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Inverse nodal and inverse spectral problems for discontinuous boundary value problems ☆

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Inverse nodal and inverse spectral problems are studied for second-order differential operators on a finite interval with discontinuity conditions inside the interval. Uniqueness theorems are proved, and a constructive procedure for the solution is provided.

1. Introduction

We study inverse nodal and inverse spectral problems for differential operators. Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, monographs [2–4,16,18,21,26,28] and the references therein). Inverse nodal problems consist in constructing operators from the given nodes (zeros) of eigenfunctions (refer to [15,19,22,23]). In the present paper we obtain some results on inverse nodal and inverse spectral problems and establish connections between them.

Consider the following boundary value problem with discontinuity conditions inside the interval:

$$-y'' + q(x)y = \lambda y, \quad 0 < x < T, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \qquad V(y) := y'(T) + Hy(T) = 0, \tag{2}$$

$$y(T/2+0) = a_1 y(T/2-0), \qquad y'(T/2+0) = a_1^{-1} y'(T/2-0) + a_2 y(T/2-0). \tag{3}$$

Here λ is the spectral parameter, $q(x)$, h, H, a_1, a_2 are real, $q(x) \in L(0,T)$, and $a_1 > 0$. Without loss of generality we assume that

$$\int_0^T q(x)\,dx = 0. \tag{4}$$

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We will consider inverse problems of recovering $q(x)$, h and H from the given spectral or nodal characteristics. The coefficients a_1 and a_2 from (3) are assumed to be known a priori and fixed. We denote the boundary value problem (1)–(3) by $B = B(q,h,H)$.

Boundary value problems with discontinuity conditions inside the interval often appear in applications. Such problems are connected with discontinuous material properties. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics (see [17,20]). Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium (see [13,24]). Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth (see [1,14]). Here the main discontinuity is caused by reflection of the shear waves at the base of the crust. Discontinuous inverse problems (in various formulations) have been considered in [5,7,12,13,24,25,27] and other works.

2. Inverse spectral problems

In this section we study the so-called incomplete inverse problem of recovering the potential $q(x)$ from a part of the spectrum of B provided that the potential is known a priori on a part of the interval. We note that for recovering $q(x)$ on the whole interval $(0,T)$ it is necessary to specify two spectra of boundary value problems with different boundary conditions (see [27]). We also note that for classical Sturm–Liouville operators incomplete inverse problems were investigated in [6,9,10].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on $[0,T/2]$ and on $[T/2,T]$. Denote $\langle y,z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (3), then

$$\langle y,z\rangle_{|x=T/2+0} = \langle y,z\rangle_{|x=T/2-0}. \tag{5}$$

Let $\varphi(x,\lambda)$ be the solution of Eq. (1) satisfying the initial conditions $y(0)=1$, $y'(0)=h$ and the matching condition (3). Then $U(\varphi)=0$. Denote $\Delta(\lambda) := -V(\varphi)$ (U and V are defined in (2)). The function $\Delta(\lambda)$ is entire in λ of order 1/2, and its zeros $\{\lambda_n\}_{n\geqslant 0}$ coincide with the eigenvalues of B. The function $\Delta(\lambda)$ is called the characteristic function for B. Since the boundary value problem B is self-adjoint, all zeros of $\Delta(\lambda)$ are real and simple.

Let $\lambda=\rho^2$, $\tau := \operatorname{Im}\rho$. For $|\lambda|\to\infty$ uniformly in x one has (see [27] or Chapter 1 in [28]):

$$\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)\,dt\right)\frac{\sin\rho x}{\rho} + o\left(\frac{1}{\rho}\exp(|\tau|x)\right), \quad x<\frac{T}{2}, \tag{6}$$

$$\varphi(x,\lambda) = \bigl(b_1\cos\rho x + b_2\cos\rho(T-x)\bigr) + \left(F_1(x)\frac{\sin\rho x}{2\rho} + F_2(x)\frac{\sin\rho(T-x)}{2\rho}\right) + o\left(\frac{1}{\rho}\exp(|\tau|x)\right), \quad x>\frac{T}{2}, \tag{7}$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + \left(h + \frac{1}{2}\int_0^x q(t)\,dt\right)\cos\rho x + o\bigl(\exp(|\tau|x)\bigr), \quad x<\frac{T}{2}, \tag{8}$$

$$\varphi'(x,\lambda) = \rho\bigl(-b_1\sin\rho x + b_2\sin\rho(T-x)\bigr) + \left(F_1(x)\frac{\cos\rho x}{2} - F_2(x)\frac{\cos\rho(T-x)}{2}\right) + o\bigl(\exp(|\tau|x)\bigr), \quad x>\frac{T}{2}, \tag{9}$$

where

$$F_1(x) = b_1\left(2h + \int_0^x q(t)\,dt\right) + a_2, \qquad F_2(x) = b_2\left(2h - \int_0^x q(t)\,dt + 2\int_0^{T/2} q(t)\,dt\right) - a_2,$$

$$b_1 = \frac{a_1 + a_1^{-1}}{2}, \qquad b_2 = \frac{a_1 - a_1^{-1}}{2}.$$

It follows from (6)–(9) that for $|\lambda|\to\infty$,

$$\Delta(\lambda) = b_1\left(\rho\sin\rho T - \frac{\omega\cos\rho T}{2} + \frac{\omega_1}{2}\right) + o\bigl(\exp(|\tau|T)\bigr), \tag{10}$$

where

$$\omega = 2H + 2h + \int_0^T q(t)\,dt + \frac{a_2}{b_1}, \qquad \omega_1 = -\frac{b_2}{b_1}\left(2H - 2h + \int_0^T q(t)\,dt - 2\int_0^{T/2} q(t)\,dt\right) - \frac{a_2}{b_1}.$$

It is easy to see that

$$\left.\begin{aligned} \omega-\omega_1 &= 2b_1^{-1}\big((b_1+b_2)H+(b_1-b_2)h+a_2\big),\\ \omega+\omega_1 &= 2b_1^{-1}\big((b_1-b_2)H+(b_1+b_2)h\big). \end{aligned}\right\} \tag{11}$$

Using (10) by the well-known method (see, for example, [4]) one has that for $n\to\infty$,

$$\rho_n:=\sqrt{\lambda_n}=\frac{\pi n}{T}+\frac{1}{2\pi n}\big(\omega+(-1)^{n-1}\omega_1\big)+o\left(\frac{1}{n}\right). \tag{12}$$

Together with B we consider a boundary value problem $\tilde B=B(\tilde q,\tilde h,\tilde H)$ of the same form but with different coefficients $\tilde q,\tilde h,\tilde H$ (we remind that the coefficients a_1 and a_2 from (3) are fixed and known a priori). We agree that if a certain symbol α denotes an object related to B, then $\tilde\alpha$ will denote an analogous object related to $\tilde B$. The following theorem has been proven by M. Horvath [11] for the Sturm–Liouville equation without interior discontinuity. We show it also holds for (1)–(3).

Theorem 1. *Fix $b\in(0,T/2]$. Let $\Lambda\subset\mathbf{N}\cup\{0\}$ be a subset of nonnegative integer numbers, and let $\Omega:=\{\lambda_n\}_{n\in\Lambda}$ be a part of the spectrum of B such that the system of functions $\{\cos 2\rho_n x\}_{n\in\Lambda}$ is complete in $L_2(0,b)$. Let $q(x)=\tilde q(x)$ a.e. on (b,T), $H=\tilde H$, and $\Omega=\tilde\Omega$. Then $q(x)=\tilde q(x)$ a.e. on $(0,T)$ and $h=\tilde h$.*

Proof. Since

$$-\varphi''(x,\lambda)+q(x)\varphi(x,\lambda)=\lambda\varphi(x,\lambda),\qquad -\tilde\varphi''(x,\lambda)+\tilde q(x)\tilde\varphi(x,\lambda)=\lambda\tilde\varphi(x,\lambda),$$
$$\varphi(0,\lambda)=\tilde\varphi(0,\lambda)=1,\qquad \varphi'(0,\lambda)=h,\qquad \tilde\varphi'(0,\lambda)=\tilde h,$$

it follows from (5) that

$$\begin{aligned}\int_0^T r(x)\varphi(x,\lambda)\tilde\varphi(x,\lambda)\,dx &= \big(\big|_0^{T/2-0}+\big|_{T/2+0}^{T}\big)\big(\varphi'(x,\lambda)\tilde\varphi(x,\lambda)-\varphi(x,\lambda)\tilde\varphi'(x,\lambda)\big)\\ &=\varphi(T,\lambda)\tilde\Delta(\lambda)-\Delta(\lambda)\tilde\varphi(T,\lambda)-(h-\tilde h),\end{aligned} \tag{13}$$

where $r(x)=q(x)-\tilde q(x)$. Taking (11) and (12) into account we get

$$h=\tilde h. \tag{14}$$

Since $\Delta(\lambda_n)=\tilde\Delta(\lambda_n)=0$ for $n\in\Lambda$, it follows from (4), (13) and (14) that

$$\int_0^b r(x)\left(\varphi(x,\lambda_n)\tilde\varphi(x,\lambda_n)-\frac12\right)dx=0,\quad n\in\Lambda. \tag{15}$$

For $x\leqslant T/2$ the following representation holds (see [4,16,18]):

$$\varphi(x,\lambda)=\cos\rho x+\int_0^x K(x,t)\cos\rho t\,dt, \tag{16}$$

where $K(x,t)$ is a continuous function which does not depend on λ. Hence

$$\varphi(x,\lambda)\tilde\varphi(x,\lambda)-\frac12=\frac12\left(\cos 2\rho x+\int_0^x V(x,t)\cos 2\rho t\,dt\right),$$

where $V(x,t)$ is a continuous function which does not depend on λ. Substituting (16) into (15) and taking the relation $\int_0^T r(x)\,dx=0$ into account, we calculate

$$\int_0^b\left(r(x)+\int_x^b V(t,x)r(t)\,dt\right)\cos 2\rho_n x\,dx=0,\quad n\in\Lambda,$$

and consequently,

$$r(x)+\int_x^b V(t,x)r(t)\,dt\quad \text{a.e. on } (0,b).$$

Since this homogeneous integral equation has only the trivial solution it follows that $r(x) = 0$ a.e. on $(0, b)$, i.e. $q(x) = \tilde{q}(x)$ a.e. on $(0, b)$. □

Theorem 1 will be used in the next section for studying inverse nodal problems. The next theorem is devoted to the particular case when $b = T/2$. In this case one has to take the whole spectrum, and the completeness can be proved. This theorem is a generalization of the result from [9] for the classical Sturm–Liouville operators.

Theorem 2. *Let $q(x) = \tilde{q}(x)$ a.e. on $(T/2, T)$, $H = \tilde{H}$ and $\lambda_n = \tilde{\lambda}_n$ for all $n \geqslant 0$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$ and $h = \tilde{h}$.*

Proof. Let us show that the system of functions $\{\cos \rho_n x\}_{n\geqslant 0}$ is complete in $L_2(0, T)$. Indeed, let $f(x) \in L_2(0, T)$ be such that

$$\int_0^T f(x) \cos \rho_n x \, dx = 0, \quad n \geqslant 0. \tag{17}$$

Consider the functions

$$F(\lambda) := \int_0^T f(x) \cos \rho x \, dx, \qquad F_0(\lambda) := \frac{F(\lambda)}{\Delta(\lambda)}.$$

Fix $\delta > 0$, and denote $G_\delta := \{\rho\colon |\rho - \rho_n| \geqslant \delta \ \forall n \geqslant 0\}$. Then (see [27])

$$\bigl|\Delta(\lambda)\bigr| \geqslant C|\rho| \exp\bigl(|\tau|T\bigr), \quad \rho \in G_\delta,$$

and consequently,

$$\bigl|F_0(\lambda)\bigr| \leqslant \frac{C}{|\rho|}, \quad \rho \in G_\delta. \tag{18}$$

On the other hand, it follows from (17) that the function $F_0(\lambda)$ is entire in λ. Together with (18) this yields $F_0(\lambda) \equiv 0$, i.e. $F(\lambda) \equiv 0$. Hence $f(x) = 0$ a.e. on $(0, T)$. Thus, the system of functions $\{\cos \rho_n x\}_{n\geqslant 0}$ is complete in $L_2(0, T)$. Applying now Theorem 1 for $b = T/2$ we obtain $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$ and $h = \tilde{h}$. □

Corollary 1. *Let $q(x) = \tilde{q}(x)$ a.e. on $(T/4, T)$, $H = \tilde{H}$ and $\lambda_{2n} = \tilde{\lambda}_{2n}$ for all $n \geqslant 0$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$ and $h = \tilde{h}$.*

Proof. It is sufficient to prove that $\{\cos(2\rho_{2n} x)\}$ is complete in $L_2(0, T/4)$. For the purpose, the reader can refer to Theorem 3 on p. 163 of [21]. □

3. Inverse nodal problems

The study of inverse nodal problem without discontinuous conditions was initiated by O.H. Hald and J.R. McLaughlin (see [8] and [19]). In this section, we consider the inverse nodal problems with discontinuous conditions. In the first part of this section we obtain uniqueness theorems and a procedure of recovering the potential $q(x)$ on the whole interval $(0, T)$ from a dense subset of nodal points. In the second part of the section we establish connections between inverse nodal and spectral problems. Using these connections and the results of Section 2, it is proved that under additional restrictions the potential can be recovered *on the whole interval* $(0, T)$ from a subset of nodal points situated only *on a part of the interval*.

The eigenfunctions of the boundary value problem B have the form $y_n(x) = \varphi(x, \lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (12) into (6) and (7) we obtain the following asymptotic formulae for $n \to \infty$ uniformly in x:

$$y_n(x) = \cos \frac{\pi n}{T} x + \frac{1}{2\pi n}\left(T\left(2h + \int_0^x q(t)\, dt\right) - \bigl(\omega + (-1)^{n-1}\omega_1\bigr)x\right) \sin \frac{\pi n}{T} x + o\left(\frac{1}{n}\right), \quad x < \frac{T}{2}, \tag{19}$$

$$\begin{aligned} y_n(x) = {} & \bigl(b_1 + (-1)^{n-1} b_2\bigr) \cos \frac{\pi n}{T} x + \frac{1}{2\pi n}\bigl(T F_1(x) + (-1)^{n-1} T F_2(x) \\ & - \bigl(\omega + (-1)^{n-1}\omega_1\bigr)\bigl(b_1 x + (-1)^{n-1} b_2 (T - x)\bigr)\bigr) \sin \frac{\pi n}{T} x + o\left(\frac{1}{n}\right), \quad x > \frac{T}{2}. \end{aligned} \tag{20}$$

For the boundary value problem B an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $y_n(x)$ has exactly n (simple) zeros inside the interval $(0, T)$, namely: $0 < x_n^1 < \cdots < x_n^n < T$. The set $X_B := \{x_n^j\}_{n\geqslant 1,\, j=\overline{1,n}}$ is called the set of nodal points of the boundary value problem B. Denote $X_B^k := \{x_{2m-k}^j\}_{m\geqslant 1,\, j=\overline{1,2m-k}}$, $k = 0, 1$. Clearly, $X_B^0 \cup X_B^1 = X_B$.

Inverse nodal problems consist in recovering the potential $q(x)$ and the coefficients h and H from the given set X_B of nodal points or from a certain its part. Denote

$$\alpha_n^j := \left(j - \frac{1}{2}\right)\frac{T}{n}.$$

Taking (19)–(20) into account, we obtain the following asymptotic formulae for nodal points as $n \to \infty$ uniformly in j:

$$x_n^j = \alpha_n^j + \frac{T}{2\pi^2 n^2}\left(T\left(2h + \int_0^{\alpha_n^j} q(t)\,dt\right) - (\omega - \omega_1)\alpha_n^j\right) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(0, \frac{T}{2}\right),\ n = 2m, \tag{21}$$

$$x_n^j = \alpha_n^j + \frac{T}{2\pi^2 n^2}\left(T\left(2h + \int_0^{\alpha_n^j} q(t)\,dt\right) - (\omega + \omega_1)\alpha_n^j\right) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(0, \frac{T}{2}\right),\ n = 2m+1, \tag{22}$$

$$x_n^j = \alpha_n^j + \frac{T}{2\pi^2 n^2}\left(T\int_0^{\alpha_n^j} q(t)\,dt - (\omega - \omega_1)\alpha_n^j + c_0\right) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(\frac{T}{2}, T\right),\ n = 2m, \tag{23}$$

$$x_n^j = \alpha_n^j + \frac{T}{2\pi^2 n^2}\left(T\int_0^{\alpha_n^j} q(t)\,dt - (\omega + \omega_1)\alpha_n^j + c_1\right) + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(\frac{T}{2}, T\right),\ n = 2m+1, \tag{24}$$

where

$$\left.\begin{aligned} c_0 &= \frac{2Ta_2}{b_1+b_2} - \frac{2Tb_2}{b_1+b_2}\int_0^{T/2} q(t)\,dt + \frac{2Th(b_1-b_2)}{b_1+b_2} + \frac{Tb_2(\omega-\omega_1)}{b_1+b_2}, \\ c_1 &= \frac{2Tb_2}{b_1-b_2}\int_0^{T/2} q(t)\,dt + \frac{2Th(b_1+b_2)}{b_1-b_2} - \frac{Tb_2(\omega+\omega_1)}{b_1-b_2}. \end{aligned}\right\} \tag{25}$$

We note that the sets X_B^k, $k = 0, 1$, are dense on $(0, T)$. Using these formulae we arrive at the following assertion.

Theorem 3. *Fix $k = 0 \vee 1$ and $x \in [0, T]$. Let $\{x_n^{j_n}\} \in X_B^k$ be chosen such that $\lim_{n\to\infty} x_n^{j_n} = x$. Then there exists a finite limit*

$$g_k(x) := \lim_{n\to\infty} \frac{2\pi^2 n}{T^2}\left(x_n^{j_n} n - \left(j_n - \frac{1}{2}\right)T\right), \tag{26}$$

and

$$\left.\begin{aligned} g_k(x) &= \int_0^x q(t)\,dt - \frac{\omega + (-1)^{k+1}\omega_1}{T}x + 2h, \quad x \leqslant \frac{T}{2}, \\ g_k(x) &= \int_0^x q(t)\,dt - \frac{\omega + (-1)^{k+1}\omega_1}{T}x + \frac{c_k}{T}, \quad x \geqslant \frac{T}{2}, \end{aligned}\right\} \tag{27}$$

where c_0 and c_1 are defined by (25)*.*

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 4. *Fix $k = 0 \vee 1$. Let $X \subset X_B^k$ be a subset of nodal points which is dense on $(0, T)$. Let $X = \tilde{X}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, $h = \tilde{h}$, $H = \tilde{H}$. Thus, the specification of X uniquely determines the potential $q(x)$ on $(0, T)$ and the coefficients of the boundary conditions. The function $q(x)$ and the numbers h, H can be constructed via the formulae*

$$q(x) = g_k'(x) - \frac{1}{T}\big(g_k(T) - g_k(0)\big), \tag{28}$$

$$h = \frac{g_k(0)}{2}, \qquad H = -\frac{g_k(T)}{2} - (-1)^k \frac{b_2}{b_1 + (-1)^k b_2}\int_0^{T/2} q(t)\,dt, \tag{29}$$

where $g_k(x)$ is calculated by (27)*.*

Proof. Formulae (28)–(29) follow from (27) and (4). Note that by (27), we have

$$g_k'(x) = p(x) - \left(\frac{\omega + (-1)^{k+1}\omega_1}{T}\right) \tag{30}$$

hence

$$g_k(T) - g_k(0) = \int_0^T p(x)\,dx - \big(\omega + (-1)^{k+1}\omega_1\big) = -\big(\omega + (-1)^{k+1}\omega_1\big). \tag{31}$$

Then (28) can by derived directly from (30) and (31). Similarly, we can derive (29). Note that if $X = \tilde{X}$, then (26) yields $g_k(x) \equiv \tilde{g}_k(x)$, $x \in [0, T]$. By virtue of (28)–(29), we get $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, $h = \tilde{h}$, $H = \tilde{H}$. □

Analogously one can prove the following more general assertion.

Theorem 5. *Let $X \subset X_B$ be dense on $(0, T)$. If $X = \tilde{X}$, then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, $h = \tilde{h}$, $H = \tilde{H}$.*

Proof. The consequence follows directly from (28)–(29). □

Alternatively, we can obtain a reconstruction formula of potential function from the nodal lengths. For $X \subset X_B$ we denote $\Lambda_X := \{n\colon\ \exists j x_n^j \in X\}$.

Definition 1. Let $X \subset X_B$. The set X is called *twin* if together with each of its points x_n^j the set X contains at least one of adjacent nodal points x_n^{j-1} or/and x_n^{j+1}.

Let us go on to the investigation of an incomplete inverse nodal problem when nodal points are given only on a part of the interval. First we will prove an auxiliary assertion.

Lemma 1. *Fix n, j. Let $x_n^j = \tilde{x}_n^j$, $x_n^{j+1} = \tilde{x}_n^{j+1}$, and let $q(x) = \tilde{q}(x)$ a.e. on (x_n^j, x_n^{j+1}). Then $\lambda_n = \tilde{\lambda}_n$.*

Proof. On the interval $x \in (x_n^j, x_n^{j+1})$ we consider the boundary value problem B_{nj} for Eq. (1) with the matching conditions (3) (if $T/2 \in (x_n^j, x_n^{j+1})$) and with the boundary conditions

$$y\big(x_n^j\big) = y\big(x_n^{j+1}\big) = 0.$$

The function $y_n(x) = \varphi(x, \lambda_n)$ is the eigenfunction of B, and simultaneously it is the eigenfunction of B_{nj}. Since $y_n(x)$ has no zeros for $x \in (x_n^j, x_n^{j+1})$, it follows that λ_n is the first eigenvalue of B_{nj}, and $y_n(x)$ is the first eigenfunction. Since $q(x) = \tilde{q}(x)$ a.e. on (x_n^j, x_n^{j+1}), one has $\lambda_n = \tilde{\lambda}_n$. □

Theorem 6. *Fix $k = 0 \vee 1$ and $b \in (0, T/4)$. Let $X \subset X_B^k \cap (b, T)$ be a dense on (b, T) twin subset of nodal points such that the system of functions $\{\cos 2\rho_n x\}_{n \in \Lambda_X}$ is complete in $L_2(0, b)$. Let $X = \tilde{X}$ and $H = \tilde{H}$. Then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, and $h = \tilde{h}$.*

Proof. Since $X = \tilde{X}$, it follows that $g_k(x) \equiv \tilde{g}_k(x)$ for $x \in (b, T)$, and consequently, $g_k'(x) = \tilde{g}_k'(x)$ a.e. on (b, T). Together with (27) this yields $q(x) - \tilde{q}(x) = d$ a.e. on (b, T), where d is a constant. Denote $q_0(x) := \tilde{q}(x) + d$, $x \in (0, T)$. Then $q(x) = q_0(x)$ a.e. on (b, T). Let $\{\lambda_n^0\}_{n \geqslant 1}$ be the spectrum of $L(q_0, \tilde{h}, \tilde{H})$. By Lemma 1, $\lambda_n = \lambda_n^0$ for $n \in \Lambda_X$. Applying Theorem 1 we obtain $q(x) = q_0(x)$ a.e. on $(0, T)$, and $h = \tilde{h}$. Thus, $q(x) = \tilde{q}(x) + d$ a.e. on $(0, T)$. Using the relation $\int_0^T q(t)\,dt = \int_0^T \tilde{q}(t)\,dt = 0$, we calculate $d = 0$, i.e. $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$. □

Theorem 7. *Fix $b \in (0, T/2)$. Let $X := X_B \cap [b, T)$. If $X = \tilde{X}$, $H = \tilde{H}$, then $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, and $h = \tilde{h}$.*

Proof. The proof is similar to that of Theorem 6. Let $X = X_1 \cup X_2$, where $X_k = X_B^k \cap (b, T)$. Fix $k = 0$ or 1, by the same arguments in the proof of Theorem 6, we have $q(x) = q_0(x) := \tilde{q}(x) + d$ in (b, T) and $h = \tilde{h}$. Applying Lemma 1, we have $\lambda_n = \tilde{\lambda}_n$, where $\{\lambda_n\}_{n \geqslant 0} = \sigma(L)$ and $\{\tilde{\lambda}_n\}_{n \geqslant 0} = \sigma(\tilde{L})$. By Theorem 1 and the fact $\int_0^T q(t)\,dt = \int_0^T \tilde{q}(t)\,dt = 0$, we obtain $d = 0$, $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$, and $h = \tilde{h}$. □

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