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A Uniform Approach to Field Quantization

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A new method for treating ordinary Bose and Fermi statistics as well as many types of parastatistics is proposed. Number operators are used to distinguish among different types of statistics, and uniqueness results for Bose, Fermi, parabose and parafermi statistics are obtained.

1. INTRODUCTION

The uniqueness results for the free Boson [13, Theorem 1′] and free Fermion [14, Theorem 2.1] fields are strikingly different. In each case the creation operators, $C(z)$, satisfy simple relations. For Boson fields

$$[C^*(z), C(y)] = \langle y, z\rangle, \tag{1.1}$$

$$[C(z), C(y)] = 0 \tag{1.2}$$

and for Fermion fields

$$[C^*(z), C(y)]_+ = \langle y, z\rangle, \tag{1.3}$$

$$[C(z), C(y)]_+ = 0. \tag{1.4}$$

Although these relations are similar in form they are dissimilar in character since (1.3) and (1.4) can be satisfied by bounded operators on a Hilbert space while (1.1) and (1.2) cannot. The creation operators for a Boson field must be unbounded and so domain questions must be considered. This complication is eliminated by replacing these relations with the Weyl relation

$$W(z)\, W(y) = e^{\frac{1}{2}i\operatorname{Im}\langle z,y\rangle} W(z+y). \tag{1.5}$$

This is formally equivalent to (1.1) and (1.2) if $W(z) = e^{iR(z)}$ where $R(z)$ is the closure of $(\frac{1}{2})^{1/2}(C(z) + C^*(z))$. Relation (1.5) is special to Bosons and there does not seem to be a way to generalize it for other types of statistics.

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We examine a different approach to (1.1) and (1.2) or similar relations involving possibly unbounded operators. For Boson and Fermion fields the operator

$$n(z) = C(z)\, C^*(z) \tag{1.6}$$

represents the number of particles in the state z when $\| z \| = 1$. In general, number operators should satisfy

$$[n(z), C(y)] = \langle y, z \rangle\, C(z). \tag{1.7}$$

When $n(z)$ is given by (1.6), (1.7) is satisfied by both Boson and Fermion fields (although only in a formal sense for Bosons). If z is a unit vector, (1.7) is formally equivalent to

$$e^{itn(z)} C(y)\, e^{-itn(z)} = C(e^{itP_z} y), \tag{1.8}$$

where P_z is the projection onto the one-dimensional space spanned by z. Since $e^{itn(z)}$ is unitary when $n(z)$ is self-adjoint, (1.8) may make sense even when the creation operators are unbounded. In fact, it actually holds for both Bosons and Fermions.

Although (1.6) and (1.8) do not distinguish between Bosons and Fermions, the requirement that a certain operator (in this case the one defined by (1.6)) represents a number operator (so that (1.8) holds), determines many of the important qualities of the statistics.

In Section 2 we use this observation to produce uniqueness theorems for Boson and Fermion fields which are very similar in form and do not use the Weyl relation. The power of this procedure is demonstrated by the simplicity with which it can be extended to other types of statistics. In Section 3 we use two other choices for $n(z)$,

$$n(z) = \tfrac{1}{2}(C(z)\, C^*(z) + C^*(z)\, C(z)) \tag{1.9}$$

and

$$n(z) = \tfrac{1}{2}(C(z)\, C^*(z) - C^*(z)\, C(z)) \tag{1.10}$$

and obtain uniqueness results for the corresponding fields. These fields, the paraboson and parafermion fields, are the best known generalizations of the Boson and Fermion fields.

Thus, by making a specific choice of $n(z)$ and requiring that (1.8) holds, we obtain uniqueness results of similar form for Boson, Fermion, paraboson and parafermion fields. Other choices for $n(z)$ are undoubtedly possible and the resulting statistics may be treated by these methods.

To set the framework for our discussion we state a few definitions.

Let H and K be complex Hilbert spaces and let C be a function from H into the set of closed, densely defined operators on K such that

$$C(z+y) \supset C(z) + C(y), \tag{1.11}$$

$$C(\gamma z) = \gamma C(z) \tag{1.12}$$

for all $z, y \in H$ and all nonzero complex numbers, γ. Then $\{H, C, K\}$ will be called a quantum structure over H. To avoid trivialities we will always assume that $C(z)$ is not identically zero for any z. If in addition, Γ is a continuous unitary representation of the unitary group on H by operators on K and v is a unit vector in K such that

$$\Gamma(U)\, C(z)\, \Gamma(U)^{-1} = C(Uz) \tag{1.13}$$

and

$$\Gamma(U)v = v, \tag{1.14}$$

then $\{H, C, K, \Gamma, v\}$ will be called an invariant quantum structure over H. An (invariant) quantum structure is called bounded if $C(z)$ is a bounded operator on K for each $z \in H$. An (invariant) quantum structure is irreducible if the only projections, P, on K such that

$$C(z)P \supset PC(z)$$

for all $z \in H$ are $P = 0$ and $P = I$. This is equivalent to the statement that the only unitary operators, V, on K such that

$$VC(z)\, V^{-1} = C(z)$$

for all $z \in H$ are scalar multiples of the identity.

Two quantum structures $\{H, C, K\}$ and $\{H, C', K'\}$ over H are unitarily equivalent if there is a unitary operator Φ from K onto K' such that for all $z \in H$,

$$C'(z) = \Phi C(z)\, \Phi^{-1}.$$

Invariant quantum structures, $\{H, C, K, \Gamma, v\}$ and $\{H, C', K', \Gamma', v'\}$ are unitarily equivalent if in addition Φ satisfies

$$v' = \Phi v$$

and for all unitary operators, U, on H

$$\Gamma'(U) = \Phi\Gamma(U)\, \Phi^{-1}.$$

Suppose that $\{H, C, K, \Gamma, v\}$ is an invariant quantum structure. If A is a (not necessarily bounded) self-adjoint operator on H then e^{itA} is a continuous one-parameter unitary group on H and by the continuity of Γ, $\Gamma(e^{itA})$ is a continuous one-parameter unitary group on K. Thus, there is a unique self-adjoint operator on K, represented by $d\Gamma(A)$, such that $\Gamma(e^{itA}) = e^{itd\Gamma(A)}$. Since $\Gamma(e^{itA})v = v$, $d\Gamma(A)v = 0$. If $A \geqslant 0$ implies that $d\Gamma(A) \geqslant 0$ then Γ or $\{H, C, K, \Gamma, v\}$ is called positive. For two commuting self-adjoint operators A and B on H, $d\Gamma(A)$ and $d\Gamma(B)$ are commuting self-adjoint operators on K since two self-adjoint operators commute if and only if their corresponding one-parameter unitary groups commute. If in addition $A \leqslant B$ and Γ is positive then $d\Gamma(A) \leqslant d\Gamma(B)$.

Suppose $\{H, C, K\}$ is a quantum structure over H, M is a closed subspace of H and P is the projection onto M. A self-adjoint operator, n, on K which is bounded from below is called a number operator for M if

$$e^{itn}C(y)\, e^{-itn} = C(e^{itP}y) \tag{1.15}$$

for all $y \in H$. If n' is another number operator for M then $e^{-itn'}e^{itn}$ commutes with each $C(z)$. When $\{H, C, K\}$ is irreducible, n' and n differ by a scalar. Since $e^{2\pi iP} = I$, $e^{2\pi in}$ commutes with each $C(z)$. Irreducibility implies that the spectrum of n is discrete and separated by integers. Thus there is at most one number operator for M which is non-negative and has 0 in its spectrum. Such a number operator will be called normalized.

The normalized number operator for M can be interpreted as the number of particles (whose state is) in the subspace M. If $nw = jw$ (that is, w has j particles in M) and $z \in M$ then $w \in \mathrm{Dom}(C(z))$ implies

$$e^{itn}C(z)w = e^{it(j+1)}C(z)w.$$

Therefore $C(z)w \in \mathrm{Dom}(n)$ and

$$nC(z)w = (j + 1)\, C(z)w.$$

Thus, $C(z)w$ has $(j + 1)$ particles in M. Under similar domain restrictions, $C^*(z)w$ has $(j - 1)$ particles in M. Similarly, if $z \in M^\perp$ then $C(z)w$ and $C^*(z)w$ have j particles in M. The interpretation of n as the number operator for M is therefore consistent with the interpretation of $C(z)$ as a creation operator and $C^*(z)$ as an annihilation operator.

Two special cases are most important. When M is the one-dimensional space spanned by a vector $z \in H$ and n is a number operator for M, then n is also called a number operator for z and

$$e^{itn}C(y)e^{-itn} = C(e^{itP_z}y). \tag{1.16}$$

When z is a unit vector, $C(z)\, C^*(z)$ is the normalized number operator for z for both Bosons and Fermions. The other special case is $M = H$. In this case a number operator for M is called a total number operator and we usually denote it by N. Since

$$e^{itN}C(z)\,e^{-itN} = e^{it}C(z)$$

and

$$e^{itN}C^*(z)\,e^{-itN} = e^{-it}C^*(z),$$

N commutes with $C(z)\,C^*(z)$ and $C^*(z)\,C(z)$.

If $\{H, C, K, \Gamma, v\}$ is an irreducible positive invariant quantum structure over H and P is a projection of H, then $d\Gamma(P)$ is a number operator for PH. Since $d\Gamma(P) \geqslant 0$ and $d\Gamma(P)v = 0$, $d\Gamma(P)$ is the normalized number operator for PH. In particular, if z is a nonzero vector in H then $d\Gamma(P_z)$ is the normalized number operator for z. $d\Gamma(I)$ is the normalized total number operator. If $v \in \mathrm{Dom}(C^*(z))$, as will usually be the case, then

$$d\Gamma(I)\,C^*(z)v = -C^*(z)v,$$

and the positivity of Γ implies that

$$C^*(z)v = 0. \tag{1.17}$$

For a given quantum structure $\{H, C, K\}$, $\mathscr{A}$ will represent the algebra with unit generated by the creation and annihilation operators. For $z \in H$, $\mathscr{A}_z$ will represent the algebra with unit generated by those creation operators $C(y)$ and those annihilation operators $C^*(y)$ for which y is either parallel or orthogonal to z. $\mathscr{A}'$ and $\mathscr{A}'_z$ will represent the subalgebras of $\mathscr{A}$ and $\mathscr{A}_z$, respectively, generated by the creation operators alone.

2. Bosons and Fermions

The two most important examples of irreducible positive invariant quantum structures are the Fock–Cook Boson quantum structure and the Fock–Cook Fermion quantum structure. These were proposed by Fock [2] in 1932 but the first rigorous mathematical work was done by Cook [1] in 1953. In Cook's paper the creation operators which we denote by $C(z)$ are denoted by $\omega_s(\phi)$ for Bosons and $\omega_A(\phi)$ for Fermions. H corresponds to $\mathfrak{R}$ and K corresponds to $\mathfrak{S}$ (Bosons) and $\mathfrak{A}$ (Fermions). $d\Gamma(A)$ corresponds to $\Omega(A)$ and v corresponds to the number 1.

As indicated in Section 1, the choice of (1.6) as the number operator does not distinguish Bosons from Fermions. We make the distinction by requiring that (1.1) or (1.3) holds in the special case $y = z$. That is, we require the creation operators to behave correctly in one dimension and let (1.8) determine the relationship between particles in different states. We first state the results when H is finite dimensional for these are particularly simple.

THEOREM 1. *Suppose H is a finite dimensional complex Hilbert space and $\{H, C, K\}$ is an irreducible quantum structure over H such that for all $z \in H$,*

$$C^*(z)\, C(z) = C(z)\, C^*(z) + \| z \|^2 \tag{2.1}$$

and

$$n(z) = C(z)\, C^*(z)$$

is a number operator for z when z is a unit vector. Then $\{H, C, K\}$ is unitarily equivalent to the Fock–Cook Boson quantum structure over H.

THEOREM 2. *Suppose H is a finite dimensional complex Hilbert space and $\{H, C, K\}$ is an irreducible quantum structure over H such that for all $z \in H$,*

$$C^*(z)\, C(z) + C(z)\, C^*(z) = \| z \|^2 \tag{2.2}$$

and

$$n(z) = C(z)\, C^*(z)$$

is a number operator for z when z is a unit vector. Then $\{H, C, K\}$ is unitarily equivalent to the Fock–Cook Fermion quantum structure over H.

Notice that these two theorems differ only in the one-dimensional relation (2.1) or (2.2). (2.2) implies that the creation operators are bounded since $C^*(z)\, C(z)$ is a positive self-adjoint operator which is less than $\| z \|^2$. Equation (2.1), of course, does not imply boundedness. In fact, (2.1) cannot be satisfied by bounded operators unless $z = 0$. We start with the proof of Theorem 1.

Proof of Theorem 1. Relation (2.1) implies that if z is a unit vector, then $n(z)$ has purely discrete spectrum with eigenvalues $0, 1, 2, \ldots$. See, for example, [7, Lemma 4.4.1]. Equation (1.8) implies that

$$e^{itn(z)} C^*(y) e^{-itn(z)} = C^*(e^{itP_z} y)$$

and so if y is orthogonal to z then $n(z)$ commutes with $n(y)$. Let $\{e_1, e_2, \ldots, e_s\}$ be an orthonormal basis for H. Let $n_j = n(e_j)$ and $P_j = P_{e_j}$. $\{n_1, n_2, \ldots, n_s\}$ is a set of commuting self-adjoint operators each with discrete spectrum so there exists a vector $w \in K$ which is simultaneously an eigenvector for each n_j. Suppose $n_j w = \alpha_j w$. Each α_j is a nonnegative integer.

$$\begin{aligned} e^{itn_j} C^*(e_i) w &= e^{itn_j} C^*(e_i)\, e^{-itn_j} e^{it\alpha_j} w \\ &= (1 + (e^{-it} - 1)\, \delta_{ij})\, e^{it\alpha_j} C^*(e_i) w. \end{aligned}$$

Therefore $C^*(e_i)w$ is in the domain of n_j and

$$n_j C^*(e_i) w = (\alpha_j - \delta_{ij})\, C^*(e_i) w.$$

$C^*(e_i)w \neq 0$ unless $\alpha_i = 0$, so

$$C^*(e_s)^{\alpha_s} \cdots C^*(e_2)^{\alpha_2} C^*(e_1)^{\alpha_1} w$$

is a (non-zero) eigenvector of each n_j with eigenvalue 0. Let v be such an eigenvector which is normalized to be a unit vector.

Let

$$N = n(e_1) + n(e_2) + \cdots + n(e_s). \tag{2.3}$$

$$\begin{aligned} e^{itN} C(z)\, e^{-itN} &= C(e^{itP_1} e^{itP_2} \cdots e^{itP_s} z) \\ &= C(e^{it(P_1 + P_2 + \cdots + P_s)} z) \\ &= C(e^{it} z). \end{aligned}$$

Thus N is a total number operator. Since $N \geqslant 0$ and $Nv = 0$, N is normalized. The normalized total number operator is unique so N is independent of the orthonormal basis chosen in the definition (2.3). If $\| z \| = 1$ and N is defined in terms of an orthonormal basis including z, we then see that $C(z)\, C^*(z)$ commutes with and is less than or equal to N. By (2.1), $C^*(z)\, C(z)$ also commutes with N and is less than or equal to $N + 1$. We now use the following lemma.

LEMMA 1. *Suppose $\{H, C, K\}$ is an irreducible quantum structure with normalized total number operator N such that for each $z \in H$, $C(z)\, C^*(z)$ and $C^*(z)\, C(z)$ are bounded on each eigenspace of N. Let v be a non-zero vector in K such that $Nv = 0$. Then v is in the domain of each operator in $\mathscr{A}$, $C^*(z)v = 0$ for each $z \in H$ and $\mathscr{D} = \mathscr{A}v$ is a dense subset of K which is a core for each $C(z)$ and $C^*(z)$. Each monomial in $\mathscr{A}$ when applied to v gives an eigenvector of N (if it is not zero) with eigenvalue equal to the number of creation operators minus the number of annihilation operators in the monomial.*

We have stated this lemma in more generality than we need here because it will be used later in a more general setting. The lemma does not assume that H is finite dimensional. We have shown that the hypotheses of the lemma are satisfied in the current situation.

Proof of Lemma 1. Let

$$K_j = \{w \in K : Nw = jw\}$$

and let Q_j be the projection onto K_j. N has spectrum in $\{0, 1, 2, \ldots\}$ so $\sum_{j=0}^{\infty} Q_j$ converges strongly to the identity. Let z be an arbitrary fixed vector in H and let $C = C(z)$. C^*C and CC^* commute with N and are bounded operators from K_j into K_j. Thus C is a bounded operator from K_j into K_{j+1} and C^* is a bounded operator from K_{j+1} into K_j. Since $v \in K_0$, $v \in \mathrm{Dom}(C^*)$ and since $C^*v \in K_{-1} = \{0\}$, $C^*v = 0$. A simple inductive argument shows that every monomial in $\mathscr{A}$ has v in its domain and maps v into K_j where j is the difference between the number of creation operators and the number of annihilation operators in the monomial.

We next show that $\mathscr{D}$ is dense. Let

$$\mathscr{D}_j = \mathscr{D} \cap K_j,$$

let P_j be the projection onto $\bar{\mathscr{D}}_j$ (the closure of $\mathscr{D}_j$) and let $P = \sum_{j=0}^{\infty} P_j$. $P_j \leqslant Q_j$ so each P_i commutes with each P_j. Since $\mathscr{D}$ is invariant under C and C^*, C maps $\bar{\mathscr{D}}_j$ into $\bar{\mathscr{D}}_{j+1}$ and C^* maps $\bar{\mathscr{D}}_{j+1}$ into $\bar{\mathscr{D}}_j$. Suppose $w \in \mathrm{Dom}(C)$.

$$Q_j C w = C Q_{j-1} w$$

so

$$Cw = \sum_{j=0}^{\infty} C Q_j w.$$

If $u \in K_i$, then a simple manipulation yields

$$\langle P C Q_j w, u \rangle = \langle C P_j w, u \rangle$$

so

$$P C Q_j w = C P_j w.$$

Thus,

$$PCw = \sum_{j=0}^{\infty} P C Q_j w = \sum_{j=0}^{\infty} C P_j w = CPw,$$

and so P is a projection such that

$$PC \subset CP.$$

Since z was arbitrary, the irreducibility implies that $P = 0$ or $P = I$. Since $v \in \mathscr{D}$, $P \neq 0$. Therefore $P = I$ and $\mathscr{D}$ is dense.

Each element of $\mathscr{D}$ is a finite linear combination of eigenvectors of N. Every K_j is invariant under $|C| = (C^*C)^{1/2}$ and $|C|$ is bounded on K_j so each eigenvector of N is an analytic vector for $|C|$. (See [8, Section X.6] for a discussion of analytic vectors.) Thus $\mathscr{D}$ is a total set of analytic vectors for $|C|$ and is therefore a core for $|C|$ [8, Theorem X.39]. $\mathscr{D}$ is therefore also a core for C. (See [11, Lemma 2.1].) A similar argument applies to C^*. ∎

We now continue the proof of Theorem 1. If $w \in \mathscr{D}$ then

$$C^*(z)\, C(z) w = C(z)\, C^*(z) w + \| z \|^2 w$$

and polarization of this yields

$$C^*(z)\, C(y) w = C(y)\, C^*(z) w + \langle y, z \rangle w. \tag{2.4}$$

This, and

$$C^*(z)v = 0 \tag{2.5}$$

imply that any vector in $\mathscr{D}$ is equal to an element in the form $A'v$ with $A' \in \mathscr{A}'$. If w_1 and $w_2 \in \mathscr{D}$, $w_1 = A_1 v$, $w_2 = A_2 v$ with $A_1, A_2 \in \mathscr{A}$, then

$$\langle w_1, w_2 \rangle = \langle A_1 v, A_2 v \rangle = \langle v, A^\dagger{}_1 A_2 v \rangle$$

where $A^\dagger{}_1$ is the formal adjoint of A_1. Since $A^\dagger{}_1 A_2 \in \mathscr{A}$, $A^\dagger{}_1 A_2 v \in \mathscr{A}'v$. Suppose $A^\dagger{}_1 A_2 v = A'v$ with $A' \in \mathscr{A}'$. A' is a sum of monomials, each of which maps v into a vector orthogonal to v unless the monomial is a multiple of the identity. Thus, the inner products of elements of $\mathscr{D}$ are determined by (2.4) and (2.5).

If $\{H, C', K'\}$ is the Fock–Cook Boson structure over H with vacuum v', then we can construct a unitary operator $\Phi\colon K \to K'$ by requiring that if $A \in \mathscr{A}$, $\Phi(Av)$ is the vector in K' obtained by replacing each $C(z)$ in A by $C'(z)$, each $C^*(z)$ by $C'^*(z)$ and v by v'. Φ is well defined because it preserves inner products. $C(z)$ and $\Phi^{-1}C'(z)\Phi$ agree on $\mathscr{D}$. $\mathscr{D}$ is a core for $C(z)$ and $\Phi\mathscr{D}$ is a core for $C'(z)$ so $C(z)$ and $\Phi^{-1}C'(z)\Phi$ are equal. ∎

The proof of Theorem 2 is similar to the one given for Theorem 1. It is easier in that the creation operators must be bounded so questions about domains do not arise.

Proof of Theorem 2. Let z be a fixed unit vector, $C = C(z)$ and $n = n(z)$. We first show that the spectrum of n contains only 0 and 1. Equation (2.2) implies that $0 \leqslant n \leqslant 1$. Differentiation of (1.16) with respect to t at $t = 0$ gives

$$[n, C] = C. \tag{2.6}$$

On the other hand, (2.2) gives

$$\begin{aligned}[n, C] &= nC - Cn \\ &= CC^*C - C^2C^* \\ &= C(I - CC^*) - C^2C^* \\ &= C - 2C^2C^*.\end{aligned}$$

Therefore $C^2C^* = Cn = 0$ so

$$C^*Cn = 0,$$

$$(I - n)n = 0.$$

Thus n has spectrum consisting solely of 0 and 1. From (2.6) it follows that

$$nC^2 = C^2(2 + n)$$

so that if w is an eigenvector of n then $C^2 w = 0$ for otherwise it would be an eigenvector of n with eigenvalue at least 2. Thus $C^2 = 0$.

Polarization of (2.2) and $C(z)^2 = 0$ give

$$C^*(z)\, C(y) + C(y)\, C^*(z) = \langle y, z \rangle, \tag{2.7}$$

$$C(z)\, C(y) + C(y)\, C(z) = 0. \tag{2.8}$$

It is well known that the irreducibility together with (2.7) and (2.8) give uniqueness. See, for example, [7, Theorems 4.14.1 and 4.15.1]. ▮

When the number of degrees of freedom is infinite (H is infinite dimensional) there are many representations of the Boson and Fermion relations. (See, for example, [3, 4].) In order to insure uniqueness we assume we have a positive invariant quantum structure.

THEOREM 3. *Suppose* $\{H, C, K, \Gamma, v\}$ *is an irreducible positive invariant quantum structure over* H *such that for all* $z \in H$,

$$C^*(z)\, C(z) = C(z)\, C^*(z) + \| z \|^2$$

and

$$n(z) = C(z)\, C^*(z)$$

is a number operator for z *when* z *is a unit vector. Then* $\{H, C, K, \Gamma, v\}$ *is unitarily equivalent to the Fock–Cook Boson invariant quantum structure over* H.

THEOREM 4. *Suppose* $\{H, C, K, \Gamma, v\}$ *is an irreducible positive invariant quantum structure over* H *such that for all* $z \in H$,

$$C^*(z)\, C(z) + C(z)\, C^*(z) = \| z \|^2$$

and

$$n(z) = C(z)\, C^*(z)$$

is a number operator for z *when* z *is a unit vector. Then* $\{H, C, K, \Gamma, v\}$ *is unitarily equivalent to the Fock–Cook Fermion invariant quantum structure over* H.

Proof of Theorem 3. $d\Gamma(P_z)$ is also a number operator for z and so by the irreducibility, for each unit vector $z \in H$ there is a scalar, $\alpha(z)$, such that

$$d\Gamma(P_z) = C(z)\, C^*(z) + \alpha(z).$$

$d\Gamma(P_z)v = 0$ so $v \in \mathrm{Dom}(d\Gamma(P_z))$ and thus $v \in \mathrm{Dom}(C^*(z))$. As noted in Section 1, the positivity of Γ implies that $C^*(z)v = 0$ so $\alpha(z) = 0$. $N = d\Gamma(I)$ is the normalized total number operator and $Nv = 0$. Since positivity implies

$$d\Gamma(P_z) \leqslant d\Gamma(I),$$

$$C(z)\, C^*(z) \leqslant N.$$

Thus, Lemma 1 is applicable. As in Theorem 1 inner products of elements of Av are determined and we have uniqueness. ▌

Proof of Theorem 4. As in Theorem 2, (2.7) and (2.8) hold. As in Theorem 3, $C^*(z)v = 0$ for all $z \in H$. It is well known that the Fermion relations together with the existence of a vacuum imply uniqueness. See, for example, [7, Theorem 4.14.1]. ▌

3. Parabosons and Parafermions

We now show that similar results hold for the paraboson and parafermion fields. The paraboson field relations

$$[[C^*(x), C(z)]_+, C(y)] = 2\langle y, x\rangle C(z), \tag{3.1}$$

$$[[C(x), C(z)]_+, C(y)] = 0 \tag{3.2}$$

and the parafermion field relations

$$[[C^*(x), C(z)], C(y)] = -2\langle y, x\rangle C(z), \tag{3.3}$$

$$[[C(x), C(z)], C(y)] = 0 \tag{3.4}$$

were introduced by Green [5] in 1953. They generalize the Boson and Fermion relations in the sense that operators which (formally) satisfy (1.1) and (1.2) also (formally) satisfy (3.1) and (3.2) while operators which satisfy (1.3) and (1.4) also satisfy (3.3) and (3.4). The most important representations of the paraboson and parafermion relations are described in [6]. See also [9]. For a fixed complex Hilbert space H and for each positive integer p, an irreducible representation of the paraboson (respectively, parafermion) relations can be constructed from p copies of the free Fock–Cook Boson (respectively, Fermion) field over H in such a way that it canonically inherits a positive invariant quantum structure from the Fock–Cook fields. We will call this the free paraboson (parafermion) invariant quantum structure of order p over H. For $p = 1$, this reduces to the Fock–Cook Boson (Fermion) invariant quantum structure. The quantum structures of different orders are inequivalent.

We will prove the following theorems.

Theorem 5. *Suppose $\{H, C, K, \Gamma, v\}$ is an irreducible positive invariant quantum structure over the infinite dimensional Hilbert space H such that for each $z \in H$, $C^*(z)\, C(z)$ and $C(z)\, C^*(z)$ commute and*

$$n(z) = \tfrac{1}{2}(C(z)\, C^*(z) + C^*(z)\, C(z))$$

is a number operator for z when z is a unit vector. Then for some positive integer p,

$\{H, C, K, \Gamma, v\}$ is unitarily equivalent to the free paraboson invariant quantum structure of order p over H.

THEOREM 6. *Suppose $\{H, C, K, \Gamma, v\}$ is a bounded, irreducible positive invariant quantum structure over H such that for each $z \in H$,*

$$n(z) = \tfrac{1}{2}(C(z)\, C^*(z) - C^*(z)\, C(z))$$

is a number operator for z when z is a unit vector. Then for some positive integer p, $\{H, C, K, \Gamma, v\}$ is unitarily equivalent to the free parafermion invariant quantum structure of order p over H.

The following general lemma will be of use.

LEMMA 2. *Suppose $\{H, C, K\}$, N and v satisfy the hypotheses of Lemma 1 and let z be a fixed unit vector in H. Suppose there exists a number operator, n, for z which has v as an eigenvector. Then for $w \in \mathscr{A}v$,*

$$[n, C(y)]w = \langle y, z\rangle\, C(z)w \tag{3.5}$$

for all $y \in H$.

Proof. It is sufficient to show that (3.5) holds for all vectors in the form $w = Av$ where A is a monomial in $\mathscr{A}_z$. For such a monomial, define the length of A to be the sum of the number of creation operators and the number of annihilation operators in A. Suppose $nv = \alpha v$. By induction on the length of A it is easy to see that w is an eigenvector of n with eigenvalue equal to α plus the number of creation operators, $C(y)$, in A with y parallel to z minus the number of annihilation operators, $C^*(y)$, with y parallel to z. If y is parallel to z,

$$nC(y)w = C(y)(n + 1)w$$

so

$$[n, C(y)]w = C(y)w,$$

while if y is orthogonal to z,

$$nC(y)w = C(y)nw$$

so

$$[n, C(y)]w = 0.$$

By representing a general $y \in H$ as a sum of vectors parallel and orthogonal to z and using the linearity of $C(\cdot)$, (3.5) follows. ∎

Proof of Theorem 5. Since $C^*(z)\, C(z)$ and $C(z)\, C^*(z)$ are positive and commute, their sum is closed so $n(z)$ is self-adjoint. $N = d\Gamma(I)$ is the normalized total number operator and $Nv = 0$. If z is a unit vector then $d\Gamma(P_z)$ and $n(z)$ are both number operators for z so there is a scalar $\alpha(z)$ such that

$$d\Gamma(P_z) = n(z) - \alpha(z).$$

Since $d\Gamma(P_z) \leqslant d\Gamma(I) = N$, $n(z)$ and thus $C(z)\,C^*(z)$ and $C^*(z)\,C(z)$ are bounded on each eigenspace of N. Thus, Lemma 1 applies. Since for every $z \in H$,

$$C^*(z)v = 0, \tag{3.6}$$
$$C^*(z)\,C(z)v = 2\alpha(z)v.$$

If U is a unitary operator on H,

$$\Gamma(U)\,C^*(z)\,C(z)v = 2\alpha(z)\,\Gamma(U)v = 2\alpha(z)v$$
$$\Gamma(U)\,C^*(z)\,C(z)\,\Gamma(U)^{-1}\,v = 2\alpha(z)v$$
$$C^*(Uz)\,C(Uz)v = 2\alpha(z)v.$$

Thus, $\alpha(z) = \alpha(Uz)$ and so $\alpha(z)$ is independent of z when z is a unit vector. Let $\alpha(z) = \frac{1}{2}p$. Then for arbitrary $z \in H$,

$$C^*(z)\,C(z)v = p \,\|\, z \,\|^2\, v. \tag{3.7}$$

Polarization of this gives

$$C^*(y)\,C(z)v = p\langle z, y\rangle v. \tag{3.8}$$

By Lemma 2, on $\mathscr{A}v$,

$$[n(z), C(y)] = \langle y, z\rangle\, C(z) \tag{3.9}$$

and polarization of this (in z) gives that on $\mathscr{A}v$, (3.1) holds. It has been shown [6, Appendix 2] that (3.1), (3.6) and (3.8) imply that p is a positive integer when H is infinite dimensional. Inner products of elements of $\mathscr{A}v$ are determined by (3.1), (3.6) and (3.8). Since $\mathscr{A}v$ is a core for $C(z)$, we may now construct a unitary equivalence (as in Theorem 1) between $\{H, C, K, \Gamma, v\}$ and the free paraboson invariant structure of order p over H. ▌

Theorem 5 should be compared with [12, Theorem 2] in which irreducibility is not assumed but instead v is required to be an analytic vector for $n(z)$ and $\mathscr{A}v$ is assumed to be dense.

We give a proof of Theorem 6 which is similar to that of Theorem 5 using Lemmas 1 and 2. These, of course, are much more powerful than are needed here since $C(z)$ is now bounded and thus the domain questions do not arise. However, these lemmas, now that they have been established, provide a short proof of Theorem 6.

Proof of Theorem 6. The relation

$$e^{itn(z)}C(z)\, e^{-itn(z)} = e^{it}C(z)$$

and its adjoint can be used to show that $n(z)$ commutes with $C(z)\, C^*(z)$ and thus $C(z)\, C^*(z)$ and $C^*(z)\, C(z)$ commute. $N = d\Gamma(I)$ and v satisfy the hypotheses of Lemma 1. As in the proof of Theorem 5,

$$d\Gamma(P_z) = n(z) - \alpha(z)$$

where $\alpha(z)$ is again independent of z but this time $\alpha(z)$ is negative. If $\alpha(z) = -\frac{1}{2}p$, (3.6) and (3.8) hold. Relation (3.9) is now satisfied on all of K and polarization gives (3.3). Again, (3.3), (3.6) and (3.8) imply that p is a positive integer [6, Appendix 1] (even when H is finite dimensional). Thus, $\{H, C, K, \Gamma, v\}$ is unitarily equivalent to the free parafermion invariant quantum structure of order p over H. ▌

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