Complex Dynamics in a Ratio-dependent Food-chain Model with Beddington-DeAngelis Functional Response

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Abstract

In this paper, a ratio-dependent food-chain model is studied, and the local stability of the positive equilibrium point of the system is analyzed. By means of computer simulation, rich system dynamics are revealed: chaos, period-doubling bifurcations, and others. From the bifurcation diagram with different bifurcation parameters, the strong influence of intra-species competition can be observed. Finally, Poincaré sections and power spectra are used to demonstrate chaotic dynamics.

Keyword: Chaos, Ratio-dependent food-chain, Poincaré section, Power spectra

1. Introduction

Ratio-dependent three-species food-chain models have been studied by many researchers, especially Gakkhar and Naji Gakkhar et al.,2002 Gakkhar et al.,2005 Gakkhar et al.,2006 Gakkhar et al.,2007 Naji et al.,2007 . They have studied these models for a long time and have shown that these models can display rich dynamics such as chaos, quasiperiodic patterns, and other complex behaviors. However, in their models, intra-species competition was seldom considered. Recently, population models with Beddington-DeAngelis functional response have been widely investigated, and these models also have complex dynamics Aziz-Alaoui et al.,2002; Dimitrov et al.,2005 Lv et al.,2008 Zhao et al.,2009 .

In 2002, Gakkhar proposed a trophic-level model based on the Berrian model Gakkhar et al.,2002 :

\[
\begin{align*}
\frac{dX_1}{dT} &= a_1 - b_1X_1 - \frac{d_1X_2}{\gamma_1 + X_1} \\
\frac{dX_2}{dT} &= a_2 - b_2X_2 - \frac{d_2X_3}{\gamma_2 + X_2} \\
\frac{dX_3}{dT} &= a_3 - b_3X_3 - \frac{d_3X_1}{\gamma_3 + X_3}
\end{align*}
\]

In this model, they didn’t consider the intra-specific competition of species \( X_1 \) and species \( X_2 \), thus simplifying the model. In the present paper, the following model is proposed:

\[
\begin{align*}
\frac{dx_1}{dt} &= a_1x_1 - b_1x_1^2 - \frac{d_1x_1x_2}{e_1 + x_1 + f_1x_2} \\
\frac{dx_2}{dt} &= a_2x_2 - \frac{x_2^2}{c_1x_1} - \frac{d_2x_1x_3}{e_2 + x_2 + f_2x_3} \\
\frac{dx_3}{dt} &= a_3x_3 - b_3x_3^2 - \frac{x_3^2}{c_2x_2}
\end{align*}
\] (1.1)

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where $x_1, x_2, x_3$ represent the densities of the three species, $\frac{b_i}{a_i} (i = 1, 3)$ represent the carrying capacity of species $x_i - x_3$, $b_i (i = 1, 3)$ represent the intra-specific competition in species $x_i - x_3$, $d_i x_i x_j / (e_i + x_i + f_i x_j)$, $d_2 x_i x_j / (e_i + x_j + f_i x_j)$ are Beddington-DeAngelis response functions, and $x_{i+1}^2 / (c_{i+1} x_i) (i = 1, 2)$ represent the dependency of the density of species $x_{i+1}$ on the density of species $x_i$. Because of the biological significance of this model, it will be studied in the space of the following system: $R^3_+ = \{(x,y,z) | x > 0, y > 0, z > 0\}$.

2. Local stability of equilibria

In this section, the model is investigated using mathematical tools. Because the model is complex, only the local stability of its equilibria will be analyzed.

By calculation, it is apparent that the system has two types of equilibria: boundary equilibrium point $E_1 = (x_1^*, x_2^*, 0)$ and positive equilibrium point $E_2 = (x_1^*, x_2^*, x_3^*)$. Next, the Jacobean matrix is used to analyze the local stability of the equilibrium points.

(1) The Jacobean matrix of the system at the equilibrium point $E_1 = (x_1^*, x_2^*, 0)$ can be analyzed using the following subsystem:

$$
\begin{align*}
\frac{dx_1}{dt} &= a_1 - b_1 x_1^2 - \frac{d_1 x_1 x_2}{e_1 + x_1 + f_1 x_2}, \\
\frac{dx_2}{dt} &= a_2 x_2 - \frac{x_2^2}{c_2 x_1}.
\end{align*}
$$

When the system degenerates to the $(x_1, x_2)$ plane, it is obvious that the equilibrium point $E_1 = (x_1^*, x_2^*, 0)$ has the same dynamic behavior as the equilibrium point $E_{11} = (x_1^*, x_2^*)$. The Jacobean matrix of the subsystem at the equilibrium point $E_{11} = (x_1^*, x_2^*)$ is

$$J(x_1^*, x_2^*) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

with

$$
\begin{align*}
G_{11} &= a_{11} e_1^2 + 2 a_{11} x_1^* e_1 + 2 a_{11} e_1 f_1 x_1^* + a_{11} x_1^{*2} + 2 a_{11} x_1^* f_1 x_2^* + a_{11} f_1^2 x_2^{*2} - 2 b_1 x_1^* e_1^2 \\
&\quad - 4 b_1 x_1^{*2} e_1 - 4 b_1 x_1^* e_1 f_1 x_2^* - 2 b_1 x_1^* f_1 x_2^* - 4 b_1 x_1^* f_1^2 x_2^{*2} - 2 b_1 x_1^* f_1^2 x_2^{*2} + 2 b_1 x_1^* f_1^2 x_2^{*2} - 2 b_1 x_1^* f_1^2 x_2^{*2} + 2 b_1 x_1^* f_1^2 x_2^{*2}, \\
&\quad - d_1 x_2^* e_1 - d_1 x_2^* f_1 (e_1 + x_1^* + f_1 x_2^*)^2 \\
G_{12} &= \frac{-d_1 x_2^* (e_1 + x_1^*)}{(e_1 + x_1^* + f_1 x_2^*)^2}, \\
G_{21} &= \frac{x_2^{*2}}{c_2 x_1^*}, \\
G_{22} &= \frac{a_{22} c_2 x_1^* - 2 x_2^*}{c_2 x_1^*}.
\end{align*}
$$

The characteristic equation of $J(x_1^*, x_2^*)$ is $\lambda^2 + \delta_1 \lambda + \delta_2 = 0$, where $\delta_1 = -(G_{11} + G_{22})$, $\delta_2 = G_{11} G_{22} - G_{12} G_{21}$. According to the Routh-Hurwitz criterion, the equilibrium point $E_{11} = (x_1^*, x_2^*)$ is locally and asymptotically stable if and only if $\delta_i > 0 (i = 1, 2)$. Therefore, $E_1 = (x_1^*, x_2^*, 0)$ is locally and asymptotically stable if and only if $\delta_i > 0 (i = 1, 2)$.

(2) The Jacobean matrix of the system at the equilibrium point $E_2 = (x_1^*, x_2^*, x_3^*)$ is:

$$
J(x_1^*, x_2^*, x_3^*) = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},
$$

with
The characteristic equation of $\lambda^3 + \sigma_1 \lambda^2 + \sigma_2 \lambda + \sigma_3 = 0$, where

\[
\sigma_1 = -(H_{11} + H_{22} + H_{33}),
\]

\[
\sigma_2 = H_{11}H_{22} - H_{12}H_{21} - H_{23}H_{32} + H_{13}H_{33},
\]

\[
\sigma_3 = H_{11}H_{23}H_{32} - H_{12}H_{21}H_{33} + H_{13}H_{22}H_{31}.
\]

According to the Routh-Hurwitz criterion, $E_2 = (x^*, y^*, z^*)$ is locally and asymptotically stable if and only $\sigma_1 > 0, \sigma_3 > 0, \sigma_2 > \sigma_3$. 

3. Numerical analysis

Because the system described above does not have an analytical solution, computer-based simulation is used to investigate its long-term dynamic behavior.

3.1. Bifurcation analysis

Bifurcation diagrams were plotted with different bifurcation parameters, varying one bifurcation parameter at a time while keeping the others fixed.

Figure 1 shows the bifurcation diagram of the system with bifurcation parameter $b_1$. The ordinate is the maximum for species $x_1$. The range of variation of $b_1$ is $0.1 \leq b_1 \leq 0.6$. From the diagram, it can be seen that the system displays period-doubling bifurcation moving to chaos, chaotic crises, periodic windows, and other complex behaviors. To make these patterns more clearly visible, Figure 1 was magnified (Figure 2). When $b_1$ slightly increases, it suddenly moves into chaos and has a long chaotic region: $[0.3, 0.43]$. After that, it goes to period-halving bifurcation and moves into chaos a second time. In the interval $[0.5, 0.585]$, the system from chaos $\rightarrow$ period $\rightarrow$ chaos several times. In the end, it displays limit cycle.
Figure 1. Bifurcation diagram with bifurcation parameter $b_1$: $0.1 \leq b_1 \leq 0.6$.

Figure 2. Magnification of Figure 1: (a) $0.1 \leq b_1 \leq 0.35$; (b) $0.35 \leq b_1 \leq 0.6$.

Figure 3. Bifurcation diagram with bifurcation parameter $b_3$: $0 \leq b_3 \leq 0.3$.

Figure 3 shows the bifurcation diagram of the system with bifurcation parameter $b_3$. The ordinate is the maximum of the species $x_3$. The range of variation of $b_3$ is $0.001 \leq b_3 \leq 0.3$. From the diagram, the complex dynamics of the system can be observed, and it can be seen that the dynamics are more complex than those with bifurcation parameter $b_1$:

period two $\rightarrow$ period four $\rightarrow$ chaos $\rightarrow$ period four $\rightarrow$ chaos $\rightarrow$ limit cycle.

In the chaotic region $[0.044, 0.124]$, complex dynamics can be found, the system from chaos $\rightarrow$ period $\rightarrow$ chaos several times and displays period windows, tangent bifurcations, period-doubling bifurcations, period-halving bifurcations and so on. To make these more clearly visible, Figure 3 was magnified (Figure 4).
Figure 4. Magnification of Figure 3: (a) $0 \leq b_3 \leq 0.085$; (b) $0.085 \leq b_3 \leq 0.17$.

Poincaré sections and power spectra (Randall et al., 1988) were then used to prove the chaotic dynamics of the system. Figure 5 displays a chaotic attractor with its Poincaré section and power spectrum. $10^5$ points were calculated to plot the Poincaré section. From the diagram, it can be seen that the Poincaré section of the chaotic attractor consists of concentrated points following an arc, so the attractor is chaotic. The power-spectrum diagram has broadband noise and irregular scattered spikes, so it also proves that the attractor is chaotic.

4. Conclusions

In this paper, complex dynamic behaviors of a system have been found. By means of bifurcation diagrams, it can be seen that these system behaviors are sensitive to the bifurcation parameters, especially those parameters which represent the interactions between species under density restrictions. When these parameters are varied, the dynamics of the system become more complex, including period-doubling bifurcations, chaos, periodic
windows, and other complex behaviors. Using Poincaré sections and power spectra, chaotic dynamics have been demonstrated.

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References