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Asymptotic expansions for Laguerre-like orthogonal polynomials

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Abstract

Asymptotic expansion for the Laguerre polynomials and for their associated functions is extended to the case of a weight function which is the product of the Laguerre weight function by a polynomial, nonnegative on the interval $[0,\infty[$.

Keywords: Asymptotic expansion; Orthogonal polynomials

1. Introduction

Let P be a nonnegative polynomial on $[0,\infty[$, let $\lambda_n(Pw_L,x)$ be the orthonormal polynomial with respect to the weight function Pw_L where w_L is the Laguerre weight function and let $q_n(Pw_L,x)$ be its associated function

$$q_n(Pw_L,x) = \int_0^\infty \frac{\lambda_n(Pw_L,t)}{x-t} P(t)w_L(t)\,\mathrm{d}t, \quad x \in \mathbb{C}\backslash[0,+\infty[.$$

The aim of this paper is to establish a uniform asymptotic expansion for $\lambda_n(Pw_L,x)$ and $q_n(Pw_L,x)$ for x in any compact subset K of $\mathbb{C}\backslash[0,\infty[$. The results given in this paper are based on the asymptotic expansion of the Laguerre polynomial given by Szegö in [8] and on the asymptotic expansion of its associated function given by Elliott in [3]. Similar ideas and methods, used for the asymptotic expansion of the error in Gauss–Laguerre quadrature formulae, can be found in [4].

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The question of asymptotics of orthogonal polynomials is very large. The present paper has found inspiration in [9], where the weight function was the Jacobi weight multiplied first by a polynomial, and second by an analytic function. As pointed out by the referees, there exists other kind of studies, namely [1], based on the studies of the recurrence relations (see more references therein) for the weight $\exp(-P(x))$, or tending to more precise results but for the special case of the Jacobi (in fact Gegenbauer) polynomials [2].

The paper is organized as follows: in Section 2, we will recall the asymptotic expansion for the Laguerre orthonormal polynomial, and we will give some notations and general results concerning orthogonal polynomials. In Sections 3 and 4, we will give an asymptotic expansion, respectively, for $\lambda_n(Pw_L,x)$ and for $q_n(Pw_L,x)$.

2. Notations and preliminaries results

In this section, we give first some notations and definitions. Then, we give an asymptotic expansion of the leading coefficient of the orthonormal polynomial $\lambda_n(Pw_L,x)$.

Everywhere in the sequel, the argument of complex numbers is defined in the interval $[0,2\pi[$ and the function $\sqrt{z}$ coincide with the real square root for z real, positive. Everywhere also, K will be the standard notation for a compact subset of $\mathbb{C}\backslash[0,\infty[$.

First, let us present the two following definitions.

Definition 1. We denote by Z the set of holomorphic functions on $\mathbb{C}\backslash[0,\infty[$, and Z_K the set of holomorphic functions on K.

Definition 2. Let X be a Banach space and let $S(X)$ be the class of all sequences $(x_n)_n, x_n \in X$ admitting an asymptotic expansion of the form

$$x_n \sim a_0 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n\sqrt{n}} + \cdots$$

where $a_0, a_1, \ldots \in X$ and a_0 not identically null. This means that for each k, the sequence $(r_{k,1}, r_{k,2}, \ldots)$, defined by the relation

$$x_n = a_0 + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n\sqrt{n}} + \cdots + \frac{a_k}{n^{k/2}} + \frac{r_{k+1,n}}{n^{(k+1)/2}},$$

is bounded.

$S(X,a_0)$ is similarly defined with a preassigned a_0.

Let $\lambda_n(w_L,x)$ be the orthonormal Laguerre polynomial with respect to the weight function $w_L(x) = x^\alpha e^{-x}$, α real, $x \in]0,\infty[$ and let $k_n(w_L)$ be its leading coefficient

$$k_n(w_L) = \frac{1}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha+1)}}.$$

The leading coefficient $k_n(w_L)$ can also be written in the following form:

$$\exists(\Psi_n(w_L))_n \in S(\mathbb{C},1) \quad \text{such that } k_n(w_L) = \frac{n^{-\alpha/2}}{\Gamma(n+1)} \Psi_n(w_L). \tag{1}$$

Adapting the Szegö's theorem given in ([8], p. 199) for the orthogonal Laguerre polynomials to the orthonormal's ones, we get the following result (the same result for Jacobi polynomials can be found in [5]).

Theorem 1. *Let* $x \in \mathbb{C}\backslash[0,\infty[$ *and let* α *be any real. We have for all compact* $K \subset \mathbb{C}\backslash[0,\infty[$ *and for* $g(x) = \frac{1}{2\sqrt{\pi}} e^{x/2}(-x)^{-(\alpha/2)-\frac{1}{4}}$

$$\exists(\Phi_n(w_L,x))_n \in S(Z_K,1) \quad \text{such that } \lambda_n(w_L,x) = (-1)^n e^{2\sqrt{-nx}} n^{-\frac{1}{4}} g(x)\Phi_n(w_L,x).$$

The remainder is uniform with respect to x in any compact set K.

Now, we will give the following theorem which is a modification of Theorem 2 given by Verlinden in [9].

Theorem 2. *Let* $p > 2$ *and let* $(x_n)_n \in S(\mathbb{C})$.

$$\text{Then} \quad \left(\sum_{k=1}^{n} \frac{x_k}{k^{p/2}}\right)_n \in S(\mathbb{C}), \quad \text{and} \quad \left(\prod_{k=1}^{n}\left(1 + \frac{x_k}{k^{p/2}}\right)\right)_n \in S(\mathbb{C}).$$

Proof. Let

$$x_n = a_0 + \sum_{h=1}^{k-1} \frac{a_h}{n^{h/2}} + \frac{r_{k,n}}{n^{k/2}}, \quad M_k = \sup_n |r_{k,n}|.$$

As x_n are bounded, the two parts of the conclusion (sum and product) are equivalent. For the same reason, as $p > 2$, the series $\sum_1^\infty x_j/j^{p/2}$ converges, and so to obtain the conclusion, we will prove that

$$\left(\sum_{j=n}^{\infty} \frac{x_j}{j^{p/2}}\right)_n \in S(\mathbb{C}).$$

This last quantity will be obtained as the following limit (this will be justified in time)

$$\sum_{j=n}^{\infty} \frac{x_j}{j^{p/2}} = \lim_{z\to 1} \sum_{j=n}^{\infty} z^{j-n} \frac{x_j}{j^{p/2}}.$$

Then

$$\sum_{j=n}^{\infty} z^{j-n} \frac{x_j}{j^{p/2}} = \sum_{j=n}^{\infty} \frac{z^{j-n}}{j^{p/2}} a_0 + \cdots + \sum_{j=n}^{\infty} \frac{z^{j-n}}{j^{(p+k-1)/2}} a_{k-1} + \sum_{j=n}^{\infty} \frac{z^{j-n} r_{k,j}}{j^{(p+k)/2}}$$

$$= \Phi(z,p/2,n)\, a_0 + \cdots + \Phi(z,(p+k-1)/2,n) a_{k-1} + \sum_{j=n}^{\infty} \frac{z^{j-n} r_{k,j}}{j^{(p+k)/2}}$$

where, for $|z| < 1, s > 0$, $x > 0$, Φ is defined by

$$\Phi(z,s,x) = \sum_{l=0}^{\infty} \frac{z^l}{(x+l)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-xt}}{1 - z e^{-t}} \, dt. \tag{2}$$

For any z in the disc of convergence, $\Phi(z,s,x)$ has an asymptotic expansion (using Watson lemma [7] $c_0(s,z)n^{-s}+\cdots$, and so $\sum_{j=n}^{\infty} z^{j-n}x_j/j^{p/2}$ has an asymptotic expansion.

In Eq. (2), the power series on the right-hand side converges for $z=1$ if $s>1$, so this series converges uniformly with respect to z in an angle with vertex 1, i.e. contained in $(|z|<1\cup\{1\})$ the disc of convergence plus the point 1 on the border of the disc. As all the involved powers $p/2,\cdots(p+k)/2$ are strictly greater than 1, this allows to take $z=1$ in the preceding formulae, and finally

$$p>2\Rightarrow\left(\sum_{j=n}^{\infty}\frac{x_j}{j^{p/2}}\right)_n\in S(\mathbb{C}),$$

which ends the proof. □

Theorem 3. *Let w be an admissible weight function on $[0,\infty[$, and let r be a real number. Let $\lambda_n'(w,x)$ be the monic orthogonal polynomial associated to the weight function w on $[0,\infty[$, $h_n(x)=(-1)^n n^{-\frac{1}{4}+r/2}\Gamma(n+1)\mathrm{e}^{2\sqrt{-nx}}$ and $\Omega_n(w,x)=\lambda_n'(w,x)/h_n(x)$.*

If $(\Omega_n(w,x))_n\in S(Z_K)$ for some compact $K\subset\mathbb{C}\backslash[0,\infty[$, then,

$$(\Gamma(n+1)n^{r/2}k_n(w))_n\in S(\mathbb{C}).$$

Proof. As the weight function w is fixed throughout the proof, it will be omitted. Let a_n' and b_n' be such that

$$x\lambda_n'(x)=\lambda_{n+1}'(x)+a_n'\lambda_n'(x)+b_n'\lambda_{n-1}'(x). \tag{3}$$

Substituting $\lambda_n(x)=k_n\lambda_n'(x)$ in the recurrence formula for the orthonormal polynomials

$$x\lambda_n(x)=c_{n+1}\lambda_{n+1}(x)+d_n\lambda_n(x)+c_n\lambda_{n-1}(x),$$

we obtain the recurrence formula for the $\lambda_n'(x)$

$$x\lambda_n'(x)=c_{n+1}\frac{k_{n+1}}{k_n}\lambda_{n+1}'(x)+d_n\lambda_n'(x)+c_n\frac{k_{n-1}}{k_n}\,\lambda_{n-1}'(x). \tag{4}$$

It follows that

$$c_{n+1}\frac{k_{n+1}}{k_n}=1,\qquad c_n\frac{k_{n-1}}{k_n}=b_n'.$$

Hence

$$k_n^2=k_0^2\prod_{j=1}^{n}\frac{1}{b_j'}. \tag{5}$$

Let us now divide Eq. (3) by $h_n=(-1)^n n^{-\frac{1}{4}+r/2}\Gamma(n+1)\mathrm{e}^{2\sqrt{-nx}}$, and let $\tilde{\Omega}_n=\mathrm{e}^{2\sqrt{-nx}}\Omega_n$. Thus we get

$$x\tilde{\Omega}_n(x)=s_n\tilde{\Omega}_{n+1}(x)+a_n'\tilde{\Omega}_n(x)+\frac{b_n'}{s_{n-1}}\,\tilde{\Omega}_{n-1}(x) \tag{6}$$

with $s_n = -(n+1)(1+1/n)^{-\frac{1}{4}+r/2}$. This relation is true for all x, so can be derived. The two relations are then considered as a linear system with respect to the constants a'_n, b'_n. In this system the value of x is of no importance as a'_n, b'_n are known to be constant. For sake of simplicity, we will write the computations for $x = -1$.

Thus, we get from (6) the following linear system with respect to a'_n, b'_n, where $\alpha_n = \tilde{\Omega}_n(-1)$ and $\beta_n = \tilde{\Omega}'_n(-1)$

$$a'_n\alpha_n + \frac{b'_n}{s_{n-1}}\,\alpha_{n-1} = -s_n\alpha_{n+1} - \alpha_n,$$

$$a'_n\beta_n + \frac{b'_n}{s_{n-1}}\,\beta_{n-1} = -s_n\beta_{n+1} - \beta_n + \alpha_n.$$

So, by elimination of a'_n, we get for b'_n, and for $n \geqslant 1$

$$\frac{b'_n}{s_{n-1}}\left(\frac{\beta_{n-1}}{\beta_n} - \frac{\alpha_{n-1}}{\alpha_n}\right) = \frac{\alpha_n}{\beta_n} - s_n\left(\frac{\beta_{n+1}}{\beta_n} - \frac{\alpha_{n+1}}{\alpha_n}\right).$$

From the definitions of the α's and β's ($\tilde{\Omega}_n(x) = \mathrm{e}^{\sqrt{-nx}}\Omega_n(x)$), we get

$$\frac{\alpha_n}{\beta_n} = \frac{\Omega_n(-1)}{-\sqrt{n}\Omega_n(-1) + \Omega'_n(-1)},$$

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{\mathrm{e}^{2\sqrt{n+1}}}{\mathrm{e}^{2\sqrt{n}}}\,\frac{\Omega_{n+1}(-1)}{\Omega_n(-1)},$$

$$\frac{\beta_{n+1}}{\beta_n} = \frac{\mathrm{e}^{2\sqrt{n+1}}}{\mathrm{e}^{2\sqrt{n}}}\,\frac{-\sqrt{n+1}\Omega_{n+1}(-1) + \Omega'_{n+1}(-1)}{-\sqrt{n}\Omega_n(-1) + \Omega'_n(-1)}.$$

The asymptotic expansion of Ω_n induces asymptotic expansions for α_n/β_n, α_{n+1}/α_n, β_{n+1}/β_n. Taking in account $s_n = -(n+1)(1+1/n)^{-1/4+r/2}$, the classical computations for formal series leads to an expansion of b'_n. The term of largest degree is n^2 (coming from $s_{n-1}s_n$), but a careful computation shows that the second term ($n^{3/2}$) is zero, and the third is rn, so finally we find that the sequence b'_n possesses an asymptotic expansion of the form

$$b'_n = n^2\left[1 + \frac{r}{n} + \sum_{j=3}^{N-1}\frac{l_j}{n^{\frac{j}{2}}} + \mathrm{O}\left(\frac{1}{n^{N/2}}\right)\right]. \tag{7}$$

Let us now consider the sequence $(c_n)_n$ defined by: $c_n = n((n+1)/n)^{r/2}$ and let us write k_n^2 given in (5) in the following form:

$$k_n^2 = \frac{k_0^2}{\prod_{j=1}^n c_j^2}\prod_{j=1}^n\frac{c_j^2}{b'_j} = \frac{k_0^2}{\Gamma(n+1)^2}(n+1)^{-r}\prod_1^n\frac{c_j^2}{b'_j}. \tag{8}$$

Since the sequence $(c_n^2)_n$ possesses an asymptotic expansion ($c_n^2 = n^2(1 + r/n + \cdots)$) we deduce from (7) that the sequence $(c_n^2/b'_n)_n$ possesses an asymptotic expansion

$$\frac{c_n^2}{b'_n} = 1 + \sum_{j=3}^{N-1}\frac{h_j}{n^{j/2}} + \mathrm{O}\left(\frac{1}{n^{N/2}}\right).$$

Using Theorem 2 and formula 8, we obtain that $(\prod_{j=1}^{n} c_j^2/b_j')_n \in S(\mathbb{C})$, and finally that

$$(\Gamma(n+1)n^{r/2}k_n)_n \in S(\mathbb{C}). \qquad \square$$

3. Asymptotic expansion of the orthonormal polynomial

In this section, we will give an uniform asymptotic expansion on compact sets $K \subset \mathbb{C}\backslash[0,\infty[$ of $\lambda_n(w_m,x)$ which is the orthonormal polynomial with respect to $w_m(x) = R_m(x)w_L(x)$ defined for $x \in [0,\infty[$ and where $(R_m)_m$ is a sequence of polynomials of degree m, nonnegative on $[0,\infty[$ and satisfying the following recurrence formula:

$$R_{m+1}(x) = (x - x_{m+1})R_m(x), \quad x_{m+1} \in]-\infty, 0],$$

or

$$R_{m+2}(x) = (x - y_{m+1})(x - \overline{y_{m+1}})R_m(x), \quad y_{m+1} \in \mathbb{C}\backslash]-\infty, 0[.$$

Using P_1 for a polynomial of degree 1 and P_2 for a polynomial of degree 2, let us first remark that the sequence (w_m) satisfies the recurrence formula

$$w_{m+1}(x) = P_1(x_{m+1},x)w_m(x),$$

or

$$w_{m+2}(x) = P_2(y_{m+1},x)w_m(x), \tag{9}$$

with $w_0(x) = w_L(x)$, $P_1(x_{m+1},x) = (x - x_{m+1})$, $P_2(y_{m+1},x) = (x - y_{m+1})(x - \overline{y_{m+1}})$, P_1 and P_2 being positive on the real positive semi-axis.

Our starting point is the Christoffel formula given by Szegö in ([8], p. 29)

$$Q(x)L_n(Qw,x) = \begin{vmatrix} \lambda_n(w,x) & \lambda_{n+1}(w,x) & \dots & \lambda_{n+m}(w,x) \\ \lambda_n(w,z_1) & \lambda_{n+1}(w,z_1) & \dots & \lambda_{n+m}(w,z_1) \\ \vdots & & & \vdots \\ \lambda_n(w,z_m) & \lambda_{n+1}(w,z_m) & \dots & \lambda_{n+m}(w,z_m) \end{vmatrix}, \tag{10}$$

where w is an admissible weight function, $Q(x) = \prod_1^m (x - z_j)$ is a nonnegative polynomial on $[0,\infty[$, $L_n(Qw,x)$ is an orthogonal polynomial with respect to the weight function Qw on $[0,\infty[$ and $\lambda_n(w,x)$ is the orthonormal polynomial with respect to the weight function w on $[0,\infty[$. In case of multiple zero z_k, the corresponding rows of the determinant are replaced by derivatives of consecutive orders of the polynomials $\lambda_n(w,x),\dots,\lambda_{n+m}(w,x)$.

Applying the Christofell's formula with $Q = P_1(x_{m+1},.)$ and $w = w_m$, we get

$$P_1(x_{m+1},x)L_n(w_{m+1},x) = \begin{vmatrix} \lambda_n(w_m,x) & \lambda_{n+1}(w_m,x) \\ \lambda_n(w_m,x_{m+1}) & \lambda_{n+1}(w_m,x_{m+1}) \end{vmatrix} \tag{11}$$

where $L_n(w_{m+1},x)$ is a (not normalized) orthogonal polynomial with respect to w_{m+1} on $[0,\infty[$.

Let $\lambda_n'(w_m,x)$ be the monic orthogonal polynomial with respect to w_m on $[0,\infty[$ and $k_n'(w_{m+1})$ the leading coefficient of $L_n(w_{m+1},x)$.

Identifying both sides of (11), we get

$$k_n'(w_{m+1}) = \begin{vmatrix} 0 & k_{n+1}(w_m) \\ \lambda_n(w_m,x_{m+1}) & \lambda_{n+1}(w_m,x_{m+1}) \end{vmatrix} = -k_{n+1}(w_m)\lambda_n(w_m,x_{m+1}).$$

Hence, for the unitary polynomial $\lambda_n'(w_{m+1},x)$, (11) becomes

$$P_1(x_{m+1},x)\lambda_n'(w_{m+1},x) = -\frac{\begin{vmatrix} \lambda_n(w_m,x) & \lambda_{n+1}(w_m,x) \\ \lambda_n(w_m,x_{m+1}) & \lambda_{n+1}(w_m,x_{m+1}) \end{vmatrix}}{k_{n+1}(w_m)\lambda_n(w_m,x_{m+1})}. \tag{12}$$

For the second case $Q = P_2(y_{m+1},.)$ and $w = w_m$, we have to share the cases where y_{m+1} is complex so the roots y_{m+1} and $\bar{y}_{m+1}$ are distinct, and the case where y_{m+1} is positive real. If y_{m+1} is real, then there is a double root for $Q = P_2(y_{m+1},.)$ and the Szegö formula has to be written with derivatives in the last row. We concentrate on the first case of two complex conjugate roots, the second one being very similar

$$P_2(y_{m+1},x)L_n(w_{m+2},x) = \begin{vmatrix} \lambda_n(w_m,x) & \lambda_{n+1}(w_m,x) & \lambda_{n+2}(w_m,x) \\ \lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1}) & \lambda_{n+2}(w_m,y_{m+1}) \\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}}) & \lambda_{n+2}(w_m,\overline{y_{m+1}}) \end{vmatrix}. \tag{13}$$

The leading coefficient is

$$k_n'(w_{m+2}) = k_{n+2}(w_m)\begin{vmatrix} \lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1}) \\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}}) \end{vmatrix}.$$

Hence, for the unitary polynomial $\lambda_n'(w_{m+2},x)$, (13) becomes

$$P_2(y_{m+1},x)\lambda_n'(w_{m+2},x) = \frac{\begin{vmatrix} \lambda_n(w_m,x) & \lambda_{n+1}(w_m,x) & \lambda_{n+2}(w_m,x) \\ \lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1}) & \lambda_{n+2}(w_m,y_{m+1}) \\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}}) & \lambda_{n+2}(w_m,\overline{y_{m+1}}) \end{vmatrix}}{k_{n+2}(w_m)\begin{vmatrix} \lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1}) \\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}}) \end{vmatrix}}. \tag{14}$$

Theorem 4. *Let $x \in K$, a compact subset of $\mathbb{C}\backslash[0,\infty[$. Let $m \in \boldsymbol{N}$, and let $(w_m)_m$ be the sequence of weight functions defined by* (9). *Then, the orthonormal polynomials with respect to the weight function w_m on $[0,\infty[$, $\lambda_n(w_m,x)$ and their leading coefficient $k_n(w_m)$ satisfy*

$$\lambda_n(w_m,x) = (-1)^n n^{-\frac{1}{4}} \mathrm{e}^{2\sqrt{-nx}} g(x)\Phi_n(w_m,x),$$

$$k_n(w_m) = \frac{n^{-(\alpha+m)/2}}{\Gamma(n+1)}\Psi_n(w_m),$$

with $g(x) = \frac{1}{2\sqrt{\pi}} e^{x/2}(-x)^{-\alpha/2-\frac{1}{4}}$, α *real*, $(\Phi_n(w_m,x))_n \in S(Z_K)$ *for all compact* K *and* $(\Psi_n(w_m))_n \in S(\mathbb{C})$. *The residuals are uniform with respect to* $x \in K$ *for all compact* K.

The uniform convergence follows from the definition of $S(Z_K)$. To avoid some repetitions, we prove the following lemma.

Lemma 1. *Let us suppose that, with the preceding notations* ($(\Phi_n(w_m,x))_n$ *and* $(\Psi_n(w_{m+1}))_n$ *are, respectively, elements of* $S(Z_K)$ *for all compact* $K \subset \mathbb{C}\backslash[0,\infty[$ *and* $S(\mathbb{C})$)

$$\lambda_n(w_m,x) = (-1)^n n^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x)\Phi_n(w_m,x),$$

$$k_n(w_m) = \frac{n^{-r/2}}{\Gamma(n+1)} \Psi_n(w_m).$$

If $w_{m+1}(x) = (x - x_{m+1})w_m(x)$, *then, there exists* $(\Phi_n(w_{m+1},x))_n$ *and* $(\Psi_n(w_{m+1}))_n$ (*respectively, elements of* $S(Z_K)$ *and* $S(\mathbb{C})$), *such that*

$$\lambda_n(w_{m+1},x) = (-1)^n n^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x)\Phi_n(w_{m+1},x),$$

$$k_n(w_{m+1}) = \frac{n^{-(r+1)/2}}{\Gamma(n+1)} \Psi_n(w_{m+1}),$$

and if $w_{m+2}(x) = (x - y_{m+1})(x - \bar{y}_{m+1})w_m(x)$, *then, there exists* $(\Phi_n(w_{m+2},x))_n$ *and* $(\Psi_n(w_{m+2}))_n$ (*respectively, elements of* $S(Z_K)$ *and* $S(\mathbb{C})$), *such that*

$$\lambda_n(w_{m+2},x) = (-1)^n n^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x)\Phi_n(w_{m+2},x),$$

$$k_n(w_{m+2}) = \frac{n^{-(r+2)/2}}{\Gamma(n+1)} \Psi_n(w_{m+2}).$$

Proof. The considered points x, y, z are supposed to be in some compact K. Throughout the proof, we will consider the functions F, G and $h_{n,j}$ defined on $\mathbb{C}\backslash[0,\infty[$ by

$$F(x,y) = \sqrt{-y} - \sqrt{-x},$$

$$G(x,y,z) = \sqrt{zy}F(z,y) + \sqrt{xz}F(x,z) + \sqrt{yx}F(y,x),$$

$$h_{n,j}(x) = \frac{e^{2\sqrt{-(n+j)x}}}{e^{2\sqrt{-nx}}}.$$

Since $\lambda_n(w_m,x) = (-1)^n n^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x)\Phi_n(w_m,x)$, we have from (12),

$$P_1(x_{m+1},x)\lambda_n'(w_{m+1},x) = \frac{(-1)^n}{k_{n+1}(w_m)} \frac{(n+1)^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x)}{\Phi_n(w_m,x_{m+1})} \Omega_n(w_m,x_{m+1};x)$$

where

$$\Omega_n(w_m,x_{m+1};x) = \begin{vmatrix} \Phi_n(w_m,x) & h_{n,1}(x)\Phi_{n+1}(w_m,x) \\ \Phi_n(w_m,x_{m+1}) & h_{n,1}(x_{m+1})\Phi_{n+1}(w_m,x_{m+1}) \end{vmatrix}.$$

The conclusion concerning the leading coefficient $k_n(w_{m+1})$ will follow from Theorem 3, so we have to prove that $(\lambda'_n/h_n)_n \in S(Z_K)$ for some h_n.

Using the recurrence assumption, $k_{n+1}(w_m)$ is written, with $(\Psi_n(w_m))_n \in S(\mathbb{C})$ as

$$k_{n+1}(w_m) = \frac{(n+1)^{-r/2}}{\Gamma(n+2)} \Psi_{n+1}(w_m).$$

From here, the computations are algebraic sums and products of series, so with no difficulties. The interesting steps are the followings.

Since $(\Phi_n(w_m,x))_n \in S(Z_K)$, there exists $(T_n(w_m,x_{m+1};x))_n \in S(Z_K)$ such that

$$\Omega_n(w_m,x_{m+1};x) = \frac{F(x,x_{m+1})}{\sqrt{n}} T_n(w_m,x_{m+1};x).$$

So, ϕ_n being the expansion of $(1+1/n)^{1/2}$, we get for the unitary polynomial $\lambda'_n(w_{m+1},x)$

$$P_1(x_{m+1},x)\lambda'_n(w_{m+1},x) = g(x)h_n(x)H_n(x),$$

$$h_n(x) = (-1)^n n^{-\frac{1}{4}+(r+1)/2}\Gamma(n+1)\mathrm{e}^{2\sqrt{-nx}},$$

$$H_n(x) = \phi_n \frac{F(x,x_{m+1})T_n(w_m,x_{m+1};x)}{\Psi_{n+1}(w_m)\Phi_n(w_m,x_{m+1})},$$

and finally, we can use Theorem 3

$$(H_n(x))_n \in S(Z_K), \quad \text{so} \quad \left(\frac{\lambda'_n(w_{m+1},x)}{h_n(x)}\right)_n \in S(Z_K)$$

and one of the required result is obtained

$$\exists (\Psi_n(w_{m+1}))_n \in S(\mathbb{C}) \text{ such that } k_n(w_{m+1}) = \frac{(n+1)^{(r+1)/2}}{\Gamma(n+1)} \Psi_n(w_{m+1}).$$

Then for the orthonormal polynomial $\lambda_n(w_{m+1},x) = k_n(w_{m+1})\lambda'_n(w_{m+1},x)$, we have

$$P_1(x_{m+1},x)\lambda_n(w_{m+1},x) = (-1)^n n^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)\left(1+\frac{1}{n}\right)^{-(r+1)/2} \Psi_n(w_{m+1})H_n(x).$$

Since the sequence $((1+\frac{1}{n})^{-(r+1)/2}\Psi_n(w_{m+1})H_n(x)/P_1(x_{m+1},x))_n \in S(Z_K)$, there exists $(\Phi_n(w_{m+1},x))_n \in S(Z_K)$, such that

$$\lambda_n(w_{m+1},x) = (-1)^n n^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)\Phi_n(w_{m+1},x).$$

Let us give a sketch of the proof for $w_{m+2}(x) = P_2(y_{m+1},x)w_m(x)$, $(y_{m+1} \in \mathbb{C})$, which goes on similarly

$$P_2(y_{m+1},x)\lambda'_n(w_{m+2},x) = \frac{(-1)^n}{k_{n+2}(w_m)} \frac{(n+1)^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)}{\Omega_n(w_m,\bar{y}_{m+1};y_{m+1})} \Omega'_n(w_m,y_{m+1};x)$$

where $\Omega_n(w_m, y; x)$ is the same as previously, and

$$\Omega'_n(w_m, y_{m+1}; x)$$
$$= \begin{vmatrix} \Phi_n(w_m,x) & h_{n,1}(x)\Phi_{n+1}(w_m,x) & h_{n,2}(x)\Phi_{n+2}(w_m,x) \\ \Phi_n(w_m,y_{m+1}) & h_{n,1}(y_{m+1})\Phi_{n+1}(w_m,y_{m+1}) & h_{n,2}(y_{m+1})\Phi_{n+2}(w_m,y_{m+1}) \\ \Phi_n(w_m,\bar{y}_{m+1}) & h_{n,1}(\bar{y}_{m+1})\Phi_{n+1}(w_m,\bar{y}_{m+1}) & h_{n,2}(\bar{y}_{m+1})\Phi_{n+2}(w_m,\bar{y}_{m+1}) \end{vmatrix}.$$

Since $(\Phi_n(w_m,x))_n \in S(Z_K)$, there exists $(T'_n(w_m, y_{m+1}, x))_n \in S(Z_K)$ such that

$$\Omega'_n(w_m, y_{m+1}; x) = \frac{G(x, y_{m+1}, \bar{y}_{m+1})}{n\sqrt{n}}\, T'_n(w_m, y_{m+1}; x).$$

Using the recurrence assumption to express $k_{n+2}(w_m)$, there exists $(\Psi_n(w_m))_n \in S(\mathbb{C})$ such that

$$k_{n+2}(w_m) = \frac{(n+2)^{-r/2}}{\Gamma(n+3)}\, \Psi_{n+2}(w_m)$$

so, ϕ'_n being the expansion of $(1+1/n)^{5/4}(1+2/n)^{(2+r)/2}$, we get for the unitary polynomial $\lambda'_n(w_{m+2}, x)$

$$P_2(y_{m+1},x)\lambda'_n(w_{m+2},x) = g(x)\tilde{h}_n(x)\tilde{H}_n(x)$$

$$\tilde{h}_n(x) = (-1)^n n^{-\frac{1}{4}+(r+2)/2}\Gamma(n+1)\mathrm{e}^{2\sqrt{-nx}}$$

$$\tilde{H}_n(x) = \phi'_n \frac{G(x, \bar{y}_{m+1}; y_{m+1})T'_n(w_m, y_{m+1}, x)}{\Psi_{n+2}(w_m)F(y_{m+1}, \bar{y}_{m+1})T_n(w_m, y_{m+1}, \bar{y}_{m+1})}.$$

Then, we use again Theorem 3

$$\exists(\Psi_n(w_{m+2}))_n \in S(\mathbb{C}) \quad \text{such that } k_n(w_{m+2}) = \frac{(n+1)^{-(r+2)/2}}{\Gamma(n+1)}\, \Psi_n(w_{m+2}).$$

Then for the orthonormal polynomial $\lambda_n(w_{m+2},x) = k_n(w_{m+2})\lambda'_n(w_{m+2},x)$, we have

$$P_2(y_{m+1},x)\lambda_n(w_{m+2},x) = (-1)^n n^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)\left(1+\frac{1}{n}\right)^{-(r+2)/2}\Psi_n(w_{m+2})\tilde{H}_n(x).$$

Since the sequence $((1+\frac{1}{n})^{-(r+2)/2}\Psi_n(w_{m+2})\tilde{H}_n(x)/P_2(y_{m+1},x))_n \in S(Z_K)$, there exists $(\Phi_n(w_{m+2},x))_n \in S(Z_K)$, such that

$$\lambda_n(w_{m+2},x) = (-1)^n n^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)\Phi_n(w_{m+2},x).$$

The result would be similar for the case of a real positive double root y_{m+1}. This ends the proof of the lemma which is used now for the □

Proof of the theorem. Taking $m=0$ in the lemma, we get the initialisation of the theorem, i.e.

$$\lambda_n(w_L,x) = (-1)^n n^{-\frac{1}{4}}\mathrm{e}^{2\sqrt{-nx}}g(x)\Phi_n(w_L,x),$$

$$k_n(w_L) = \frac{n^{-\alpha/2}}{\Gamma(n+1)}\, \Psi_n(w_L)$$

and so inductively, we get the result for all m

$$\lambda_n(w_m,x) = (-1)^n n^{-\frac{1}{4}} e^{2\sqrt{-nx}} g(x) \Phi_n(w_m,x),$$

$$k_n(w_m) = \frac{n^{-(\alpha+m)/2}}{\Gamma(n+1)} \Psi_n(w_m). \qquad \square$$

4. Asymptotic expansion of the associated function

In this section, we will give the uniform asymptotic expansion on compact subsets of $\mathbb{C}\backslash[0,\infty[$ of the associated function $q_n(w_m,x)$ given by

$$q_n(w_m,x) = \int_0^\infty \frac{\lambda_n(w_m,t)}{x-t} w_m(t)\,dt, \quad x \in \mathbb{C}\backslash[0,\infty[,$$

where the sequence of weight functions $(w_m)_m$ is given by (9).

Let us first give the expression of the uniform asymptotic expansion for the associated function $q_n(w_L,x)$ on compact sets of $\mathbb{C}\backslash[0,\infty[$.

To do this, following the notations given by Elliott in [3] we will set

$$z_n = 4\left(n + \frac{\alpha+1}{2}\right), \quad \xi^2 = -x.$$

We have for $\alpha \geqslant 0$ and $x \in \mathbb{C}\backslash[0,\infty[$,

$$q_n(w_L,x) = (-1)^{n+1} \frac{2^{1+\alpha}}{z_n^{\alpha/2}} \sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}} (-x)^{\alpha/2} e^{-x/2} V_{n,N}(x), \tag{15}$$

with

$$V_{n,N}(x) = K_\alpha\left(\sqrt{-z_n x}\right) \sum_{j=0}^{N-1} \frac{A_j(\xi)}{z_n^j} - K_{\alpha+1}\left(\sqrt{-z_n x}\right) \sum_{j=0}^{N-1} \frac{B_j(\xi)}{z_n^{j+\frac{1}{2}}} + O\left(\frac{|A_N(\xi)|}{z_n^N}\right) \tag{16}$$

where the O-term holds uniformly on any compact sets of $\mathbb{C}\backslash[0,\infty[$ and K_α are the Bessel functions of the second kind. The functions A_j and B_j are defined on $\mathbb{C}\backslash[0,\infty[$ by

$$A_0(\xi) = 1,$$

$$2B_s(\xi) = -A_s'(\xi) + \int_0^\xi \left\{ t^2 A_s(t) - \frac{(2\alpha+1)}{t} A_s'(t) \right\} dt,$$

$$2A_{s+1}(\xi) = \frac{(2\alpha+1)}{\xi} B_s(\xi) - B_s'(\xi) + \int_0^\xi t^2 B_s(t)\,dt + r_{s+1},$$

where the constants r_{s+1} are chosen so that

$$A_{s+1}(0) = 0, \quad \text{for } s = 0,1,2,\ldots\,.$$

Theorem 5. *Let* $x\in\mathbb{C}\backslash[0,\infty[$ *and let* $\alpha\geqslant 0$. *Let, also,* $h(x)=\sqrt{\pi}e^{-x/2}(-x)^{(\alpha/2)-\frac{1}{4}}$. *Then, for all compact* K, *subset of* $\mathbb{C}\backslash[0,\infty[$, *there exists* $(\Sigma_n(w_L,x))_n\in S(Z_K,1)$, *such that*

$$q_n(w_L,x)=(-1)^{n+1}e^{2\sqrt{-nx}}n^{-\frac{1}{4}}h(x)\Sigma_n(w_L,x).$$

The residuals are uniform with respect to $x\in K$.

Proof. Let K and $\alpha\geqslant 0$. With $V_{n,N}$ given by (16), we have from [3], uniformly on K

$$\forall N\in\mathbb{N},\quad q_n(w_L,x)=(-1)^{n+1}\frac{2^{1+\alpha}}{z_n^{\alpha/2}}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}(-x)^{\alpha/2}e^{-x/2}V_{n,N}(x).$$

Let us set $\Sigma_n(w_L,x)=q_n(w_L,x)/(-1)^{n+1}e^{-2\sqrt{-nx}}n^{-\frac{1}{4}}h(x)$. Then, we get

$$\forall N\in\mathbb{N},\quad \Sigma_n(w_L,x)\frac{2^{1+\alpha}}{\sqrt{\pi}}\,\frac{n^{\frac{1}{4}}}{z_n^{\alpha/2}}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}(-x)^{\frac{1}{4}}\,\frac{V_{n,N}(x)}{e^{-2\sqrt{-nx}}}. \tag{17}$$

From ([6], p. 215) we have since $\frac{\pi}{2}\leqslant\arg\sqrt{-x}\leqslant\frac{3\pi}{2}$,

$$K_\alpha\left(\sqrt{-z_nx}\right)\sim\sqrt{\frac{\pi}{2\sqrt{-z_nx}}}\,e^{-\sqrt{-z_nx}}{}_2F_0\left(\frac{1}{2}+\alpha,\frac{1}{2}-\alpha;-\frac{1}{2\sqrt{-z_nx}}\right),\quad |z_n|\to\infty$$

with ${}_2F_0(a,b;z)$ given by

$${}_2F_0(a,b;z)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!}\,z^j.$$

Then, we have, uniformly on K,

$$\forall N\in\mathbb{N},\quad \frac{2(-x)^{\frac{1}{4}}e^{2\sqrt{-nx}}}{\sqrt{\pi}}K_\alpha\left(\sqrt{-z_nx}\right)=\frac{\sqrt{2}}{z_n^{\frac{1}{4}}}\,\frac{e^{-\sqrt{-z_nx}}}{e^{-2\sqrt{-nx}}}\left[\sum_{j=0}^{N}\frac{a_j(x)}{z_n^{k/2}}+O\left(\frac{1}{z_n^{(N+1)/2}}\right)\right]$$

with $a_j(x)=(-1)^j[(\frac{1}{2}+\alpha)_j(\frac{1}{2}-\alpha)_j/(j!)2^j(-x)^{j/2}]$.

On the other hand, we have, uniformly on K,

$$\forall N\in\mathbb{N},\quad \frac{e^{-\sqrt{-z_nx}}}{e^{-2\sqrt{-nx}}}=1+\sum_{j=1}^{N}\frac{b_j(\sqrt{-x})}{n^{j/2}}+O\left(\frac{1}{n^{(N+1)/2}}\right)$$

where the b_j are polynomials of degree j. Hence, we get uniformly on the compact K,

$$\forall N\in\mathbb{N},\quad \frac{2(-x)^{\frac{1}{4}}e^{-2\sqrt{-nx}}}{\sqrt{\pi}}K_\alpha\left(\sqrt{-z_nx}\right)=\frac{1}{n^{\frac{1}{4}}}\left[\sum_{j=0}^{N}\frac{c_j(x)}{n^{j/2}}+O\left(\frac{1}{n^{(N+1)/2}}\right)\right],$$

with $c_0(x)=a_0(x)=1$.

We also have

$$\forall N\in\mathbb{N},\quad \frac{2^\alpha n^{\frac{1}{4}}}{z_n^{\alpha/2}}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}=n^{\frac{1}{4}}\left[1+\sum_{j=1}^{N}\frac{c_j}{n^{j/2}}+O\left(\frac{1}{n^{(N+1)/2}}\right)\right].$$

Therefore, using the Taylor expansion of the functions $A_j(\xi)$, we get

$$\frac{2^{\alpha+1}}{\sqrt{\pi}}\,\frac{n^{\frac{1}{4}}}{z_n^{\alpha/2}}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\,\frac{K_\alpha\left(\sqrt{-z_n x}\right)}{(-x)^{-\frac{1}{4}}\mathrm{e}^{-2\sqrt{-nx}}}\sum_{j=0}^{N-1}\frac{A_j(\xi)}{z_n^j}$$

$$=1+\sum_{j=1}^{N}\frac{d_j(x)}{n^{k/2}}+\mathrm{O}\left(\frac{1}{n^{(N+1)/2}}\right), \tag{18}$$

where the d_k are holomorphic functions on $\mathbb{C}\backslash[0,\infty[$ and the O-term holds uniformly on K.

By the same way, we prove that

$$\frac{2^{\alpha+1}}{\sqrt{\pi}}\,\frac{n^{\frac{1}{4}}}{z_n^{\alpha/2}}\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}\,\frac{K_{\alpha+1}\left(\sqrt{-z_n x}\right)}{(-x)^{-\frac{1}{4}}\mathrm{e}^{-2\sqrt{-nx}}}\sum_{j=0}^{N-1}\frac{B_j(\xi)}{z_n^{j+1}}=\sum_{j=0}^{N}\frac{f_j(x)}{n^{(j+1)/2}}+\mathrm{O}\left(\frac{1}{n^{(N+1)/2}}\right), \tag{19}$$

where the f_k are holomorphic functions on $\mathbb{C}\backslash[0,\infty[$ and the O-term holds uniformly on K.

Hence, we get the result from (17)–(19). □

Now, to give asymptotic expansion for $q_n(w_m,x)$, we must as in the previous section, discuss two cases.

First case: $w_{m+1}(x)=P_1(x_{m+1},x)w_m(x)$. Then, multiplying (12) by $k_n(w_{m+1})$, we get

$$P_1(x_{m+1},x)\lambda_n(w_{m+1},t)=-\frac{k_n(w_{m+1})}{k_{n+1}(w_m)}\,\frac{\begin{vmatrix}\lambda_n(w_m,t) & \lambda_{n+1}(w_m,t)\\ \lambda_n(w_m,x_{m+1}) & \lambda_{n+1}(w_m,x_{m+1})\end{vmatrix}}{\lambda_n(w_m,x_{m+1})}.$$

Now, multiplying the previous equality by $w_m(t)/(x-t)$ and integrating from zero to infinity, we get

$$q_n(w_{m+1},x)=-\frac{k_n(w_{m+1})}{k_{n+1}(w_m)}\,\frac{\begin{vmatrix}q_n(w_m,x) & q_{n+1}(w_m,x)\\ \lambda_n(w_m,x_{m+1}) & \lambda_{n+1}(w_m,x_{m+1})\end{vmatrix}}{\lambda_n(w_m,x_{m+1})}. \tag{20}$$

Second case: $w_{m+2}(x)=P_2(y_{m+1},x)w_m(x)$. Then, multiplying (14) by $k_n(w_{m+2})$, we get similarly

$$q_n(w_{m+2},x)=\frac{k_n(w_{m+2})}{k_{n+2}(w_m)}\,\frac{\begin{vmatrix}q_n(w_m,x) & q_{n+1}(w_m,x) & q_{n+2}(w_m,x)\\ \lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1}) & \lambda_{n+2}(w_m,y_{m+1})\\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}}) & \lambda_{n+2}(w_m,\overline{y_{m+1}})\end{vmatrix}}{\begin{vmatrix}\lambda_n(w_m,y_{m+1}) & \lambda_{n+1}(w_m,y_{m+1})\\ \lambda_n(w_m,\overline{y_{m+1}}) & \lambda_{n+1}(w_m,\overline{y_{m+1}})\end{vmatrix}}. \tag{21}$$

The analogous result is obtained for y_{m+1} real. We are now ready to give the last result, the asymptotic expansion of the associated function

Theorem 6. *Let $\alpha \geqslant 0$ and let $(w_m)_m$ be the sequence of weight functions defined by* (9).

Then, with $h(x) = \sqrt{\pi}(-x)^{\alpha/2-\frac{1}{4}}\mathrm{e}^{-x/2}$, for all compact K, subset of $\mathbb{C}\backslash[0,\infty[$, there exists $(\Sigma_n(w_m,x))_n \in S(Z_K)$ such that the associated function $q_n(w_m,x)$ satisfies

$$q_n(w_m,x) = (-1)^{n+1} n^{-\frac{1}{4}} \mathrm{e}^{-2\sqrt{-nx}} h(x) \Sigma_n(w_m,x).$$

The residuals are uniform with respect to $x \in K$.

Proof. To avoid repetition, we prove the result only for the case $w_{m+1}(x) = P_1(x_{m+1},x)w_m(x)$. The case $w_{m+2}(x) = P_2(y_{m+1},x)w_m(x)$ can be proved with the same arguments. Throughout the proof, we will use $F(x,y) = \sqrt{-y} - \sqrt{-x}$ and we will consider a compact set K, subset of $\mathbb{C}\backslash[0,\infty[$.

For $m = 0$, the result comes from Theorem 5.

The initial step to go from $w_0 = w_L$ to w_1 is the same as the general one. So let us suppose that $q_n(w_m,x) = (-1)^{n+1} n^{-\frac{1}{4}} \mathrm{e}^{-2\sqrt{-nx}} h(x) \Sigma_n(w_m,x)$, with $(\Sigma_n(w_m,x))_n \in S(Z_K)$.

Since $\lambda_n(w_m,x) = (-1)^n n^{-\frac{1}{4}} \mathrm{e}^{2\sqrt{-nx}} g(x) \Phi_n(w_m,x)$, we get

$$q_n(w_{m+1},x) = (-1)^{n+1} \frac{k_n(w_{m+1})}{k_{n+1}(w_m)} (n+1)^{-\frac{1}{4}} \mathrm{e}^{-\sqrt{-nx}} h(x) \frac{\Omega''_n(w_m,x_{m+1};x)}{\Phi_n(w_m,x_{m+1})},$$

with

$$\Omega''_n(w_m,x_{m+1};x) = -\begin{vmatrix} \Sigma_n(w_m,x) & \dfrac{\mathrm{e}^{-2\sqrt{-(n+1)x}}}{\mathrm{e}^{-2\sqrt{-nx}}}\,\Sigma_{n+1}(w_m,x) \\ \Phi_n(w_m,x_{m+1}) & \dfrac{\mathrm{e}^{2\sqrt{-(n+1)x_{m+1}}}}{\mathrm{e}^{2\sqrt{-nx_{m+1}}}}\,\Phi_{n+1}(w_m,x_{m+1}). \end{vmatrix}$$

On one hand, we have from Theorem 4, $k_n(w_{m+1})/k_{n+1}(w_m) = \sqrt{n}\Theta_n(w_{m+1})$ with $(\Theta_n(w_{m+1}))_n \in S(\mathbb{C})$.

On the other hand,

$$\Omega''_n(w_m,x_{m+1};x) = \frac{F(x,x_{m+1})}{\sqrt{n}} \tilde{\Omega}''_n(w_{m+1},x),$$

where $(\tilde{\Omega}''_n(w_{m+1},x))_n \in S(Z_K)$.

Hence, there exists $(\Sigma_n(w_{m+1},x))_n \in S(Z_K)$, such that

$$q_n(w_{m+1},x) = (-1)^{n+1} n^{-\frac{1}{4}} \mathrm{e}^{-2\sqrt{-nx}} h(x) \Sigma_n(w_{m+1},x). \qquad \square$$

5. Conclusion

In this paper, it was shown that an asymptotic expansion for the orthonormal polynomials with respect to the Laguerre weight function w_L and for its associated function, valid outside the interval $[0,\infty[$, can be extended to the weight functions Pw_L where P is a nonnegative polynomial on $[0,\infty[$. The natural question arising now is: what about asymptotics when the polynomial P is replaced by an holomorphic function. The answer to this question will be given in a future paper.

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