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Advances in Mathematics 226 (2011) 4796–4815

ADVANCES IN Mathematics

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On the K-stability of complete intersections in polarized manifolds

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Received 7 February 2008; accepted 17 December 2010

Available online 8 January 2011

Communicated by Gang Tian

Abstract

We consider the problem of existence of constant scalar curvature Kähler metrics on complete intersections of sections of vector bundles. In particular we give general formulas relating the Futaki invariant of such a manifold to the weight of sections defining it and to the Futaki invariant of the ambient manifold. As applications we give a new Mukai–Umemura–Tian like example of Fano 5-fold admitting no Kähler–Einstein metric, and a strong evidence of K-stability of complete intersections in Grassmannians.

Keywords: Futaki invariant; Complete intersection; K-stability; Constant scalar curvature Kähler metric; Kähler–Einstein metric; Fano manifold

1. Introduction

The problem of determining which manifolds admit a Kähler constant scalar curvature metric (Kcsc), and in which Kähler classes, is by now a central one in differential geometry and it has been approached with a variety of geometric and analytical methods.

A classical result due to Matsushima and Lichnerowicz [19,16] shows that a such a manifolds have a reductive identity component of the automorphisms group, a condition unsensitive

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.12.018

of the Kähler class where we look for the Kcsc metric. In the eighties Futaki [11], later generalized by Calabi [3], introduced an invariant, since then called *the Futaki invariant*, sensitive of the Kähler class. The deep nature of this invariant has stimulated a great amount of research. While it can be used directly to show that a manifold M does not have a Kcsc metric in a Kähler class, a more refined analysis, mainly due to Ding and Tian [5], Tian [27], Paul and Tian [22] and Donaldson [7], has led to relate this invariant on a manifold M to the existence of Kcsc metric on *any* manifold degenerating in a suitable sense to M.

This idea has been formalized in precise conjectures due to Tian ([26], Conjecture 1.4 in [27] and Conjecture 6.2 in [28]) and Donaldson [7] relating the existence of such metrics to the K-stability of the polarized manifold. The key point relevant for our paper is that the knowledge of the Futaki invariant gives informations on the existence of Kcsc metrics on the manifolds on which the calculations are carried on *and also* on any Kähler manifold degenerating on it.

The problem of calculating explicitely the Futaki invariant of a polarized manifold has then got further importance. Its original analytical definition is extremely hard to use, since requires an explicit knowledge of the Ricci potential and of the Kähler metric, data which are almost always missing. On the other hand it led to the discovery of the so-called *localization formulae* [12,25] which have been a very useful tool in this problem. Yet, they require an explicit knowledge of the space of holomorphic vector fields and of the Kähler metric which is again very hard to have.

Finally Donaldson [7] gave a purely algebraic interpretation of the Futaki invariant, extending it to singular varieties and schemes, which is the one we use in this paper and that will be recalled in Section 2. Let us just recall at this point that the Futaki invariant is defined for a polarized scheme (M, L) endowed with a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ that linearizes on L (hence a holomorphic vector field η_ρ). We will then denote thorough this paper such a structure by (M, L, ρ) and by $F(M, L, \rho)$ the Futaki invariant of η_ρ in the class $c_1(L)$ of this triple.

We can now describe our result. We assume that we are given a polarized variety (M, L) endowed with a $\mathbb{C}^\times$-action that linearizes on L. If $X \subset M$ is an invariant complete intersection of sections of holomorphic vector bundles $E_1, \dots, E_s$ on M, we will show that is possible to express $F(X, L|_X, \rho)$ in terms of the weights of sections defining X and holomorphic invariants of the bundles E_j's and L.

In this paper we make explicit the formula in two relevant cases: the first, when L is the anti-canonical bundle K_M^{-1} of M and all E_j's are isomorphic to a fixed vector bundle E such that $\det E$ is a (rational) multiple of L as linearized vector bundle; the second, when each E_j is isomorphic to some power L^{r_j} of the polarizing line bundle. We do not state the formula for the general case, but it can be recovered through some calculations from Lemmata 5.2 and 5.3.

Let us consider the first case. Let E be a $\mathbb{C}^\times$-linearized holomorphic vector bundle on a smooth Fano manifold M such that $(\det E)^q = K_M^{-p}$ for some integers p, q. For each $j \in \{1, \dots, s\}$ let $\sigma_j \in H^0(M, E)$ be a non-zero holomorphic semi-invariant section, in other words there exists $\alpha_j \in \mathbb{Z}$ such that $\rho(t) \cdot \sigma_j = t^{\alpha_j} \sigma_j$. Thus the zero locus $X_j = \sigma_j^{-1}(0)$ is ρ-invariant and $L = \det E$ restrict to a linearized ample line bundle on X_j. Consider the intersection $X = \bigcap_{j=1}^s X_j$ and assume that $\dim(X) = n - sk$, being $k = \mathrm{rank}(E)$. Moreover, by adjunction, X is a possibly singular Fano variety if $q - ps > 0$. Our first result is the following

Theorem 1.1. *Under the above conventions and assumptions we have*

$$F(X, L|_X, \rho) = \frac{ps - q}{2q} \frac{a_0(X, L|_X)}{d_0(X, L|_X)} - \frac{k}{2} \sum_{j=1}^{s} \alpha_j, \tag{1}$$

where $d_0(X, L|_X)$ and $a_0(X, L|_X)$ are respectively the degree of $(X, L|_X)$ and its equivariant analogue (see Definition 2.1) and can be computed by means of holomorphic invariants of E and the quantity $\sum_{j=1}^{s} \alpha_j$.

The above theorem gives a significant simplification of the Donaldson version of the Futaki invariant (Definition 2.1) in that the above formula involves only a_0 and d_0 and not a_1 and d_1 which are in general much harder to compute. In general, in the anti-canonical class we have a formula involving just a_0 and d_0 only if we fix the linearization (see Remark 2.2). Nevertheless we notice that in theorem above the linearization is free to vary.

It is also important to notice that $\sum_{j=1}^{s} \alpha_j$ is nothing but the Mumford weight of the plane $P = \operatorname{span}\{\sigma_j\} \in Gr(s, H^0(M, E))$. With an additional hypothesis on the linearization of the given $\mathbb{C}^\times$-action on E, theorem above gives the following

Corollary 1.2. *Under the above conventions and assumptions, if the $\mathbb{C}^\times$-linearization on E satisfies $\int_M c_k^G(E)^s c_1^G(E)^{n-sk+1} = 0$, then*

$$F\bigl(X, K_X^{-1}, \rho\bigr) = -CT \sum_{j=1}^{s} \alpha_j, \tag{2}$$

where

$$C = \left(2p(n - sk + 1) \int_M c_k(E)^s c_1(E)^{n-sk}\right)^{-1} > 0$$

and

$$T = kp(n - sk + 1) \int_M c_k(E)^s c_1(E)^{n-sk} - (q - ps) \int_M c_k(E)^{s-1} c_{k-1}(E) c_1(E)^{n-sk+1}$$

are characteristic numbers of E (independent of the $\mathbb{C}^\times$-linearization).

The interest in the above corollary is twofold. On the one hand it relates two very natural, and a priori unrelated, invariants of the manifold X in a completely general setting. On the other hand it generalizes a special case, proved by completely different ad hoc arguments by Tian [27], used to produce the first (and up to now the only) examples of smooth Fano manifolds with discrete automorphism group without Kähler–Einstein metrics.

Another application of our study is that if (M, L) is a complex Grassmannian anti-canonically polarized and P is a *generic* subspace of $H^0(M, E)$, in a sense explained in Section 6, then X_P degenerates onto an X_{P_0} whose Futaki invariant is positive, hence hinting at the K-stability of this type of manifolds. In particular this gives strong evidence to K-stability of these manifolds if their moduli space is discrete.

Of course the above corollary rises the question whether T has a specific sign. We do not believe in general this to be the case, but we describe some classes of examples for which we can conclude, thanks to a theorem of Beltrametti, Schneider and Sommese [1], that T is indeed positive (see also Remark 3.4).

Our second type of results comes from looking at classes different form the canonical one. We will restrict ourselves to the case when the bundles where to choose the sections are all line bundles and are all (possibly varying) powers of a fixed line bundle L. Thus if L is sufficiently positive we can embed M in a projective space $\mathbb{P}^N$ and X is the intersection of M with a number of hypersurfaces. We are then interpreting our results in terms of Kcsc metrics in $c_1(L)$. This situation has been previously studied by Lu [17] in the case when the ambient manifold is a projective space (see also [18]). Again our result has a computational interest in that it makes very easy to calculate the Futaki invariant for a great variety of manifolds, but also a conceptual one that we underline in the following

Corollary 1.3. *Let (M, L) be an n-dimensional polarized manifold endowed with a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ and a linearization on L. For each $j \in \{1, \dots, s\}$ consider a section $\sigma_j \in H^0(M, L^r)$ such that $\rho(t) \cdot \sigma_j = t^{\alpha_j}\sigma_j$ for some $\alpha_j \in \mathbb{Z}$. Let $X = \bigcap_{j=1}^s \sigma^{-1}(0)$. Suppose $\dim(X) = n - s$, then*

$$F(X, L|_X, \rho) = F(M, L, \rho) - C\mu(X, M, L, \rho),$$

where $\mu(X, M, L, \rho)$, is the Chow weight of the polarized manifold (see Section 4 for the definition) and $C \in \mathbb{R}$.

If moreover $\mathrm{Pic}(M) = \mathbb{Z}$ then $C \geqslant 0$ with equality if and only if $M \simeq \mathbb{P}^n$ and $L = \mathcal{O}(1)$. In particular, if $\mathrm{Pic}(M) = \mathbb{Z}$, M has a Kcsc in $c_1(L)$ and $Y \subset M$ is a non-necessarily invariant complete intersection of sections of L^r which is K-semi-stable, then Y is Chow semi-stable as a submanifold of M.

The relevance of this last statement is that the conclusion is *not about asymptotic Chow stability* of $(Y, L|_Y)$, which is known to be related by a result of Donaldson [6] to the existence of Kcsc metrics. For example, even in the very special case of hypersurfaces of projective spaces, this gives strong further evidence of their K-semi-stability (cf. Tian [24]). We believe that the assumption on $\mathrm{Pic}(M)$ should be unnecessary for the same conclusion to hold, yet our proof requires at present this type of hypothesis.

Having dropped the assumption on the smoothness of X we can use our formulae for singular varieties which arise as central fiber of test configurations. We give in Section 6 an explicit example of this situation with a central fiber of our type with non-positive Futaki invariant, hence producing non-Kcsc manifolds (the degenerating ones).

Another explicit application of our formulae comes when looking at the quintic Del Pezzo threefold, X_5, for which it was not known whether it admits a Kcsc metric. In fact our analysis shows that it is K-stable, when confining to those test configurations whose central fibers are still manifolds of the type considered in our paper. While we believe a complete algebraic proof of its K-stability is then at hand, showing that every other test configuration do not destabilizes, we remark that very recently X_5 (which is rigid in moduli) has proven to be Kähler–Einstein. This follows by the argument used by Donaldson to prove that the Mukai–Umemura threefold is Kähler–Einstein, or directly by the work of Cheltsov and Shramov [4].

Unfortunately the other Fano threefolds with $\mathrm{Pic} = \mathbb{Z}$ for which the existence of a canonical metric is unknown, when smooth do not have continuous automorphisms. If we take singular ones defined by sections of the appropriate bundles with non-positive Futaki invariant, we still cannot find test configurations with smooth general fibers. We leave this important problem for further research.

2. Preliminaries

Let us recall the fundamental concepts leading to the Tian–Yau–Donaldson conjecture. We will follow essentially Donaldson's language [7] which is more suitable for our purposes, though, as mentioned in the introduction, the reader is referred to the papers [5,22,26] and [27] for the different approaches leading to these problems.

Definition 2.1. Let (V, L) be an n-dimensional polarized variety or scheme. Given a one parameter subgroup $\rho : \mathbb{C}^\times \to \mathrm{Aut}(V)$ with a linearization on L and denoted by $w(V, L)$ the weight of the $\mathbb{C}^\times$-action induced on $\bigwedge^{\mathrm{top}} H^0(V, L)$, we have the following asymptotic expansions as $k \gg 0$:

$$w\big(V, L^m\big) = a_0(V, L)m^{n+1} + a_1(V, L)m^n + O\big(m^{n-1}\big), \tag{3}$$

$$h^0\big(V, L^m\big) = d_0(V, L)m^n + d_1(V, L)m^{n-1} + O\big(m^{n-2}\big). \tag{4}$$

The (normalized) *Futaki invariant* of the action is

$$F(V, L, \rho) = \frac{a_0(V, L)\, d_1(V, L)}{d_0(V, L)^2} - \frac{a_1(V, L)}{d_0(V, L)}.$$

Remark 2.2. Is not difficult to see that the Futaki invariant is unchanged if we replace L with some tensor power L^r, moreover it is independent of the linearization chosen on L. Unlike the general case, when V is smooth and $L = K_V^{-1}$ is the canonical bundle there is a natural linearization of the $\mathbb{C}^\times$-action ρ on L induced by the (holomorphic) tangent map

$$d\rho : TM \to TM.$$

In this case we will call L the *anti-canonical linearized bundle*.

We observe that the Futaki invariant of a polarized manifold (V, L) assumes a simple form when L is the anti-canonical linearized line bundle. Indeed, by the equivariant Riemann–Roch theorem we get $d_0(V, K_V^{-1}) = \int_V \frac{c_1(V)^n}{n!}$, $d_1(V, K_V^{-1}) = \int_V \frac{c_1(V)^n}{2(n-1)!}$, $a_0(V, K_V^{-1}) = \int_V \frac{c_1^G(V)^{n+1}}{(n+1)!}$, $a_1(V, K_V^{-1}) = \int_V \frac{c_1^G(V)^{n+1}}{2n!}$ (where c_1^G denotes the equivariant first Chern class), whence

$$F\big(V, K_V^{-1}, \rho\big) = -\frac{1}{2}\frac{a_0(V, L)}{d_0(V, L)}.$$

The relevance of the Futaki invariant is related to the definition of K-stability. To introduce it we need the following

Definition 2.3. A *test configuration* of a polarized manifold (X, L) consists of a polarized scheme $(\mathcal{X}, \mathcal{L})$ endowed with a $\mathbb{C}^\times$-action that linearizes on $\mathcal{L}$ and a flat $\mathbb{C}^\times$-equivariant map $\pi : \mathcal{X} \to \mathbb{C}$ such that $(\pi^{-1}(1), \mathcal{L}|_{\pi^{-1}(1)}) \simeq (X, L^r)$ for some $r > 0$.

When (X, L) has a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$, a test configuration where $\mathcal{X} = X \times \mathbb{C}$ and $\mathbb{C}^\times$ acts on $\mathcal{X}$ diagonally trought ρ is called *product configuration*.

Definition 2.4. The pair (X, L) is *K-stable* if for each test configuration for (X, L) the Futaki invariant of the induced action on $(\pi^{-1}(0), \mathcal{L}|_{\pi^{-1}(0)})$ is greater than or equal to zero, with equality if and only if we have a product configuration.

Finally we remark that the apparently different definition of K-stability given in [7] is due to the different choice of the sign in the definition of the Futaki invariant.

3. The case $(\det E)^q \simeq K_M^{-p}$

Theorem 3.1. *Let (M, L), with $L = K_M^{-1}$, be an n-dimensional anti-canonically polarized Fano manifold endowed with a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ and a linearization on L. Let E be a rank k linearized vector bundle on M such that $(\det E)^q \simeq L^p$ as linearized bundles for some p. For each $j \in \{1, \dots, s\}$ consider a non-zero section $\sigma_j \in H^0(M, E)$ such that $\rho(t) \cdot \sigma_j = t^{\alpha_j} \sigma_j$ for some $\alpha_j \in \mathbb{Z}$ and set $X = \bigcap_{j=1}^s \sigma_j^{-1}(0)$. If $\dim(X) = n - sk$, then we have*

$$F(X, L|_X, \rho) = \frac{ps - q}{2q} \frac{a_0(X, L|_X)}{d_0(X, L|_X)} - \frac{k}{2} \sum_{j=1}^s \alpha_j, \tag{5}$$

where

$$a_0(X, L|_X) = \int_M \frac{c_k^G(E)^s c_1^G(L)^{n-sk+1}}{(n - sk + 1)!} - \sum_{j=1}^s \alpha_j \int_M \frac{c_k(E)^{s-1} c_{k-1}(E) c_1(M)^{n-sk+1}}{(n - sk + 1)!},$$

$$d_0(X, L|_X) = \int_M \frac{c_k(E)^s c_1(M)^{n-sk}}{(n - sk)!}.$$

Remark 3.2. Clearly the linearization of E is fixed from the one of L thanks to the hypothesis $(\det E)^q \simeq L^p$ as linearized bundles. The latter is crucial to get the compact formula (5). Indeed α_j and $a_0(X, L|_X)$ depend on the linearization of E and L respectively, but on the other hand $F(X, L|_X, \rho)$ is independent of the linearization of L.

Proof of Theorem 3.1. Since $c_{sk}(E^{\oplus s}) = c_k(E)^s$, $c_1(E^{\oplus s}) = s c_1(E)$ and by hypothesis $q c_1(E) = c_1(L^p) = p c_1(M)$, by Lemma 5.2 we get

$$d_0(X) = \int_M \frac{c_k(E)^s c_1(M)^{n-sk}}{(n - sk)!},$$

$$d_1(X) = \left(1 - \frac{p}{q} s\right) \int_M \frac{c_k(E)^s c_1(M)^{n-sk}}{2(n - sk - 1)!} = \frac{(q - ps)(n - sk)}{2q} d_0(X).$$

Since $F(X, L|_X, \rho)$ is independent of the linearization on L, we are free to change it to make easier the calculations. In particular we choose on $L \simeq K_M^{-1}$ the natural linearization coming from the lifting of the $\mathbb{C}^\times$-action on the holomorphic tangent bundle TM. This gives $c_1^G(L) = c_1^G(M)$, where c_1^G denotes the equivariant first Chern class (in the Cartan model of the equivariant cohomology of M). To preserve the hypothesis we have to vary accordingly the linearization of E to have $qc_1^G(E) = c_1^G(L^p) = pc_1^G(M)$. Finally, by relations $c_{sk}^G(E^{\oplus s}) = c_k^G(E)^s$, $c_1^G(E^{\oplus s}) = sc_1^G(E)$ and Lemma 5.3 we have

$$a_0(X) = \int_M \frac{c_k^G(E)^s c_1^G(M)^{n-sk+1}}{(n-sk+1)!} - \sum_{j=1}^s \alpha_j \int_M \frac{c_k(E)^{s-1} c_{k-1}(E) c_1(M)^{n-sk+1}}{(n-sk+1)!},$$

$$\begin{aligned} a_1(X) &= \left(1 - \frac{p}{q} s\right) \int_M \frac{c_k^G(E)^s c_1^G(M)^{n-sk+1}}{2(n-sk)!} + \sum_{j=1}^s k\alpha_j \int_M \frac{c_k(E)^s c_1(M)^{n-sk}}{2(n-sk)!} \\ &\quad - \left(1 - \frac{p}{q} s\right) \sum_{j=1}^s \alpha_j \int_M \frac{c_k(E)^{s-1} c_{k-1}(E) c_1(M)^{n-sk+1}}{2(n-sk)!} \\ &= \frac{(q-ps)(n-sk+1)}{2q} a_0(X, L_X) + \frac{k}{2} d_0(X, L_X) \sum_{j=1}^s \alpha_j. \end{aligned}$$

Thus, by Definition 2.1 we get

$$\begin{aligned} F(X, L|_X, \rho) &= \frac{(q-ps)(n-sk)}{2q} \frac{a_0(X, L|_X)}{d_0(X, L|_X)} - \frac{(q-ps)(n-sk+1)}{2q} \frac{a_0(X, L|_X)}{d_0(X, L|_X)} - \frac{k}{2} \sum_{j=1}^s \alpha_j \\ &= \frac{ps-q}{2q} \frac{a_0(X, L|_X)}{d_0(X, L|_X)} - \frac{k}{2} \sum_{j=1}^s \alpha_j. \qquad \square \end{aligned}$$

When E has the right linearization, the Futaki invariant of X is a multiple of the weight $\sum_{j=1}^s \alpha_j$ of $P = \operatorname{span}\{\sigma_1, \dots, \sigma_s\}$. Indeed we have the following

Corollary 3.3. *In the situation of Theorem* 3.1*, if the choosen linearization on E satisfies $\int_M c_k^G(E)^s c_1^G(E)^{n-sk+1} = 0$ then*

$$F(X, L|_X, \rho) = -CT \sum_{j=1}^s \alpha_j, \tag{6}$$

where $C = (2p(n-sk+1)\int_M c_k(E)^s c_1(E)^{n-sk})^{-1} > 0$ and

$$T = kp(n-sk+1) \int_M c_k(E)^s c_1(E)^{n-sk} - (q-ps) \int_M c_k(E)^{s-1} c_{k-1}(E) c_1(E)^{n-sk+1}.$$

Proof. Substituting the expressions of $a_0(X, L|_X)$ and $d_0(X, L|_X)$ on (5) we get

$$F(X, L|_X, \rho) = -C\left((q-ps) \int_M c_k^G(E)^s c_1^G(E)^{n-sk+1} + T \sum_{j=1}^s \alpha_j\right),$$

and formula (6) follows immediatly by hypothesis.

To show the positivity of the constant C is enough to observe that L_X is ample and, by definition of $d_0(X, L|_X)$, the constant $1/C$ is a positive multiple of the degree of $(X, L|_X)$. $\square$

Remark 3.4. Establishing the positivity of the constant T is a problem quite delicate. At least when E is ample, one would apply the theory of Fulton and Lazarsfeld [10] to conclude that $T > 0$. This is true when $q - ps \leqslant 0$ (i.e., by adjunction formula, when X is not Fano), but unfortunately this is not true in general because the polynomial in the Chern classes defining T is not numerically positive. Nevertheless, if E is very ample (i.e. the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is very ample), then by a theorem of Beltrametti, Schneider and Sommese [1] we get the bound

$$T \geqslant k^{n-sk+1}\bigl(p(n+1) - kq\bigr),$$

that already gives a good number of examples, some of which are described in the last section.

4. The case $E_j \simeq L^{r_j}$

Now we turn to consider the second case mentioned in the introduction. In particular we allow $L \neq K_M^{-1}$, but we consider sections $\sigma_j \in H^0(M, L^{r_j})$ in some tensor power of the polarizing bundle L. We have the following

Theorem 4.1. *Let (M, L) be an n-dimensional polarized manifold endowed with a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ and a linearization on L. For each $j \in \{1, \dots, s\}$ consider a section $\sigma_j \in H^0(M, L^{r_j})$ such that $\rho(t) \cdot \sigma_j = t^{\alpha_j}\sigma_j$ for some $\alpha_j \in \mathbb{Z}$. Let $X = \bigcap_{j=1}^s \sigma^{-1}(0)$. Suppose $\dim(X) = n - s$, then we have*

$$F(X, L|_X, \rho) = F(M, L, \rho) + \frac{1}{2}\left(-\sum_{j=1}^s\left(\frac{\alpha_j}{r_j} - \frac{a_0}{d_0}\right)r_j + \frac{\frac{2d_1}{n\,d_0} - \sum_{j=1}^s r_j}{n+1-s}\sum_{j=1}^s\left(\frac{\alpha_j}{r_j} - \frac{a_0}{d_0}\right)\right), \tag{7}$$

where $a_0 = a_0(M, L) = \int_M \frac{c_1^G(L)^{n+1}}{(n+1)!}$, $d_0 = d_0(M, L) = \int_M \frac{c_1(L)^n}{n!}$ and $d_1 = d_1(M, L) = \int_M \frac{c_1(L)^{n-1}c_1(M)}{2(n-1)!}$.

Proof. Since $c_s(L^{r_1} \oplus \dots \oplus L^{r_s}) = (\prod_{j=1}^s r_j)c_1(L)^s$, $c_1(L^{r_1} \oplus \dots \oplus L^{r_s}) = \sum_{j=1}^s r_j c_1(L)$, by Lemma 5.2 we get

$$d_0(X) = \left(\prod_{j=1}^s r_j\right)\int_M \frac{c_1(L)^n}{(n-s)!} = \frac{d_0 n!}{(n-s)!}\prod_{j=1}^s r_j,$$

$$\begin{aligned}d_1(X) &= \left(\prod_{j=1}^s r_j\right)\int_M \frac{(c_1(M) - \sum_{j=1}^s r_j c_1(L))c_1(L)^{n-1}}{2(n-s-1)!}\\ &= \left(\frac{2d_1}{n\,d_0} - \sum_{j=1}^s r_j\right)\frac{d_0 n!}{2(n-s-1)!}\prod_{j=1}^s r_j = \frac{n-s}{2}\left(\frac{2d_1}{n\,d_0} - \sum_{j=1}^s r_j\right)d_0(X),\end{aligned}$$

and analogously by Lemma 5.3

$$a_0(X) = \left(\int_M \frac{c_1^G(L)^{n+1}}{(n-s+1)!} - \sum_{j=1}^s \frac{\alpha_j}{r_j} \int_M \frac{c_1(L)^n}{(n-s+1)!} \right) \prod_{j=1}^s r_j$$

$$= \frac{a_0(n+1)!}{(n-s+1)!} \prod_{j=1}^s r_j - \frac{\sum_{j=1}^s \frac{\alpha_j}{r_j}}{n-s+1} d_0(X),$$

$$a_1(X) = \left(\int_M \frac{(c_1^G(M) - \sum_{j=1}^s r_j c_1^G(L)) c_1^G(L)^n}{2(n-s)!} + \sum_{j=1}^s \alpha_j \int_M \frac{c_1(L)^n}{2(n-s)!} \right.$$

$$\left. - \sum_{j=1}^s \frac{\alpha_j}{r_j} \int_M \frac{(c_1(M) - \sum_{j=1}^s r_j c_1(L)) c_1(L)^{n-1}}{2(n-s)!} \right) \prod_{j=1}^s r_j$$

$$= \frac{a_1 n!}{(n-s)!} \prod_{j=1}^s r_j - \frac{a_0(n+1)! \sum_{j=1}^s r_j}{2(n-s)!} \prod_{j=1}^s r_j + \frac{1}{2} d_0(X) \sum_{j=1}^s \alpha_j - \frac{1}{n-s} d_1(X) \sum_{j=1}^s \frac{\alpha_j}{r_j}.$$

Thus $F(X, L|_X, \rho)$ equals to

$$\left(\frac{n+1}{n-s+1} \frac{a_0}{d_0} - \frac{\sum_{j=1}^s \frac{\alpha_j}{r_j}}{n-s+1} \right) \frac{d_1(X)}{d_0(X)} - \frac{a_1}{d_0} + \frac{n+1}{2} \frac{a_0}{d_0} \sum_{j=1}^s r_j - \frac{1}{2} \sum_{j=1}^s \alpha_j + \frac{1}{n-s} \frac{d_1(X)}{d_0(X)} \sum_{j=1}^s \frac{\alpha_j}{r_j}$$

$$= \left(\frac{a_0}{d_0} + \frac{s}{n-s+1} \frac{a_0}{d_0} + \frac{\sum_{j=1}^s \frac{\alpha_j}{r_j}}{(n-s)(n-s+1)} \right) \frac{d_1(X)}{d_0(X)} - \frac{a_1}{d_0} + \frac{n+1}{2} \frac{a_0}{d_0} \sum_{j=1}^s r_j - \frac{1}{2} \sum_{j=1}^s \alpha_j$$

$$= \frac{a_0\, d_1}{d_0^2} - \frac{n}{2} \sum_{j=1}^s r_j - \frac{s}{2} \left(\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j \right) \frac{a_0}{d_0} + \frac{s}{2} \frac{n-s}{n-s+1} \left(\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j \right) \frac{a_0}{d_0}$$

$$+ \frac{\frac{1}{2} (\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j) \sum_{j=1}^s \frac{\alpha_j}{r_j}}{n-s+1} - \frac{a_1}{d_0} + \frac{n+1}{2} \frac{a_0}{d_0} \sum_{j=1}^s r_j - \frac{1}{2} \sum_{j=1}^s \alpha_j$$

$$= \frac{a_0\, d_1}{d_0^2} - \frac{a_1}{d_0} - \frac{\frac{1}{2} (\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j) s \frac{a_0}{d_0}}{n-s+1} + \frac{\frac{1}{2} (\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j) \sum_{j=1}^s \frac{\alpha_j}{r_j}}{n-s+1}$$

$$+ \frac{1}{2} \frac{a_0}{d_0} \sum_{j=1}^s r_j - \frac{1}{2} \sum_{j=1}^s \alpha_j$$

$$= \frac{a_0\, d_1}{d_0^2} - \frac{a_1}{d_0} + \frac{1}{2} \frac{\frac{2\, d_1}{n\, d_0} - \sum_{j=1}^s r_j}{n-s+1} \sum_{j=1}^s \left(\frac{\alpha_j}{r_j} - \frac{a_0}{d_0} \right) - \frac{1}{2} \sum_{j=1}^s \left(\frac{\alpha_j}{r_j} - \frac{a_0}{d_0} \right) r_j$$

and we are done. □

When $M = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ Theorem 4.1 gives the following result due to Z. Lu on complete intersections [17]:

Corollary 4.2. *Let $X \subset \mathbb{P}^n$ be an $(n-s)$-dimensional subvariety defined by homogeneous polynomials $F_1, \dots, F_s$ of degree $r_1, \dots, r_s$ respectively. Let $\rho : \mathbb{C}^\times \to SL(n+1)$ be a one parameter subgroup such that*

$$\rho(t) \cdot F_j = t^{\alpha_j} F_j, \quad j = 1, \dots, s$$

for some $\alpha_1, \dots, \alpha_s \in \mathbb{Z}$. Then we have

$$F\big(X, \mathcal{O}_X(1), \rho\big) = \frac{1}{2}\left(-\sum_{j=1}^{s} \alpha_j + \frac{n+1-\sum_{j=1}^{s} r_j}{n+1-s} \sum_{j=1}^{s} \frac{\alpha_j}{r_j}\right).$$

Proof. Since $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \simeq \mathbb{C}[z_0, \dots, z_n]_m$ then

$$h^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big) = \binom{n+m}{m} = \frac{1}{n!} m^n + \frac{n(n+1)}{2n!} m^{n-1} + O\big(m^{n-2}\big),$$

thus $\frac{2d_1}{n\,d_0} = n+1$. Moreover, taking on $\mathcal{O}_{\mathbb{P}^n}$ the unique linearization induced by $SL(n+1)$ we get $w(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$, and in particular $a_0 = 0$. □

The formula (7) becomes simpler if all the r_j's are equal. Moreover in this case $F(X, L|_X, \rho)$ has a nice expression in term of the so-called "Chow weight" of $(X, L|_X)$, whose definition, essentially due to Mumford [21], is the following

Definition 4.3. In the situation of Definition 2.1, let $X \subset V$ be an s-codimensional invariant subvariety. Thus $L|_X$ is a linearized line bundle and we have the asymptotic expansions

$$w\big(X, L|_X^m\big) = a_0(X, L|_X) m^{n-s+1} + O\big(m^{n-s}\big),$$
$$h^0\big(X, L|_X^m\big) = d_0(X, L|_X) m^{n-s} + O\big(m^{n-s-1}\big).$$

The *Chow weight* of X with respect the chosen one-parameter subgroup of $\mathrm{Aut}(V)$ is

$$\mu(X, V, L, \rho) = \frac{a_0(V, L)}{d_0(V, L)} - \frac{a_0(X, L|_X)}{d_0(X, L|_X)}.$$

If $Y \subset V$ is a non-necessarily invariant subvariety, we define $\mu(Y, V, L, \rho)$ to be the Chow weight of the flat limit $\lim_{t\to 0} \rho(t) \cdot Y$. For any reductive subgroup $G \subset \mathrm{Aut}(V)$, we say that Y is *Chow semi-stable w.r.t. G* if $\mu(Y, V, L, \rho) \leqslant 0$ for all one-parameter subgroups ρ of G.

Remark 4.4. Is not difficult to see that in the case $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, the definition above of Chow-stability reduces to the Mumford's one.

Corollary 4.5. *In the situation of Theorem* 4.1*, if $r_j = r$ for all j, then X is a section of M determined by the linear system $P = \operatorname{span}(\sigma_1, \dots, \sigma_s) \subset H^0(M, L^r)$. In this case we have*

$$\begin{aligned} F(X, L|_X, \rho) &= F(M, L, \rho) + \frac{\frac{2d_1}{n\,d_0} - r(n+1)}{2(n+1-s)} \sum_{j=1}^{s} \left(\frac{\alpha_j}{r} - \frac{a_0}{d_0} \right) \\ &= F(M, L, \rho) - C\mu(X, M, L, \rho), \end{aligned}$$

where $C = r\frac{n+1}{2} - \frac{d_1(M,L)}{n\,d_0(M,L)}$.

If moreover $\operatorname{Pic}(M) = \mathbb{Z}$ then $C \geqslant 0$ with equality if and only if $M \simeq \mathbb{P}^n$ and $L = \mathcal{O}(1)$.

Proof. The first equation is an obvious consequence of (7) when $r_j = r$ for all j. The second is a consequence of

$$\mu(X, M, L, \rho) = \frac{1}{n+1-s} \sum_{j=1}^{s} \left(\frac{\alpha_j}{r_j} - \frac{a_0}{d_0} \right),$$

which follows from definition of Chow weight 4.3 and from formulae for $a_0(X, L|_X)$ and $d_0(X, L|_X)$ in Lemmata 5.2 and 5.3.

To prove the non-negativity of C, recall that by Kobayashi and Ochiai [15, Theorem 1.1] if $c_1(M)$ is proportional to $c_1(L)$, which is guaranteed by the assumption on $\operatorname{Pic}(M)$, then $c_1(M) \leqslant (n+1)c_1(L)$ with equality if and only if $(M, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Thus we have

$$\frac{d_1(M, L)}{n\,d_0(M, L)} = \frac{\int_M c_1(L)^{n-1} c_1(M)}{2 \int_M c_1(L)^n} \leqslant \frac{n+1}{2}$$

and the statement follows. $\square$

Remark 4.6. In the case $\operatorname{Pic}(M) = \mathbb{Z}$ and $F(M, L, \rho) = 0$ (e.g. when M admits a Kcsc metric in $c_1(L)$), by the corollary above we get that Chow-instability of a non-necessarily invariant complete intersection $X \subset M$ implies K-instability of X.

5. Proofs of fundamental lemmata

Lemma 5.1. *Let B be a holomorphic vector bundle of rank b on a manifold M, then*

$$\sum_{p=0}^{k} (-1)^p\, ch\left(\bigwedge^p B^* \right) = c_b(B)\, td(B)^{-1}.$$

Proof. It is Lemma 18 in [2] (see also [9, Example 3.2.5]). Let $\alpha_1, \dots, \alpha_b$ be Chern roots of B. Since $ch(\bigwedge^p B^*) = \sum_{1 \leqslant i_1 < \dots < i_p \leqslant b} e^{-(\alpha_{i_1} + \dots + \alpha_{i_p})}$, then we have

$$\sum_{p=0}^{k} (-1)^p\, ch\left(\bigwedge^p B^* \right) = \sum_{p=0}^{b} (-1)^p \sum_{1 \leqslant i_1 < \dots < i_p \leqslant b} e^{-(\alpha_{i_1} + \dots + \alpha_{i_p})}$$

$$= \prod_{i=1}^{b} \left(1 - e^{-\alpha_i}\right)$$
$$= \prod_{i=1}^{b} \alpha_i \prod_{i=1}^{b} \frac{1 - e^{-\alpha_i}}{\alpha_i},$$

and the statement is proved. □

Lemma 5.2. *Let (M, L) be an n-dimensional polarized manifold and let $E_1, \dots, E_s$ be a collection of holomorphic vector bundles on M. Set $k_j = \operatorname{rank}(E_j)$, $B = E_1 \oplus \cdots \oplus E_s$ and $b = \operatorname{rank}(B) = \sum_{j=1}^{s} k_j$. For each $j \in \{1, \dots, s\}$ consider a non-zero section $\sigma_j \in H^0(M, E_j)$ and set $\sigma = (\sigma_1, \dots, \sigma_s) \in H^0(M, B)$ and $X = \sigma^{-1}(0)$. If $\dim(X) = n - b$ we have the asymptotic expansion as $k \to +\infty$*

$$h^0\left(X, L|_X^m\right) = d_0(X) m^{n-b} + d_1(X) m^{n-b-1} + O\left(m^{n-b-2}\right),$$

where

$$d_0(X) = \int_M \frac{c_b(B) c_1(L)^{n-b}}{(n-b)!},$$
$$d_1(X) = \int_M \frac{c_b(B)(c_1(M) - c_1(B)) c_1(L)^{n-b-1}}{2(n-b-1)!}$$

(here $c_b(B) = \prod_{j=1}^{s} c_{k_j}(E_j)$ and $c_1(B) = \sum_{j=1}^{s} c_1(E_j)$).

Proof. Let $\mathcal{O}_X$ be the structure sheaf of X. By assumption σ is a regular section, so the Koszul complex

$$0 \to \bigwedge^{b} B^* \to \bigwedge^{b-1} B^* \to \cdots \to B^* \to \mathcal{O}_M \to \mathcal{O}_X \to 0$$

induced by σ is exact. Tensoring by L^m preserves the exactness, thus

$$\chi\left(X, L|_X^m\right) = \sum_{p=0}^{b} (-1)^p \chi\left(M, L^m \otimes \bigwedge^{p} B^*\right)$$

and by the Hirzebruch–Riemann–Roch theorem we get

$$\chi\left(X, L|_X^m\right) = \sum_{p=0}^{b} (-1)^p \int_M ch\left(\bigwedge^{p} B^*\right) e^{m c_1(L)}\, td(M)$$

$$= \int_M ch\left(\sum_{p=0}^{b} (-1)^p \bigwedge^p B^* \right) e^{mc_1(L)}\, td(M)$$

$$= \int_M c_b(B)\, td(B)^{-1} e^{mc_1(L)}\, td(M),$$

where second equality follows by elementary properties of the Chern character, and the last one holds by Lemma 5.1.

As $k \to +\infty$ we have the expansion

$$\chi\big(X, L|_X^m\big) = m^{n-b} \int_M \frac{c_b(B)c_1(L)^{n-b}}{(n-b)!} + m^{n-b-1} \int_M \frac{c_b(B)(c_1(M) - c_1(B))c_1(L)^{n-b-1}}{2(n-b-1)!} + O\big(m^{n-b-2}\big),$$

where we used $td(M) = 1 + \frac{1}{2}c_1(M) + \cdots$ and $td(B)^{-1} = 1 - \frac{1}{2}c_1(B) + \cdots$ (dots representing terms of degree greater then one). Finally the equality $h^0(X, L|_X^m) = \chi(X, L|_X^m)$ follows by ampleness of L. $\square$

Lemma 5.3. *Let (M, L) be an n-dimensional polarized manifold endowed with a $\mathbb{C}^\times$-action $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ and a linearization on L. Let $E_1, \dots, E_s$ be a collection of linearized vector bundles on M. Set $k_j = \mathrm{rank}(E_j)$, $B = E_1 \oplus \cdots \oplus E_s$ and $b = \mathrm{rank}(B) = \sum_{j=1}^s k_j$. For each $j \in \{1, \dots, s\}$ consider a non-zero section $\sigma_j \in H^0(M, E_j)$ such that $\rho(t) \cdot \sigma_j = t^{\alpha_j}\sigma_j$ for some $\alpha_j \in \mathbb{Z}$, and set $\sigma = (\sigma_1, \dots, \sigma_s) \in H^0(M, B)$ and $X = \sigma^{-1}(0)$. If $\dim(X) = n - b$, then we have the asymptotic expansion as $k \to +\infty$*

$$w^0\big(X, L|_X^m\big) = a_0(X)m^{n-b+1} + a_1(X)m^{n-b} + O\big(m^{n-b-1}\big),$$

where

$$a_0(X) = \int_M \frac{c_b^G(B)c_1^G(L)^{n-b+1}}{(n-b+1)!} - \sum_{j=1}^s \alpha_j \int_M \frac{c_b(B)c_{k_j-1}(E_j)c_1(L)^{n-b+1}}{(n-b+1)!\,c_{k_j}(E_j)},$$

$$a_1(X) = \int_M \frac{c_b^G(B)(c_1^G(M) - c_1^G(B))c_1^G(L)^{n-b}}{2(n-b)!} + \sum_{j=1}^s k_j\alpha_j \int_M \frac{c_b(B)c_1(L)^{n-b}}{2(n-b)!} - \sum_{j=1}^s \alpha_j \int_M \frac{c_b(B)c_{k_j-1}(E_j)(c_1(M) - c_1(B))c_1(L)^{n-b}}{2(n-b)!\,c_{k_j}(E_j)}$$

(here $c_b^G(B) = \prod_{j=1}^s c_{k_j}^G(E_j)$ and $c_1^G(B) = \sum_{j=1}^s c_1^G(E_j)$).

Proof. It is very similar to the previous on the dimension of $H^0(X, L|_X^m)$. Since sections σ_j are only semi-invariant, they do not give rise to equivariant sequences of bundles, but to overcame the problem we can initially change the linearization of each E_j and go back to original one at the end of computations. Denoted by $\mathbb{C}_\beta$ the trivial line bundle on M with linearization $t \cdot u = t^\beta u$, for each $j \in \{1, \dots, s\}$ let

$$F_j = E_j \otimes \mathbb{C}_{-\alpha_j}.$$

In this way, each $\sigma_j \in H^0(M, E_j)$ is an invariant section.

Now consider the rank $b = \sum_{j=1}^s k_j$, $\mathbb{C}^\times$-linearized vector bundle $F = \bigoplus_{j=1}^s F_j$, and let $\sigma \in H^0(M, F)$ be the holomorphic section defined by $\sigma = (\sigma_1, \dots, \sigma_s)$. Clearly σ is invariant and we have $X = \sigma^{-1}(0)$. Let $\mathcal{O}_X$ be the structure sheaf of X. By assumption σ is a regular section, so the Koszul complex

$$0 \to \bigwedge^b F^* \to \bigwedge^{b-1} F^* \to \cdots \to F^* \to \mathcal{O}_M \to \mathcal{O}_X \to 0$$

induced by σ is exact and equivariant. Tensoring by L^m preserves the exactness and equivariance, thus

$$\chi^G\big(X, L^m|_X\big) = \sum_q (-1)^q \operatorname{tr}\big(e^{it} | H^q\big(X, L^m|_X\big)\big) = \sum_{p=0}^b (-1)^p \chi^G\left(M, L^m \otimes \bigwedge^p F^*\right)$$

and by the equivariant Riemann–Roch theorem we get

$$\begin{aligned}
\chi^G\big(X, L|_X^m\big) &= \sum_{p=0}^b (-1)^p \int_M ch^G\left(\bigwedge^p F^*\right) e^{mc_1^G(L)}\, td^G(M) \\
&= \int_M ch^G\left(\sum_{p=0}^b (-1)^p \bigwedge^p F^*\right) e^{mc_1^G(L)}\, td^G(M) \\
&= \int_M c_b^G(F)\, td^G(F)^{-1} e^{mc_1^G(L)}\, td^G(M),
\end{aligned}$$

where the last equality holds by Lemma 5.1. Since the right part of the equivariant Riemann–Roch theorem is a power series convergent in some neighborhood of zero of the Lie algebra of the acting group, to get the trace of the generator of the action on the virtual space $\bigoplus_q (-1)^q H^q(X, L|_X^m)$, is sufficient to take the "linear term" of the integrand. Explicitly, as $m \to +\infty$ we have $H^q(X, L|_X^m) = 0$ for $q > 0$ by ampleness of L, and we get the expansion

$$\begin{aligned}
w^0\big(X, L|_X^m\big) = {}& m^{n-b+1} \int_M \frac{c_b^G(F) c_1^G(L)^{n-b+1}}{(n-b+1)!} + m^{n-b} \int_M \frac{c_b^G(F)(c_1^G(M) - c_1^G(F)) c_1^G(L)^{n-b}}{2(n-b)!} \\
&+ O\big(m^{n-b-1}\big),
\end{aligned} \tag{8}$$

where we used $td(M) = 1 + \frac{1}{2} c_1(M) + \cdots$ and $td(F)^{-1} = 1 - \frac{1}{2} c_1(F) + \cdots$ (dots representing terms of degree greater then one). Finally we have to come back to original linearization of E_j's.

Since $F_i = E_i \otimes \mathbb{C}_{-\alpha_i}$, by the Cartan model of the equivariant cohomology of M, is easy to see that $c_1^G(F_j) = c_1^G(E_j) - k_j\alpha_j$ and $c_{k_j}^G(F_j) = \sum_{p=0}^{k_j}(-\alpha_j)^{k_j-p}c_p^G(E_j)$, whence

$$c_b^G(F) = \prod_{j=1}^{s} c_{k_j}^G(F_j) = \prod_{j=1}^{s}\sum_{p=0}^{k_j}(-\alpha_j)^{k_j-p}c_p^G(E_j)$$

$$= c_b^G(B)\left(1 - \sum_{j=1}^{s}\alpha_j \frac{c_{k_j-1}^G(E_j)}{c_{k_j}^G(E_j)} + \cdots\right),$$

$$c_1^G(F) = \sum_{j=1}^{s} c_1^G(F_j) = c_1^G(E) - \sum_{j=1}^{s} k_j\alpha_j,$$

and substituting in (8) we are done. □

6. Applications and examples

In this section we show some consequences of Theorems 3.1 and 4.1. In particular we use those theorems to calculate the Futaki invariant of central fibers of test configurations arising from degenerations of linear sections of vector bundles.

More precisely consider an n-dimensional polarized manifold (M, L) endowed with a one-parameter subgroup of automorphisms $\rho : \mathbb{C}^\times \to \operatorname{Aut}(M)$ that linearizes on L. Let $P = \operatorname{span}(\eta_1, \ldots, \eta_s) \subset H^0(M, E)$ be an s-dimensional linear system of a rank k linearized holomorphic vector bundle E on M. Assume that $X_P = \bigcap_{j=1}^s \eta_j^{-1}(0)$ is ks-codimensional. The ρ-action on P gives naturally a test configuration for the variety $(X_P, L|_{X_P})$ as follows. Let $P_t = \rho(t) \cdot P$ and let $\mathcal{X}$ be the closure of $\{(x,t) \in M \times \mathbb{C}^\times \mid x \in X_{P_t}\}$ in $M \times \mathbb{C}$. The projection on the second factor induces a flat morphism $\pi : \mathcal{X} \to \mathbb{C}$. Let $X_{P_0} = \bigcap_{j=1}^s \sigma_j^{-1}(0)$, where $P_0 = \operatorname{span}(\sigma_1, \ldots, \sigma_s) = \lim_{t\to 0}\rho(t) \cdot P$ with σ_j's semi-invariant. By the uniqueness of the flat limit [13, Proposition 9.8] we have $\pi^{-1}(0) = X_{P_0}$.

6.1. A Mukai–Umemura–Tian like example with singular central fibre

Consider the Grassmannian $M = G(4,6)$ of 4-planes in $\mathbb{C}^6$ polarized with $L = \bigwedge^2 Q$, being Q the universal quotient bundle. Since the Kodaira map induced by L is the Plüker embedding $M \hookrightarrow \mathbb{P}^{14}$, for each $\eta_1, \eta_2, \eta_3 \in H^0(M, L)$ linearly independent, the subvariety $X = \bigcap_{j=1}^3 \eta_j^{-1}(0)$ is a section of $G(4,6)$ with a 3-codimensional subspace in $\mathbb{P}^{14}$. The general X arising in this way is a Fano 5-fold.

Let $\rho : \mathbb{C}^\times \to SL(6)$ be the subgroup generated by $\operatorname{diag}(-5, -3, -1, 1, 3, 5)$ and consider $P_\epsilon = \operatorname{span}\{\eta_1, \eta_2, \eta_3\} \subset H^0(M, L)$ where

$$\eta_1 = e_{16} + e_{25} + e_{34}, \qquad \eta_2 = e_{15} + e_{24} + \varepsilon e_{46}, \qquad \eta_3 = e_{26} + e_{35} + \varepsilon e_{45}$$

and we identify $H^0(M, L) \simeq \bigwedge^2 \mathbb{C}^6$.

By local calculations it is easy to see that X_{P_0} is $\mathbb{C}^\times$-invariant and is singular at points $e_2 \wedge e_3 \wedge e_5 \wedge e_6$ and $e_1 \wedge e_2 \wedge e_4 \wedge e_5$. On the other hand, for $\varepsilon \neq 0$ the variety X_{P_ε} is non-singular but not invariant.

Now let

$$\sigma_1 = e_{16} + e_{25} + e_{34}, \qquad \sigma_2 = e_{15} + e_{24}, \qquad \sigma_3 = e_{26} + e_{35}.$$

We have $P_0 = \operatorname{span}\{\sigma_1, \sigma_2, \sigma_3\}$, moreover $\rho(t) \cdot P_\varepsilon$ tends to P_0 as $t \to 0$. Thus, following the construction shown at the start of this section, there is a test configuration of $(X_{P_\varepsilon}, L|_{X_{P_\varepsilon}})$ with central fibre $(X_{P_0}, L|_{X_{P_0}})$. Since

$$\rho(t) \cdot \sigma_1 = \sigma_1, \qquad \rho(t) \cdot \sigma_2 = t^{-2}\sigma_2, \qquad \rho(t) \cdot \sigma_3 = t^2\sigma_3,$$

by Corollary 4.5 we get

$$F(X_{P_0}, L|_{X_{P_0}}, \rho) = C(0 - 2 + 2) = 0,$$

where we used $F(M, L, \rho) = 0$ and $a_0(M, L) = 0$.

Hence by [27] we proved the following

Proposition 6.1. *For each $\varepsilon \neq 0$ the manifold X_{P_ε} is not K-stable, hence is not Kähler–Einstein.*

6.2. The quintic Del Pezzo threefold

Consider the Grassmannian $M = G(2,5)$ of planes in $\mathbb{C}^5$ polarized with $L = \bigwedge^3 Q$, where Q is the universal quotient bundle. As well-known the Kodaira map induced by L is the Plüker embedding $M \hookrightarrow \mathbb{P}^9$. Thus for each $\sigma_1, \sigma_2, \sigma_3 \in H^0(M, L)$ linearly independent, the subvariety $X = \bigcap_{j=1}^3 \sigma_j^{-1}(0)$ is a section of $G(2,5)$ with a 3-codimensional subspace in $\mathbb{P}^9$. The general X arising in this way is the quintic Del Pezzo threefold [14], in particular it is Fano.

Proposition 6.2. *Each degeneration of X induced by a one-parameter subgroup $\rho : \mathbb{C}^\times \to$ $\operatorname{Aut}(M)$ has non-negative Futaki invariant.*

Proof. Consider the isomorphism $H^0(M, L) \simeq \bigwedge^3 \mathbb{C}^5$ given by $\bigwedge^3 \mathbb{C}^5 \ni v \mapsto \sigma_v \in H^0(M, L)$ where

$$\sigma_v(E) = v + \bigwedge^2 \mathbb{C}^5 \wedge E \in \bigwedge^3 \bigl(\mathbb{C}^5/E\bigr),$$

for all $E \in M$. Thus we can identify σ_j with $u_j \in \bigwedge^3 \mathbb{C}^5$.

We recall that each automorphism of M comes from the action of an element of $SL(5)$ on $\mathbb{C}^5$. Thus we can consider $\rho : \mathbb{C}^\times \to SL(5)$. Let $(e_1, \ldots, e_5)$ be a basis of eigenvectors and let $\nu_1, \ldots, \nu_5 \in \mathbb{Z}$ be the weights of ρ. We have $\nu_1 + \cdots + \nu_5 = 0$ and we can suppose without loss $\nu_1 \leqslant \cdots \leqslant \nu_5$.

Now, since u_j's are general we can also suppose

$$u_1 = \sum_{1 \leqslant i < j < k \leqslant 5} c_1^{ijk} e_{ijk},$$

$$u_2 = \sum_{1 \leqslant i<j<k \leqslant 5, i+j+k \geqslant 7} c_2^{ijk} e_{ijk},$$

$$u_3 = \sum_{1 \leqslant i<j<k \leqslant 5, i+j+k \geqslant 8} c_3^{ijk} e_{ijk}$$

and $c_\ell^{ijk} \neq 0$.

The action induced by ρ on $\bigwedge^3 \mathbb{C}^5$ gives a weak order ($\preccurlyeq$) on the basis $(e_{ijk} \mid 1 \leqslant i < j < k \leqslant 5)$ as follows: we define $e_{i_1 j_1 k_1} \preccurlyeq e_{i_2, j_2, k_2}$ if $\nu_{i_1} + \nu_{j_1} + \nu_{k_1} \leqslant \nu_{i_2} + \nu_{j_2} + \nu_{k_2}$. Obviously $e_{123} \preccurlyeq e_{124} \preccurlyeq e_{134} \preccurlyeq e_{234}$ and $e_{125} \preccurlyeq e_{ij5}$ for all $i < j$. Thus $\operatorname{span}(u_1, u_2, u_3)$ tends to $\operatorname{span}(v_1, v_2, v_3)$ under the action of $\rho(t)$ as $t \to 0$, where

$$v_1 = \min\{e_{123}, e_{125}\} = e_{123},$$

$$v_2 = \min\{e_{124}, e_{125}\} = e_{124},$$

$$v_3 = \min\{e_{134}, e_{125}\}.$$

Let α_j be the weight of v_j. We have: $\alpha_1 = \nu_1 + \nu_2 + \nu_3$, $\alpha_2 = \nu_1 + \nu_2 + \nu_4$ and $\alpha_3 = \min\{\nu_1 + \nu_3 + \nu_4, \nu_1 + \nu_2 + \nu_5\}$. In both cases that can occur is easy to check that

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant 0.$$

Finally let $X_0 = \lim_{t \to 0} \rho(t) \cdot X = \bigcap_{j=1}^3 \sigma_{v_j}^{-1}(0)$. By Corollary 4.5 we get

$$F(X_0, L|_{X_0}, \rho) = -\frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3) \geqslant 0,$$

where we used that $F(M, L, \rho) = 0$, $a_0(M, L) = 0$ and $\frac{2d_1}{n d_0} = 5$ (the latter follows from the general fact that if $L^q = K_M^{-1}$ then $\frac{2d_1(M,L)}{n d_0(M,L)} = q$). $\square$

The result above is an evidence to the K-stability of the quintic Del Pezzo threefold X_5. In this specific case the above discussion can be strengthened by observing:

- The complex structure of $X = X_5$ is rigid [14, Corollary 3.4.2], hence it cannot be used as central fiber of a non-product test configuration.
- Donaldson's proof of the existence of Kähler–Einstein metric on the Mukai–Umemura manifold X_{22} [8] covers also this case, hence $X = X_5$ is indeed Kähler–Einstein and so K-stable. As showed in [20], the manifolds X_{22} and X_5 share all the properties involved in his argument. In particular, X_5 has a very ample anti-canonical bundle; there is a holomorphic action of $PSL(2, \mathbb{C})$ on X_5 and a point $x_0 \in X_5$ with stabilizer the octahedral group $\Gamma \subset SO(3) \subset PSL(2, \mathbb{C})$; there is a $PSL(2, \mathbb{C})$-invariant anti-canonical section with reduced zero-set that has at worst cusp-like singularities and is the complement of the $PSL(2, \mathbb{C})$-orbit of x_0. With the same facts about X_{22} (the only, unimportant, difference is the stabilizer Γ, which is the icosahedral group in this case) Donaldson proved that the Tian's α-invariant is greater or equal then $\frac{5}{6}$. That is true for the X_5 as well and the proof is word by word the same. The existence of a Kähler–Einstein metric follows by the theorem of Tian on α-invariant [23].

On the other hand, by completely algebraic methods, the α-invariant of X_5 and X_{22} has been proved very recently to be equal to $\frac{5}{6}$ by Cheltsov and Shramov [4].

6.3. General complete intersections in Grassmannians

Following a construction given by Tian [25], we generalize Proposition 6.2 to general intersections of some exterior power of the universal quotient bundle on the Grassmannian.

As will be clear from the proof the generality condition depends on the one-parameter subgroup ρ.

Proposition 6.3. *Let $(M, L) = (G(k, N), K^{-1}_{G(k,N)})$ be the Grassmannian of k-planes in $\mathbb{C}^N$ anti-canonically polarized. Suppose $k(N-k) > N-1$ to avoid trivialities. Fix a one-parameter subgroup $\rho : \mathbb{C}^\times \to \mathrm{Aut}(M)$ and a linearization to L. Denoted by Q the universal quotient bundle on M, let $E = \bigwedge^\ell Q$ endowed with a linearization on E such that $(\det E)^N \simeq L^{\binom{N-k}{\ell}}$ as linearized bundles. Let $P \subset H^0(M, E)$ be a general d-dimensional subspace such that $X_P = \bigcap_{\sigma\in P} \sigma^{-1}(0)$ has dimension $k(N-k) - d\binom{N-k}{\ell} > 0$.*

X_P is Fano if and only if $N - d\binom{N-k}{\ell} > 0$. In this case we have

$$F(X_{P_0}, L|_{X_{P_0}}, \rho) > 0,$$

where $P_0 = \lim_{t\to 0} \rho(t) \cdot P$.

Proof. Take on E and L the unique linearizations induced by $SL(N)$. Consider the induced representation of ρ on $H^0(M, E)$ and fix a basis of semi-invariant sections $\sigma_1, \dots, \sigma_{h^0(E)}$. Thus for each $j \in \{1, \dots, h^0(E)\}$ there is a unique $\alpha_j \in \mathbb{Z}$ such that $t \cdot \sigma_j = t^{\alpha_j}\sigma_j$. We can suppose without loss

$$\alpha_i \leqslant \alpha_j \quad \text{if } i < j. \tag{9}$$

Let $\eta_1, \dots, \eta_d$ be a basis of P. Since P is general we can suppose

$$\eta_1 = \sum_{j=1}^{h^0(E)} c_{1j}\sigma_j,$$

$$\eta_2 = \sum_{j=2}^{h^0(E)} c_{2j}\sigma_j,$$

$$\vdots$$

$$\eta_d = \sum_{j=d}^{h^0(E)} c_{dj}\sigma_j,$$

where $c_{ii} \neq 0$ for all $i \in \{1, \dots, d\}$. Thus the limit of P under the action of ρ is the plane $P_0 = \mathrm{span}(\sigma_1, \dots, \sigma_d)$. In the chosen linearization ρ acts on $H^0(M, E)$ as a subgroup of $SL(h^0(E))$, thus $\sum_{j=1}^{h^0(E)} \alpha_j = 0$. Hence, by (9) and non-triviality of ρ we have

$$\alpha(P) = \sum_{j=1}^{d} \alpha_j < 0. \tag{10}$$

Since P is general, X_P is smooth. Moreover, by the adjunction formula and the hypothesis on E we get

$$c_1(X_P) = \iota^* c_1(M) - d\iota^* c_1(E) = \left(1 - \frac{d}{N}\binom{N-k}{\ell}\right)\iota^* c_1(M),$$

where $\iota : X \hookrightarrow M$ is the inclusion. This prove the Fano condition.

By the localization theorem for equivariant cohomology is not hard to see that

$$\int_{G(k,N)} c^G_{\binom{N-k}{\ell}}\left(\bigwedge^{\ell} Q\right)^d c^G_1\left(\bigwedge^{\ell} Q\right)^{k(N-k)-d\binom{N-k}{\ell}+1} = 0.$$

Hence, by Corollary 3.3 we get

$$F(X_{P_0}, L|_{X_{P_0}}, \rho) = -CT \sum_{j=1}^{d} \alpha_j,$$

where $C > 0$ and

$$T > \big(k(N-k) + 1 - N\big)\binom{N-k}{\ell}^{k(N-k)-d\binom{N-k}{\ell}+2} > 0.$$

Actually $\bigwedge^{\ell} Q$ is not very ample, however in this case we can apply [1, Proposition 1] to get the first inequality above. □

Acknowledgments

Part of this work has been carried out in Fall 2007 during the visit of the second author at the Princeton University, whose hospitality is gratefully acknowledged. It is a great pleasure to thank G. Tian for many enlightening discussions. Thanks also to Y. Rubinstein and J. Stoppa for many important conversations. The first author was partially supported by the ERC grant 207573 "Vectorial Problems". The second author was partially supported by the Marie Curie IOF grant 255579 (CAMEGEST).

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