols acting on the unweighted Bergman space over the unit disc $\mathbb{D} \subset \mathbb{C}$ has been answered before in [10]. Theorem A below gives the result of Theorem 6 in [10]:

Theorem A. *(See [10].) Let $\psi, \varphi \in L^\infty(\mathbb{D}, dA)$ where dA is the usual area measure on $\mathbb{D}$. Let φ be a non-trivial radial function. If the Toeplitz operators T_ψ and T_φ commute on the Bergman space, then ψ is a radial function.*

In order to prove Theorem A, the authors use an expansion of ψ into an L^2-convergent series. Then the commutator condition $[T_\psi, T_\varphi] = 0$ can be converted into a functional equation for the Mellin transform of φ and coefficient functions of that expansion. From this equation the result follows, but as an essential ingredient in the argument the Blaschke condition for the possible distribution of zeros of non-vanishing bounded holomorphic functions on $\mathbb{D}$ (or a right half-plane) is used. Since here we consider Toeplitz operators with i.g. unbounded symbols acting on a function space over the complex plane, we cannot use such type of arguments and this fact causes the main complications. However, we can prove a result similar to Theorem A above.

Let $\mathcal{S}$ be a space of measurable complex valued functions on the complex plane and of at most polynomial growth at infinity (see Definition 4.1), then we show:

Theorem B. *Let $u, v \in \mathcal{S}$ and assume that u is radial and non-constant. If the Toeplitz operators T_u and T_v commute on the Segal–Bargmann space $H^2(\mathbb{C}, d\mu)$, then v is a radial function.*

The restriction to symbols in the space $\mathcal{S}$ in Theorem B is necessary. We provide an example of a unitary Toeplitz operator T_f with radial symbol $f \notin \mathcal{S}$ acting on $H^2(\mathbb{C}, d\mu)$ such that $[T_f, T_g] = 0$ where g is a bounded non-radial function on the complex plane; cf. Example 5.8. We do not know whether a similar effect is possible in case of the Toeplitz operators with certain unbounded symbols acting on the Bergman space over the unit disc $\mathbb{D}$.

In the last part of this paper we discuss commuting Toeplitz operators with polynomial symbols acting on $H^2(\mathbb{C}^n, d\mu)$. It has been shown in [3,9] that the corresponding Toeplitz operators form an algebra under composition and the symbol of the product of two operators can be calculated as a Moyal-type product. In case of dimension $n > 1$ and for polynomials p and q where p is non-constant and radial it is shown that the condition $[T_p, T_q] = 0$ is equivalent with q belonging to a certain subspace $\mathcal{P}_1$ of all polynomials (see (5.3) for the definition of $\mathcal{P}_1$). In particular, the space $\mathcal{P}_1$ strictly includes all polynomials that are radial in each component.

In Section 2 we define a generalized Segal–Bargmann space by replacing the usual Gaussian density by a suitable radial function, cf. [2,5,15]. In this setting we explain the notion of Toeplitz operators and state our main question on commuting Toeplitz operators. In the case of the dimension $n = 1$ we convert the commutator condition on Toeplitz operators into a functional equation for Mellin transforms of the symbols and their components, respectively; cf. Proposition 2.4.

In Section 3 we specialize to the classical Segal–Bargmann space of Gaussian square integrable entire functions on the complex plane. The construction in [3] of an algebra of operators containing Toeplitz operators with a certain type of unbounded symbols is recalled. As an important feature, the Berezin transform is one-to-one on this algebra.

Section 4 contains the proof of Theorem B which is deduced from a detailed discussion of the functional equation we have obtained in Proposition 2.4. In particular, we need to analyze the Mellin convolutions of certain functions in Propositions 4.8 and 4.9.

In Section 5 we discuss commuting Toeplitz operators with polynomial symbols acting on $H^2(\mathbb{C}^n, d\mu)$ where $n > 1$. Using the product structure of $\mathbb{C}^n$ it is easy to produce commuting Toeplitz operators. Finally, we give a counterexample to Theorem B in case of $u \notin \mathcal{S}$ in the end of Section 5.

2. Preliminaries

Throughout this section let φ be a nonnegative integrable radial function on $\mathbb{C}^n$. With $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we write $\bar{z} := (\bar{z}_1, \dots, \bar{z}_n)$ for its complex conjugate. By dv we denote the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and additionally we suppose that φ satisfies the following two conditions:

$$\hat{\varphi}(k) := \int_{\mathbb{C}^n} |z|^{2k} \varphi(z)\, dv(z) < \infty, \qquad \limsup_k \sqrt[k]{\hat{\varphi}(k)} = \infty$$

for every $k = 0, 1, \dots$ and with the Euclidean norm $|\cdot|$ on $\mathbb{C}^n$. Let $\mathscr{F}_\varphi$ be the set of all entire functions in $L^2(\mathbb{C}^n, \varphi\, dv)$. Then it is known that $\mathscr{F}_\varphi$ is a closed linear subspace of $L^2(\mathbb{C}^n, \varphi\, dv)$ with the inner product

$$\langle f, g \rangle_\varphi = \int_{\mathbb{C}^n} f \bar{g} \varphi\, dv$$

and the usual L^2-norm $\|f\|_\varphi = \sqrt{\langle f, f \rangle_\varphi}$ where $f, g \in L^2(\mathbb{C}^n, \varphi\, dv)$. In fact, $\mathscr{F}_\varphi$ is a reproducing kernel Hilbert space and the corresponding reproducing kernel $K_\varphi(z, w)$ can be given by

$$K_\varphi(z, w) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{(n - 1 + |\alpha|)!}{(n-1)!\alpha!} \frac{z^\alpha \bar{w}^\alpha}{\hat{\varphi}(|\alpha|)} = \sum_{k=0}^{\infty} \frac{(n)_k}{k! \hat{\varphi}(k)} (z \cdot \overline{w})^k \tag{2.1}$$

where $z \cdot \overline{w} := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $(n)_k := n(n+1)\cdots(n+k-1)$ denotes the usual *Pochhammer symbol*. Also, the notation $\mathbb{N}_0^n$ denotes the set of all n-tuples of nonnegative integers. Note that by Stirling's formula we have the asymptotics:

$$\frac{(n)_k}{k!} \sim k^{n-1}(n-1)! \tag{2.2}$$

as $k \to \infty$. Therefore, it follows from the above assumptions on $\hat{\varphi}(k)$ that (2.1) converges uniformly on every compact subsets of $\mathbb{C}^n \times \mathbb{C}^n$. See [15] for details and related facts.

Let P_φ be the orthogonal projection from $L^2(\mathbb{C}^n, \varphi\, dv)$ onto $\mathscr{F}_\varphi$. Let $u : \mathbb{C}^n \to \mathbb{C}$ be a measurable function. Then, for all

$$f \in \mathcal{D}_u := \bigl\{ g \in \mathscr{F}_\varphi \colon\ u \cdot g \in L^2\bigl(\mathbb{C}^n, \varphi\, dv\bigr) \bigr\}$$

the Toeplitz operator T_u^φ with symbol u is defined by

$$T_u^\varphi f := P_\varphi(uf).$$

Note that in general T_u^φ is an unbounded linear operator on $\mathcal{D}_u \subset \mathscr{F}_\varphi$. Clearly, in case of $u \in L^\infty(\mathbb{C}^n)$ the Toeplitz operator T_u^φ is bounded with $\|T_u^\varphi\| \leqslant \|u\|_\infty$. In the following we are considering products of two Toeplitz operators. Hence, we restrict ourself to spaces of measurable complex valued symbols $\mathcal{S}$ which have the following property:

Assumption. There is a dense subspace $\mathcal{H}_\mathcal{S} \subset \bigcap_{u \in \mathcal{S}} \mathcal{D}_u \subset \mathscr{F}_\varphi$ which is invariant under all Toeplitz operators T_u^φ with symbol $u \in \mathcal{S}$.

Remark 2.1. In case of $\mathcal{S} = L^\infty(\mathbb{C}^n)$ we can choose $\mathcal{H}_\mathcal{S} = \mathscr{F}_\varphi$. If φ is a Gaussian density we construct a space $\mathcal{S}$ containing unbounded symbols and a corresponding invariant subspace $\mathcal{H}_\mathcal{S}$ in the next section.

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we put $e_\alpha(z) := z^\alpha \|z^\alpha\|_\varphi^{-1}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Note, that $\{e_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ forms a dense subset of $\mathscr{F}_\varphi$. We have:

Lemma 2.2. *Let $u \in \mathcal{S}$ be radial and assume that $e_\beta \in \mathcal{H}_\mathcal{S}$ for all $\beta \in \mathbb{N}_0^n$. Moreover, assume that for all $m \in \mathbb{N}$*

$$u(z) \sum_{k=0}^{\infty} \frac{k^{n-1}}{\hat{\varphi}(k)} |z|^{k+m} \in L^1\bigl(\mathbb{C}^n, \varphi\, dv\bigr). \tag{2.3}$$

Then, we have $P_\varphi(u\bar{z}^\beta) = 0$ for $\beta \neq 0$ and T_u^φ is diagonal with respect to the orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbb{N}_0}$. More precisely, $T_u^\varphi e_\beta = \widetilde{u}(\beta) e_\beta$ where

$$\widetilde{u}(\beta) := \frac{1}{\hat{\varphi}(|\beta|)} \int_{\mathbb{C}^n} u(z)|z|^{2|\beta|}\varphi(z)\, dv(z).$$

Note that $\widetilde{u}(\beta)$ only depends on $|\beta|$.

Proof. Let $\rho \in \{\bar{z}^\beta, z^\beta\}$ with $\beta \in \mathbb{N}_0^n$. Using the expression (2.1) of the reproducing kernel function, we see

$$\begin{aligned} P_\varphi(u\rho)(w) &= \int_{\mathbb{C}^n} u(z)\rho(z)K_\varphi(w,z)\varphi(z)\,dv(z) \\ &= \int_{\mathbb{C}^n} u(z)\rho(z)\sum_{k=0}^\infty \frac{(n)_k}{k!\hat{\varphi}(k)}(w\cdot\bar{z})^k\varphi(z)\,dv(z) \\ &= \sum_{\alpha\in\mathbb{N}_0^n}\frac{(n)_{|\alpha|}}{\alpha!}\frac{w^\alpha}{\hat{\varphi}(|\alpha|)}\int_{\mathbb{C}^n} u(z)\rho(z)\bar{z}^\alpha\varphi(z)\,dv(z). \end{aligned} \tag{2.4}$$

Here we have used (2.2) and (2.3) together with Lebesgue's dominated convergence theorem in order to interchange the integration and summation. In case of $\rho(z) = \bar{z}^\beta$ where $\beta \neq 0$ the first assertion follows from $\int_{\mathbb{C}^n} u(z)\bar{z}^{\beta+\alpha}\varphi(z)\,dv(z) = 0$ for all $\alpha \in \mathbb{N}_0^n$.

In order to prove the second assertion choose $\rho(z) = z^\beta$ and let $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ denote the (real) $(2n-1)$-dimensional unit sphere with the usual measure $d\sigma_{2n-1}$. Using the relation $\int_{\mathbb{S}^{2n-1}} |\zeta^\beta|^2\,d\sigma_{2n-1}(\zeta) = 2\beta!\pi^n/(n+|\beta|-1)!$, we have

$$\begin{aligned} \int_{\mathbb{C}^n} uz^\beta\bar{z}^\alpha\varphi\,dv &= \delta_{\alpha,\beta}\int_{\mathbb{S}^{2n-1}} \left|\zeta^\beta\right|^2 d\sigma_{2n-1}(\zeta)\int_0^\infty r^{2(n+|\beta|)-1}u(r)\varphi(r)\,dr \\ &= \delta_{\alpha,\beta}\frac{\beta!(n-1)!}{(n+|\beta|-1)!}\int_{\mathbb{C}^n} |z|^{2|\beta|}u(z)\varphi(z)\,dv(z), \end{aligned}$$

and the second assertion follows from (2.4). The proof is complete. □

Let $u \in \mathcal{S}$ be separately radial, i.e. u only depends on $|z_1|, |z_2|, \ldots, |z_n|$ and assume that (2.3) holds. Then for all $\alpha \in \mathbb{N}_0^n$ we have:

(a) $\int_{\mathbb{C}^n} u(z)\bar{z}^{\beta+\alpha}\varphi(z)\,dv(z) = 0$, in case of $\beta \neq 0$,
(b) $\int_{\mathbb{C}^n} u(z)z^\alpha\bar{z}^\beta\varphi(z)\,dv(z) = c_\alpha\delta_{\alpha,\beta}$ where $c_\alpha \in \mathbb{C}$ is a suitable number.

In fact, the relations (a) and (b) follow from the invariance of both integrals under the linear transformation $U(z) := iz$. By the same argument as before $P_\varphi(u\bar{z}^\beta) = 0$ for $\beta \neq 0$ and T_u^φ is diagonal with respect to the orthonormal basis $\{e_\alpha\}_{\alpha\in\mathbb{N}_0^n}$. Due to this observation it holds:

Proposition 2.3. *Let $u, v \in \mathcal{S}$ be separately radial, then we have $T_u^\varphi T_v^\varphi = T_v^\varphi T_u^\varphi$ on* span$\{e_\alpha \mid \alpha \in \mathbb{N}_0^n\}$.

In the following we assume that all functions $u \in \mathcal{S}$ fulfill condition (2.3). Let us specialize now to the complex one-dimensional case. With our notations before recall that

$$\widetilde{u}(k) = \frac{1}{\hat{\varphi}(k)}\int_{\mathbb{C}} u(w)|w|^{2k}\varphi(w)\,dv.$$

Assume that $u \in \mathcal{S}$ is non-constant radial and (2.3) is fulfilled. Let $v \in \mathcal{S}$ be of the form

$$v(z) = v\big(re^{i\theta}\big) = \sum_{j=-\infty}^{\infty} v_j(r)e^{ij\theta}, \quad z := re^{i\theta} \tag{2.5}$$

where each v_j can be interpreted as a radial function on $\mathbb{C}$ with (2.3). Moreover, we assume that the sum in (2.5) converges in the topology of $L^2(\mathbb{C}, \varphi\, dv)$. Suppose that the Toeplitz operators T_u^φ and T_v^φ commute

$$T_u^\varphi T_v^\varphi = T_v^\varphi T_u^\varphi : \operatorname{span}\left\{e_k(z) = \frac{z^k}{\sqrt{\hat{\varphi}(k)}} \;\Big|\; k \in \mathbb{N}_0\right\} \subset \mathcal{H}_\mathcal{S} \to \mathcal{H}_\mathcal{S}. \tag{2.6}$$

Since u is radial, we have by Lemma 2.2

$$T_u^\varphi e_k = \widetilde{u}(k)e_k$$

for all $k \in \mathbb{N}_0$. By Lemma 2.2 and our assumptions on v_j it follows that

$$\begin{aligned}
T_v^\varphi T_u^\varphi e_k(z) &= \widetilde{u}(k) P_\varphi[ve_k](z) \\
&= \widetilde{u}(k) \sum_{j=-\infty}^{\infty} P_\varphi\big[v_j e^{ij\theta} e_k\big](z) \\
&= \frac{\widetilde{u}(k)}{\sqrt{\hat{\varphi}(k)}} \sum_{j=-\infty}^{\infty} P_\varphi\big[v_j r^k e^{i(j+k)\theta}\big](z) \\
&= \frac{\widetilde{u}(k)}{\sqrt{\hat{\varphi}(k)}} \sum_{j \geqslant -k} \frac{z^{j+k}}{\hat{\varphi}(j+k)} \int_\mathbb{C} v_j(w)|w|^{j+2k}\varphi(w)\, dv(w),
\end{aligned} \tag{2.7}$$

for every $z \in \mathbb{C}$. Moreover, we have

$$\begin{aligned}
T_v^\varphi e_k(z) &= \frac{1}{\sqrt{\hat{\varphi}(k)}} \sum_{j=-\infty}^{\infty} P_\varphi\big[v_j r^k e^{i(j+k)\theta}\big](z) \\
&= \frac{1}{\sqrt{\hat{\varphi}(k)}} \sum_{j \geqslant -k} \frac{z^{j+k}}{\hat{\varphi}(j+k)} \int_\mathbb{C} v_j(w)|w|^{j+2k}\varphi(w)\, dv(w).
\end{aligned}$$

It follows from Lemma 2.2 again and the assumption on $u \in \mathcal{S}$ that

$$\begin{aligned}
T_u^\varphi T_v^\varphi e_k(z) &= \frac{1}{\sqrt{\hat{\varphi}(k)}} \sum_{j \geqslant -k} \frac{P_\varphi[uw^{j+k}](z)}{\hat{\varphi}(j+k)} \int_\mathbb{C} v_j(w)|w|^{j+2k}\varphi(w)\, dv(w) \\
&= \frac{1}{\sqrt{\hat{\varphi}(k)}} \sum_{j \geqslant -k} \frac{z^{j+k}\widetilde{u}(j+k)}{\hat{\varphi}(j+k)} \int_\mathbb{C} v_j(w)|w|^{j+2k}\varphi(w)\, dv(w).
\end{aligned} \tag{2.8}$$

Since the operators T_u^φ and T_v^φ are commuting by assumption, it follows from (2.7) and (2.8) that for all integers $k \geqslant 0$, j with $j+k \geqslant 0$

$$\left[\widetilde{u}(k) - \widetilde{u}(j+k)\right] \int_{\mathbb{C}} v_j(w)|w|^{j+2k} \varphi(w)\, dv(w) = 0,$$

or equivalently

$$\left[\widetilde{u}(k) - \widetilde{u}(j+k)\right] \int_0^\infty v_j(r) r^{j+2k+1} \varphi(r)\, dr = 0. \tag{2.9}$$

Note that

$$\widetilde{u}(k) = \frac{1}{\hat{\varphi}(k)} \int_{\mathbb{C}} u(w)|w|^{2k} \varphi(w)\, dv = \frac{2\pi}{\hat{\varphi}(k)} \int_0^\infty u(r) r^{2k+1} \varphi(r)\, dr.$$

Thus, $\widetilde{u}(k)$ can be expressed as values of the Mellin transform of $u\varphi$. Given a (suitable) function ψ on the half line $(0,\infty)$, the Mellin transform $\mathcal{M}[\psi](z)$ of the complex parameter z is defined by

$$\mathcal{M}[\psi](z) = \int_0^\infty \psi(t) t^{z-1}\, dt.$$

Recall that each $\mathcal{M}[\psi]$ for suitable ψ is complex analytic on a strip in the complex plane parallel to the imaginary axis. Moreover, the Mellin transform $\mathcal{M}$ is injective. For all $k \in \mathbb{N}_0$ one can write

$$\hat{\varphi}(k) = 2\pi \mathcal{M}[\varphi](2k+2)$$

and hence

$$\widetilde{u}(k) = \frac{1}{\hat{\varphi}(k)} \int_{\mathbb{C}} u(w)|w|^{2k} \varphi(w)\, dv = \frac{\mathcal{M}[u\varphi](2k+2)}{\mathcal{M}[\varphi](2k+2)}.$$

We can rewrite (2.9) by using the Mellin transform and summarizing the above calculations we have shown:

Proposition 2.4. *Let $u, v \in \mathcal{S}$ such that u is non-constant and radial. Assume that v can be written in the form* (2.5) *which converges in $L^2(\mathbb{C}, \varphi\, dv)$. Under the assumption* (2.6) *it follows*

$$\left[\frac{\mathcal{M}[u\varphi](2k+2)}{\mathcal{M}[\varphi](2k+2)} - \frac{\mathcal{M}[u\varphi](2k+2j+2)}{\mathcal{M}[\varphi](2k+2j+2)}\right] \mathcal{M}[v_j\varphi](j+2k+2) = 0 \tag{2.10}$$

for all integers $k \geqslant 0$ and j with $j+k \geqslant 0$.

3. An enveloping algebra for Toeplitz operators on the Segal–Bargmann space

In the remaining part of this paper we specialize our analysis to the case of a standard Gaussian weight-function φ in order to study Eq. (2.10) in a precise way. On $\mathbb{C}^n$ we consider the normalized Gaussian measure $d\mu$ given by $d\mu = \varphi\, dv$ where φ is defined by $\varphi(z) := \pi^{-n} e^{-|z|^2}$, cf. [2,4]. Then $\mathscr{F}_\varphi = H^2(\mathbb{C}^n, d\mu)$ is called the *Segal–Bargmann space* and it has the reproducing kernel:

$$K_\varphi = K : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} : (z, w) \mapsto K(z, w) = e^{z\cdot\overline{w}}.$$

To keep the notations short we will write $P := P_\varphi$ for the projection and $T_u := T_u^\varphi$ for the Toeplitz operator with symbol u. Since in the following we are considering products of i.g. unbounded Toeplitz operators on $\mathscr{F}_\varphi$, we must carefully choose the space of symbols to obtain densely defined operators on an invariant domain of definition. We follow the construction in [3] and for completeness we give a short summary here.

We write $\mathcal{M}(\mathbb{C}^n)$ for the space of measurable complex valued functions on $\mathbb{C}^n$. For $c \in \mathbb{R}$ we set

$$\mathcal{D}_c := \big\{ f \in \mathcal{M}\big(\mathbb{C}^n\big) \colon \exists d > 0 \text{ such that } \big|f(z)\big| \leqslant d \exp\big(c|z|^2\big) \text{ a.e.}\big\}$$

and we define a space of symbols by

$$\operatorname{Sym}_{>0}\big(\mathbb{C}^n\big) := \bigcap_{j=1}^{\infty} \mathcal{D}_{\frac{1}{j}},$$

which is a $*$-*algebra* under the complex conjugation and pointwise multiplication, cf. [3]. Clearly, this space contains all essentially bounded functions, but also functions having a polynomial or even linear exponential growth at infinity. Consider the following sequence $(c_j)_{j\in\mathbb{N}_0}$ of positive real numbers:

$$c_j := \frac{1}{2} - \frac{1}{2j+2}$$

and put $\mathcal{H}_j := \mathcal{D}_{c_j} \cap \mathscr{F}_\varphi$.[2] Then we obtain a scale of Banach spaces

$$\mathbb{C} \cong \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_j \subset \mathcal{H}_{j+1} \cdots \subset \mathcal{H} := \bigcup_{j\in\mathbb{N}} \mathcal{H}_j \subset \mathscr{F}_\varphi, \tag{3.1}$$

where the norm $\|\cdot\|_j$ of $\mathcal{H}_j$ is given by $\|f\|_j := \|\exp\{-c_j|\cdot|^2\} f\|_{L^\infty(\mathbb{C}^n)}$. It is well known that the last inclusion $\mathcal{H} \subset \mathscr{F}_\varphi$ is dense in the topology of $\mathscr{F}_\varphi$.

Given two linear spaces X and Y we write $L(X, Y)$ for all linear operators from X to Y. If in addition X and Y are normed spaces we denote by $\mathcal{L}(X, Y)$ the subspace of all bounded linear

[2] The specific choice of the sequence $(c_j)_j$ is needed in Proposition 3.2.

operators. As usual we shortly write $L(X) := L(X, X)$ and $\mathcal{L}(X) := \mathcal{L}(X, X)$. With $k \in \mathbb{N}_0$ and the notations in (3.1) we define

$$\mathcal{L}_k(\mathcal{H}) := \left\{A \in L(\mathcal{H}):\ A_{|\mathcal{H}_j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_{j+k}) \text{ for all } j \in \mathbb{N}_0\right\}.$$

We say that operators in $\mathcal{L}_k(\mathcal{H})$ act on the scale (3.1) by an order shift k.

Definition 3.1. The space of operators acting on (3.1) by a *finite order shift* is given by

$$\mathcal{L}^{\mathrm{fos}}(\mathcal{H}) := \bigcup_{k\in\mathbb{N}_0} \mathcal{L}_k(\mathcal{H}).$$

Since for $A_{k_\ell} \in \mathcal{L}_{k_\ell}(\mathcal{H})$ where $\ell \in \{1, 2\}$, $A_{k_1} \circ A_{k_2} \in \mathcal{L}_{k_1+k_2}(\mathcal{H})$ we see that $\mathcal{L}^{\mathrm{fos}}(\mathcal{H})$ in fact defines an algebra of linear operators on $\mathcal{H}$. The normalized reproducing kernel of $H^2(\mathbb{C}^n, d\mu)$ is

$$k_z(u) = \exp\left\{u \cdot \bar{z} - \frac{|z|^2}{2}\right\}, \quad z, u \in \mathbb{C}^n.$$

By a straightforward calculation we have $k_z \in \mathcal{H}$ for all $z \in \mathbb{C}^n$ and for all operators $A \in \mathcal{L}^{\mathrm{fos}}(\mathcal{H})$ we can define the *Berezin transform* of A as usual by

$$\sim : \mathcal{L}^{\mathrm{fos}}(\mathcal{H}) \to C^{\omega}\big(\mathbb{C}^n\big) : A \mapsto \widetilde{A}(z) = \langle Ak_z, k_z\rangle,$$

where $C^{\omega}(\mathbb{C}^n)$ denotes the space of all real analytic functions on $\mathbb{C}^n$ and $\langle\cdot,\cdot\rangle$ is the inner product of $L^2(\mathbb{C}^n, d\mu)$. It has been shown in [3]:

Proposition 3.2. *Let $f, g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ and T_f the Toeplitz operator on $\mathscr{F}_\varphi$. Then we have*:

(a) *The restriction of T_f to $\mathcal{H}$ defines an element in the algebra $\mathcal{L}^{\mathrm{fos}}(\mathcal{H})$. In particular, the product $T_g T_f$ exists as a densely defined operator on $\mathscr{F}_\varphi$.*
(b) *The Berezin transform $\sim : \mathcal{L}^{\mathrm{fos}}(\mathcal{H}) \to C^{\omega}(\mathbb{C}^n)$ is one-to-one.*

From this we obtain a simple result which in the case of bounded symbols $f, g \in L^\infty(\mathbb{C}^n)$ directly follows from $T_{\bar{f}} = T_f^*$.

Lemma 3.3. *Let $f, g \in \mathrm{Sym}_{>0}(\mathbb{C}^n)$ such that $[T_f, T_g] = 0$, then $[T_{\bar{f}}, T_{\bar{g}}] = 0$.*

Proof. Note that the Berezin transform of $T_f T_g$ is given by

$$\begin{aligned}\widetilde{T_f T_g}(z) &= \langle f \cdot T_g k_z, k_z\rangle \\ &= e^{-|z|^2} \int_{\mathbb{C}^n\times\mathbb{C}^n} f(u)g(w)e^{w\cdot\bar{z}+\overline{w}\cdot u+\bar{u}\cdot z}\, d\mu(u, w),\end{aligned}$$

where the existence of the integrals for all $z \in \mathbb{C}^n$ follows from $T_g k_z \in \mathcal{H}$. Hence, Proposition 3.2 shows that the relation $[T_f, T_g] = 0$ is equivalent to

$$0 = \int_{\mathbb{C}^n \times \mathbb{C}^n} \big[f(u)g(w) - f(w)g(u)\big] e^{w\cdot\bar{z} + \overline{w}\cdot u + \bar{u}\cdot z} \, d\mu(u, w),$$

for all $z \in \mathbb{C}^n$. After applying the complex conjugation to this equation and using the transform $(u, w) \mapsto (w, u)$ we obtain

$$0 = \int_{\mathbb{C}^n \times \mathbb{C}^n} \big[\bar{f}(u)\bar{g}(w) - \bar{f}(w)\bar{g}(u)\big] e^{w\cdot\bar{z} + \overline{w}\cdot u + \bar{u}\cdot z} \, d\mu(u, w),$$

which (again, by Proposition 3.2) implies that $[T_{\bar{f}}, T_{\bar{g}}] = 0$. □

4. Commuting Toeplitz operators on the complex plane

We consider the case of dimension $n = 1$ and define a symbol space for the Toeplitz operators in consideration:

Definition 4.1. Let $\mathcal{S}$ be the subspace of measurable functions with at most polynomial growth at infinity:

$$\mathcal{S} := \big\{ f : \mathbb{C} \to \mathbb{C}\text{: } \exists C,\ m > 0 \text{ such that } \big|f(z)\big| \leqslant C\big(1 + |z|\big)^m \text{ for all } z \in \mathbb{C} \big\}.$$

From $\mathcal{S} \subset \mathrm{Sym}_{>0}(\mathbb{C})$ and Proposition 3.2 it is clear that $\{T_{f_{|\mathcal{H}}} \mid f \in \mathcal{S}\} \subset \mathcal{L}^{\mathrm{fos}}(\mathcal{H})$ and therefore all finite products of Toeplitz operators with symbols in $\mathcal{S}$ are well defined at least as densely defined operators with domain $\mathcal{H}$.

The Gaussian density φ is given by $\varphi(z) := \pi^{-1} e^{-|z|^2}$ and for $\mathrm{Re}(z) > 0$ its Mellin transform can be expressed by the usual Gamma function as

$$\mathcal{M}[\varphi](z) = \frac{1}{\pi} \int_0^\infty e^{-t^2} t^{z-1} \, dt = \frac{1}{2\pi} \int_0^\infty e^{-t} t^{\frac{z}{2}-1} \, dt = \frac{1}{2\pi} \Gamma\left(\frac{z}{2}\right).$$

From this one has $\hat{\varphi}(k) = 2\pi \mathcal{M}[\varphi](2k + 2) = \Gamma(k + 1) = k!$, and therefore it is easy to check that the condition (2.3) is fulfilled for all $u \in \mathcal{S}$.

We need the following simple observation.

Lemma 4.2. *Each $v \in \mathcal{S}$ has an $L^2(\mathbb{C}, d\mu)$-convergent expansion of the form*

$$v\big(re^{i\theta}\big) = \sum_{j=-\infty}^{\infty} v_j(r) e^{ij\theta}, \quad z = re^{i\theta}. \tag{4.1}$$

By interpreting v_j as radial functions on the complex plane we have $v_j \in \mathcal{S}$ for all $j \in \mathbb{Z}$.

Proof. Let $v \in \mathcal{S}$, then

$$\|v\|^2_{L^2(\mathbb{C},d\mu)} = \frac{1}{\pi}\int_0^\infty\int_0^{2\pi}\left|v\left(re^{i\theta}\right)\right|^2 d\theta\, re^{-r^2}\, dr < \infty.$$

By Fubini's theorem, it follows $\Phi_{v,r}(\lambda) := v(r\lambda) \in L^2(\mathbb{S}^1)$ for a.e. $r > 0$. For such $r > 0$ we can expand $\Phi_{v,r}$ into an $L^2(\mathbb{S}^1)$-convergent Fourier-series:

$$\Phi_{v,r}\left(e^{i\theta}\right) = \sum_{j\in\mathbb{Z}} v_j(r)e^{ij\theta} \tag{4.2}$$

with (measurable) coefficients $v_j(r) = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})e^{-ij\theta}\, d\theta$. In particular, there are $C, m > 0$ such that

$$\left|v_j(r)\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|v\left(re^{i\theta}\right)\right| d\theta \leqslant C\left(1+r^2\right)^m,$$

which shows that, interpreted as a radial function on the complex plane, v_j defines an element in $\mathcal{S}$ for all $j \in \mathbb{Z}$. Finally, for all $\varepsilon > 0$ we can choose a finite set $J \subset \mathbb{Z}$ such that

$$\int_{\mathbb{C}}\left|\sum_{j\notin J} v_j(r)e^{ij\theta}\right|^2 d\mu(z) = 2\sum_{j\notin J}\int_0^\infty \left|v_j(r)\right|^2 re^{-r^2}\, dr < \varepsilon.$$

This shows the $L^2(\mathbb{C}, d\mu)$-convergence of the series in (4.1). □

Now, we analyze Eq. (2.10) in Proposition 2.4. Let $u, v \in \mathcal{S}$ and $j \in \mathbb{Z}$, then we put

$$F_j(z) = \Phi_j(z)\Theta_j(z) \tag{4.3}$$

where $\Theta_j(z) := \mathcal{M}[v_j\varphi](j + 2z + 2)$ and

$$\Phi_j(z) := 2\pi\left[\frac{\mathcal{M}[u\varphi](2z+2)}{\Gamma(z+1)} - \frac{\mathcal{M}[u\varphi](2z+2j+2)}{\Gamma(z+1+j)}\right].$$

Since the Gamma function does not have zeros in the complex plane, it follows that:

Lemma 4.3. *$F_j(z)$ is holomorphic on the half-plane* $\mathrm{Re}(z) > \max\{-1, -j-1\}$.

Note that with these notations the condition (2.10) can be written as $F_j(k) = 0$ for all integers $k \geqslant 0$ and j with $j + k \geqslant 0$. The following example shows that $\Phi_j(k) = 0$ for all $k \in \mathbb{N}$ is possible in case of a non-vanishing symbol u of exponential growth. Hence, with such u the relation (2.10) is fulfilled independently of the choice of v_j where $j \in \mathbb{N}$.

Example 4.4. Consider $u(t) := e^{\lambda t^2}$ where $\lambda \in \mathbb{C}$ and $\mathrm{Re}(\lambda) < 1$. Then we have

$$\begin{aligned}\mathcal{M}[u\varphi](z) &= \frac{1}{\pi}\mathcal{M}\big[e^{(\lambda-1)t^2}\big](z)\\ &= \frac{1}{\pi}\int_0^\infty e^{-(1-\lambda)t^2} t^{z-1}\,dt = \frac{1}{2\pi}(1-\lambda)^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right).\end{aligned}$$

Therefore

$$\Phi_j(z) = (1-\lambda)^{-z-1}\big[1-(1-\lambda)^{-j}\big].$$

Fix $\lambda := \lambda_n \in \mathbb{C}$ where $n \in \mathbb{N}$ such that $(1-\lambda_n)^{-1} = e^{i\frac{2\pi}{n}}$ i.e. $\lambda_n = 1 - e^{-i\frac{2\pi}{n}}$. Moreover, let n be sufficiently large with $\mathrm{Re}(\lambda_n) < 1$. It follows:

$$\Phi_j(k) = e^{2\pi i\frac{k+1}{n}}\big[1-e^{2\pi i\frac{j}{n}}\big].$$

Let $j = n$, then $\Phi_j(k) = 0$ for all $k \geqslant 0$ with $j + k \geqslant 0$. Note that in case of $\lambda \neq 0$ we have $\mathrm{Re}(\lambda_n) = 1 - \cos(\frac{2\pi}{n}) > 0$ and $u(t)$ is of exponential growth at infinity.

Now, we define the function:

$$\begin{aligned}\widetilde{\Phi}_j(z) &:= \frac{1}{2\pi}F_j(z)\Gamma(z+1)\\ &= \Bigg[\mathcal{M}[u\varphi](2z+2) - \underbrace{\prod_{\ell=1}^{j}(z+\ell)^{-1}\mathcal{M}[u\varphi](2z+2j+2)}_{=:H_j(z)}\Bigg]\mathcal{M}[v_j\varphi](2z+2+j).\end{aligned} \tag{4.4}$$

We wish to express the function $H_j(z)$ and finally $\widetilde{\Phi}_j(z)$ as a Mellin transform. Recall that for a suitable holomorphic function $\widetilde{\Psi}(z)$ on a right half-plane $\mathrm{Re}(z) > \delta$ the *inverse Mellin transform* is given by

$$\big\{\mathcal{M}^{-1}\widetilde{\Psi}\big\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\widetilde{\Psi}(s)\,ds$$

whenever the integral exists. Here we put $\widetilde{\Psi}_j(z) = \widetilde{\Psi}(z) := \prod_{\ell=1}^{j}(z+\ell)^{-1}$ with $\mathrm{Re}(z) > -1$. Then we have

Lemma 4.5. *The inverse Mellin transform* $m_j(x) := \{\mathcal{M}^{-1}\widetilde{\Psi}_j\}(x)$ *has support in* $[0,1]$. *Moreover* $m_j(x) = O(x^\alpha)$ *for all* $\alpha < 1$ *as* $x \to 0$.

Proof. In case of $j = 1$ it is known that:

$$m_1(x) = \begin{cases} x, & \text{if } 0 < x < 1, \\ 0, & \text{if } x > 1 \end{cases}$$

and the assertion directly follows. Now, we assume that $j \in \{2, 3, \ldots\}$. With $c > -1$ we have

$$m_j(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \prod_{\ell=1}^{j} \frac{1}{(s+\ell)}\, ds = \frac{1}{2\pi i x^c} \int_{\mathbb{R}} x^{-it} \prod_{\ell=1}^{j} \frac{1}{(c+\ell+it)}\, dt.$$

Since $\widetilde{\Psi}_j(z)$ is holomorphic on $\mathrm{Re}(z) > -1$ the above integral is independent of $c > -1$. Now:

$$\left|m_j(x)\right| \leqslant \frac{1}{2\pi x^c} \int_{\mathbb{R}} \prod_{\ell=1}^{j} \frac{1}{\sqrt{(c+\ell)^2 + t^2}}\, dt. \tag{4.5}$$

As $c \to \infty$ the integral on the right-hand side tends to zero. In particular, it is bounded as a function of c by some $\eta > 0$. Hence, for all $c > 0$

$$0 \leqslant \left|m_j(x)\right| \leqslant \frac{\eta}{2\pi x^c},$$

and this shows that $m_j(x) = 0$ for $x > 1$. Now, we study the behavior of $m_j(x)$, $j \geqslant 2$ as $x \to 0$. Since the integral (4.5) converges for $c > -1$, the assertion follows. □

Recall that for suitable functions $f, g : \mathbb{R}^+ \to \mathbb{C}$ and $x > 0$ the *Mellin convolution* is defined by

$$(f * g)(x) := \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y}.$$

Definition 4.6. We define a space $\mathcal{A}$ of complex valued measurable functions on $\mathbb{R}^+$ by

$$\mathcal{A} := \Big\{ u(x)\colon \mathbb{R}^+ \to \mathbb{C}\colon \exists C,\ c > 0 \text{ and } \exists \rho, \eta \geqslant 0 \text{ such that } \left|u(x)\right| \leqslant \frac{c}{x^\rho}$$
$$\text{for all } x \in (0,1] \text{ and } \left|u(x)\right| \leqslant C x^\eta \text{ for all } x \in [1,\infty) \Big\}.$$

In the following we will often identify radial functions on $\mathbb{C}$ and functions on $\mathbb{R}^+$ in the obvious way. In this sense we have $\mathcal{S} \subset \mathcal{A}$ (see Definition 4.1).

We need a simple technical lemma:

Lemma 4.7. *Let $\rho \geqslant 0$. Then $g(x) := e^{x^2} \int_x^\infty e^{-y^2} y^{\rho-1}\, dy$ is of order $O(x^\rho)$ as $x \to \infty$.*

Proof. Let $2n + 1$ with $n \in \mathbb{N}_0$ be the smallest odd number $\geqslant \rho - 1$. With a real parameter $a > 0$ we see that there is a polynomial $P(x)$ of degree $2n$ such that

$$\int_x^\infty e^{-y^2} y^{2n+1}\,dy = (-1)^n \frac{d^n}{da^n}\left\{\int_x^\infty e^{-ay^2} y\,dy\right\}_{|a=1}$$
$$= (-1)^n \frac{d^n}{da^n}\left\{\frac{e^{-ax^2}}{2a}\right\}_{|a=1}$$
$$= P(x)e^{-x^2}.$$

Therefore, we obtain in the case $x \geqslant 1$:

$$0 \leqslant g(x) \leqslant e^{x^2}\int_x^\infty e^{-y^2} y^{2n+1}\,dy \leqslant \left|P(x)\right| \leqslant Dx^{2n}.$$

Finally, note that $2n \leqslant \rho$ due to our choice of n. The proof is complete. □

Proposition 4.8. *With $u, v \in \mathcal{A}$ we write $f_u(x) := u(x)e^{-x^2}$ and $f_v(x) := v(x)e^{-x^2}$. Then the Mellin convolution $(f_u * f_v)(x)$ exists for all $x \geqslant 0$ and there is $h \in \mathcal{A}$ such that*

$$(f_u * f_v)(x) = h(x)e^{-x}, \quad x \in \mathbb{R}^+.$$

Proof. We choose $C_w, c_w > 0$ and $\rho_w, \eta_w \geqslant 0$ such that

$$\left|w(x)\right| \leqslant \frac{c_w}{x^{\rho_w}} \quad \text{for all } x \in (0,1], \qquad \left|w(x)\right| \leqslant C_w x^{\eta_w} \quad \text{for all } x \in [1,\infty),$$

where $w \in \{u, v\}$. Let $x > 1$, then

$$\left|(f_u * f_v)(x)\right| \leqslant \int_0^1 \underbrace{\left|f_u(y)\right|}_{\leqslant \frac{c_u}{y^{\rho_u}}} \underbrace{\left|f_v\left(\frac{x}{y}\right)\right|}_{\leqslant C_v \frac{x^{\eta_v}}{y^{\eta_v}} e^{-\frac{x^2}{y^2}}} \frac{dy}{y}$$
$$+ \int_1^x \underbrace{\left|f_u(y)\right|}_{\leqslant C_u y^{\eta_u} e^{-y^2}} \underbrace{\left|f_v\left(\frac{x}{y}\right)\right|}_{\leqslant C_v \frac{x^{\eta_v}}{y^{\eta_v}} e^{-\frac{x^2}{y^2}}} dy + \int_x^\infty \underbrace{\left|f_u(y)\right|}_{\leqslant C_u y^{\eta_u} e^{-y^2}} \underbrace{\left|f_v\left(\frac{x}{y}\right)\right|}_{\leqslant c_v \frac{y^{\rho_v}}{x^{\rho_v}}} \frac{dy}{y}.$$

Therefore, we have

$$\left|(f_u * f_v)(x)\right| \leqslant C_v c_u x^{\eta_v} \int_0^1 y^{-\rho_u - \eta_v - 1} e^{-\frac{x^2}{y^2}}\,dy + C_u C_v x^{\eta_v} \int_1^x y^{\eta_u - \eta_v} e^{-y^2 - \frac{x^2}{y^2}}\,dy$$
$$+ C_u c_v x^{-\rho_v} \int_x^\infty y^{\eta_u + \rho_v - 1} e^{-y^2}\,dy. \tag{4.6}$$

We estimate the three integrals on the right as functions of x as $x \to \infty$. Using the transformation rule in the first equality below and Lemma 4.7, we find

$$x^{\eta_v} \int_0^1 y^{-\rho_u-\eta_v-1} e^{-\frac{x^2}{y^2}}\, dy = x^{-\rho_u} \int_x^\infty r^{\rho_u+\eta_v-1} e^{-r^2}\, dr \leqslant D_1 x^{\eta_v} e^{-x^2},$$

where $D_1 > 0$ is a suitable constant independent of $x > 1$. Now, we look at the last term on the right of (4.6). Applying Lemma 4.7 again, it follows

$$x^{-\rho_v} \int_x^\infty y^{\eta_u+\rho_v-1} e^{-y^2}\, dy \leqslant D_2 x^{\eta_u} e^{-x^2}.$$

In order to estimate the middle term on the right of (4.6) note that for $y \in [1, x]$

$$y^2 + \frac{x^2}{y^2} \geqslant 2x \geqslant x + y.$$

This implies that

$$\begin{aligned} x^{\eta_v} \int_1^x y^{\eta_u-\eta_v} e^{-y^2-\frac{x^2}{y^2}}\, dy &\leqslant x^{\eta_u+\eta_v} \int_1^x e^{-y^2-\frac{x^2}{y^2}}\, dy \\ &\leqslant x^{\eta_u+\eta_v} e^{-x} \int_1^x e^{-y}\, dy \\ &\leqslant x^{\eta_u+\eta_v} e^{-x-1}. \end{aligned}$$

Summarizing these estimates, we have with a suitable constant $C > 0$ and $x > 1$:

$$\left|(f_u * f_v)(x)\right| \leqslant C\left(x^{\eta_v} e^{-x^2} + x^{\eta_u} e^{-x^2} + x^{\eta_v+\eta_u} e^{-x}\right).$$

In case of $x \in [1, \infty)$ we define $h(x) := e^x (f_u * f_v)(x)$. Then we have

$$\left|h(x)\right| \leqslant C_h x^{\eta_v+\eta_u}$$

with a suitable constant $C_h > 0$.

Next, we estimate the Mellin convolution $(f_u * f_v)(x)$ with $x \in (0, 1]$. We can assume that $\rho_u \geqslant \rho_v$:

$$\big|(f_u * f_v)(x)\big| \leqslant \int_0^x \underbrace{\big|u(y)\big|}_{\leqslant \frac{c_u}{y^{\rho_u}}} \underbrace{\left|v\left(\frac{x}{y}\right)\right|}_{C_v \frac{x^{\eta_v}}{y^{\eta_v}}} e^{-y^2-\frac{x^2}{y^2}} \frac{dy}{y} +$$

$$+ \int_x^1 \underbrace{\big|u(y)\big|}_{\leqslant \frac{c_u}{y^{\rho_u}}} \underbrace{\left|v\left(\frac{x}{y}\right)\right|}_{c_v \frac{y^{\rho_v}}{x^{\rho_v}}} e^{-y^2-\frac{x^2}{y^2}} \frac{dy}{y} + \int_1^\infty \underbrace{\big|u(y)\big|}_{\leqslant C_u y^{\eta_u}} \underbrace{\left|v\left(\frac{x}{y}\right)\right|}_{\leqslant c_v \frac{y^{\rho_v}}{x^{\rho_v}}} e^{-y^2-\frac{x^2}{y^2}} \frac{dy}{y}.$$

Therefore, we have

$$\big|(f_u * f_v)(x)\big| \leqslant c_u C_v x^{\eta_v} \int_0^x y^{-\rho_u-\eta_v-1} e^{-\frac{x^2}{y^2}}\, dy$$
$$+ \frac{c_u c_v}{x^{\rho_v}} \int_x^1 y^{\rho_v-\rho_u-1} e^{-\frac{x^2}{y^2}}\, dy + \frac{C_u c_v}{x^{\rho_v}} \int_1^\infty y^{\eta_u+\rho_v-1} e^{-y^2}\, dy. \tag{4.7}$$

We estimate the three terms on the right separately. Applying the transformation rule we have as $x \to 0$:

$$x^{\eta_v} \int_0^x y^{-\rho_u-\eta_v-1} e^{-\frac{x^2}{y^2}}\, dy = x^{-\rho_u} \int_1^\infty r^{\rho_u+\eta_v-1} e^{-r^2}\, dr = O\big(x^{-\rho_u}\big).$$

As for the second integral, we have

$$x^{-\rho_v} \int_x^1 y^{\rho_v-\rho_u-1} e^{-\frac{x^2}{y^2}}\, dy = x^{-\rho_u} \int_x^1 r^{\rho_u-\rho_v-1} e^{-r^2}\, dr$$
$$= \begin{cases} O(x^{-\rho_u}), & \text{if } \rho_u > \rho_v, \\ O(x^{-\rho_u} \log \frac{1}{x}), & \text{if } \rho_u = \rho_v. \end{cases}$$

Since the last term on the right of (4.7) is of order $O(x^{-\rho_v})$ as $x \to 0$ we see that there is a constant $C > 0$ such that

$$\big|(f_u * f_v)(x)\big| \leqslant C\left(x^{-\rho_u} \log \frac{1}{x} + x^{-\rho_v}\right).$$

Again, we set $h(x) = e^x (f_u * f_v)(x)$ with $x \in (0, 1]$. Then there is $c_h > 0$ such that $|h(x)| \leqslant \frac{c_h}{x^{\rho_u+\rho_v+1}}$ and hence $h \in \mathcal{A}$ as desired. The proof is complete. □

Let $u, v \in \mathcal{A}$ and in addition assume that the support $\operatorname{supp} v$ of v is contained in $[0, 1]$. As before we write $f_u(x) := u(x)e^{-x^2}$. Then we have:

Proposition 4.9. *Under the above assumptions there is $h \in \mathcal{A}$ such that*

$$(f_u * v)(x) = h(x)e^{-x^2}, \quad x \in \mathbb{R}^+.$$

Proof. We use the same notations as in the proposition before. Since $\operatorname{supp} v \subset [0, 1]$, we have for $x \geqslant 1$ and together with Lemma 4.7

$$\begin{aligned}\left|(f_u * v)(x)\right| &\leqslant \int_x^\infty \underbrace{\left|f_u(y)\right|}_{\leqslant C_u y^{\eta_u} e^{-y^2}} \underbrace{\left|v\left(\frac{x}{y}\right)\right|}_{\leqslant c_v \frac{y^{\rho_v}}{x^{\rho_v}}} \frac{dy}{y} \\ &\leqslant \frac{C_u c_v}{x^{\rho_v}} \int_x^\infty y^{\eta_u+\rho_v-1} e^{-y^2}\, dy \leqslant C x^{\eta_u} e^{-x^2},\end{aligned}$$

where $C > 0$ is a suitable constant. We set $h(x) := e^{x^2}(f_u * v)(x)$ in case of $x \geqslant 1$. In case of $x \in (0, 1]$ we can apply the same calculation as in the proof of Proposition 4.8. □

Now, we return to (4.4). Let $j \in \mathbb{N}$ and as before we write $\widetilde{\Psi}_j(z) = \prod_{\ell=1}^j (z+\ell)^{-1}$. According to Lemma 4.5 there is a function $\widetilde{m}_j : \mathbb{R}^+ \to \mathbb{C}$ with $\operatorname{supp} \widetilde{m}_j \subset [0, 1]$ such that

$$\mathcal{M}[\widetilde{m}_j](z) = \widetilde{\Psi}_j(z).$$

Now, the transformation $y := \sqrt{x}$ shows

$$\widetilde{\Psi}_j(z) = \int_0^1 \widetilde{m}_j(x) x^{z-1}\, dx = 2\int_0^1 \widetilde{m}_j\big(y^2\big) y^{2z-1}\, dy = \mathcal{M}[m_j](2z),$$

where we write $m_j(y) := 2\widetilde{m_j}(y^2)$. Clearly, it holds $\operatorname{supp} m_j \subset [0, 1]$. Hence (4.4) can be written in the form:

$$\widetilde{\Phi}_j(z) = \big\{\mathcal{M}\big[ux^2\varphi\big](2z) - \mathcal{M}[m_j](2z)\mathcal{M}\big[ux^{2j+2}\varphi\big](2z)\big\}\mathcal{M}\big[v_j x^{2+j}\varphi\big](2z).$$

Assume $u, v_j \in \mathcal{A}$, then we have $ux^2, ux^{2j+2}, v_j x^{2+j} \in \mathcal{A}$. Due to Proposition 4.9, there is a function $h \in \mathcal{A}$ such that

$$m_j * \big(ux^{2j+2}\varphi\big) = he^{-x^2}.$$

According to the *Mellin convolution theorem* we have for $\operatorname{Re}(z)$ sufficiently large

$$\mathcal{M}[m_j](2z) \cdot \mathcal{M}\big[ux^{2j+2}\varphi\big](2z) = \mathcal{M}\big[m_j * \big(ux^{2j+2}\varphi\big)\big](2z) = \mathcal{M}\big[he^{-x^2}\big](2z)$$

and with this notations

$$\widetilde{\Phi}_j(z) = \big[\mathcal{M}\big[ux^2\varphi\big](2z) - \mathcal{M}\big[he^{-x^2}\big](2z)\big]\mathcal{M}\big[v_j x^{2+j}\varphi\big](2z).$$

Next, we can apply Proposition 4.8 to see that there are functions $g_1, g_2 \in \mathcal{A}$ such that

(i) $[(ux^2\varphi) * (v_j x^{2+j}\varphi)](x) = g_1(x)e^{-x}$,
(ii) $[(he^{-x^2}) * (v_j x^{2+j}\varphi)](x) = g_2(x)e^{-x}$.

Again, we can use the *Mellin convolution theorem* in order to see that

$$\begin{aligned}\widetilde{\Phi}_j(z) &= \mathcal{M}\big[\big(ux^2\varphi\big) * \big(v_j x^{2+j}\varphi\big)\big](2z) - \mathcal{M}\big[\big(he^{-x^2}\big) * \big(v_j x^{2+j}\varphi\big)\big](2z)\\ &= \mathcal{M}\big[(g_1 - g_2)e^{-x}\big](2z).\end{aligned}$$

We have shown with $g := g_1 - g_2$:

Proposition 4.10. *With the notations in* (4.4) *there is* $g \in \mathcal{A}$ *such that* $\widetilde{\Phi}_j(z) = \mathcal{M}[ge^{-x}](2z)$.

The next proposition is essential in our proof. It is a replacement for the Blaschke condition which is used by the authors of [10] to prove Theorem A of the introduction.

Proposition 4.11. *Let* $u \in \mathcal{A}$ *and* $a \in (0, 2]$. *For fixed number* $m_0 \in \mathbb{N}$ *the condition*:

$$\int_0^\infty u(t)e^{-t}t^{ak}\,dt = 0, \tag{4.8}$$

for all $k \geqslant m_0$ *implies that* $u = 0$ *a.e. on* $\mathbb{R}^+$.

Proof. Since with $m_0 \in \mathbb{N}_0$ we have

$$\int_0^\infty u(t)e^{-t}t^{a(m_0+k)}\,dt = t^{m_0 a}\int_0^\infty u(t)e^{-t}t^{ak}\,dt = 0$$

for all $k \in \mathbb{N}$. Hence, we can assume that u is integrable over [0, 1] and that $m_0 = 0$. First, we consider the case $a \in (0, 2)$. The transformation $r = t^a$ in the integral (4.8) implies for all $k \in \mathbb{N}_0$:

$$\frac{1}{a}\int_0^\infty \underbrace{u\big(r^{\frac{1}{a}}\big)e^{-r^{\frac{1}{a}}}r^{\frac{1}{a}-1}}_{=:h(r)} r^k\,dr = 0.$$

With $x \in \mathbb{R}$ consider the sum:

$$0 = \sum_{k=0}^\infty (-1)^k \underbrace{\int_0^\infty h(r)\frac{(x\sqrt{r}\,)^{2k}}{(2k)!}\,dr}_{=0}$$

$$= \int_0^\infty h(r)\cos(x\sqrt{r})\,dr$$

$$= 2\int_0^\infty rh(r^2)\cos(xr)\,dr. \tag{4.9}$$

In the second equality we have used $\sum_{k=0}^{m}\frac{(x\sqrt{r})^{2k}}{(2k)!} \leqslant e^{x\sqrt{r}}$ for all $m \in \mathbb{N}$ and $\frac{1}{a} > \frac{1}{2}$ together with Lebesgue dominated convergence theorem in order to interchange the sum and integral. We extend $s(r) := rh(r^2) \in L^1(\mathbb{R}^+)$ by zero to an integrable function on the real line. Then we have

$$0 = \int_{\mathbb{R}} s(r)\left(e^{ixr} + e^{-ixr}\right)dr = \int_{\mathbb{R}} \left[s(r) + s(-r)\right]e^{ixr}\,dr.$$

Since the Fourier transform is one-to-one on $L^1(\mathbb{R})$ it follows $s(r) + s(-r) = 0$ a.e. on $\mathbb{R}$. Since $s(-r) = 0$ in case of $r > 0$ we have $s(r) = 0$ a.e. on $\mathbb{R}^+$ and this implies that $u = 0$, a.e. on $\mathbb{R}^+$.

Next, we consider the case $a = 2$. We can exchange the integration and summation in (4.9) at least in the case $|x| < 1$. Hence, we have for all $|x| < 1$ and with our former notations

$$0 = \int_0^\infty \left[s(r) + s(-r)\right]e^{ixr}\,dr.$$

Let us replace x by the complex variable $z = \sigma + it$. From $|e^{izr}| = e^{-tr}$ and

$$0 \leqslant \left|s(r)\right| = \left|rh\left(r^2\right)\right| = \left|u(r)e^{-r}\right| \leqslant C_u r^{\eta_u} e^{-r}$$

with some positive $C_u > 0$ and $\eta_u > 0$ it follows that

$$F(z) = \int_{\mathbb{R}} \left[s(r) + s(-r)\right]e^{izr}\,dr$$

defines a holomorphic function on $\mathcal{N} := \{z \in \mathbb{C} \mid -1 < \mathrm{Im}(z) = t < 1\}$. Since $F(x) = 0$ whenever $|x| < 1$ we have $F \equiv 0$ on $\mathcal{N}$. In particular, the Fourier transform of $s(r) + s(-r)$ vanishes identically. As before we obtain $u(r) = 0$ a.e. on $\mathbb{R}^+$. □

The next example shows that Proposition 4.11 fails in the case of $a > 2$.

Example 4.12. Consider $g_a(r) := \sin(ar)e^{-r}$ where $a > 0$. Then:

$$\mathcal{M}[g_a](z) = \frac{1}{2i}\int_0^\infty e^{-(1-ia)r} r^{z-1}\,dr - \frac{1}{2i}\int_0^\infty e^{-(1+ia)r} r^{z-1}\,dr$$

$$= \frac{1}{2i}(1-ia)^{-z}\Gamma(z) - \frac{1}{2i}(1+ia)^{-z}\Gamma(z)$$

$$= \frac{1}{2i}\left(1+a^2\right)^{-\frac{z}{2}}\left[e^{iz\arctan a} - e^{-iz\arctan a}\right]\Gamma(z)$$
$$= \left(1+a^2\right)^{-\frac{z}{2}}\sin(z\arctan a)\Gamma(z).$$

Let $\beta > 2$, then $\mathcal{M}[g_a](\beta z)$ vanishes exactly for $\beta z \arctan a = k\pi$ with $k \in \mathbb{Z}$. Choose a with $\arctan a = \frac{\pi}{\beta} < \frac{\pi}{2}$, then $\mathcal{M}[g_a](\beta k) = 0$ for all $k \in \mathbb{N}$.

In our arguments below we need some well-known estimates on the Gamma function:

Lemma 4.13. *There are constants $A > 0$ and $C > 0$ independent of z, such that for* $\operatorname{Re} z = \sigma > A$

$$\frac{\Gamma(\sigma)}{|\Gamma(z)|} \leqslant C\left[\frac{\sigma^2}{\sigma^2+t^2}\right]^{\frac{\sigma}{2}-\frac{1}{4}} \exp\left\{|t|\arctan\frac{|t|}{\sigma}\right\}. \tag{4.10}$$

In particular, we have for $\operatorname{Re} z = \sigma > A > 1$

$$\frac{\Gamma(\sigma)}{|\Gamma(z)|} \leqslant C\exp\left\{\frac{\pi}{2}|t|\right\}. \tag{4.11}$$

Proof. We write $z = \sigma + it \in \mathbb{C}$ and we use a well-known formula on the asymptotic expansion of $\log\Gamma(z+1)$ (cf. [18, p. 279]):

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z - \frac{1}{2}\log(2\pi) + \psi(z),$$

where $\lim_{|z|\to\infty}\psi(z) = 0$. Consider the line $L_\sigma := \{z \in \mathbb{C} \mid \operatorname{Re}(z) = \sigma\}$, which for $\sigma > 0$ is parametrized by

$$L_\sigma = \left\{z := \sqrt{\sigma^2+t^2}\exp\left\{i\arctan\frac{t}{\sigma}\right\} \;\middle|\; t \in \mathbb{R}\right\}.$$

If we insert such $z \in L_\sigma$ into the above formula, we obtain

$$\Gamma(z) = \exp\left\{\left(z - \frac{1}{2}\right)\left(\log\sqrt{\sigma^2+t^2} + i\arctan\frac{t}{\sigma}\right) - z - \frac{1}{2}\log 2\pi + \psi(z)\right\}.$$

Therefore, there is $\rho: \mathbb{C} \to \mathbb{R}$ such that $\lim_{|z|\to\infty}\rho(z) = 1$ and

$$\left|\Gamma(z)\right| = \rho(z)\exp\left\{\left(\sigma - \frac{1}{2}\right)\log\sqrt{\sigma^2+t^2} - t\arctan\frac{t}{\sigma} - \sigma - \frac{1}{2}\log 2\pi\right\}$$
$$= \rho(z)\frac{e^{-\sigma}}{\sqrt{2\pi}}\left(\sigma^2+t^2\right)^{\frac{\sigma}{2}-\frac{1}{4}}\exp\left\{-|t|\arctan\frac{|t|}{\sigma}\right\}, \tag{4.12}$$

which gives the desired result. The proof is complete. □

Let $u \in \mathcal{A}$ and assume that

$$\Psi_j(z) := \frac{\mathcal{M}[u\varphi](2z+2)}{\Gamma(z+1)}$$

is periodic on $\mathrm{Re}(z) \geqslant A$ with period $j \in \mathbb{N}$. Therefore, it can be considered as an entire function on the complex plane. With the notations in the proof of Proposition 4.8 we have for $\sigma = \mathrm{Re}(z) \geqslant \frac{\rho_u}{2}$

$$\begin{aligned}
\left|\mathcal{M}[u\varphi](2z+2)\right| &\leqslant \frac{1}{\pi}\int_0^\infty \left|u(t)\right| e^{-t^2} t^{2\sigma+1}\,dt \\
&\leqslant \frac{c_u}{\pi}\int_0^1 e^{-t^2} t^{2\sigma-\rho_u+1}\,dt + \frac{C_u}{\pi}\int_1^\infty e^{-t^2} t^{2\sigma+\eta_u+1}\,dt \\
&\leqslant \frac{c_u}{2\pi}\Gamma\left(\sigma - \frac{\rho_u}{2} + 1\right) + \frac{C_u}{2\pi}\Gamma\left(\sigma + \frac{\eta_u}{2} + 1\right) \\
&\leqslant C\Gamma\left(\sigma + \frac{\eta_u}{2} + 1\right),
\end{aligned}$$

where $C > 0$ is a suitable constant. Without lost of generality we can assume that η_u is an even integer. According to (4.11) in Lemma 4.13, it follows

$$\begin{aligned}
\left|\Psi_j(z)\right| &\leqslant C\frac{\Gamma(\sigma + \frac{\eta_u}{2} + 1)}{|\Gamma(z+1)|} \\
&\leqslant C\left(\sigma + \frac{\eta_u}{2}\right)\left(\sigma + \frac{\eta_u}{2} - 1\right)\cdots(\sigma+1)\frac{\Gamma(\sigma)}{|\Gamma(z)|} \\
&\leqslant C_1\big[|\sigma| + 1\big]^m \exp\left\{\frac{\pi}{2}|t|\right\}
\end{aligned}$$

where $C_1 > 0$ and $m \in \mathbb{N}$ are sufficiently large. Let $z = \sigma + it \in \mathbb{C}$ and choose $k \in \mathbb{Z}$ with

$$\frac{\rho_u}{2} + j \geqslant \sigma + jk \geqslant \frac{\rho_u}{2}.$$

By the periodicity of $\Psi_j(z)$, we have

$$\left|\Psi_j(z)\right| = \left|\Psi_j(z+jk)\right| \leqslant C_1\left[\frac{\rho_u}{2} + j + 1\right]^m \exp\left\{\frac{\pi}{2}|t|\right\} \leqslant C_2 \exp\left\{\frac{\pi}{2}|z|\right\}.$$

We have proved:

Lemma 4.14. *If the entire function Ψ_j is periodic of period j, then Ψ_j has linear exponential growth as $|z| \to \infty$.*

The next proposition characterizes the class of all periodic entire functions of at most linear exponential growth at infinity.

Proposition 4.15. *Let f be an entire function of period $j \in \mathbb{N}$. Assume that there are $A, B > 0$ such that*

$$\left|f(z)\right| \leqslant A e^{B|z|}.$$

Then f is a trigonometric polynomial of the following type:

$$f(z) = \sum_{\ell=-n}^{n} a_\ell e^{\frac{2\pi i \ell z}{j}}, \quad a_\ell \in \mathbb{C}. \tag{4.13}$$

Proof. Let $w = re^{2\pi i\varphi} \in \mathbb{C}^*$ with $r > 0$ and $\varphi \in [0, 1)$ and define

$$g(w) := f\left(-\frac{ji}{2\pi}\log w\right) = f\left(j\varphi - \frac{ji}{2\pi}\log r\right).$$

Note that g is an entire function on the complex plane since f has the period $j \in \mathbb{N}$. If n is an integer greater than B we have in case of $r > 1$

$$\left|w^{nj} g(w)\right| \leqslant Ar^{nj}\exp\{nj\varphi + nj\log r\} = \widetilde{A}r^{2nj} = \widetilde{A}|w|^{2nj}.$$

In case of $0 < r \leqslant 1$, we have $|w^{nj}g(w)| \leqslant Ar^{nj}\exp\{nj\varphi + nj\log r^{-1}\} = \widetilde{A}$, where $\widetilde{A} > 0$ is a suitable constant. Therefore

$$w^{nj}g(w) = \begin{cases} O(|w|^{2nj}), & \text{as } |w| \to \infty, \\ O(1), & \text{as } |w| \to 0. \end{cases}$$

The second estimate shows that $w^{nj}g(w)$ has a removable singularity at 0. We remove it. The first inequality shows that $w^{nj}g(w)$ is a polynomial of degree $\leqslant 2nj$. Therefore,

$$f(z) = g\left(e^{\frac{2\pi i z}{j}}\right)$$

is a trigonometric polynomial of the type (4.13). $\square$

So, we see from Lemma 4.14 and Proposition 4.15 that Ψ_j must be a trigonometric polynomial of the form

$$\begin{aligned}\Psi_j(z) &= \sum_{\ell=-n}^{n} a_\ell e^{\frac{2\pi i\ell z}{j}} \\ &= \sum_{\substack{\ell=-n\\|4\ell|>j}}^{n} a_\ell e^{\frac{2\pi i\ell z}{j}} + \sum_{\substack{\ell=-n\\|4\ell|\leqslant j}}^{n} a_\ell e^{\frac{2\pi i\ell z}{j}} \\ &=: \Psi_j^+(z) + \Psi_j^-(z).\end{aligned} \tag{4.14}$$

First, we observe that the trigonometric polynomial $\Psi_j^-(z)$ can be written in the form:

$$\Psi_j^-(z) = \frac{\mathcal{M}[w\varphi](2z+2)}{\Gamma(z+1)}$$

with a suitable function w (which is of exponential growth as $|z| \to \infty$). This follows from our calculations in Example 4.4. More precisely, for $\lambda \in \mathbb{C}$ and $\mathrm{Re}(\lambda) < 1$

$$\mathcal{M}\big[e^{\lambda r^2}\varphi\big](z) = \frac{1}{2\pi}(1-\lambda)^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right).$$

Therefore:

$$\frac{\mathcal{M}[2\pi(1-\lambda)e^{\lambda r^2}\varphi](2z+2)}{\Gamma(z+1)} = (1-\lambda)^{-z}.$$

If we choose $1-\lambda = e^{-\frac{2\pi i\ell}{j}}$ where $|4\ell| < j$, then we have $\mathrm{Re}(\lambda) = 1 - \cos\frac{2\pi\ell}{j} < 1$ and with the definition $w(r) := 2\pi(1-\lambda)e^{\lambda r^2}$ it follows that

$$\frac{\mathcal{M}[w\varphi](2z+2)}{\Gamma(z+1)} = e^{\frac{2\pi i\ell z}{j}}.$$

We still need to consider the case $|4\ell| = j$, which means that $e^{-\frac{2\pi i\ell}{j}} = \pm i$. Let $4\ell = -j$, then we have in case of $-1 < \mathrm{Re}(z) < 0$:

$$\frac{\mathcal{M}[-2ie^{it^2}](2z+2)}{\Gamma(z+1)} = e^{\frac{\pi i z}{2}}.$$

The case $4\ell = j$ can be treated in a similar way. Note that in these calculations and for $\lambda \neq 0$ we have $\mathrm{Re}(\lambda) > 0$, which implies that $w \notin \mathcal{A}$. With our notations in (4.14) we prove:

Proposition 4.16. *Let $u \in \mathcal{A}$ such that*

$$\Psi_j(z) = \frac{\mathcal{M}[u\varphi](2z+2)}{\Gamma(z+1)}$$

is an entire function of period $j \in \mathbb{N}$. Then u must be a constant function.

Proof. By what was said before $\Psi_j(z)$ is a trigonometric polynomial of the form (4.14) and using the assumption and the notations above we can write

$$\mathcal{M}[u\varphi](2z+2) = \Gamma(z+1)\big[\Psi_j^+(z) + \Psi_j^-(z)\big].$$

First we show that Ψ_j^+ must vanish identically. Otherwise there is $\ell_0 \in \{-n, \dots, 0, \dots, n\}$ such that $|\ell_0| = \max\{|\ell|\colon a_\ell \neq 0\} > \frac{j}{4}$. To fix a particular case let us assume that $\ell_0 < 0$. It is easy to check that

$$\Psi_j^+(ir) \sim a_{\ell_0} e^{-\frac{2\pi\ell_0 r}{j}}, \quad \text{as } r \to \infty.$$

From (4.12), we known that $\Gamma(ir+1) \sim const \cdot \sqrt{1+r^2} \exp\{-r \arctan r\}$ as $r \to \infty$. Therefore, we conclude from $-\frac{2\pi\ell_0}{j} > \frac{\pi}{2} = \lim_{r\to\infty} \arctan r$ that $r \mapsto \Gamma(ir+1)\Psi_j^+(ir)$ has growth of exponential order as $r \mapsto \infty$. A similar argument shows that $r \mapsto \Gamma(ir+1)\Psi_j^-(ir)$ can have at most (linear) polynomial growth as $r \to \infty$. Finally, note that $r \mapsto \mathcal{M}[u\varphi](2ir+2)$ is bounded:

$$\left|\mathcal{M}[u\varphi](2ir+2)\right| \leqslant \int_0^\infty |u\varphi|(t)t^2\,dt < \infty.$$

Hence, we must find u with

$$\mathcal{M}[u\varphi](2z+2) = \Gamma(z+1)\Psi_j^-(z). \tag{4.15}$$

Due to our calculations above we can find $\lambda_\ell, a_\ell \in \mathbb{C}$ with $1 \geqslant \mathrm{Re}(\lambda_\ell) \geqslant 0$ such that for all $z \in \mathbb{C}$ with $-1 < \mathrm{Re}(z) < 0$:

$$\Gamma(z+1)\Psi_j^-(z) = \mathcal{M}\left[\varphi \sum_{\ell=-n}^{n} a_\ell e^{\lambda_\ell r^2}\right](2z+2).$$

The Mellin transform is one-to-one and therefore we conclude that (4.15) is uniquely solved by

$$u(r) = \sum_{\ell=-n}^{n} a_\ell e^{\lambda_\ell r^2}.$$

Since $u \in \mathcal{A}$ is of at most polynomial growth at infinity, we have $a_\ell = 0$ if $\lambda_\ell \neq 0$ and u must be constant. □

Now, we can formulate and prove our main theorem in this section:

Theorem 4.17. *Let $u, v \in \mathcal{S}$ and assume that u is radial and non-constant. If $T_u T_v = T_v T_u$ on $\mathcal{H}$, cf.* (3.1)*, then v is a radial function.*

We remark that the condition $u, v \in \mathcal{S}$ is essential. A counter example to the above statement in case of $u \notin \mathcal{S}$ and even bounded v is given in Example 5.8.

Proof of Theorem 4.17. According to Lemma 4.2 there is an expansion of v:

$$v\left(re^{i\theta}\right) = \sum_{j=-\infty}^{\infty} v_j(r)e^{ij\theta}, \quad z = re^{i\theta}, \tag{4.16}$$

which is convergent in $L^2(\mathbb{C}, d\mu)$ and it holds $v_j \in \mathcal{A}$ for all $j \in \mathbb{Z}$. Assume, that v is not radial, then there is $j \in \mathbb{Z} \setminus \{0\}$ such that $v_j(r) \neq 0$. Using Lemma 3.3 we can assume that $j \in \mathbb{N}$. Using the notation in (4.3) we see that the holomorphic function Θ_j does not vanish on a right half-plane. Proposition 2.4 shows that

$$F_j(k) = \Phi_j(k)\Theta_j(k) = 0$$

for all $k \in \mathbb{N}$. From this we have $\widetilde{\Phi}_j(k) = 0$ for $k \in \mathbb{N}$ where

$$\widetilde{\Phi}_j(z) := \frac{1}{2\pi} F_j(z) \Gamma(z+1)$$

is the holomorphic function on $\mathrm{Re}(z) > -1$ in (4.4). According to Proposition 4.10 there is $g \in \mathcal{A}$ such that

$$\widetilde{\Phi}_j(z) = \mathcal{M}\big[ge^{-x}\big](2z)$$

and therefore $\widetilde{\Phi}_j(k) = \int_0^\infty g(x)e^{-x}x^{2k-1}\,dx = 0$ for all $k \in \mathbb{N}$. Using Proposition 4.11 with $a = 2$ it follows that $g = 0$ a.e. on $\mathbb{R}^+$ and therefore

$$\widetilde{\Phi}_j(z) = \frac{1}{2\pi} \Phi_j(z) \Theta_j(z) \Gamma(z+1) = 0$$

on $\mathrm{Re}(z) > -1$. Consequently, we have

$$\Phi_j(z) = 2\pi \left[\frac{\mathcal{M}[u\varphi](2z+2)}{\Gamma(z+1)} - \frac{\mathcal{M}[u\varphi](2z+2j+2)}{\Gamma(z+1+j)} \right] \equiv 0,$$

which shows that

$$\Psi_j(z) := \frac{\mathcal{M}[u\varphi](2z+2)}{\Gamma(z+1)}$$

is an entire function on the complex plane of period $j \in \mathbb{N}$. Finally, using Proposition 4.16 it follows that u is constant, which gives a contradiction and the proof is finished. □

As a direct consequence of Theorem 4.17 we remark:

Corollary 4.18. *Let $v \in \mathcal{S}$ and assume that the Toeplitz operator T_v is diagonal with respect to the standard orthonormal basis $\{(j!)^{-\frac{1}{2}} z^j\colon\ j \in \mathbb{N}_0\}$. Then v has to be a radial function.*

Proof. Fix a non-constant radial function $u \in S$. Due to our assumption on T_v we have $[T_u, T_v] = 0$ and the assertion follows from Theorem 4.17. □

Note that Corollary 4.18 can also be proved directly. Let Rv denote the *radialization* of the symbol v. In case the Toeplitz operator T_v is diagonal, it can be checked that $T_{Rv} = T_v$, which implies that $T_{Rv-v} = 0$. It follows that $v = Rv$ is a radial function.

5. Discussion on $\mathbb{C}^n$

We use the notations of Section 3, and we write $\mathbb{P}[z, \bar{z}]$ for the space of all polynomials in the complex variables $z = (z_1, \ldots, z_n)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$. For simplicity we only consider Toeplitz operators T_f with symbols $f \in \mathbb{P}[z, \bar{z}]$. In general, such operators are unbounded but as a common dense domain of definition we can choose:

$$\mathcal{D} := \mathbb{P}[z, \bar{z}] \cap H^2\big(\mathbb{C}^n, d\mu\big)$$

which is the space of all holomorphic polynomials on $\mathbb{C}^n$. Moreover, one can easily check that $\mathcal{D}$ is invariant under all Toeplitz operators T_f with $f \in \mathbb{P}[z, \bar{z}]$. The linear space:

$$\mathcal{O} := \{T_f \colon \mathcal{D} \to \mathcal{D} \mid f \in \mathbb{P}[z, \bar{z}]\}$$

is also invariant under the operator product. More precisely, it holds (cf. [3,9]):

Lemma 5.1. *Let $p, q \in \mathbb{P}[z, \bar{z}]$, then the product $T_p T_q$ is a Toeplitz operator $T_{p \sharp q}$, where the symbol $p \sharp q$ can be calculated as*

$$p \sharp q = \sum_{\gamma \in \mathbb{N}_0^n} \frac{(-1)^{|\gamma|}}{\gamma!} \partial_\gamma p \cdot \bar{\partial}_\gamma q. \tag{5.1}$$

Here we write $\partial_\gamma := \frac{\partial^{|\gamma|}}{\partial z^\gamma}$ and $\bar{\partial}_\gamma := \frac{\partial^{|\gamma|}}{\partial \bar{z}^\gamma}$ where $\gamma \in \mathbb{N}_0^n$.

By using this result, we can easily construct non-radial commuting Toeplitz operators in case of dimension $n \geqslant 2$. Consider the linear subspace $\mathcal{P}_0$ of $\mathbb{P}[z, \bar{z}]$ defined by

$$\mathcal{P}_0 := \left\{ p \in \mathbb{P}[z, \bar{z}] \colon \exists m \in \mathbb{N},\ \exists a_\alpha \in \mathbb{C} \text{ such that } p(z) = \sum_{j=0}^{m} \sum_{|\alpha|=j} a_\alpha \left|z^\alpha\right|^2 \right\}.$$

Lemma 5.2. *The space* $(\mathcal{P}_0, \sharp)$ *is a commutative $\sharp$-sub-algebra of* $(\mathbb{P}[z, \bar{z}], \sharp)$.

Proof. First assume that $p, q \in \mathcal{P}_0$ have the form $p(z) = |z^\alpha|^2$ and $q(z) = |z^\beta|^2$ where $\alpha, \beta \in \mathbb{N}_0^n$. Note that

$$\begin{aligned} \frac{\partial^{|\gamma|} p}{\partial z^\gamma} \cdot \frac{\partial^{|\gamma|} q}{\partial \bar{z}^\gamma} &= \left\{ \frac{\alpha!}{(\alpha - \gamma)!} z^{\alpha-\gamma} \bar{z}^\alpha \right\} \cdot \left\{ \frac{\beta!}{(\beta - \gamma)!} z^\beta \bar{z}^{\beta-\gamma} \right\} \\ &= \frac{\alpha! \beta!}{(\alpha - \gamma)!(\beta - \gamma)!} \left|z^{\alpha+\beta-\gamma}\right|^2 \\ &= \frac{\partial^{|\gamma|} q}{\partial z^\gamma} \cdot \frac{\partial^{|\gamma|} p}{\partial \bar{z}^\gamma}. \end{aligned}$$

By linearity it follows that $\mathcal{P}_0$ is closed under the $\sharp$-product and the $\sharp$-multiplication is commutative. □

As an immediate consequence of Lemmas 5.1 and 5.2 we find that

$$\mathcal{O}_0 := \{T_p \colon p \in \mathcal{P}_0\}$$

is a commutative sub-algebra in $\mathcal{O}$. Let $p, q \in \mathbb{P}[z, \bar{z}]$ be of degree k and m with principal parts p_k and q_m, respectively. We see from Lemma 5.1 that the relation $[T_p, T_q] = T_p T_q - T_q T_p = 0$ is equivalent to

$$0 = p \sharp q - q \sharp p = \sum_{\ell=0}^{\infty} (-1)^\ell \sum_{|\gamma|=\ell} \frac{1}{\gamma!} \{\partial_\gamma p \cdot \bar{\partial}_\gamma q - \partial_\gamma q \cdot \bar{\partial}_\gamma p\}.$$

In general, the right-hand side is a polynomial of degree $m+k-2$ having the principal part:

$$PP(p \sharp q - q \sharp p) = \sum_{j=1}^{n} \left\{ \frac{\partial p_k}{\partial z_j} \frac{\partial q_m}{\partial \bar{z}_j} - \frac{\partial q_m}{\partial z_j} \frac{\partial p_k}{\partial \bar{z}_j} \right\}.$$

Corollary 5.3. *A necessary condition for* $[T_p, T_q] = T_p T_q - T_q T_p = 0$ *is*

$$0 = \sum_{j=0}^{n} \left\{ \frac{\partial p_k}{\partial z_j} \frac{\partial q_m}{\partial \bar{z}_j} - \frac{\partial q_m}{\partial z_j} \frac{\partial p_k}{\partial \bar{z}_j} \right\}. \tag{5.2}$$

Consider the space $\mathbb{P}_a[z] := \{p \in \mathbb{P}[z, \bar{z}] \mid p \text{ is complex analytic}\}$, which we also have denoted by $\mathcal{D}$ before. As a simple consequence of Corollary 5.3 we have:

Proposition 5.4. *Let* $n = 1$ *and assume that* $p \in \mathbb{P}_a[z]$ *is non-constant. If* $q \in \mathbb{P}[z, \bar{z}]$ *such that* $[T_p, T_q] = 0$*, then* $q \in \mathbb{P}_a[z]$.

Proof. Using our former notations we conclude from Corollary 5.3 and $\frac{\partial p_k}{\partial \bar{z}} = 0$ that $\frac{\partial p_k}{\partial z} \cdot \frac{\partial q_m}{\partial \bar{z}} = 0$. If p_k is non-constant we see that $\frac{\partial q_m}{\partial \bar{z}} = 0$, which implies that $q_m \in \mathbb{P}_a[z]$. Now, we can apply the same argument to p and $q - q_m$, using $[T_p, T_{q-q_m}] = 0$ to show that q_{m-1} (the principal part of $q - q_m$) is in $\mathbb{P}_a[z]$. The assertion follows by induction. □

Next, we consider the space of radial symmetric polynomials:

$$\mathbb{P}_{\mathrm{rad}}[z, \bar{z}] := \left\{ p\left(|z|^2\right) \in \mathbb{P}[z, \bar{z}] \;\middle|\; p = p(r) \text{ is a polynomial in one variable } r \right\}$$

and the space $\mathcal{P}_1$ of all polynomials $p \in \mathbb{P}[z, \bar{z}]$ which fulfill the invariance $p(e^{it} z) = p(z)$ for all $t \in \mathbb{R}$ and $z \in \mathbb{C}^n$:

$$\mathcal{P}_1 := \left\{ p(z) = \sum_{\alpha, \beta \in \mathbb{N}_0^n} a_{\alpha,\beta} z^\alpha \bar{z}^\beta \in \mathbb{P}[z, \bar{z}] \;\middle|\; a_{\alpha,\beta} = 0 \text{ if } |\alpha| \neq |\beta| \right\}. \tag{5.3}$$

We clearly have $\mathbb{P}_{\mathrm{rad}}[z, \bar{z}] \subset \mathcal{P}_0 \subset \mathcal{P}_1$ and all the inclusions are strict. Generalizing Lemma 5.2 we can prove the following:

Lemma 5.5. *Let* $p \in \mathbb{P}_{\mathrm{rad}}[z, \bar{z}]$ *and* $q \in \mathcal{P}_1$*, then the Toeplitz operators* T_p *and* T_q *commute.*

Proof. Let $\ell \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{N}_0^n$. Then it follows by a direct calculation:

$$\frac{\partial^{|\gamma|}}{\partial z^\gamma} |z|^{2\ell} \cdot \frac{\partial^{|\gamma|}}{\partial \bar{z}^\gamma} \left(z^\alpha \bar{z}^\beta\right) = \binom{\ell}{|\gamma|} \binom{\beta}{\gamma} |\gamma|! \gamma! z^\alpha \bar{z}^\beta |z|^{2(\ell - |\gamma|)},$$

$$\frac{\partial^{|\gamma|}}{\partial z^\gamma} \left(z^\alpha \bar{z}^\beta\right) \cdot \frac{\partial^{|\gamma|}}{\partial \bar{z}^\gamma} |z|^{2\ell} = \binom{\ell}{|\gamma|} \binom{\alpha}{\gamma} |\gamma|! \gamma! z^\alpha \bar{z}^\beta |z|^{2(\ell - |\gamma|)}.$$

Hence, we have

$$|z|^{2\ell} \sharp z^\alpha \bar{z}^\beta = z^\alpha \bar{z}^\beta \sum_{\gamma \in \mathbb{N}_0^n} \binom{\ell}{|\gamma|}\binom{\beta}{\gamma} |\gamma|!(-1)^{|\gamma|}|z|^{2(\ell-|\gamma|)},$$

$$z^\alpha \bar{z}^\beta \sharp |z|^{2\ell} = z^\alpha \bar{z}^\beta \sum_{\gamma \in \mathbb{N}_0} \binom{\ell}{|\gamma|}\binom{\alpha}{\gamma} |\gamma|!(-1)^{|\gamma|}|z|^{2(\ell-|\gamma|)}.$$

In order to see that $|z|^{2\ell} \sharp z^\alpha \bar{z}^\beta = z^\alpha \bar{z}^\beta \sharp |z|^{2\ell}$ in case of $|\alpha| = |\beta|$ note that $\sum_{|\gamma|=m} \binom{\beta}{\gamma} = \binom{|\beta|}{m}$ for $m \in \mathbb{N}_0$. The assertion now follows by linearity of the $f \sharp g$-product in f and g. □

Finally, we state the converse result. A similar statement in case of Toeplitz operators acting on Bergman spaces over the unit ball has been given in [12].

Proposition 5.6. *Let $p \in \mathbb{P}_{\mathrm{rad}}[z, \bar{z}]$ be non-constant and $q \in \mathbb{P}[z, \bar{z}]$ such that $[T_q, T_p] = 0$, then $q \in \mathcal{P}_1$.*

Proof. Let $p(z) = \sum_{j=0}^k a_j |z|^{2j}$ be non-constant with principal part $p_k(z) = a_k |z|^{2k}$. The general form of q is $q(z) = \sum_{j=0}^m \sum_{|\alpha|+|\beta|=j} a_{\alpha,\beta} z^\alpha \bar{z}^\beta = \sum_{j=0}^m q_j(z)$. We define the differential operators

$$L_j := \bar{z}_j \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_j},$$

where $j = 1, \dots, n$. Then condition (5.2) can be written in the form:

$$0 = |z|^{2k-2} \sum_{j=1}^n L_j q_m,$$

which implies that

$$0 = \sum_{j=1}^n \sum_{|\alpha|+|\beta|=m} a_{\alpha,\beta} L_j z^\alpha \bar{z}^\beta = \sum_{j=1}^n \sum_{|\alpha|+|\beta|=m} a_{\alpha,\beta} (\alpha_j - \beta_j) z^\alpha \bar{z}^\beta$$
$$= \sum_{|\alpha|+|\beta|=m} a_{\alpha,\beta} \big(|\alpha| - |\beta|\big) z^\alpha \bar{z}^\beta.$$

It follows that $a_{\alpha,\beta} = 0$ for $|\alpha| \neq |\beta|$ and therefore $q_m \in \mathcal{P}_1$. According to our assumptions and due to Lemma 5.5 the Toeplitz operator T_{q-q_m} also commutes with T_p and we can use induction to finish the proof. □

We give a generalization of Corollary 4.18 to Toeplitz operators on $H^2(\mathbb{C}^n, d\mu)$, $n > 1$ with polynomial symbols:

Corollary 5.7. *Let $q \in \mathbb{P}[z, \bar{z}]$ and assume that T_q is diagonal with respect to the standard orthonormal basis $\mathcal{B} := [e_\alpha := (\alpha!)^{-\frac{1}{2}} z^\alpha \colon \alpha \in \mathbb{N}_0^n]$. Then $q \in \mathcal{P}_0$, i.e. q is radial in all components $z_1, \dots, z_n$.*

Proof. Let $p_j(z) := |z_j|^2$ for $j = 1, \dots, n$ such that $p_j \in \mathcal{P}_0$. Then the operators T_{p_j} are diagonal with respect to the basis $\mathcal{B}$. Since by assumption T_q is diagonal with respect to $\mathcal{B}$ as well, it follows that $[T_{p_j}, T_q] = 0$ and (5.2) shows that $L_j q_m = 0$ for all $j = 1, \dots, n$. In the same way as in the proof of Proposition 5.6 we obtain that $q \in \mathcal{P}_0$. $\square$

There are bounded Toeplitz operators with radial symbols of exponential growth at infinity that are commuting with Toeplitz operators having non-radial bounded symbol. The construction is closely related to our observations in Example 4.4:

Example 5.8. On $\mathbb{C}^n$ we consider the radial function $g_\lambda(z) := e^{\lambda |z|^2}$ where $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) < \frac{1}{2}$. The latter condition ensures that the Toeplitz operator T_{g_λ} is densely defined on $H^2(\mathbb{C}^n, d\mu)$. Next, we consider the composition operators

$$U_\gamma : \mathcal{D}_\gamma \subset H^2\bigl(\mathbb{C}^n, d\mu\bigr) \to H^2\bigl(\mathbb{C}^n, d\mu\bigr) : f \mapsto f(\overline{\gamma} z)$$

where $\gamma \in \mathbb{C}$. By using the reproducing kernel $K(z, w) = e^{z \cdot \overline{w}}$ of $H^2(\mathbb{C}^n, d\mu)$ we see that the Berezin transform $\widetilde{U_\gamma}(z)$ of U_γ is given by

$$\widetilde{U_\gamma}(z) = e^{-|z|^2} \bigl\langle U_\gamma K(\cdot, z), K(\cdot, z)\bigr\rangle = e^{(\gamma - 1)|z|^2}.$$

On the other hand, the Berezin transform of T_{g_λ} is given by

$$\begin{aligned} \widetilde{T_{g_\lambda}}(z) &= e^{-|z|^2} \int_{\mathbb{C}^n} g_\lambda(u) e^{z \cdot \bar{u} + \bar{z} \cdot u} \, d\mu(u) \\ &= \pi^{-n} e^{-|z|^2} \int_{\mathbb{C}^n} e^{z \cdot \bar{u} + \bar{z} \cdot u - (1-\lambda)|u|^2} \, dv(u) \\ &= (1 - \lambda)^{-1} e^{\lambda (1-\lambda)^{-1} |z|^2}. \end{aligned}$$

Since the Berezin transform is one-to-one on (suitable) operators we find that

$$(1 - \lambda)^{-1} U_{(1-\lambda)^{-1}} = T_{g_\lambda}.$$

For all $m \in \mathbb{N}$ we define $\lambda_m := 1 - e^{-\frac{2i\pi}{m}}$. By choosing m sufficiently large, it follows $\mathrm{Re}\, \lambda_m < \frac{1}{2}$. Then $T_{g_{\lambda_m}}$ is well defined and

$$e^{\frac{2i\pi}{m}} U_{e^{\frac{2i\pi}{m}}} = T_{g_{\lambda_m}}. \tag{5.4}$$

Note that by this equation the Toeplitz operator $T_{g_{\lambda_m}}$ is unitary with unbounded symbol. We shortly write $V_k := U_{e^{\frac{2i\pi}{k}}}$ where $k \in \mathbb{Z}$ and remark that for any bounded symbol f

$$V_k T_f V_{-k} = T_{V_k f}. \tag{5.5}$$

It immediately follows from (5.5) and (5.4) that $T_{g_{\lambda_m}}$ commutes with all Toeplitz operators with symbols that are invariant under V_m. To give an explicit example, choose $n = 1$ and $m = 8$ such that $\operatorname{Re}\lambda_m = 1 - \cos(\frac{\pi}{4}) = 1 - \frac{\sqrt{2}}{2} < \frac{1}{2}$. Then $T_{g_{\lambda_8}}$ commutes with T_f where f is the bounded non-radial function $f(z) = \frac{z^8}{|z|^8}$.

Problem. Is there an extension of the results in Section 5 to Toeplitz operators with arbitrary measurable symbols that have at most polynomial growth at infinity?

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