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Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball

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Abstract

This work is a continuation of the recent study by the authors on approximation theory over the sphere and the ball. The main results define new Sobolev spaces on these domains and study polynomial approximations for functions in these spaces, including simultaneous approximation by polynomials and the relation between the best approximation of a function and its derivatives.

Keywords: Approximation by polynomials; Sobolev space; Modulus of smoothness; K-functional; Sphere; Ball

1. Introduction

In a recent work [3], the authors defined new moduli of smoothness and K-functionals on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ and the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \le 1\}$ of $\mathbb{R}^d$, where $\|x\|$ denotes the usual Euclidean norm, and used them to characterize the best approximation by polynomials. This work is a continuation of [3] and studies polynomial approximation in Sobolev spaces.

The new modulus of smoothness on the sphere is defined in terms of forward differences in the angle $\theta_{i,j}$ of the polar coordinates on the (x_i, x_j) planes, and it is essentially the maximum

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over all possible angles of the moduli of smoothness of one variable in $\theta_{i,j}$. There are $\binom{d}{2}$ such angles, which are clearly redundant as a coordinate system. Nevertheless, our new definition effectively reduces a large part of problems in approximation theory on $\mathbb{S}^d$ to problems on $\mathbb{S}^1$, which allows us to tap into the rich resources of trigonometric approximation theory for ideas and tools and adopt them for problems on the sphere. These same angles also become indispensable for our new definition of moduli of smoothness on the unit ball $\mathbb{B}^d$. In fact, our moduli on $\mathbb{B}^d$ are defined as the maximum of moduli of smoothness of one variable in these angles and of one additional term that takes care of the boundary behavior. We had two ways to define the additional term, the first one is deduced from the results on the sphere and the second one is the direct extension of the Ditzian–Totik modulus of smoothness on $[-1, 1]$, both of which capture the boundary behavior of the unit ball and permit both direct and inverse theorem for the best approximation. For $d = 1$, approximation by polynomials on $\mathbb{B}^1 = [-1, 1]$ is often deduced from approximation by trigonometric polynomials on the circle $\mathbb{S}^1$ by projecting even functions and their approximations on $\mathbb{S}^1$ to $[-1, 1]$. This procedure can be adopted to higher dimension by projection functions on $\mathbb{S}^d$ onto $\mathbb{B}^d$, and this is how our first modulus of smoothness on $\mathbb{B}^d$ was defined. It should be mentioned that our new moduli of smoothness on the sphere and on the ball are computable; in fact, the computation is not much harder than what is needed for computing classical modulus of smoothness of one variable. A number of examples were given in [3].

In the present paper we continue the work in this direction, study best approximation of functions and their derivatives. On the sphere, our result will be given in terms of differential operators $D_{i,j} = x_i\partial_j - x_j\partial_i$, which can be identified as partial derivatives with respect to $\theta_{i,j}$. We shall define Sobolev spaces and Lipschitz spaces in terms of $D_{i,j}$ on these two domains and study approximation by polynomials in these spaces, including simultaneous approximation of functions and their derivatives. The study is motivated by a question from Kendall Atkinson (cf. [1]) about the numerical solution of a Poisson equation.

The paper is organized as follows. The main results in [3] on the unit sphere will be recalled in Section 2 and the new results on the sphere will be developed in Section 3. The results in [3] on the unit ball will be recollected and further clarified in Section 4. Finally, the new results on the ball are developed in Section 5.

Throughout this paper we denote by $c, c_1, c_2, \ldots$ generic constants that may depend on fixed parameters and their values may vary from line to line. We write $A \lesssim B$ if $A \le cB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Polynomial approximation on the sphere: Recent progress

In this section we recall recent progress on polynomial approximation on the sphere as developed in [3].

Let $L^p(\mathbb{S}^{d-1})$ be the L^p-space with respect to the usual Lebesgue measure $d\sigma$ on $\mathbb{S}^{d-1}$ with norm denoted by $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{S}^{d-1})}$ for $1 \le p \le \infty$, where for $p = \infty$, we replace L^∞ by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the uniform norm.

2.1. Polynomial spaces and spherical harmonics

We denote by Π_n^d the space of polynomials of total degree n in d variables, and by $\Pi_n(\mathbb{S}^{d-1}) := \Pi_n^d|_{\mathbb{S}^{d-1}}$ the space of all polynomials in Π_n^d restricted on $\mathbb{S}^{d-1}$. In the following we shall write Π_n^d for $\Pi_n^d(\mathbb{S}^{d-1})$ whenever it causes no confusion. The quantity of best approximation is then defined by

$$E_n(f)_p := \inf_{g\in \Pi_{n-1}^d} \|f-g\|_p, \quad 1 \le p \le \infty. \tag{2.1}$$

Let $\mathcal{H}_n^d$ denote the space of spherical harmonics of degree n on $\mathbb{S}^{d-1}$, which are the restrictions of homogeneous harmonic polynomials to $\mathbb{S}^{d-1}$. Let Δ_0 be the Laplace–Beltrami operator on the sphere, defined by

$$\Delta_0 f(x) := \Delta F(x), \quad x \in \mathbb{S}^{d-1}, \qquad F(y) = f\left(\frac{y}{\|y\|}\right), \tag{2.2}$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator and it acts on the variables y. Then the spherical harmonics are the eigenfunctions of Δ_0,

$$\Delta_0 Y = -n(n+d-2)Y, \quad Y \in \mathcal{H}_n^d. \tag{2.3}$$

The reproducing kernel of the space $\mathcal{H}_n^d$ in $L^2(\mathbb{S}^{d-1})$ is given by the zonal harmonic

$$Z_{n,d}(x,y) := \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \quad \lambda = \frac{d-2}{2}, \tag{2.4}$$

where $\langle x, y\rangle$ denotes the Euclidean dot product of $x, y \in \mathbb{R}^d$ and C_n^λ is the Gegenbauer polynomial with index λ, normalized by $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$. The orthogonal projection $\operatorname{proj}_n : L^2(\mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d$ is an integral operator given by

$$\operatorname{proj}_n f(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Z_{n,d}(\langle x,y\rangle) d\sigma(y), \tag{2.5}$$

where $\omega_d = \int_{\mathbb{S}^{d-1}} d\sigma$ is the surface area of $\mathbb{S}^{d-1}$. Let η be a C^∞-function on $[0,\infty)$ with the properties that $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ for $x \ge 2$. We define

$$V_n f(x) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \operatorname{proj}_k f(x) = \int_{\mathbb{S}^{d-1}} f(y) K_n(\langle x,y\rangle)\, d\sigma(y), \tag{2.6}$$

for $x \in \mathbb{S}^{d-1}$ and $n = 1, 2, \ldots$, where

$$K_n(t) := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^\lambda(t), \quad t \in [-1,1].$$

Then $V_n f \in \Pi_{2n}^d$, $V_n f = f$ for all $f \in \Pi_n^d$, and for $f \in L^p(\mathbb{S}^{d-1})$ or $C(\mathbb{S}^{d-1})$,

$$\|V_n f - f\|_p \le c E_n(f)_p, \quad 1 \le p \le \infty. \tag{2.7}$$

2.2. *A class of differential operators on* $\mathbb{S}^{d-1}$

One of the main tools in our study is a class of differential operators $D_{i,j}$, $1 \le i \ne j \le d$, that commute with the Laplace–Beltrami operator. Throughout this paper we denote by $e_1, \ldots, e_d$ the following orthonormal basis in $\mathbb{R}^d$,

$$e_1 = (1,0,\ldots,0), e_2 = (0,1,0,\ldots,0), \ldots, e_d = (0,\ldots,0,1). \tag{2.8}$$

(We could, of course, choose another orthonormal basis in $\mathbb{R}^d$ with respect to which we set the coordinates, and in many cases, our discussion below relies on the choice of such an orthonormal basis.) Let $SO(d)$ denote the group of rotations on $\mathbb{R}^d$. For $1 \le i \ne j \le d$ and $t \in \mathbb{R}$, we denote by $Q_{i,j,t}$ the rotation by the angle t in the (x_i, x_j)-plane, oriented such that the rotation from the vector e_i to the vector e_j is assumed to be positive. For example, the action of the rotation $Q_{1,2,t} \in SO(d)$ is given by

$$\begin{aligned} Q_{1,2,t}(x_1, \dots, x_d) &= (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \cdots, x_d) \\ &= (s \cos(\phi + t), s \sin(\phi + t), x_3, \dots, x_d), \end{aligned} \tag{2.9}$$

where $(x_1, x_2) = s(\cos \phi, \sin \phi)$, and other $Q_{i,j,t}$ are defined likewise.

To each $Q \in SO(d)$ corresponds an operator $L(Q)$ in the space $L^2(\mathbb{S}^{d-1})$, defined by $L(Q)f(x) := f(Q^{-1}x)$ for $x \in \mathbb{S}^{d-1}$, which is a group representation of $SO(d)$. The infinitesimal operator of $L(Q_{i,j,t})$ has the form

$$D_{i,j} := \frac{\partial}{\partial t}\left[L(Q_{i,j,t})\right]\Big|_{t=0} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \le i < j \le d. \tag{2.10}$$

In particular, it is easy to verify that, taking $(i, j) = (1, 2)$ as an example,

$$D_{1,2}^r f(x) = \left(-\frac{\partial}{\partial \phi}\right)^r f(s \cos \phi, s \sin \phi, x_3, \dots, x_d). \tag{2.11}$$

The following useful observation, which asserts that $D_{i,j}^r f$ is independent of smooth extensions of f, is a simple consequence of (2.11):

Proposition 2.1. *Let $x_0 \in \mathbb{S}^{d-1}$, and let F and G be two smooth functions on an open neighborhood $U \subset \mathbb{R}^d$ of x_0 which coincide on $U \cap \mathbb{S}^{d-1}$. Then $D_{i,j}^r F(x_0) = D_{i,j}^r G(x_0)$.*

The operators $D_{i,j}$ are connected to the usual tangential partial derivatives according to the following formula [3, (3.15)]:

$$\frac{\partial}{\partial x_j}\left[f\left(\frac{x}{\|x\|}\right)\right]_{\|x\|=1} = \partial_j f - x_j \sum_{i=1}^d x_i \partial_i f = -\sum_{\{i:1\le i\ne j\le d\}} x_i D_{i,j} f. \tag{2.12}$$

The operators $D_{i,j}$ are also closely related to the Laplace–Beltrami operator Δ_0. In fact, Δ_0 satisfies the following decomposition [3, (2.6)],

$$\Delta_0 = \sum_{1\le i<j\le d} D_{i,j}^2. \tag{2.13}$$

Furthermore, each operator $D_{i,j}$ in this decomposition commutes with Δ_0. In particular, by (2.3), this implies that the spaces of spherical harmonics on $\mathbb{S}^{d-1}$ are invariant under $D_{i,j}$.

2.3. *Moduli of smoothness and K-functionals on $\mathbb{S}^{d-1}$*

For $1 \le i < j \le d$ and $\theta \in [-\pi, \pi]$, we define the rth difference operator $\triangle_{i,j,\theta}^r$ by

$$\triangle_{i,j,\theta}^r := (I - T_{Q_{i,j,\theta}})^r = \sum_{k=0}^r (-1)^k \binom{r}{k} T_{Q_{i,j,k\theta}},$$

where $T_Q f(x) := f(Qx)$ for $Q \in SO(d)$. This differential operator can be expressed in terms of the usual forward difference as, for example, for $(i, j) = 1(, 2)$,

$$\triangle_{1,2,\theta}^r f(x) = \overrightarrow{\triangle}_\theta^r f\left(x_1 \cos(\cdot) - x_2 \sin(\cdot), x_1 \sin(\cdot) + x_2 \cos(\cdot), x_3, \ldots, x_d\right), \tag{2.14}$$

where $\overrightarrow{\triangle}_\theta^r$ is acted on the variable $(\cdot)$, and is evaluated at $t = 0$. The following new modulus of smoothness was recently introduced in [3, Definition 2.2]:

Definition 2.2. For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, define

$$\omega_r(f, t)_p := \sup_{|\theta| \le t} \max_{1 \le i < j \le d} \left\| \Delta_{i,j,\theta}^r f \right\|_p. \tag{2.15}$$

It appears that this modulus of smoothness is not rotationally invariant; that is, in general, $\omega_r(f, t)_p \ne \omega_r(T_\rho f, t)_p$ for $\rho \in SO(d)$. A similar comment applies to the K-functional defined in Definition 2.4 below.

This modulus of smoothness enjoys most of the properties of classical moduli of smoothness [3, Proposition 2.7], and it permits both direct and inverse theorems [3, Theorem 3.4]:

Theorem 2.3. *For $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$,*

$$E_n(f)_p \le c\, \omega_r(f, n^{-1})_p, \quad 1 \le p \le \infty. \tag{2.16}$$

On the other hand,

$$\omega_r(f, n^{-1})_p \le c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_{k-1}(f)_p, \quad 1 \le p \le \infty. \tag{2.17}$$

A new K-functional on $\mathbb{S}^{d-1}$ was defined in terms of the differential operators $D_{i,j}$ in [3, Definition 2.4]:

Definition 2.4. For $r \in \mathbb{N}_0$ and $t \ge 0$,

$$K_r(f, t)_p := \inf_{g \in C^r(\mathbb{S}^{d-1})} \left\{ \|f - g\|_p + t^r \max_{1 \le i < j \le d} \|D_{i,j}^r g\|_p \right\}. \tag{2.18}$$

As in the classical setting, our K-functional and moduli of smoothness are equivalent [3, Theorem 3.6].

Theorem 2.5. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < t < 1$,*

$$\omega_r(f, t)_p \sim K_r(f, t)_p, \quad 1 \le p \le \infty.$$

Remark 2.1. In the case when $r = 1, 2$ and $1 < p < \infty$, our modulus of smoothness and K-functional are equivalent to a rotationally invariant modulus [3, Corollary 3.11]. Whether the equivalence holds in general remains open.

Finally, we point out that there are several well studied moduli of smoothness on $\mathbb{S}^{d-1}$ (see, for instance, [4,8]). One of the advantages of our new modulus is that it reduces many problems in approximation on $\mathbb{S}^{d-1}$ to the corresponding problems of trigonometric approximation of one variable, the latter is classical and well studied. Another advantage is that our modulus is relatively easier to compute, as demonstrated in Part 3 of [3].

3. Sobolev spaces and simultaneous approximation on $\mathbb{S}^{d-1}$

The classical Sobolev space W_p^r on $\mathbb{S}^{d-1}$ is defined via the fractional order Laplace–Beltrami operator (see, for example, [2,7,8]):

$$W_p^r := \left\{ f \in L^p(\mathbb{S}^{d-1}) : \ \|f\|_{W_p^r} := \|f\|_p + \|(-\Delta_0)^{r/2} f\|_p < \infty \right\}, \tag{3.1}$$

where $(-\Delta_0)^{r/2}$ denotes the fractional order Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ defined in the distributional sense, which satisfies, in particular,

$$(-\Delta_0)^{r/2} Y = (n(n+d-2))^{r/2} Y, \quad Y \in \mathcal{H}_n^d, \tag{3.2}$$

according to (2.3). We shall introduce a new Sobolev type space on $\mathbb{S}^{d-1}$ in this section and then study approximation by polynomials for functions in this new space. Our new Sobolev space is defined via the differential operators $D_{i,j}$, $1 \le i < j \le d$, which, by (2.13), are more primitive than Δ_0. First, however, we need a lemma.

Lemma 3.1. *For $f, g \in C^1(\mathbb{S}^{d-1})$ and $1 \le i \ne j \le d$,*

$$\int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) d\sigma(x) = -\int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) d\sigma(x). \tag{3.3}$$

Proof. By the rotation invariance of the Lebesgue measure $d\sigma$, we obtain, for any $\theta \in [-\pi, \pi]$,

$$\int_{\mathbb{S}^{d-1}} f(x) g(Q_{i,j,-\theta} x) \, d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(Q_{i,j,\theta} x) g(x) \, d\sigma(x).$$

Differentiating both sides of this identity with respect to θ and evaluating the resulted equation at $\theta = 0$ lead to the desired Eq. (3.3). $\square$

The Eq. (3.3) allows us to define distributional derivatives $D_{i,j}^r$ on $\mathbb{S}^{d-1}$ for $r \in \mathbb{N}$ via the identity,

$$\int_{\mathbb{S}^{d-1}} D_{i,j}^r f(x) g(x) d\sigma(x) = (-1)^r \int_{\mathbb{S}^{d-1}} f(x) D_{i,j}^r g(x) d\sigma(x), \quad g \in C^\infty(\mathbb{S}^{d-1}).$$

We can now define our new Sobolev space on the sphere.

Definition 3.2. For $r \in \mathbb{N}$ and $1 \le p \le \infty$, we define the Sobolev space $\mathcal{W}_p^r \equiv \mathcal{W}_p^r(\mathbb{S}^{d-1})$ to be the space of functions $f \in L^p(\mathbb{S}^{d-1})$ whose distributional derivatives $D_{i,j}^r f$, $1 \le i < j \le d$, all belong to $L^p(\mathbb{S}^{d-1})$, with norm

$$\|f\|_{\mathcal{W}_p^r(\mathbb{S}^{d-1})} := \|f\|_p + \sum_{1 \le i < j \le d} \|D_{i,j}^r f\|_p,$$

where $L^p(\mathbb{S}^{d-1})$ is replaced by $C(\mathbb{S}^{d-1})$ when $p = \infty$.

The following proposition compares the new Sobolev space with the classical one defined in (3.1):

Proposition 3.3. *For $1 < p < \infty$ and $r = 1$ or 2, or $p = 2$ and $r \in \mathbb{N}$, one has*

$$W_p^r = \mathcal{W}_p^r \quad \textit{and} \quad \|f\|_{\mathcal{W}_p^r} \sim \|f\|_{W_p^r}. \tag{3.4}$$

In general, for $r \geq 3$ and $1 < p < \infty$,

$$W_p^r \subset \mathcal{W}_p^r \quad \textit{and} \quad \|f\|_{\mathcal{W}_p^r} \lesssim \|f\|_{W_p^r}. \tag{3.5}$$

Proof. (3.5) is an immediate consequence of (3.13) of [3], whereas (3.4) for the case of $1 < p < \infty$ and $r = 1, 2$ follows directly from (3.17) of [3]. Thus, it remains to show that for all $f \in C^r(\mathbb{S}^{d-1})$ and $r \in \mathbb{N}$,

$$\|(-\triangle_0)^{r/2} f\|_2 \lesssim \sum_{1 \leq i < j \leq d} \|D_{i,j}^r f\|_2. \tag{3.6}$$

Without loss of generality, we may assume that $\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = 0$. For $j \in \mathbb{N}$, define $\theta_j f = \sum_{2^{j-1} \leq k < 2^j} \operatorname{proj}_k f$. Since $\operatorname{proj}_k f \in \mathcal{H}_k^d$, it is evident that $\theta_j f$ is orthogonal to $\theta_l f$ if $j \neq l$. Hence, we have

$$\|(-\triangle_0)^{r/2} f\|_2^2 = \sum_{k=1}^{\infty} \|\theta_k\big((-\triangle_0)^{r/2} f\big)\|_2^2 \sim \sum_{k=1}^{\infty} 2^{2kr} \|\theta_k f\|_2^2,$$

where the equivalence follows from the fact that Δ_0 commutes with proj_k and (3.2), which implies, by [3, Corollary 3.7] with $p = 2$ and the definition in (2.18),

$$\|(-\triangle_0)^{r/2} f\|_2^2 \lesssim \sum_{k=1}^{\infty} 2^{2kr} K_r(\theta_k f, 2^{-k})_2^2 \lesssim \max_{1 \leq i < j \leq d} \sum_{k=1}^{\infty} \|D_{i,j}^r(\theta_k f)\|_2^2$$

$$\sim \max_{1 \leq i < j \leq d} \sum_{k=1}^{\infty} \|\theta_k(D_{i,j}^r f)\|_2^2 \sim \max_{1 \leq i < j \leq d} \|D_{i,j}^r f\|_2^2$$

where we have used the fact that $D_{i,j}^r$ commutes with proj_k (see, for example, (3.9) below). This proves (3.6) and completes the proof. □

Remark 3.1. One interesting question is if our Sobolev spaces $\mathcal{W}_p^r$ are rotationally invariant. For $r = 1, 2$ and $1 < p < \infty$, or $r \geq 3$ and $p = 2$, it indeed is according to Proposition 3.3. This question is pertinent to Remark 2.1.

Theorem 3.4. *If $r \in \mathbb{N}$, $f \in \mathcal{W}_p^r$, and $1 \leq p \leq \infty$, then*

$$E_{2n}(f)_p \leq c n^{-r} \max_{1 \leq i < j \leq d} E_n(D_{i,j}^r f)_p. \tag{3.7}$$

Furthermore, $V_n f$, defined by (2.6), provides the near best simultaneous approximation for all $D_{i,j}^r f$, $1 \leq i < j \leq d$, in the sense that

$$\|D_{i,j}^r(f - V_n f)\|_p \leq c E_n(D_{i,j}^r f)_p, \quad 1 \leq i < j \leq d. \tag{3.8}$$

Proof. Applying (3.3) to the function $g : [-1, 1] \mapsto \mathbb{R}$ and using

$$D_{i,j}^{(x)} g(\langle x, y\rangle) = g'(\langle x, y\rangle)(x_i y_j - x_j y_i) = -D_{i,j}^{(y)} g(\langle x, y\rangle),$$

it follows immediately that

$$D_{i,j}^{(x)} \int_{\mathbb{S}^{d-1}} f(y) g(\langle x, y\rangle) d\sigma(y) = \int_{\mathbb{S}^{d-1}} D_{i,j} f(y) g(\langle x, y\rangle) d\sigma(y). \tag{3.9}$$

Consequently, by (2.4) and (2.6), we see that $V_n D_{i,j}^r = D_{i,j}^r V_n$. Thus, using Theorems 2.3 and 2.5, we obtain

$$\begin{aligned} E_{2n}(f)_p &= E_{2n}(f - V_n f)_p \le c K_r(f - V_n f, n^{-1})_p \\ &\le c n^{-r} \max_{1\le j<j\le d} \|D_{i,j}^r (f - V_n f)\|_p \\ &= c n^{-r} \max_{1\le j<j\le d} \|D_{i,j}^r f - V_n(D_{i,j}^r f)\|_p \\ &\le c n^{-r} \max_{1\le i<j\le d} E_n(D_{i,j}^r f)_p, \end{aligned}$$

where we used (2.18) in the third step, the fact that $V_n D_{i,j}^r = D_{i,j}^r V_n$ in the fourth step, and (2.7) in the last step. This proves (3.7). The inequality (3.8) follows immediately from the above proof. $\square$

Next we define a Lipschitz space on the sphere and consider approximation in such a space.

Definition 3.5. For $r \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in [0, 1)$, we define the Lipschitz space $\mathcal{W}_p^{r,\alpha} \equiv \mathcal{W}_p^{r,\alpha}(\mathbb{S}^{d-1})$ to be the space of all functions $f \in \mathcal{W}_p^r$ with

$$\|f\|_{\mathcal{W}_p^{r,\alpha}(\mathbb{S}^{d-1})} := \|f\|_p + \max_{1\le i<j\le d} \sup_{0<|\theta|\le 1} \frac{\|\Delta_{i,j,\theta}^{\ell}(D_{i,j}^r f)\|_p}{|\theta|^\alpha} < \infty,$$

where ℓ is a fixed positive integer, for example, $\ell = 1$.

Clearly, $\mathcal{W}_p^r(\mathbb{S}^{d-1}) = \mathcal{W}_p^{r,0}(\mathbb{S}^{d-1})$. Our next theorem gives an equivalent characterization of the space $\mathcal{W}_p^{r,\alpha}$ for $\alpha \in (0, 1)$. For the same set of parameters as in the definition of $\mathcal{W}_p^{r,\alpha}$, we define the space

$$H_p^{r+\alpha} := \left\{ f \in L^p(\mathbb{S}^{d-1}) : \|f\|_{H_p^{r+\alpha}} := \|f\|_p + \sup_{0<t\le 1} \frac{\omega_{r+1}(f,t)_p}{t^{r+\alpha}} < \infty \right\}.$$

Theorem 3.6. *If $r \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in (0, 1)$, then $\mathcal{W}_p^{r,\alpha} = H_p^{r+\alpha}$, and moreover,*

$$\|f\|_{\mathcal{W}_p^{r,\alpha}} \sim \|f\|_{H_p^{\alpha+r}} \sim \|f\|_p + \sup_{n\ge 1} n^{r+\alpha} E_n(f)_p.$$

Proof. To prove that $f \in \mathcal{W}_p^{r,\alpha}$ implies $f \in H_p^{r+\alpha}$ and $\|f\|_{H_p^{r+\alpha}} \le c\|f\|_{\mathcal{W}_p^{r,\alpha}}$, it suffices to show that for $f \in \mathcal{W}_p^r$ and $\ell \in \mathbb{N}$,

$$\|\triangle_{i,j,\theta}^{r+\ell} f\|_p \le c|\theta|^r \|\triangle_{i,j,\theta}^{\ell}(D_{i,j}^r f)\|_p. \tag{3.10}$$

Using Lemma 2.6(ii) in [3], we have

$$\| \triangle^{r+\ell}_{i,j,\theta} f\|_p = \left\| \triangle^r_{i,j,\theta}(\triangle^\ell_{i,j,\theta} f)\right\|_p \le c|\theta|^r \left\| D^r_{i,j}(\triangle^\ell_{i,j,\theta} f)\right\|_p.$$

However, from (2.14) and (2.11), a quick computation shows that $\triangle_{i,j,\theta} D_{i,j} = D_{i,j} \triangle_{i,j,\theta}$, hence, by iteration, $\triangle^\ell_{i,j,\theta} D^r_{i,j} = D^r_{i,j} \triangle^\ell_{i,j,\theta}$. As a result,

$$\|D^r_{i,j} \triangle^\ell_{i,j,\theta} f\|_p = \| \triangle^\ell_{i,j,\theta} D^r_{i,j} f\|_p.$$

Together, these two displayed equations yield (3.10).

Conversely, assume $f \in H^{r+\alpha}_p$. We first show that $D^r_{i,j} f \in L^p(\mathbb{S}^{d-1})$. For $g \in C^\infty(\mathbb{S}^{d-1})$ and $G_x(t) := g(Q_{i,j,t}x)$, we can write [3, (4.8)]

$$\triangle^r_{i,j,\theta}(g)(x) = \int_0^\theta \cdots \int_0^\theta G^{(r)}_x(t_1 + \cdots + t_r)\, dt_1 \cdots dt_r,$$

which implies, in particular, that

$$\lim_{\theta\to 0} \frac{\triangle^r_{i,j,\theta}(g)(x)}{\theta^r} = G^{(r)}_x(0) = D^r_{i,j} g(x). \tag{3.11}$$

Thus, by the definition of the distributional derivative $D^r_{i,j} f$, it follows that, for $g \in C^\infty(\mathbb{S}^{d-1})$,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} [D^r_{i,j} f(x)] g(x)\, d\sigma &= (-1)^r \int_{\mathbb{S}^{d-1}} f(x) D^r_{i,j} g(x)\, d\sigma \\ &= (-1)^r \lim_{\theta\to 0} \int_{\mathbb{S}^{d-1}} f(x) \frac{\triangle^r_{i,j,\theta}\, g(x)}{\theta^r}\, d\sigma \\ &= (-1)^r \lim_{\theta\to 0} \int_{\mathbb{S}^{d-1}} \frac{\triangle^r_{i,j,-\theta}\, f(x)}{\theta^r} g(x)\, d\sigma,\end{aligned}$$

where the last step uses the rotation invariance of the Lebesgue measure $d\sigma$. However, by the Marchaud inequality (Proposition 2.7 of [3]), for $f \in H^{r+\alpha}_p$,

$$\omega_r(f,t)_p \le c_\ell t^r \int_t^1 \frac{\omega_{r+\ell}(f,u)_p}{u^{r+1}} du \le c t^r \|f\|_{H^{r+\alpha}_p}.$$

Hence, by Hölder's inequality, we deduce with $\frac{1}{p} + \frac{1}{p'} = 1$ that

$$\left| \int_{\mathbb{S}^{d-1}} [D^r_{i,j} f(x)] g(x)\, d\sigma(x)\right| \le c\|f\|_{H^{r+\alpha}_p} \|g\|_{p'},$$

which implies, upon taking supreme over all g with $\|g\|_{p'} \le 1$ that $D^r_{i,j} f \in L^p(\mathbb{S}^{d-1})$. Next we note that for $\ell \in \mathbb{N}$,

$$\| \triangle^\ell_{i,j,\theta}(D^r_{i,j} f)\|_p \le c \int_0^\theta \| \triangle^{r+\ell}_{i,j,u} f\|_p \frac{du}{u^{r+1}}, \tag{3.12}$$

which follows from the analogue result for trigonometric functions [5, (7.1)] as in the proof of Lemma 2.6 of [3]. Consequently, it follows that

$$\|f\|_{\mathcal{W}^{r,\alpha}_p} \le \|f\|_p + c \max_{1\le i<j\le d} \sup_{0\le|\theta|\le 1} \frac{1}{|\theta|^\alpha} \int_0^\theta \| \triangle^{r+\ell}_{i,j,u} f\|_p \frac{du}{u^{r+1}} \le c\|f\|_{H^{r+\alpha}_p}$$

since $0 < \alpha < 1$.

Finally, to complete the proof, we observe that the equivalence

$$\|f\|_{H_p^{\alpha+r}} \sim \|f\|_p + \sup_{n\ge 1} n^{r+\alpha} E_n(f)_p \tag{3.13}$$

follows directly from (2.16) and (2.17). □

Theorem 3.6 implies the following:

Corollary 3.7. *If $r \in \mathbb{N}$, $\alpha \in [0, 1)$, $f \in \mathcal{W}_p^{r,\alpha}$, and $1 \le p \le \infty$, then*

$$E_n(f)_p \le cn^{-r-\alpha} \|f\|_{\mathcal{W}_p^{r,\alpha}}.$$

Remark 3.2. Since the best approximation $E_n(f)_p$ is rotationally invariant, by (3.13), the spaces $H_p^{\alpha+r}$ are rotationally invariant, whereas by Theorem 3.6, our Lipschitz spaces are also rotationally invariant when $1 \le p \le \infty$, $r \in \mathbb{N}$ and $\alpha \in (0, 1)$. It remains unclear whether or not the space $W_p^{r,0}(\mathbb{S}^{d-1})$ is rotationally invariant when $r \ge 3$ or $p = 1, \infty$ and $r = 1, 2$.

4. Polynomial approximation on $\mathbb{B}^d$: Recent Progress

In this section, we recall recent progress on polynomial approximation on $\mathbb{B}^d$ as developed in [3]. For $\mu \ge 0$, let W_μ denote the weight function on $\mathbb{B}^d$ defined by

$$W_\mu(x) := (1 - \|x\|^2)^{\mu-1/2}. \tag{4.1}$$

For $1 \le p < \infty$ we denote by $\|f\|_{p,\mu}$ the norm for the weighted L^p space $L^p(\mathbb{B}^d, W_\mu)$,

$$\|f\|_{p,\mu} := \left(\int_{\mathbb{B}^d} |f(x)|^p W_\mu(x)dx\right)^{1/p}, \tag{4.2}$$

and $\|f\|_{\infty,\mu} := \|f\|_\infty$ for $f \in C(\mathbb{B}^d)$. When we need to emphasis that the norm is taken over $\mathbb{B}^d$, we write $\|f\|_{p,\mu} = \|f\|_{L^p(\mathbb{B}^d,W_\mu)}$. For $f \in L^p(\mathbb{B}^d, W_\mu)$, $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$, $p = \infty$, the best approximation by polynomials is defined by

$$E_n(f)_{p,\mu} := \inf_{p\in\Pi_n^d} \|f - p\|_{p,\mu}.$$

4.1. Weighted orthogonal polynomial expansions on $\mathbb{B}^d$

Let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of degree n with respect to the weight function W_μ on $\mathbb{B}^d$. We denote by $P_n^\mu(x, y)$ the reproducing kernel of $\mathcal{V}_n^d(W_\mu)$ in $L^2(\mathbb{B}^d, W_\mu)$. It is shown in [9, Theorem 2.6] that

$$P_n^\mu(x, y) = \int_{\mathbb{S}^{m-1}} Z_{n,d+m}\left(\langle x, y\rangle + \sqrt{1 - \|y\|^2}\,\langle x', \xi\rangle\right) d\sigma(\xi) \tag{4.3}$$

for any $x, y \in \mathbb{B}^d$ and $(x, x') \in \mathbb{S}^{d+m-1}$, where $Z_{n,d}(t)$ is the zonal harmonic defined in (2.4) and $\mu = \frac{m-1}{2}$. For η being a C^∞-function on $[0,\infty)$ that satisfies the properties as defined in Section 1.1, we define an operator

$$V_n^\mu f(x) := a_\mu \int_{\mathbb{B}^d} f(y)K_n^\mu(x, y)W_\mu(y)dy, \quad x \in \mathbb{B}^d, \tag{4.4}$$

where a_μ is the normalization constant of W_μ and

$$K_n^\mu(x,y) := \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) P_k^\mu(x,y). \tag{4.5}$$

This operator plays the same role as $V_n f$ in the study on $\mathbb{S}^{d-1}$. In particular, $V_n^\mu f \in \Pi_{2n}^d$ and $\|f - V_n^\mu f\|_{p,\mu} \le c E_n(f)_{p,\mu}$, $1 \le p \le \infty$.

The spaces $\mathcal{V}_n^d(W_\mu)$ of orthogonal polynomials are also the eigenspaces of the following second order differential operator:

$$\mathcal{D}_\mu := \sum_{i=1}^d (1-x_i^2)\partial_i^2 - 2\sum_{1\le i<j\le d} x_i x_j \partial_i \partial_j - (d+2\mu)\sum_{i=1}^d x_i \partial_i. \tag{4.6}$$

Indeed, elements of $\mathcal{V}_n^d(W_\mu)$ satisfy (cf. [6, p. 38])

$$\mathcal{D}_\mu P = -n(n+d+2\mu-1)P \quad \text{for all } P \in \mathcal{V}_n^d(W_\mu). \tag{4.7}$$

It was shown in [3, Proposition 7.1] that the differential operator D_μ can be decomposed as a sum of second order differential operators:

$$\mathcal{D}_\mu = \sum_{i=1}^d D_{i,i}^2 + \sum_{1\le i<j\le d} D_{i,j}^2 = \sum_{1\le i\le j\le d} D_{i,j}^2, \tag{4.8}$$

where the operators $D_{i,j}^2$ for $1 \le i < j \le d$ are defined as in (2.10), and

$$D_{i,i}^2 \equiv D_{\mu,i,i}^2 := [W_\mu(x)]^{-1}\partial_i \left[(1-\|x\|^2)W_\mu(x)\right]\partial_i, \quad 1 \le i \le d. \tag{4.9}$$

For the rest of the paper, we will always set $\varphi(x) = \sqrt{1-\|x\|^2}$. We have the following useful estimates:

Proposition 4.1. *If* $1 < p < \infty$, *and* $g \in C^2(\mathbb{B}^d)$ *then*

$$\|\mathcal{D}_\mu g\|_{p,\mu} \sim \sum_{1\le i\le j\le d} \|D_{i,j}^2 g\|_{p,\mu}. \tag{4.10}$$

Furthermore, if $r \in \mathbb{N}$, $1 < p < \infty$, *and* $g \in C^{2r}(\mathbb{B}^d)$ *then*

$$c_1\|\varphi^{2r}\partial_i^{2r} g\|_{p,\mu} \le \|D_{i,i}^{2r} g\|_{p,\mu} \le c_2\|\varphi^{2r}\partial_i^{2r} g\|_{p,\mu} + c_2\|g\|_{p,\mu}, \quad 1 \le i \le d. \tag{4.11}$$

Proof. (4.10) was proved in [3, Theorem 7.3]. In the case when $r = 1$, (4.11) was shown in [3, Theorem 7.4], and the proof there works equally well for $r \ge 1$. □

4.2. Moduli of smoothness and K-functionals

Two moduli of smoothness and their equivalent K-functionals on $\mathbb{B}^d$ were introduced in [3]. Our first modulus on $\mathbb{B}^d$ was defined via an extension $\widetilde{f}$ of a function $f : \mathbb{B}^d \to \mathbb{R}$ to $\mathbb{B}^{d+1}$, defined by

$$\widetilde{f}(x, x_{d+1}) := f(x), \quad (x, x_{d+1}) \in \mathbb{B}^{d+1}, x \in \mathbb{B}^d. \tag{4.12}$$

More precisely, it is defined as follows [3, Definition 5.3].

Definition 4.2. For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{B}^d, W_\mu)$, $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$ for $p = \infty$, define

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta|\le t} \Big\{ \max_{1\le i<j\le d} \|\triangle_{i,j,\theta}^r f\|_{L^p(\mathbb{B}^d,W_\mu)}, \max_{1\le i\le d} \|\triangle_{i,d+1,\theta}^r \widetilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \Big\}, \tag{4.13}$$

where for $m = 1$, $\| \triangle_{i,d+1,\theta}^r \widetilde{f}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}$ is replaced by $\| \triangle_{i,d+1,\theta}^r \widetilde{f}\|_{L^p(\mathbb{S}^d)}$.

This definition doesn't appear to be rotationally invariant. This same comment applies to Definitions 4.3 and 5.7 below.

In the case when $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$, we have established in [3, Theorem 5.5] the direct theorem, that is, the Jackson inequality

$$E_n(f)_{p,\mu} \le c\,\omega_r(f,n^{-1})_{p,\mu}, \quad 1 \le p \le \infty, \tag{4.14}$$

and the corresponding inverse theorem,

$$\omega_r(f,n^{-1})_{p,\mu} \le c\,n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu}, \tag{4.15}$$

in terms of this new modulus of smoothness. Moreover, it was also shown in [3, Theorem 5.8] that the modulus of smoothness $\omega_r(f,t)_{p,\mu}$ is equivalent to the following K-functional:

$$K_r(f,t)_{p,\mu} := \inf_{g\in C^r(\mathbb{B}^d)} \Big\{ \|f-g\|_{L^p(\mathbb{B}^d,W_\mu)} + t^r \max_{1\le i<j\le d} \|D_{i,j}^r g\|_{L^p(\mathbb{B}^d,W_\mu)} + t^r \max_{1\le i\le d} \|D_{i,d+1}^r \widetilde{g}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \Big\} \tag{4.16}$$

where if $m = 1$, then $\|D_{i,d+1}^r \widetilde{g}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}$ is replaced by $\|D_{i,d+1}^r \widetilde{g}\|_{L^p(\mathbb{S}^d)}$.

The second modulus of smoothness for a function on the unit ball introduced in [3] can be considered as a higher-dimensional analogue of the classical Ditzian–Totik modulus on the interval $[-1, 1]$. In the unweighted case, this modulus is defined as follows [3, Definition 6.7]:

Definition 4.3. For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{B}^d, W_\mu)$, $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$ for $p = \infty$, define

$$\widehat{\omega}_r(f,t)_p := \sup_{0<|h|\le t} \Big\{ \max_{1\le i<j\le d} \|\triangle_{i,j,h}^r f\|_p, \max_{1\le i\le d} \|\widehat{\triangle}_{h\varphi e_i}^r f\|_p \Big\}, \tag{4.17}$$

where $\widehat{\Delta}_h$ denotes the central difference and

$$\widehat{\Delta}_{h\varphi e_i}^r f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x) e_i\right),$$

and we assume $\widehat{\Delta}_{h\varphi e_i}^r = 0$ if either of the points $x \pm r\frac{h\varphi(x)}{2} e_i$ does not belong to $\mathbb{B}^d$.

The weighted version $\widehat{\omega}_r(f,t)_{p,\mu}$ of the above modulus can also be defined, but is more complicated. Both direct and inverse theorems were established in terms of $\widehat{\omega}_r(f,t)_p$

in [3, Theorem 6.13]. Furthermore, it was shown in [3, Theorem 6.10] that the modulus of smoothness $\widehat{\omega}_r(f,t)_p$ is equivalent to the following K-functional

$$\widehat{K}_r(f,t)_{p,\mu} := \inf_{g\in C^r(\mathbb{B}^d)}\Big\{\|f-g\|_{p,\mu} + t^r \max_{1\le i<j\le d}\|D_{i,j}^r g\|_{p,\mu} + t^r \max_{1\le i\le d}\|\varphi^r \partial_i^r g\|_{p,\mu}\Big\}$$

in the sense that, for the equivalence between $\widehat{K}_r(f,t)_p$ and $\omega_r(f,t)_p$ in the unweighted case,

$$c^{-1}\widehat{\omega}^r(f,t)_p \le \widehat{K}_r(f,t)_p \le c\,\widehat{\omega}^r(f,t)_p + c\,t^r\|f\|_p.$$

The two K-functionals, hence their equivalent moduli of smoothness, are connected as shown in [3, Theorem 6.2].

Theorem 4.4. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. Let $f \in L^p(\mathbb{B}^d, W_\mu)$ if $1 \le p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. We further assume that r is odd when $p = \infty$. Then*

$$\widehat{K}_1(f,t)_{p,\mu} \sim K_1(f,t)_{p,\mu}, \tag{4.18}$$

and for $r \ge 1$, there is a $t_r > 0$ such that

$$K_r(f,t)_{p,\mu} \le c\,\widehat{K}_r(f,t)_{p,\mu} + c\,t^r\|f\|_{p,\mu}, \quad 0 < t < t_r. \tag{4.19}$$

Remark 4.1. As in Remark 2.1, one could ask if these new moduli of smoothness and K-functions depend on the orthonormal basis of the Euclidean space.

Finally, we point out that it was shown in [3] that both moduli $\widehat{\omega}_r(f,t)_{p,\mu}$ and $\omega_r(f,t)_{p,\mu}$ enjoy most of the properties of classical moduli of smoothness and they are computable as demonstrated in Part 3 of [3]. In comparison, the only other modulus of smoothness [10] on the unit ball that is strong enough to characterize the best approximation is hardly computable.

4.3. Representation of the term $D_{i,d+1}^r \widetilde{f}$

The term $D_{i,d+1}^r \widetilde{g}$ appears in the definition of our first K-functional $K_r(f,t)_{p,\mu}$ in (4.16) on the ball, where $\widetilde{g}(x, x_{d+1}) = g(x)$ as in (4.12). Notice that $\widetilde{f}$ is a function in $x \in \mathbb{R}^d$, but the operator $D_{i,d+1} = x_i\partial_{d+1} - x_{d+1}\partial_i$ involves x_{d+1}, so that $D_{i,d+1}^r\widetilde{f}$ is indeed a function of (x, x_{d+1}) in $\mathbb{B}^{d+1}$. The following lemma gives an explicit formula of this term in terms of f.

Lemma 4.5. *Assume that $(y, y_{d+1}) = s(x, x_{d+1}) \in \mathbb{B}^{d+1}$ with $s = \|(y, y_{d+1})\| > 0$, $x \in \mathbb{B}^d$ and $x_{d+1} = \varphi(x) \ge 0$. If $f \in C^r(\mathbb{B}^d)$, then*

$$(D_{i,d+1}^r\widetilde{f})(y, y_{d+1}) = \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)^r\Big[f(sx)\Big], \quad 1 \le i \le d.$$

Proof. The proof uses induction. For $r = 1$, we have

$$D_{i,d+1}\widetilde{f}(y, y_{d+1}) = (y_i\partial_{d+1} - y_{d+1}\partial_i)f(y) = -y_{d+1}\partial_i f(y).$$

Hence, using the fact that $\frac{\partial}{\partial x_i}[f(sx)] = s(\partial_i f)(sx)$ we have

$$(D_{i,d+1}\widetilde{f})(sx, sx_{d+1}) = -sx_{d+1}(\partial_i f)(sx) = -s\varphi(x)(\partial_i f)(sx) = -\varphi(x)\frac{\partial}{\partial x_i}[f(sx)].$$

Let $F_r(x, x_{d+1}) = D^r_{i,d+1}\widetilde{f}(x, x_{d+1})$. Assume that the result has been established for r. Then $F_r(sx, s\varphi(x)) = (-\varphi\partial_i)^r[f(sx)]$. By definition,

$$\begin{aligned} F_{r+1}(sx, sx_{d+1}) &= (D_{i,d+1}F_r)(sx, sx_{d+1}) \\ &= sx_i(\partial_{d+1}F_r)(sx, sx_{d+1}) - sx_{d+1}(\partial_i F_r)(sx, sx_{d+1}). \end{aligned} \tag{4.20}$$

On the other hand, taking the derivative by the chain rule shows that

$$\begin{aligned} \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)^{r+1}[f(sx)] &= \left(-\varphi(x)\frac{\partial}{\partial x_i}\right)[F_r(sx, s\varphi(x))] \\ &= -s\varphi(x)(\partial_i F_r)(sx, s\varphi(x)) + sx_i(\partial_{d+1}F_r)(sx, s\varphi(x)), \end{aligned}$$

which is the same as the right hand side of (4.20) with $x_{d+1} = \varphi(x)$. $\square$

Lemma 4.6. *The function $D^r_{i,d+1}\widetilde{f}(x, x_{d+1})$ is even in x_{d+1} if r is even, and odd in x_{d+1} if r is odd.*

Proof. For $r = 1$, $D_{i,d+1}\widetilde{f}(x, x_{d+1}) = -x_{d+1}\partial_i f(x)$ is clearly odd in x_{d+1}. And

$$D^2_{i,d+1}\widetilde{f}(x, x_{d+1}) = -x_i\partial_i f(x) + x^2_{d+1}\partial_i^2 f(x)$$

is even in x_{d+1}. The general case follows from induction upon using (4.20). $\square$

Recall that our K-functional $K_r(f,t)_{p,\mu}$ with $\mu = \frac{m-1}{2}$ in (4.16) is defined. when $m = 1$, with $\|D^r_{i,d+1}\widetilde{g}\|_{L^p(\mathbb{S}^d)}$ in place of $\|D^r_{i,d+1}\widetilde{g}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}$. Hence, as a consequence of the above lemmas, we conclude the following:

Proposition 4.7. *For $g \in C^r(\mathbb{B}^d)$ and the Chebyshev weight W_0 on $\mathbb{B}^d$, we have*

$$\|D^r_{i,d+1}\widetilde{g}\|_{L^p(\mathbb{S}^d)} = \|(\varphi\partial_i)^r g\|_{L^p(\mathbb{B}^d,W_0)}. \tag{4.21}$$

Proof. Let $\mathbb{S}^d_+ = \{x \in \mathbb{S}^d : x_{d+1} \ge 0\}$. By Lemma 4.6 we only need to consider $\mathbb{S}^d_+$ when dealing with $D^r_{i,d+1}\widetilde{g}$. By Lemma 4.5 with $s = 1$, we then obtain

$$\begin{aligned} \int_{\mathbb{S}^d} \left|D^r_{i,d+1}\widetilde{g}(x, x_{d+1})\right|^p d\sigma(x, x_{d+1}) &= 2\int_{\mathbb{S}^d_+} \left|(\varphi(x)\partial_i)^r g(x)\right|^p d\sigma(x, x_{d+1}) \\ &= \int_{\mathbb{B}^d} \left|(\varphi(x)\partial_i)^r g(x)\right|^p \frac{dx}{\sqrt{1-\|x\|^2}}, \end{aligned}$$

which is what we want to prove. $\square$

In general, using polar coordinates, Lemmas 4.5 and 4.6, we can deduce

Proposition 4.8. *If $g \in C^r(\mathbb{B}^d)$, $\mu = \frac{m-1}{2}$ and $m > 1$, then for $1 \le p < \infty$,*

$$\begin{aligned} &\|D^r_{i,d+1}\widetilde{g}\|^p_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \\ &\quad = \int_0^1 s^d(1-s^2)^{\mu-1}\int_{\mathbb{B}^d} |(\varphi(x)\partial_i)^r[g(sx)]|^p \frac{dx}{\sqrt{1-\|x\|^2}}ds; \end{aligned} \tag{4.22}$$

whereas for $p = \infty$, we have

$$\max_{y\in\mathbb{B}^{d+1}} |D^r_{i,d+1}\widetilde{g}(y)| = \max_{x\in\mathbb{B}^d, 0\le s\le 1} \left|\left(\varphi(x)\frac{\partial}{\partial x_i}\right)^r [g(sx)]\right|.$$

5. Sobolev spaces and simultaneous approximation on $\mathbb{B}^d$

We start with the definition of a Sobolev space on $\mathbb{B}^d$.

Definition 5.1. For $1 \le p \le \infty$, $f \in C^r(\mathbb{B}^d)$, and $r \in \mathbb{N}$, we define

$$\|f\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)} := \|f\|_{p,\mu} + \sum_{1\le i<j\le d} \|D_{i,j}^r f\|_{p,\mu} + \sum_{i=1}^{d} \|\varphi^r \partial_i^r f\|_{p,\mu}, \tag{5.1}$$

and define $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$ to be the completion of $C^r(\mathbb{B}^d)$ with respect to the norm $\|\cdot\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)}$.

Remark 5.1. Since convergence in the norm $\|\cdot\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)}$ implies convergence in the weighted L^p-norm $\|\cdot\|_{p,\mu}$, we may assume that $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu) \subset L^p(\mathbb{B}^d, W_\mu)$ when $p < \infty$, and $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu) \subset C(\mathbb{B}^d)$ when $p = \infty$. As a consequence, we can also extend the definitions of the operators $D_{i,j}^r f$ and $\varphi^r \partial_i^r f$ to the whole space $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$.

The following proposition follows readily from (6.15) and (6.16) of [3] and Proposition 4.1:

Proposition 5.2. *If $f \in C^r(\mathbb{B}^d)$, $\mu \ge 0$ and $1 \le p \le \infty$, then*

$$\sum_{1\le i\le d} \|D_{i,d+1}^r \widetilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-\frac{1}{2}})} \le c\|f\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)}. \tag{5.2}$$

Furthermore, if $f \in C^{2r}(\mathbb{B}^d)$ and $1 < p < \infty$ then

$$\|f\|_{\mathcal{W}_p^{2r}(\mathbb{B}^d, W_\mu)} \sim \sum_{1\le i\le j\le d} \|D_{i,j}^{2r} f\|_{p,\mu}. \tag{5.3}$$

Theorem 5.3. *Let $\mu = \frac{m-1}{2}$ with $m \in \mathbb{N}$. For $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$, $1 \le p \le \infty$,*

$$E_{2n}(f)_{p,\mu} \le cn^{-r}\Big[\max_{1\le i<j\le d} E_n(D_{i,j}^r f)_{p,\mu} + \max_{1\le i\le d} E_n(D_{i,d+1}^r \widetilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-\frac{1}{2}})}\Big]. \tag{5.4}$$

Furthermore, $V_n^\mu f$, defined by (4.4), provides the near best simultaneous approximation for all $D_{i,j}^r f$, $1 \le i < j \le d+1$ in the sense that

$$\|D_{i,j}^r(f - V_n^\mu f)\|_{p,\mu} \le cE_n(D_{i,j}^r f)_{p,\mu} \quad 1 \le i < j \le d$$
$$\|D_{i,d+1}^r(\widetilde{f} - \widetilde{V_n^\mu f})\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-\frac{1}{2}})} \le cE_n(D_{i,d+1}^r \widetilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-\frac{1}{2}})}, \quad 1 \le i \le d.$$

Proof. For f defined on $\mathbb{B}^d$, we define $F(x, x') := f(x)$, $x \in \mathbb{B}^d$, $(x, x') \in \mathbb{S}^{d+m-1}$. By [3, Lemma 5.2], $(V_n F)(x, x') = V_n^\mu f(x)$. Furthermore, by [3, Lemma 5.7], $K_r(f, n^{-1})_{L^p(\mathbb{B}^d, W_\mu)} \sim K_r(F, n^{-1})_{L^p(\mathbb{S}^{d+m-1})}$. Hence, it follows that

$$\begin{aligned}
E_{2n}(f)_{L^p(\mathbb{B}^d)} &\le cK_r(f - V_n^\mu f, n^{-1})_{L^p(\mathbb{B}^d, W_\mu)} \\
&\le cK_r(F - V_n F, n^{-1})_{L^p(\mathbb{S}^{d+m-1})} \\
&\le cn^{-r} \max_{1\le i<j\le d+m} \|D_{i,j}^r(F - V_n F)\|_{L^p(\mathbb{S}^{d+m-1})} \\
&= cn^{-r} \max_{1\le i<j\le d+1} \|D_{i,j}^r F - D_{i,j}^r V_n F\|_{L^p(\mathbb{S}^{d+m-1})},
\end{aligned}$$

where the last step follows from the fact that $V_n(D_{i,d+k}^r F)$ depends on x_j, $1 \le j \le d$, and x_{d+k}, which implies that we only need to consider $1 \le i < j \le d+1$.

Denote by $V_{n,d}^{\mu}$ the operator (4.4) associated with W_μ on $\mathbb{B}^d$ and $\widetilde{f}(x, x_{d+1}) = f(x)$. By Lemma 5.2 of [3],

$$V_n F(x, x') = V_{n,d+1}^{\mu-1/2} \widetilde{f}(x, x_{d+1}) = V_{n,d}^{\mu} f(x). \tag{5.5}$$

Since $D_{i,j}^r V_n = V_n D_{i,j}^r$ on the sphere, it follows that, for $1 \le i, j \le d$,

$$D_{i,j}^r V_{n,d}^{\mu} f(x) = D_{i,j}^r (V_n F)(x, x') = V_n D_{i,j}^r F(x, x') = V_{n,d}^{\mu} D_{i,j}^r f(x).$$

Consequently, it follows from [3, (5.8)] that

$$\begin{aligned} \|D_{i,j}^r F - D_{i,j}^r (V_n F)\|_{L^p(\mathbb{S}^{d+m-1})} &= c\|D_{i,j}^r f - D_{i,j}^r V_{n,d}^{\mu} f(x)\|_{L^p(\mathbb{B}^d, W_\mu)} \\ &= c\|D_{i,j}^r f - V_{n,d}^{\mu}(D_{i,j}^r f)(x)\|_{L^p(\mathbb{B}^d, W_\mu)} \\ &\le c E_n(D_{i,j}^r f)_{L^p(\mathbb{B}^d, W_\mu)}. \end{aligned}$$

Whereas for $D_{i,d+1}^r F$ term, we have for $1 \le i \le d$,

$$\begin{aligned} D_{i,d+1}^r V_{n,d+1}^{\mu-1/2} \widetilde{f}(x, x_{d+1}) &= D_{i,d+1}^r (V_n F)(x, x') \\ &= V_n (D_{i,d+1}^r F)(x, x') = V_{n,d+1}^{\mu-1/2} D_{i,j}^r \widetilde{f}(x, x_{d+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} \|D_{i,d+1}^r F - D_{i,d+1}^r V_n F\|_{L^p(\mathbb{S}^{d+m-1})} &= c\|D_{i,j}^r f - V_{n,d+1}^{\mu-1/2} D_{i,j}^r f(x)\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \\ &\le c E_n(D_{i,j}^r \widetilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}. \end{aligned}$$

This proves (5.4). The conclusion that $V_n^{\mu} f$ is the near best simultaneous approximation follows from the above proof and (5.5). □

Corollary 5.4. *Let $\mu = \frac{m-1}{2}$ with $m \in \mathbb{N}$. If $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$, $1 \le p \le \infty$, then*

$$E_n(f)_{p,\mu} \le c n^{-r} \|f\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)}.$$

Proof. This follows immediately from (5.1), (5.2) and Theorem 5.3. □

In the next corollary, we replace $D_{i,d+1}^r \widetilde{f}$ term in (5.4) by ordinary derivatives of f. First we consider the Chebyshev weight W_0 on $\mathbb{B}^d$ (with $\mu = 0$).

Corollary 5.5. *For $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_0)$, $r \in \mathbb{N}$ and $1 \le p \le \infty$,*

$$E_{2n}(f)_{p,0} \le c n^{-r} \max_{1 \le i < j \le d} E_n(D_{i,j}^r f)_{p,0} + c n^{-r} \max_{1 \le i \le d} E_n\left((\varphi \partial_i)^r f\right)_{p,0}.$$

Proof. By (4.21), for all $f \in C^r(\mathbb{B}^d)$,

$$(\varphi(x)\partial_i)^r f(x) = D_{i,d+1}^r \widetilde{f}(x, x_{d+1}),$$

where $x \in \mathbb{B}^d$ and $x_{d+1} = \varphi(x)$. The desired conclusion then follows. □

For $\mu > 0$, including the case $\mu = 1/2$ (the constant weight function), however, the best that we can do is the following:

Corollary 5.6. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. For $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$, $1 \le p \le \infty$,*

$$E_{2n}(f)_{p,\mu} \le cn^{-r} \max_{1\le i<j\le d} E_n(D_{i,j}^r f)_{p,\mu}$$
$$+ cn^{-r} \max_{1\le i\le d}\Big[\max_{1\le j<\frac{r+1}{2}} E_{n-r}(\partial_i^j f)_{p,\mu} + \max_{\frac{r+1}{2}\le j\le r} E_{n-r}(\partial_i^j f)_{p,\mu+(j-\frac{r}{2})p}\Big].$$

Proof. It was shown in Lemma 6.4 of [3] that

$$D_{i,d+1}^r \widetilde{f}(x, x_{d+1}) = \sum_{j=1}^{r} p_{j,r}(x_i, x_{d+1})\partial_i^j f(x), \quad x \in \mathbb{B}^d, (x, x_{d+1}) \in \mathbb{B}^{d+1},$$

where $p_{j,r}$ is a polynomial of degree $\le j$. Since $E_n(f)$ is subadditive, it follows that

$$E_n(D_{i,d+1}^r \widetilde{f})_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \le \sum_{j=1}^{r} \inf_{g\in\Pi_{n-r}^d} \left\| p_{j,r}(\partial_i^j \widetilde{f} - \widetilde{g})\right\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}. \tag{5.6}$$

However, using (6.11) and (6.12) of [3], we have, for $(x, x_{d+1}) \in \mathbb{B}^{d+1}$,

$$|p_{j,r}(x, x_{d+1})| \le \begin{cases} c, & \text{if } 1 \le j < \dfrac{r+1}{2}, \\ c|x_{d+1}|^{2j-r}, & \text{if } \dfrac{r+1}{2} \le j \le r. \end{cases}$$

Thus, by (5.6), we deduce

$$E_n(D_{i,d+1}^r \widetilde{f})_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})} \le c \max_{1\le j<\frac{r+1}{2}} \inf_{g\in\Pi_{n-r}^d} \|\partial_i^j f - g\|_{p,\mu}$$
$$+ c \max_{\frac{r+1}{2}\le j\le r} \inf_{g\in\Pi_{n-r}^d} \|\partial_i^j f - g\|_{p,\mu+(j-\frac{r}{2})p}.$$

The desired conclusion then follows from Theorem 5.3. $\square$

It remains to be seen if $D_{i,d+1}^r \widetilde{f}$ term in (5.4) can be bounded by a term that involves only $(\varphi\partial)^r f$ in the case of $\mu > 0$.

Similar to the case of $\mathbb{S}^{d-1}$, we can also define a Lipschitz space on the ball.

Definition 5.7. For $r \in \mathbb{N}$, $\alpha \in [0, 1)$, and $1 \le p \le \infty$, we define $\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)$ to be the space of all functions $f : \mathbb{B}^d \to \mathbb{R}$ with finite norm

$$\|f\|_{\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d,W_\mu)} := \|f\|_{p,\mu} + \max_{1\le i<j\le d} \sup_{0<|\theta|\le 1} |\theta|^{-\alpha}\|\triangle_{i,j,\theta}^{\ell}(D_{i,j}^r f)\|_{p,\mu}$$
$$+ \max_{1\le i\le d} \sup_{0<|\theta|\le 1} |\theta|^{-\alpha}\|\triangle_{i,d+1,\theta}^{\ell}(D_{i,d+1}^r \widetilde{f})\|_{L^p(\mathbb{B}^d,W_{\mu-1/2})}$$

with the usual change when $p = \infty$, where ℓ is a fixed positive integer, say $\ell = 1$.

We can also give an equivalent characterization of the space $\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)$ in terms of our modulus of smoothness. For the same set of parameters as in the definition of $\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)$, we define a space

$$H_p^{r+\alpha}(\mathbb{B}^d, W_\mu) := \left\{ f \in L^p(\mathbb{B}^d, W_\mu) : \sup_{0<t\le 1} \frac{\omega_{r+\ell}(f,t)_{p,\mu}}{t^{r+\alpha}} < \infty \right\}.$$

Theorem 5.8. *Let $\mu = \frac{m-1}{2}$. If $r \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in (0,1)$, then*

$$\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu) = H_p^{\alpha+r}(\mathbb{B}^d, W_\mu)$$

and

$$\|f\|_{\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)} \sim \|f\|_{H_p^{\alpha+r}(\mathbb{B}^d, W_\mu)} \sim \|f\|_p + \sup_{n\ge 1} n^{r+\alpha} E_n(f)_{p,\mu}.$$

Proof. This follows from (4.14), (4.15), and Theorem 3.6 since for $F(x,x') := f(x)$, $(x,x') \in \mathbb{S}^{d+m-1}$ and $x \in \mathbb{B}^d$, we have $\omega(F,t)_{L^p(\mathbb{S}^{d+m-1})} \sim \omega_r(f,t)_{p,\mu}$ by Lemma 5.4 of [3]. $\square$

We could also define a Lipschitz space that uses central differences of $\partial_i^r f$ in place of $D_{i,d+1}^r \widetilde{f}$ in $\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)$, so that it is equivalent to an analogue of $H_p^{r+\alpha}(\mathbb{B}^d, W_\mu)$ with $\omega_{r+\ell}(f,t)_{p,\mu}$ in place of $\widehat{\omega}_{r+\ell}(f,t)_{p,\mu}$.

As a consequence of the last theorem and the Jackson estimate, we have:

Corollary 5.9. *Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. If $r \in \mathbb{N}$, $\alpha \in [0,1)$, $f \in \mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)$, and $1 \le p \le \infty$, then*

$$E_n(f)_{p,\mu} \le c n^{-r-\alpha} \|f\|_{\mathcal{W}_p^{r,\alpha}(\mathbb{B}^d, W_\mu)}. \tag{5.7}$$

Let us point out that for $f \in C^r(\mathbb{B}^d)$ the traditional definition of the Lipschitz continuity takes the form, for $0 < \alpha < 1$,

$$|\partial^\beta f(x) - \partial^\beta f(y)| \le c\|x-y\|^\alpha, \quad \beta \in \mathbb{N}^d, |\beta| = r \tag{5.8}$$

for all $x, y \in \mathbb{B}^d$. Let us denote by $\mathrm{Lip}_{r,\alpha}$ the space of all $C^r(\mathbb{B}^d)$ functions that satisfy (5.8). From the definition of $D_{i,j}$ it follows readily that

$$\mathrm{Lip}_{r,\alpha} \subset \mathcal{W}_\infty^{r,\alpha}.$$

Hence, the estimate (5.7) holds for the functions in $\mathrm{Lip}_{r,\alpha}$. On the other hand, our definition of $\mathcal{W}_p^{r,\alpha}$ is more general than $\mathrm{Lip}_{r,\alpha}$ as the following example shows.

Example. Let $f_\alpha(x) = (1 - \|x\|^2 + \|x - x_0\|^2)^\alpha$ on $\mathbb{B}^d$ with a fixed $x_0 \in \mathbb{S}^{d-1}$. Assume $1/2 < \alpha < 1$. Then by [3, Example 10.1], $\omega_r(f_\alpha, t)_\infty \sim t^{2\alpha}$, so that by Theorem 5.8, $f_\alpha \in \mathcal{W}_\infty^{1,2\alpha-1}(\mathbb{B}^d)$. On the other hand, setting $x_0 = (1,0,0,\ldots,0)$ shows that $f_\alpha(x) = (1 - x_1^2 + (1-x_1)^2)^\alpha = 2^\alpha(1-x_1)^\alpha$, whose first partial derivative is unbounded on $\mathbb{B}^d$ so that it is not an $\mathrm{Lip}_{1,\alpha}$ function. We note that $D_{i,j}\tilde{f}_\alpha \in C(\mathbb{B}^{d+1})$ for all $1 \le i < j \le d+1$.

Remark 5.2. According to Theorem 5.8, for $\alpha \in (0,1)$, $r \in \mathbb{N}$, and $1 \le p \le \infty$, our Lipschitz space $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$ is rotationally invariant. Similar to Remark 3.1, it remains to be seen whether our Sobolev spaces $\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$ are rotationally invariant in the general case.

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