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Asymptotically Diagonal Delay Differential Systems

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1. INTRODUCTION

A recent monograph by Eastham [8] contains a survey of the results on the asymptotic integration of ordinary differential equations. One of the classical results is the following theorem.

THEOREM A (The Hartman–Wintner Theorem). *Consider the system*

$$x' = (D(t)) + R(t))x, \tag{1.1}$$

where D, R are continuous $n \times n$ matrix functions on $[0, \infty)$. Suppose that $D(t)$ is a diagonal matrix,

$$D(t) = \operatorname{diag}\{d_1(t), d_2(t), \dots, d_n(t)\},$$

and suppose there exists a $\delta > 0$ such that

$$\left|d_i(t) - d_j(t)\right| \geq \delta \tag{1.2}$$

on some interval $[t_0, \infty)$ for each $i \neq j$, $i, j \in \{1, 2, \dots, n\}$. Let

$$R \in L^p \qquad \text{for some } p \in (1, 2]. \tag{1.3}$$

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Then Eq. (1.1) *has a fundamental system of the form*

$$X(t) = (I + F(t))\exp\left(\int_{t_0}^{t} (D(s) + \operatorname{diag}\{R(s)\})\, ds\right) \qquad \text{as } t \to \infty, \tag{1.4}$$

where I denotes the identity matrix and F is a matrix function, $F(t) \to 0$ *as* $t \to \infty$.

In an effort to extend the Hartman–Wintner theorem to delay differential systems Haddock and Sacker [11] conjectured an asymptotic formula for the solutions of the equation

$$x'(t) = \left(\tilde{A} + A(t)\right)x(t) + B(t)x(t-\tau), \tag{1.5}$$

where $\tau > 0$, $\tilde{A} = \operatorname{diag}\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$ with $\tilde{a}_i \neq \tilde{a}_j$ for $i \neq j$ and A, B are L^2-matrix functions. They stated that there exists a matrix function F, $F(t) \to 0$ as $t \to \infty$, such that for each solution x of (1.5) there exist a constant vector c and a vector function f, $f(t) \to 0$ as $t \to \infty$, such that

$$x(t) = (I + F(t))\exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right)(c + f(t)), \tag{1.6}$$

where

$$\Lambda(t) = \tilde{A} + \operatorname{diag}\{A(t)\} + \operatorname{diag}\{B(t)\}e^{-\tilde{A}\tau}.$$

Haddock and Sacker [11] proved formula (1.6) in the scalar case (with $F \equiv 0$). In the case of "quasi-triangular" systems formula (1.6) was proved by the first two authors [2]. A complete proof of the conjecture was given by Ai [1]. For further extension allowing $A, B \in L^p$ with $p > 2$, we refer to the work of Cassel and Hou [4].

In the above-mentioned works the delay differential equation is considered as a perturbation of an (autonomous) ordinary differential equation. The situation becomes more complicated when the limiting equation is a delay differential equation (see [3, 5, 12]). A first result of this type concerning the scalar equation

$$x'(t) = (\tilde{a} + a(t))x(t) + \left(\tilde{b} + b(t)\right)x(t-\tau) \tag{1.7}$$

($\tau > 0$, $\tilde{a}, \tilde{b} = \text{const.}$, $a(t), b(t)$ "small") can be found in the classical book by Bellman and Cooke [3]. Assuming that the characteristic equation

$$\lambda = \tilde{a} + \tilde{b}e^{-\lambda\tau} \tag{1.8}$$

of the limiting equation

$$x'(t) = \tilde{a}x(t) + \tilde{b}x(t-\tau)$$

has a dominant real root and under additional technical assumptions, they proved an asymptotic formula for the solutions of (1.7) (formula (1.13) below). Recently, the last two authors established the following improvement of [3, Theorem 9.2].

THEOREM B [10, Theorem 1]. *Assume that in Eq.* (1.7),

$$e^{-\tilde{a}\tau}\tilde{b}\tau > -\frac{1}{e}, \tag{1.9}$$

$$a(t) \to 0, \qquad b(t) \to 0 \text{ as } t \to \infty, \tag{1.10}$$

$$a \in L^2, \qquad b \in L^2, \tag{1.11}$$

and

$$a(t) - \frac{1}{\tau}\int_{t-\tau}^{t} a(s)\,ds \in L^1, \qquad b(t) - \frac{1}{\tau}\int_{t-\tau}^{t} b(s)\,ds \in L^1. \tag{1.12}$$

Then the following statements are valid.

(i) *Equation* (1.8) *has a unique real solution* $\tilde{\lambda}$ *in the interval* $(\tilde{a} - 1/\tau, \infty)$.

(ii) *For every solution x of* (1.7) *the limit*

$$\xi[x] = \lim_{t\to\infty} x(t)\exp\left(-\int_{t_0}^{t}\lambda(s)\,ds\right) \tag{1.13}$$

exists and is finite, where

$$\lambda(t) = \tilde{\lambda} + \left(1 + (\tilde{\lambda} - \tilde{a})\tau\right)^{-1}\left(a(t) + e^{-\tilde{\lambda}\tau}b(t)\right).$$

(iii) *There exists a solution x of* (1.7) *for which* $\xi[x] \neq 0$.

We remark that condition (1.12) is actually independent of τ. A necessary and sufficient condition for a locally integrable function a to satisfy relation (1.12) is that it can be written in the form $a = \alpha + \bar{\alpha}$, where α is a continuously differentiable function with $\alpha' \in L^1$ and $\bar{\alpha} \in L^1$. Thus, (1.12) is satisfied if $a' \in L^1$ (mod L^1). The proof of the above result is given in the Appendix.

The aim of the present paper is to give an extension of Theorem B to a system of the form

$$x'(t) = \big(\tilde{A} + A(t)\big)x(t) + \big(\tilde{B} + B(t)\big)x(t - \tau), \tag{1.14}$$

where $\tilde{A}$, $\tilde{B}$ are constant diagonal matrices and A, B are ‘‘small’’ matrix functions. The main theorem concerning Eq. (1.14) is formulated in Section 3. The method developed in this paper is new compared to those used in the above-mentioned works on asymptotic integration of systems [1, 2, 4, 11]. Our method is based on previous qualitative results showing that if the delay term is ‘‘small’’ then the solutions of the delay differential system are asymptotic to some member of an n-parameter family of special solutions. (Results of this type were initiated by Ryabov [17]. For further extension see [7, 9, 15, 16].) Roughly speaking, this means that there exist n linearly independent solutions which asymptotically characterize the other solutions. These linearly independent solutions form a fundamental system of a related ordinary differential equation. In general, we do not know the special solutions, therefore it is important to find a way to determine the above-mentioned ordinary differential equation in terms of the coefficients and the delays. This result which is of independent interest is formulated in Theorem 2.4 in Section 2. In Section 3, we use the above result to reduce the problem of asymptotic integration for Eq. (1.14) to a similar problem for an ordinary differential equation to which the Hartman–Wintner theorem (Theorem A) can be applied.

2. ASYMPTOTIC PROPERTIES OF LINEAR SYSTEMS WITH SMALL DELAYS

Let R^n denote the n-dimensional space of real vectors with any convenient norm $|\cdot|$. The induced norm of a matrix M (with real entries) is given by

$$|M| = \sup\{|Mx|: x \in R^n, |x| = 1\}.$$

Given $\tau > 0$, denote by C the space of continuous functions from $[-\tau, 0]$ into R^n. With the usual supremum norm,

$$\|\phi\| = \sup_{-\tau \le s \le 0} |\phi(s)|, \qquad \phi \in C,$$

C is a Banach space. Adopting the usual notation, if x is a continuous function on some set which includes $[t - \tau, t]$, the new function $x_t \in C$ is defined by

$$x_t(s) = x(t + s) \qquad \text{for } -\tau \le s \le 0.$$

Consider the nonautonomous delay differential system

$$x'(t) = L(t, x_t), \tag{2.1}$$

where $L: R \times C \to R^n$. Assume L to be continuous, with $L(t, \cdot)$ linear on C, and for $(t, \phi) \in R \times C$,

$$|L(t, \phi)| \le K\|\phi\|; \qquad K = \text{const}.$$

It is known [13] that for any $(t_0, \phi) \in R \times C$, Eq. (2.1) has a unique solution x on $[t_0 - \tau, \infty)$ with initial condition $x_{t_0} = \phi$.

The following notion was introduced by Ryabov [17].

DEFINITION. By a *special solution* of (2.1), we mean a solution $\bar{x}$ of (2.1) defined on the whole real line and such that

$$|\bar{x}(t)|e^{t/\tau} \text{ is bounded for } t < 0.$$

Let us introduce some known results for Eq. (2.1).

THEOREM 2.1 [7, Theorem 1]. *If*

$$K\tau e < 1, \tag{2.2}$$

then for every $(t_0, x_0) \in R \times R^n$, *Eq.* (2.1) *has a unique special solution* $\bar{x}$ *such that* $\bar{x}(t_0) = x_0$.

Fix $t_0 \in R$. Let X denote the matrix function defined by

$$X(t) = \operatorname{col}(\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_n(t)), \qquad t \in R,$$

where $\bar{x}_i$ $(1 \le i \le n)$ denotes the special solution of (2.1) such that $\bar{x}_i(t_0) = e_i$; e_i being the ith unit vector in R^n. Clearly, X satisfies the matrix equation

$$X'(t) = L(t, X_t), \qquad t \in R$$
$$X(t_0) = I,$$

moreover,

$$|X(t)|e^{t/\tau} \text{ is bounded for } t < t_0.$$

The special solution $\bar{x}$ of (2.1) passing through (t_0, c) has the form $\bar{x}(t) = X(t)c$, $t \in R$.

We shall call X the *special matrix solution* of (2.1). In the following theorem we summarize some useful properties of X.

THEOREM 2.2 [7, Theorem 3]. *Suppose condition* (2.2) *holds. Let* $t_0 \in R$ *be fixed. If* X *is the special matrix solution of* (2.1) *defined as above, then*

$$\det X(t) \neq 0 \qquad \text{for all } t \in R \tag{2.3}$$

and

$$|X(s)X^{-1}(t)| \leq e^{\lambda_0(t-s)} \qquad \text{for all } s \leq t, \tag{2.4}$$

where λ_0 *is the unique real solution of the equation*

$$\lambda = Ke^{\lambda\tau}$$

in the interval $(0, 1/\tau)$.

The interest for special solutions will be clear from the next theorem which shows that every solution of (2.1) approaches (exponentially) asymptotically some special solution.

THEOREM 2.3. *Suppose condition* (2.2) *holds. Then for every solution* x *of* (2.1) *on* $[t_0 - \tau, \infty)$ *the limit*

$$\xi = \lim_{t\to\infty} X^{-1}(t)x(t) \in R^n \tag{2.5}$$

exists, and

$$\sup_{t \geq t_0} |x(t) - X(t)\xi| e^{t/\tau} < \infty. \tag{2.6}$$

The constant vector ξ *given by* (2.5) *is the only one satisfying* (2.6).

Theorem 2.3 is a special case of a more general result concerning nonlinear equations due to J. Jarník and J. Kurzweil [15]. For a simple proof of (2.5) see Driver [7, Theorem 4].

Remark. If the functional L is defined in t only on the half-line $[t_0, \infty)$, we can put $L(t, \phi) = 0$ for $t < t_0$. (This might make $L(t, \phi)$ discontinuous at $t = t_0$, but that does not cause any problem.) Then one can easily see that the special matrix solution X of (2.1) is determined by

$$X(t) = I \qquad \text{for } t \in (-\infty, t_0].$$

In [9, 16] it was shown that in this case the asymptotic formulae (2.5) and (2.6) are true even under weaker assumptions than (2.2) (see [9, Theorem 5.3; 16, Theorem 2.2]).

The above results can be interpreted as follows. Under the "smallness" condition (2.2), the special matrix solution X of (2.1) is a fundamental system of solutions of an ordinary differential equation

$$x' = M(t)x. \tag{2.7}$$

That is, Eq. (2.7) contains the information about the asymptotic behavior of the solutions of (2.1). Theorem 2.3 is only of theoretical importance because in most cases we do not know the special solutions. Therefore it is of great interest to be able to construct the mentioned ordinary differential equation without knowledge of the special solutions using only the functional L. The next theorem, the main result of this section, shows how.

THEOREM 2.4. *Suppose condition* (2.2) *holds. Define a sequence of matrix functions* $\{M_k(t,s)\}_{k=0}^{\infty}$ *as follows: Put*

$$M_0(t,s) = L(s,\bar{I})$$

for $s \le t$, *where* $\bar{I}$ *is the matrix function defined by* $\bar{I}(t) = I$ *for* $-\tau \le t \le 0$, *and*

$$M_{k+1}(t,s) = -L\left(s, \int_{s+\cdot}^{t} M_k(t,u)\,du\right)$$

for $s \le t$ *and* $k = 0, 1, \ldots$. *Then the special matrix solution* X *of* (2.1) *is a fundamental system of the ordinary differential equation* (2.7), *where*

$$M(t) = \sum_{k=0}^{\infty} M_k(t,t) \qquad \text{for } t \in R, \tag{2.8}$$

the last series being uniformly convergent on R.

Proof. First we show that the series (2.8) converges uniformly on R. Let $t \in R$ be fixed. Clearly

$$|M_0(t,s)| \le K, \qquad s \le t.$$

Assume for induction that

$$|M_k(t,s)| \le \frac{K^{k+1}}{k!}(t - s + k\tau)^k, \qquad s \le t \tag{2.9}$$

for some k. Then for $s \le t$,

$$\begin{aligned} |M_{k+1}(t,s)| &\le K \sup_{-\tau \le \theta \le 0} \left| \int_{s+\theta}^{t} M_k(t,u)\,du \right| \le K \int_{s-\tau}^{t} |M_k(t,u)|\,du \\ &\le K \int_{s-\tau}^{t} \frac{K^{k+1}}{k!} (t-u+k\tau)^k\,du \\ &= \frac{K^{k+2}}{k!} \left[-\frac{(t-u+k\tau)^{k+1}}{k+1} \right]_{u=s-\tau}^{u=t} \\ &\le \frac{K^{k+2}}{(k+1)!} (t-s+(k+1)\tau)^{k+1}. \end{aligned}$$

Thus, (2.9) is confirmed for all $k = 0, 1, \ldots$.

From (2.9), we get

$$|M_k(t,t)| \le K(K\tau)^k \frac{k^k}{k!}, \qquad t \in R,\ k = 0, 1, \ldots . \tag{2.10}$$

We have

$$\frac{k^k}{k!} \le \sum_{i=0}^{\infty} \frac{k^i}{i!} = e^k \qquad \text{for all } k = 0, 1, \ldots, \tag{2.11}$$

and (2.10) and (2.11) give

$$|M_k(t,t)| \le K(K\tau e)^k \qquad \text{for } t \in R \text{ and } k = 0, 1, \ldots,$$

which by virtue of (2.2) implies the uniform convergence of the series (2.8) on R.

Now let us define a sequence by

$$N_0(t,s) = L(s, X_s) X^{-1}(t)$$

for $s \le t$ and

$$N_{k+1}(t,s) = -L\left(s, \int_{s+\cdot}^{t} N_k(t,u)\,du \right)$$

for $s \le t$ and $k = 0, 1, \ldots$. We shall show by induction that

$$M_k(t,s) + N_{k+1}(t,s) = N_k(t,s), \qquad s \le t \tag{2.12}$$

for $k = 0, 1, \dots$. For $k = 0$, we have

$$\begin{aligned} M_0(t,s) + N_1(t,s) &= L(s, \bar{I}) - L\left(s, \int_{s+\cdot}^{t} L(u, X_u) X^{-1}(t)\, du\right) \\ &= L(s, \bar{I}) - L\left(s, \int_{s+\cdot}^{t} X'(u) X^{-1}(t)\, du\right) \\ &= L(s, \bar{I}) - L\big(s, (X(t) - X_s) X^{-1}(t)\big) \\ &= L(s, X_s) X^{-1}(t) = N_0(t,s). \end{aligned}$$

Assume that (2.12) holds for some k. Then

$$\begin{aligned} &M_{k+1}(t,s) + N_{k+2}(t,s) \\ &\quad = -L\left(s, \int_{s+\cdot}^{t} M_k(t,u)\, du\right) - L\left(s, \int_{s+\cdot}^{t} N_{k+1}(t,u)\, du\right) \\ &\quad = -L\left(s, \int_{s+\cdot}^{t} \big(M_k(t,u) + N_{k+1}(t,u)\big)\, du\right) \\ &\quad = -L\left(s, \int_{s+\cdot}^{t} N_k(t,u)\, du\right) = N_{k+1}(t,s). \end{aligned}$$

Thus, (2.12) holds for all $k = 0, 1, \dots$.

From (2.12), we obtain

$$M_k(t,t) + N_{k+1}(t,t) = N_k(t,t) \qquad \text{for } t \in R \text{ and } k = 0, 1, \dots . \tag{2.13}$$

Now we prove that

$$X'(t) = \left(\sum_{i=0}^{k} M_i(t,t) + N_{k+1}(t,t)\right) X(t) \qquad \text{for } t \in R \tag{2.14}$$

for all $k = 0, 1, \dots$. By virtue of (2.13) it suffices to verify (2.14) for $k = 0$. We have

$$\big(M_0(t,t) + N_1(t,t)\big) X(t) = N_0(t,t) X(t) = L(t, X_t) = X'(t)$$

for all $t \in R$. Consequently, (2.14) holds for all k.

Our next aim is to show that $N_k(t,t) \to 0$ as $k \to \infty$. It is easy to show that if (2.2) is fulfilled and λ_0 is the unique root of $\lambda = Ke^{\lambda\tau}$ in $(0, 1/\tau)$, then for $\lambda \in (\lambda_0, 1/\tau)$ we have

$$K\frac{e^{\lambda\tau}}{\lambda} < 1. \tag{2.15}$$

Let $\lambda \in (\lambda_0, 1/\tau)$. We shall show by induction that

$$|N_k(t,s)| \le \frac{K^{k+1}}{\lambda^k}\exp[\lambda(t-s+(k+1)\tau)], \qquad s \le t \tag{2.16}$$

for $k = 0, 1, \dots$. Using (2.4) of Theorem 2.2, we get

$$|N_0(t,s)| \le K\|X_s X^{-1}(t)\| = K \sup_{-\tau\le\theta\le 0} |X(s+\theta)X^{-1}(t)| \le Ke^{\lambda(t-s+\tau)}.$$

Assume that (2.16) holds for some k. Then

$$\begin{aligned}|N_{k+1}(t,s)| &\le K \sup_{-\tau\le\theta\le 0}\left|\int_{s+\theta}^{t} N_k(t,u)\,du\right| \\ &\le K\int_{s-\tau}^{t}\frac{K^{k+1}}{\lambda^k}\exp[\lambda(t-u+(k+1)\tau)]\,du \\ &\le \frac{K^{k+2}}{\lambda^{k+1}}\exp[\lambda(t-s+(k+2)\tau)].\end{aligned}$$

Thus, (2.16) is confirmed for all $k = 0, 1, \dots$.

From (2.16), we get

$$|N_k(t,t)| \le Ke^{\lambda\tau}\left(K\frac{e^{\lambda\tau}}{\lambda}\right)^k, \qquad t \in R,\ k = 0, 1, \dots,$$

which in view of (2.15) implies that $N_k(t,t) \to 0$ uniformly on R as $k \to \infty$. The proof now can be completed by letting $k \to \infty$ in (2.14).

3. THE MAIN THEOREM

We shall deal with the equation

$$x'(t) = \left(\tilde{A} + A(t)\right)x(t) + \left(\tilde{B} + B(t)\right)x(t-\tau), \tag{3.1}$$

where τ is a positive constant,

(H$_1$) $\tilde{A}$, $\tilde{B}$ are constant diagonal matrices,

$$\tilde{A} = \operatorname{diag}\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}, \qquad \tilde{B} = \operatorname{diag}\{\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n\},$$

such that for each $i = 1, 2, \dots, n$,

$$\left(|\tilde{a}_i| + |\tilde{b}_i|\right)\tau < \frac{1}{e}, \tag{3.2}$$

so that the equation

$$\lambda = \tilde{a}_i + \tilde{b}_i e^{-\lambda\tau} \tag{3.3}$$

has a unique real root $\tilde{\lambda}_i$ in the interval $(\tilde{a}_i - 1/\tau, \infty)$. Furthermore, suppose that

$$\tilde{\lambda}_i \neq \tilde{\lambda}_j \qquad \text{for each } i \neq j,\ i, j \in \{1, 2, \dots, n\}. \tag{3.4}$$

(H$_2$) $A(\cdot) = (a_{ij}(\cdot))_{1 \le i, j \le n}$, $B(\cdot) = (b_{ij}(\cdot))_{1 \le i, j \le n}$ are continuous matrix functions on $[0, \infty)$ such that

$$\alpha(t) := \sup_{s \ge t} |A(s)| \in L^2, \qquad \beta(t) := \sup_{s \ge t} |B(s)| \in L^2, \tag{3.5}$$

moreover, for each $i = 1, 2, \dots, n$,

$$a_{ii}(t) - \frac{1}{\tau}\int_{t-\tau}^{t} a_{ii}(s)\, ds \in L^1, \qquad b_{ii}(t) - \frac{1}{\tau}\int_{t-\tau}^{t} b_{ii}(s)\, ds \in L^1. \tag{3.6}$$

Now we formulate the main result of the paper.

THEOREM 3.1. *Let hypotheses* (H$_1$) *and* (H$_2$) *be fulfilled. Then there exists a matrix function F, $F(t) \to 0$ as $t \to \infty$, such that the following statements are valid.*

(A) *For every constant vector c, the function x defined by*

$$x(t) = (I + F(t))\exp\left(\int_0^t \Lambda(s)\, ds\right)c,$$

where

$$\Lambda(t) = \tilde{\Lambda} + \left[I + (\tilde{\Lambda} - \tilde{A})\tau\right]^{-1}\left[\operatorname{diag}\{A(t)\} + \operatorname{diag}\{B(t)\}e^{-\tilde{\Lambda}\tau}\right]$$

and

$$\tilde{\Lambda} = \operatorname{diag}\left\{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n\right\},$$

is a solution of (3.1).

(B) *For every solution x of* (3.1) *there exist a constant vector c and a vector function f, $f(t) \to 0$ as $t \to \infty$, such that*

$$x(t) = (I + F(t))\exp\left(\int_0^t \Lambda(s)\,ds\right)(c + f(t)) \tag{3.7}$$

and

$$\left| x(t) - (I + F(t))\exp\left(\int_0^t \Lambda(s)\,ds\right)c \right| e^{t/\tau} \quad \text{is bounded for } t \to \infty \tag{3.8}$$

(*with Λ defined as in statement* (A)).

Remarks. (i) It is obvious that if $\tilde{a}_i + \tilde{b}_i \neq \tilde{a}_j + \tilde{b}_j$ for each $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$, then condition (3.4) is fulfilled for any sufficiently small τ.

(ii) Evidently, condition (3.5) is satisfied if and only if

$$|A(t)| = O(\gamma(t)), \qquad |B(t)| = O(\gamma(t)) \qquad \text{as} \qquad t \to \infty$$

for some nonincreasing function $\gamma \in L^2$.

(iii) As we already mentioned in the Introduction, condition (3.6) is fulfilled if and only if, on some interval $[t_1, \infty)$, a_{ii} and b_{ii} can be written in the form

$$a_{ii} = \alpha_i + \overline{\alpha}_i, \qquad b_{ii} = \beta_i + \overline{\beta}_i,$$

where α_i, β_i are continuously differentiable functions with $\alpha_i', \beta_i' \in L^1$ and $\overline{\alpha}_i, \overline{\beta}_i \in L^1$ (see the Proposition in the Appendix).

(iv) It follows from the proof of Theorem 3.1 below that the constant c determined by (3.7) is the only one satisfying (3.8).

(v) If $\tilde{B} = 0$ then $\tilde{\Lambda} = \tilde{A}$ and (3.7) reduces to the asymptotic formula (1.6).

Before we present the proof of Theorem 3.1, we need some lemmas regarding L^p-functions.

LEMMA 3.1 [11, Lemma 2.1]. *If $f \in L^p([t_0, \infty), R)$ for some $p > 0$, then for any $\tau > 0$,*

$$\int_{t-\tau}^{t} |f(s)|\,ds \in L^p\left([t_0 + \tau, \infty), R\right)$$

and

$$\int_{t-\tau}^{t} |f(s)|\, ds \to 0 \qquad \text{as } t \to \infty.$$

LEMMA 3.2. *Let $f \in L^2([t_0, \infty), [0, \infty))$ be a nonincreasing function. Then for every $\tau > 0$ and $\kappa > 1$ there exists a nonincreasing function $g \in L^2([t_0, \infty), [0, \infty))$ so that*

$$f(t) \le g(t) \qquad \textit{for } t \ge t_0 \tag{3.9}$$

and

$$g(t-\tau) \le \kappa g(t) \qquad \textit{for } t \ge t_0 + \tau. \tag{3.10}$$

Proof. Put

$$a_k = f\big(t_0 + (k-1)\tau\big) \qquad \text{for } k = 1, 2, \dots .$$

Then

$$a_{k+1} \le f(t) \le a_k \qquad \text{for } t \in \big[t_0 + (k-1)\tau, t_0 + k\tau\big], k = 1, 2, \dots .$$

Consequently

$$\tau a_{k+1}^2 \le \int_{t_0+(k-1)\tau}^{t_0+k\tau} f^2(t)\, dt, \qquad k = 1, 2, \dots .$$

and hence

$$\sum_{k=2}^{\infty} a_k^2 \le \frac{1}{\tau} \int_{t_0}^{\infty} f^2(t)\, dt < \infty.$$

Define

$$b_1 = a_1$$

and

$$b_k = \max\left\{ a_k, \frac{b_{k-1}}{\kappa} \right\}$$

for $k = 2, 3, \dots$. Clearly

$$a_k \le b_k \qquad \text{for } k = 1, 2, \dots$$

and

$$b_{k-1} \le \kappa b_k \qquad \text{for } k = 2, 3, \dots .$$

It follows by easy induction that

$$b_{k+1} \le b_k \qquad \text{for } k = 1, 2, \dots,$$

that is, the sequence $\{b_k\}_{k=1}^{\infty}$ is nonincreasing.

Now we show that

$$\sum_{k=1}^{\infty} b_k^2 < \infty.$$

Evidently,

$$b_j^2 \le \frac{b_{j-1}^2}{\kappa^2} + a_j^2, \qquad j = 2, 3, \dots .$$

Summing the last inequality from $j = 2$ to $j = k$, we obtain

$$\sum_{j=2}^{k} b_j^2 \le \frac{1}{\kappa^2} \sum_{j=1}^{k-1} b_j^2 + \sum_{j=2}^{k} a_j^2,$$

or

$$\left(1 - \frac{1}{\kappa^2}\right) \sum_{j=2}^{k} b_j^2 + \frac{1}{\kappa^2} b_k^2 - \frac{1}{\kappa^2} b_1^2 \le \sum_{j=2}^{k} a_j^2.$$

Hence

$$\sum_{j=2}^{\infty} b_j^2 \le \frac{\kappa^2}{\kappa^2 - 1} \left(\frac{b_1^2}{\kappa^2} + \sum_{j=2}^{\infty} a_j^2 \right) < \infty.$$

Define

$$g(t) = b_k \qquad \text{for } t \in \left[t_0 + (k-1)\tau, t_0 + k\tau\right], k = 1, 2, \dots .$$

Obviously, g satisfies both conditions (3.9) and (3.10). Moreover,

$$\int_{t_0}^{\infty} g^2(t)\, dt = \sum_{k=1}^{\infty} \int_{t_0+(k-1)\tau}^{t_0+k\tau} g^2(t)\, dt = \tau \sum_{k=1}^{\infty} b_k^2 < \infty$$

which completes the proof.

Proof of Theorem 3.1. Since the assumptions of the theorem are independent of the given norm in R^n, we may (and do) consider the maximum norm, that is, $|x| = \max_{1 \le i \le n} |x_i|$ for $x = \operatorname{col}(x_1, x_2, \dots, x_n) \in R^n$. Then (3.2) is equivalent to

$$\left(|\tilde{A}| + |\tilde{B}|\right)\tau < \frac{1}{e}. \tag{3.11}$$

Find constants κ and $\kappa_*, 1 < \kappa < \kappa_*$, so that

$$\bigl(|\tilde{A}| + |\tilde{B}|\bigr)\kappa_*^2\tau < \frac{1}{e},$$

and then $\epsilon > 0$ so that

$$\bigl(|\tilde{A}| + |\tilde{B}| + \epsilon\bigr)\kappa_*^2\tau < \frac{1}{e}. \tag{3.12}$$

(Such constants certainly exist.) Obviously, from (H_2),

$$\alpha(t) \to 0, \qquad \beta(t) \to 0, \qquad \text{as } t \to \infty. \tag{3.13}$$

Let t_0 be so large that

$$\alpha(t) + \beta(t) < \epsilon \qquad \text{for } t \geq t_0. \tag{3.14}$$

Define

$$\hat{A}(t) = \begin{cases} \tilde{A} + A(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0, \end{cases}$$

$$\hat{B}(t) = \begin{cases} \tilde{B} + B(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0. \end{cases}$$

Let X and Y be the special matrix solutions of the equations

$$x'(t) = \hat{A}(t)x(t) + \hat{B}(t)x(t-\tau) \tag{3.15}$$

and

$$y'(t) = \operatorname{diag}\{\hat{A}(t)\}y(t) + \operatorname{diag}\{\hat{B}(t)\}y(t-\tau), \tag{3.16}$$

respectively, with $X(t_0) = Y(t_0) = I$. That is, X and Y are the matrix solutions of the equations

$$x'(t) = \bigl(\tilde{A} + A(t)\bigr)x(t) + \bigl(\tilde{B} + B(t)\bigr)x(t-\tau) \qquad \text{for } t \geq t_0 \tag{3.17}$$

and

$$y'(t) = \bigl(\tilde{A} + \operatorname{diag}\{A(t)\}\bigr)y(t) + \bigl(\tilde{B} + \operatorname{diag}\{B(t)\}\bigr)y(t-\tau) \qquad \text{for } t \geq t_0 \tag{3.18}$$

with initial conditions

$$X(t) = I \qquad \text{for } t \leq t_0 \tag{3.19}$$

and

$$Y(t) = I \qquad \text{for } t \le t_0. \tag{3.20}$$

Denoting $K = |\tilde{A}| + |\tilde{B}| + \epsilon$, we have

$$\sup_{t \in R} \left[\left| \hat{A}(t) \right| + \left| \hat{B}(t) \right| \right] \le K,$$

and (cf. (3.12)) $K\tau e < 1$. By Theorem 2.3 (see the Remark after Theorem 2.3) we conclude that for every solution x of (3.17) the limit

$$c[x] = \lim_{t \to \infty} X^{-1}(t) x(t) \in R^n \tag{3.21}$$

exists and

$$\sup_{t \ge t_0} |x(t) - X(t)c[x]| e^{t/\tau} < \infty. \tag{3.22}$$

We remark that for the special solution $\bar{x}(t) = X(t)c$ ($c \in R^n$ arbitrary) we have $c[\bar{x}] = c$. Analogous formulae hold for the solutions of (3.18).

Define

$$C_0(t, s) = \hat{A}(s) + \hat{B}(s),$$

$$D_0(t, s) = \operatorname{diag}\{\hat{A}(s)\} + \operatorname{diag}\{\hat{B}(s)\}$$

for $s \le t$, and

$$C_{k+1}(t, s) = -\hat{A}(s) \int_s^t C_k(t, u)\, du - \hat{B}(s) \int_{s-\tau}^t C_k(t, u)\, du,$$

$$D_{k+1}(t, s) = -\operatorname{diag}\{\hat{A}(s)\} \int_s^t D_k(t, u)\, du - \operatorname{diag}\{\hat{B}(s)\} \int_{s-\tau}^t D_k(t, u)\, du$$

for $s \le t$ and $k = 0, 1, \dots$. By Theorem 2.4, the special matrix solutions X and Y are fundamental systems of the ordinary differential equations

$$x' = C(t) x \tag{3.23}$$

and

$$y' = D(t) y, \tag{3.24}$$

where

$$C(t) = \sum_{k=0}^{\infty} C_k(t, t)$$

and

$$D(t) = \sum_{k=0}^{\infty} D_k(t, t)$$

for $t \in R$. Clearly, $D_k(t, s)$ and hence $D(t)$ is a diagonal matrix.

We shall prove the following claims.

CLAIM 1. $C - D \in L^2$.

CLAIM 2. $\operatorname{diag}\{C - D\} \in L^1$.

CLAIM 3. $\exp(-\int_{t_0}^{t} \Lambda(s)\, ds) Y(t) \to \Delta_1$ *as* $t \to \infty$ *where* Δ_1 *is an invertible diagonal matrix.*

CLAIM 4. $D(t) \to \tilde{\Lambda}$ *as* $t \to \infty$.

Proof of Claim 1. By Lemma 3.2, there exists a positive nonincreasing function $\gamma \in L^2$ such that

$$\alpha(t) + \beta(t) \le \gamma(t) \qquad \text{for } t \ge t_0 \tag{3.25}$$

and

$$\gamma(t-\tau) \le \kappa\gamma(t) \qquad \text{for } t \ge t_0. \tag{3.26}$$

It will be convenient to define $\gamma(t) = 0$ for $t < t_0$.

We shall prove that

$$|C_k(t,s) - D_k(t,s)| \le \frac{\kappa_*}{\kappa_* - \kappa} \frac{(K\kappa_*)^k}{k!} \gamma(s)(t - s + k\tau)^k \tag{3.27}$$

for all $t_0 \le s \le t$ and $k = 0, 1, \ldots$. To prove (3.27), we shall use an induction argument and the following estimate from the proof of Theorem 2.4 (cf. (2.9)),

$$|C_k(t,s)| \le \frac{K^{k+1}}{k!}(t - s + k\tau)^k \tag{3.28}$$

for all $t_0 \le s \le t$ and $k = 0, 1, \ldots$.

Evidently

$$|C_0(t,s) - D_0(t,s)| \le \gamma(s)$$

for $t_0 \le s \le t$ and $k = 0, 1, \ldots$.

For $t_0 \le s \le t$ and $k = 0, 1, \ldots,$ we have

$$\begin{aligned} &C_{k+1}(t,s) - D_{k+1}(t,s) \\ &\quad = -\left(\tilde{A} + \operatorname{diag}\{A(s)\}\right)\int_s^t (C_k(t,u) - D_k(t,u))\, du \\ &\qquad -\left(\tilde{B} + \operatorname{diag}\{B(s)\}\right)\int_{s-\tau}^t (C_k(t,u) - D_k(t,u))\, du \\ &\qquad -(A(s) - \operatorname{diag}\{A(s)\})\int_s^t C_k(t,u)\, du \\ &\qquad -(B(s) - \operatorname{diag}\{B(s)\})\int_{s-\tau}^t C_k(t,u)\, du. \end{aligned}$$

So the validity of (3.27) for some k implies

$$
\begin{aligned}
&\left|C_{k+1}(t,s) - D_{k+1}(t,s)\right| \\
&\quad \le K\int_{s-\tau}^{t} \left|C_k(t,u) - D_k(t,u)\right| du + \gamma(s)\int_{s-\tau}^{t} \left|C_k(t,u)\right| du \\
&\quad \le K\int_{s-\tau}^{t} \frac{\kappa_*}{\kappa_* - \kappa}\frac{(K\kappa_*)^k}{k!}\gamma(u)(t-u+k\tau)^k\, du \\
&\qquad + \gamma(s)\int_{s-\tau}^{t}\frac{K^{k+1}}{k!}(t-u+k\tau)^k\, du \\
&\quad \le \left(\frac{(K\kappa_*)^{k+1}}{(\kappa_*-\kappa)k!}\gamma(s-\tau) + \frac{K^{k+1}}{k!}\gamma(s)\right)\int_{s-\tau}^{t}(t-u+k\tau)^k\, du \\
&\quad \le \frac{(K\kappa_*)^{k+1}}{k!}\left(\frac{\kappa}{\kappa_*-\kappa} + 1\right)\gamma(s)\int_{s-\tau}^{t}(t-u+k\tau)^k\, du \\
&\quad \le \frac{\kappa_*}{\kappa_*-\kappa}\frac{(K\kappa_*)^{k+1}}{(k+1)!}\gamma(s)(t-s+(k+1)\tau)^{k+1},
\end{aligned}
$$

where we used (3.25)–(3.28) and the monotonicity of γ. Thus, (3.27) is confirmed.

Writing $s = t$ in (3.27) and taking into account (2.11), we obtain

$$\left|C_k(t,t) - D_k(t,t)\right| \le \frac{\kappa_*}{\kappa_* - \kappa}(K\kappa_*, \tau e)^k \gamma(t)$$

for $t \ge t_0$ and $k = 0, 1, \dots$. Since $K\kappa_*\tau e < 1$ (cf. (3.12)), we have

$$|C(t) - D(t)| \le \sum_{k=0}^{\infty}\left|C_k(t,t) - D_k(t,t)\right| \le \frac{\kappa_*}{\kappa_* - \kappa}(1 - K\kappa_*\tau e)^{-1}\gamma(t)$$

for $t \ge t_0$. Claim 1 now follows from the fact that $\gamma \in L^2$.

Proof of Claim 2. We shall show by induction that

$$\left|\operatorname{diag}\{C_k(t,s) - D_k(t,s)\}\right| \le \frac{\kappa K^{k-1}\kappa_*^{2k+1}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)k!}\gamma^2(t)(t-s+k\tau)^k \tag{3.29}$$

for all $t_0 \le s \le t$ and $k = 0, 1, \dots$. To prove (3.29), we shall use estimate (3.27) and the identity

$$\begin{aligned}
&\operatorname{diag}\{C_{k+1}(t,s) - D_{k+1}(t,s)\} \\
&\quad = -\tilde{A}\int_s^t \operatorname{diag}\{C_k(t,u) - D_k(t,u)\}\,du \\
&\qquad - \tilde{B}\int_{s-\tau}^t \operatorname{diag}\{C_k(t,u) - D_k(t,u)\}\,du \\
&\qquad - \operatorname{diag}\left\{A(s)\int_s^t (C_k(t,u) - D_k(t,u))\,du\right\} \\
&\qquad - \operatorname{diag}\left\{B(s)\int_{s-\tau}^t (C_k(t,u) - D_k(t,u))\,du\right\}. \qquad (3.30)
\end{aligned}$$

Evidently

$$\operatorname{diag}\{C_0(t,s) - D_0(t,s)\} = 0$$

for $s \le t$. Thus, (3.29) holds for $k = 0$. Assuming that (3.29) holds for some k, we have (by (3.30) and (3.27))

$$\begin{aligned}
&\left|\operatorname{diag}\{C_{k+1}(t,s) - D_{k+1}(t,s)\}\right| \\
&\quad \le \left(|\tilde{A}| + |\tilde{B}|\right)\int_{s-\tau}^t \left|\operatorname{diag}\{C_k(t,u) - D_k(t,u)\}\right| du \\
&\qquad + (|A(s)| + |B(s)|)\int_{s-\tau}^t |C_k(t,u) - D_k(t,u)|\,du \\
&\quad \le K\int_{s-\tau}^t \frac{\kappa K^{k-1}\kappa_*^{2k+1}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)k!}\gamma^2(u)(t-u+k\tau)^k\,du \\
&\qquad + \gamma(s)\int_{s-\tau}^t \frac{K^k\kappa_*^{k+1}}{(\kappa_* - \kappa)k!}\gamma(u)(t-u+k\tau)^k\,du \\
&\quad \le \left(\frac{\kappa K^k\kappa_*^{2k+1}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)k!}\gamma^2(s-\tau)\right. \\
&\qquad \left. + \frac{K^k\kappa_*^{2k+1}}{(\kappa_* - \kappa)k!}\gamma(s)\gamma(s-\tau)\right)\int_{s-\tau}^t (t-u+k\tau)^k\,du \\
&\quad \le \left(\frac{\kappa^3K^k\kappa_*^{2k+1}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)k!} + \frac{\kappa K^k\kappa_*^{2k+1}}{(\kappa_* - \kappa)k!}\right)\gamma^2(s) \\
&\qquad \times\int_{s-\tau}^t (t-u+k\tau)^k\,du
\end{aligned}$$

$$= \frac{\kappa K^k \kappa_*^{2k+3}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)k!} \gamma^2(s) \int_{s-\tau}^{t} (t - u + k\tau)^k \, du$$

$$\leq \frac{\kappa K^k \kappa_*^{2k+3}}{(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)(k+1)!} \gamma^2(s)(t - u + (k+1)\tau)^{k+1}$$

for $t_0 \leq s \leq t$. Consequently, (3.29) is valid for all $k = 0, 1, \ldots$.

Taking $s = t$ in (3.29), we obtain

$$\left|\operatorname{diag}\{C_k(t,t) - D_k(t,t)\}\right| \leq \frac{\kappa\kappa_*}{K(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)} \frac{(K\kappa_*^2 k\tau)^k}{k!} \gamma^2(t)$$
$$\leq \frac{\kappa\kappa_*}{K(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)} (K\kappa_*^2 \tau e)^k \gamma^2(t)$$

for all $t \geq t_0$ and $k = 0, 1, \ldots$. Since $K\kappa_*^2 \tau e < 1$ (cf. (3.12)), we have

$$\left|\operatorname{diag}\{C(t) - D(t)\}\right| = \left|\sum_{k=0}^{\infty} \operatorname{diag}\{C_k(t,t) - D_k(t,t)\}\right|$$
$$\leq \sum_{k=0}^{\infty} \left|\operatorname{diag}\{C_k(t,t) - D_k(t,t)\}\right|$$
$$\leq \frac{\kappa\kappa_*}{K(\kappa_* - \kappa)(\kappa_*^2 - \kappa^2)(1 - K\kappa_* \tau e)} \gamma^2(t)$$

for $t \geq t_0$ which completes the proof of Claim 2.

Proof of Claim 3. Y is a diagonal matrix,

$$Y(t) = \operatorname{diag}\{y_1(t), y_2(t), \ldots, y_n(t)\}.$$

For each $i = 1, 2, \ldots, n$, y_i is the solution of the equation

$$y'(t) = (\tilde{a}_i + a_{ii}(t))y(t) + (\tilde{b}_i + b_{ii}(t))y(t - \tau) \qquad \text{for } t \geq t_0 \quad (3.31)$$

with initial condition

$$y_i(t) = 1 \qquad \text{for } t_0 - \tau \leq t \leq t_0.$$

By Theorems 2.3 and 2.4, y_i is positive (cf. (2.3)) and for every solution y of (3.31) the limit

$$\xi[y] = \lim_{t\to\infty} \frac{y(t)}{y_i(t)} \in R$$

exists. By Theorem B, for every solution y of (3.31) the limit

$$\eta[y] = \lim_{t\to\infty} y(t)\exp\left(-\int_{t_0}^{t} \lambda_i(s)\, ds\right) \in R$$

exists, where

$$\lambda_i(t) = \tilde{\lambda}_i + \left[1 + \left(\tilde{\lambda}_i - \tilde{a}_i\right)\tau\right]^{-1}\left[a_{ii}(t) + e^{-\tilde{\lambda}_i\tau} b_{ii}(t)\right].$$

Moreover, there exists a solution y of (3.31) for which $\eta[y] \neq 0$. Denote

$$\eta_i := \eta[y_i] \qquad \text{for } i = 1, 2, \dots, n.$$

Then for every solution y of (3.31), we have

$$\eta[y] = \lim_{t\to\infty} \frac{y(t)}{y_i(t)} \frac{y_i(t)}{\exp\left(\int_{t_0}^{t} \lambda_i(s)\, ds\right)} = \eta_i\, \xi[y].$$

Consequently

$$\eta_i \neq 0 \qquad \text{for } i = 1, 2, \dots, n. \tag{3.32}$$

Otherwise, $\eta[y] = 0$ for every solution y of (3.31) which is a contradiction. Therefore

$$\Delta_1 = \lim_{t\to\infty} \left[Y(t)\exp\left(-\int_{t_0}^{t} \Lambda(s)\, ds\right)\right] = \operatorname{diag}\{\eta_1, \eta_2, \dots, \eta_n\}$$

is invertible.

Proof of Claim 4. Let

$$D(t) = \operatorname{diag}\{d_1(t), d_2(t), \dots, d_n(t)\}.$$

Using the notation from the proof of Claim 3, we have

$$y_i'(t) = d_i(t) y_i(t)$$

for $t \geq t_0$ and $i = 1, 2, \ldots, n$. Hence

$$d_i(t) = \frac{y_i'(t)}{y_i(t)} = \tilde{a}_i + a_{ii}(t) + \big(\tilde{b}_i + b_{ii}(t)\big)\frac{y_i(t-\tau)}{y_i(t)}$$

$$= \tilde{a}_i + a_{ii}(t) + \big(\tilde{b}_i + b_{ii}(t)\big)\frac{y_i(t-\tau)\exp\left(-\int_{t_0}^{t-\tau}\lambda_i(s)\,ds\right)}{y_i(t)\exp\left(-\int_{t_0}^{t}\lambda_i(s)\,ds\right)}$$

$$\times \exp\left(-\int_{t-\tau}^{t}\lambda_i(s)\,ds\right).$$

Using (3.14), (3.32), and Lemma 3.1, we get

$$d_i(t) = \tilde{a}_i + o(1) + \big(\tilde{b}_i + o(1)\big)\frac{\eta_i + o(1)}{\eta_i + o(1)}e^{-\tilde{\lambda}_i\tau + o(1)}$$

$$= \tilde{a}_i + \tilde{b}_i e^{-\tilde{\lambda}_i\tau} + o(1) = \tilde{\lambda}_i + o(1)$$

as $t \to \infty$ which completes the proof of Claim 4.

We complete the proof of Theorem 3.1. Rewrite Eq. (3.23) as

$$x' = (D(t) + R(t))x, \tag{3.23$'$}$$

where $R = C - D$. Claim 1 implies (1.3) with $p = 2$. Further, from Claim 4 and (3.4) it follows that (1.2) is satisfied. That is, all the hypotheses of the Hartman–Wintner theorem are fulfilled. Consequently, Eq. (3.23)$'$ has a fundamental system $\tilde{X}$ of the form

$$\tilde{X}(t) = \big(I + \tilde{F}(t)\big)\exp\left(\int_{t_0}^{t}(D(s) + \operatorname{diag}\{R(s)\})\,ds\right)$$

$$= \big(I + \tilde{F}(t)\big)Y(t)\exp\left(\int_{t_0}^{t}\operatorname{diag}\{C(s) - D(s)\}\,ds\right)$$

$$= \big(I + \tilde{F}(t)\big)\exp\left(\int_{t_0}^{t}\Lambda(s)\,ds\right)\left[\exp\left(-\int_{t_0}^{t}\Lambda(s)\,ds\right)Y(t)\right]$$

$$\times \exp\left(\int_{t_0}^{t}\operatorname{diag}\{C(s) - D(s)\}\,ds\right),$$

where $\tilde{F}$ is a matrix function, $\tilde{F}(t) \to 0$ as $t \to \infty$.

By Claim 3

$$\exp\left(-\int_{t_0}^{t} \Lambda(s)\,ds\right) Y(t) = \Delta_1 + D_1(t),$$

where D_1 is a diagonal matrix function, $D_1(t) \to 0$ as $t \to \infty$.

By Claim 2

$$\exp\left(\int_{t_0}^{t} \operatorname{diag}\{C(s) - D(s)\}\,ds\right) = \Delta_2 + D_2(t),$$

where Δ_2 $(= \exp(\int_{t_0}^{\infty} \operatorname{diag}\{C(s) - D(s)\}\,ds))$ is an invertible diagonal matrix and D_2 is a diagonal matrix function, $D_2(t) \to 0$ as $t \to \infty$.

Consequently

$$\tilde{X}(t) = \left(I + \tilde{F}(t)\right)\exp\left(\int_{t_0}^{t} \Lambda(s)\,ds\right)\left(\Delta_1\Delta_2 + D_3(t)\right),$$

where D_3 is a diagonal matrix function, $D_3(t) \to 0$ as $t \to \infty$.

Consider now the fundamental system $\overline{X}$ of (3.23) defined by

$$\overline{X}(t) = \tilde{X}(t)\Delta_2^{-1}\Delta_1^{-1}.$$

Then

$$\overline{X}(t) = \left(I + \tilde{F}(t)\right)\exp\left(\int_{t_0}^{t} \Lambda(s)\,ds\right)\left(I + D_4(t)\right),$$

where D_4 is a diagonal matrix function, $D_4(t) \to 0$ as $t \to \infty$.

Taking into account that $\Lambda(t)$ and $D_4(t)$ are diagonal matrices, we have

$$\begin{aligned}\overline{X}(t) &= \left(I + \tilde{F}(t)\right)\left(I + D_4(t)\right)\exp\left(\int_{t_0}^{t} \Lambda(s)\,ds\right) \\ &= (I + F(t))\exp\left(\int_{t_0}^{t} \Lambda(s)\,ds\right),\end{aligned}$$

where F is a matrix function, $F(t) \to 0$ as $t \to \infty$. Since $\overline{X}$ and X are fundamental systems of the same equation ((3.23)), there exists an invertible matrix C so that

$$X(t) = \overline{X}(t)C,$$

that is,

$$X(t) = (I + F(t))\exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right)C.$$

The statements of the theorem now follow from the asymptotic representations (3.21) and (3.22). The proof of the theorem is complete.

APPENDIX

We give a proof of the result mentioned in the Introduction.

PROPOSITION. *Let $\tau > 0$. A necessary and sufficient condition for a locally integrable function u: $[t_0 - \tau, \infty) \to R$ to be such that*

$$u(t) - \frac{1}{\tau}\int_{t-\tau}^{t} u(s)\, ds \in L^1\big([t_0, \infty), R\big) \tag{4.1}$$

is that, for some $t_1 \geq t_0$, u can be decomposed on $[t_1, \infty)$ as a sum

$$u = f + g, \tag{4.2}$$

where $f \in C^1([t_1, \infty), R)$ with $f' \in L^1([t_1, \infty), R)$ and $g \in L^1([t_1, \infty), R)$.

Proof. Sufficiency. Let u have the form (4.2). According to Lemma 3.1, $\int_{t-\tau}^{t} g(s)\, ds \in L^1$. Consequently, it suffices to show that

$$f(t) - \frac{1}{\tau}\int_{t-\tau}^{t} f(s)\, ds \in L^1.$$

But,

$$\begin{aligned} \left| f(t) - \frac{1}{\tau}\int_{t-\tau}^{t} f(s)\, ds \right| &= \frac{1}{\tau}\left| \int_{t-\tau}^{t} (f(t) - f(s))\, ds \right| \\ &\leq \frac{1}{\tau}\int_{t-\tau}^{t} \left| \int_{s}^{t} f'(u)\, du \right| ds \leq \int_{t-\tau}^{t} |f'(u)|\, du, \end{aligned}$$

and, by Lemma 3.1 again, $\int_{t-\tau}^{t} |f'(u)|\, du \in L^1$ which proves the sufficiency part.

Necessity. It will be proved in two steps. First we prove the necessity in the special case when

$$u \in C^1\big([t_0 - \tau, \infty, R\big). \tag{4.3}$$

Then we show that the general case can be reduced to the previous one.

Step 1. Suppose that u fulfills (4.3) and (4.1). Without loss of generality, we may assume that $t_0 = 0$. Define

$$h(t) = u(t) - \frac{1}{\tau}\int_{t-\tau}^{t} u(s)\,ds, \qquad t \geq 0. \tag{4.4}$$

By assumptions, h is continuously differentiable on $[0, \infty)$ and $h \in L^1$. Differentiating (4.4), we obtain

$$u'(t) - \frac{1}{\tau}(u(t) - u(t-\tau)) = h'(t), \qquad t \geq 0,$$

a nonhomogeneous delay differential equation. By the variation-of-constants formula [13]

$$u(t) = v(t) + \int_0^t w(t-s)h'(s)\,ds, \qquad t \geq 0, \tag{4.5}$$

where v is the solution of the homogeneous equation

$$v'(t) = \frac{1}{\tau}(v(t) - v(t-\tau)), \qquad t \geq 0 \tag{4.6}$$

with initial condition

$$v(t) = u(t) \qquad \text{for } -\tau \leq t \leq 0,$$

and w is the fundamental solution of (4.6), i.e., w is the solution of (4.6) with initial condition

$$w(t) = \begin{cases} 0 & \text{for } -\tau \leq t < 0 \\ 1 & \text{for } t = 0 \end{cases}.$$

A simple analysis of the characteristic equation

$$\lambda = \frac{1}{\tau}(1 - e^{-\lambda\tau}) \tag{4.7}$$

of Eq. (4.6) shows that zero is a double root of (4.7) and all other roots have negative real parts. Therefore (cf. [3, Chap. 4]) both solutions v and w can be written in the form

$$v(t) = c_1 + c_2 t + o(e^{-\alpha t}) \qquad \text{as } t \to \infty$$

$$w(t) = \gamma_1 + \gamma_2 t + o(e^{-\alpha t}) \qquad \text{as } t \to \infty$$

for some c_i, $\gamma_i \in R$, $i = 1, 2$ and $\alpha > 0$. Substituting the above asymptotic representations of v and w into (4.5), we get

$$u(t) = c_1 + \gamma_1(h(t) - h(0)) + (c_2 - \gamma_2 h(0))t + \gamma_2 \int_0^t h(s)\,ds + k(t)$$

with

$$k(t) = k_1(t) + \int_0^t k_2(t-s)h'(s)\,ds,$$

where $k_i(t) = o(e^{-\alpha t})$ as $t \to \infty$, $i = 1, 2$. Since $h' \in L^1$, we have $k \in L^1$. Consequently, u can be written in the form

$$u(t) = f(t) + g(t) + ct,$$

where $f \in C^1$ with $f' \in L^1$, $g \in L^1$ and $c \in R$. From the sufficiency part, we know that $\tilde{u} = f + g$ satisfies the desired property, that is,

$$\tilde{u}(t) - \frac{1}{\tau}\int_{t-\tau}^t \tilde{u}(s)\,ds \in L^1.$$

Since, by assumption, the same is true for u, it is necessary that $ct = u(t) - \tilde{u}(t)$ satisfies the property too, that is,

$$ct - \frac{c}{\tau}\left(\frac{t^2}{2} - \frac{(t-\tau)^2}{2}\right) = \frac{c\tau}{2} \in L^1$$

which implies that $c = 0$. Therefore $u = f + g$ with $f \in C^1$, f' and g in L^1, which is the desired result.

Step 2. Consider now the general case. That is, we assume that u is only locally integrable and satisfies (4.1). Define

$$u_0(t) = u(t)$$

and

$$u_{i+1}(t) = \frac{1}{\tau}\int_{t-\tau}^t u_i(s)\,ds$$

for $i = 0, 1, 2$ and $t \geq t_0 + i\tau$. We also introduce

$$h_i(t) = u_i(t) - u_{i+1}(t)$$

for $i = 0, 1, 2$ and $t \geq t_0 + i\tau$. Obviously

$$h_{i+1}(t) = \frac{1}{\tau}\int_{t-\tau}^t h_i(s)\,ds \tag{4.8}$$

for $i = 0, 1$ and $t \geq t_0 + (i+1)\tau$. By assumption (4.1),

$$h_0(t) = u_0(t) - u_1(t) = u(t) - \frac{1}{\tau}\int_{t-\tau}^{t} u(s)\, ds \in L^1$$

which, together with (4.8), implies (cf. Lemma 3.1)

$$h_i \in L^1 \qquad \text{for } i = 0, 1, 2. \tag{4.9}$$

Clearly, $u_1 \in C([t_0, \infty), R)$ and $u_2 \in C^1([t_0 + \tau, \infty), R)$. Furthermore,

$$u_2(t) - \frac{1}{\tau}\int_{t-\tau}^{t} u_2(s)\, ds = u_2(t) - u_3(t) = h_2(t) \in L^1.$$

According to the previous part of the proof (Step 1), on some $[t_1, \infty)$, u_2 can be written as

$$u_2 = f + h,$$

where $f \in C^1$ with $f' \in L^1$ and $h \in L^1$. Hence

$$u = u_0 = u_1 + h_0 = u_2 + h_1 + h_0 = f + h + h_1 + h_0 = f + g,$$

where $g = h + h_1 + h_0 \in L^1$ (cf. (4.9)).

The proof of the Proposition is complete.

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