Discrete Optimization 6 (2009) 64–78

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Facet-inducing inequalities for chromatic scheduling polytopes based on covering cliques

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A R T I C L E   I N F O

Article history:
Received 4 October 2006
Received in revised form 1 September 2008
Accepted 4 September 2008
Available online 17 October 2008

Keywords:
Bandwidth allocation
Polyhedral combinatorics
Facets
Separation

A B S T R A C T

Chromatic scheduling polytopes arise as solution sets of the bandwidth allocation problem in certain radio access networks supplying wireless access to voice/data communication networks to customers with individual communication demands. This bandwidth allocation problem is a special chromatic scheduling problem; both problems are $\mathcal{NP}$-complete and, furthermore, there exist no polynomial-time algorithms with a fixed quality guarantee for them. As algorithms based on cutting planes are shown to be successful for many other combinatorial optimization problems, the goal is to apply such methods to the bandwidth allocation problem. For that, knowledge on the associated polytopes is required. The present paper contributes to this issue, introducing new classes of valid inequalities based on variations and extensions of the covering-clique inequalities presented in [J. Marenco, Chromatic scheduling polytopes coming from the bandwidth allocation problem in point-to-multipoint radio access systems, Ph.D. Thesis, Universidad de Buenos Aires, Argentina, 2005; J. Marenco, A. Wagler, Chromatic scheduling polytopes coming from the bandwidth allocation problem in point-to-multipoint radio access systems, Annals of Operations Research 150-1 (2007) 159–175]. We discuss conditions ensuring that these inequalities define facets of chromatic scheduling polytopes, and we show that the associated separation problems are $\mathcal{NP}$-complete.

1. Introduction

The purpose of a Point-to-Multipoint radio access system (PMP-system) is to supply wireless access to voice/data communication networks. Base stations form the access points to the backbone network and customer terminals are linked to base stations by means of radio signals.

There are two main differences between PMP-systems and cellular phone networks. Firstly, each customer is provided a fixed antenna and is assigned to a certain sector of a base station (see Fig. 1(a)). Secondly, the customers have individual communication demands of consecutive channels, hence the task is to assign frequency intervals instead of single channels (see Fig. 1c). A central issue is that a link connecting a customer terminal and a base station may be subject to interference from another link using the same frequency: links to customers of the same sector must not use the same frequency (since they are served by the same antenna) and, in addition, some links of customers in different sectors may also cause interferences (due to power and direction of the transmitted signals), see the links in Fig. 1(b).

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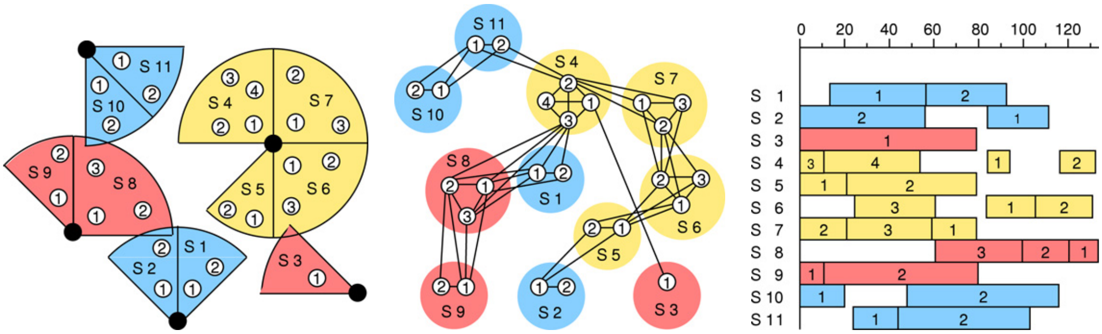
doi:10.1016/j.disopt.2008.09.002



(a) Sectorization. (b) Graph. (c) Frequency plan.

Fig. 1. Bandwidth allocation in Point-to-Multipoint radio access systems. (a) Base stations (black dots) and division of customers (white dots) into sectors. (b) Associated interference graph. (c) Feasible solution (each row corresponds to a sector and the horizontal axis represents the frequency spectrum).

To maintain the links in PMP-systems, some specific part of the radio frequency spectrum has to be used. This typically causes capacity problems and, therefore, it is necessary to reuse frequencies. The *bandwidth allocation problem in PMP-systems* has to be solved in order to guarantee an interference-free communication. The goal is to assign a frequency interval within the available radio frequency spectrum to each customer (see Fig. 1(c)), taking into account the individual communication demands and possible interference.

The input of this problem is given as follows. Let $\mathcal{T} = \{t_1, \ldots, t_n\}$ be the set of all customer terminals, and let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a partition of $\mathcal{T}$ into sectors, providing the information which sector serves each terminal. Let $d = (d_1, \ldots, d_n)$ be the vector of communication demands associated with the customer terminals, indicating that customer $t_i \in \mathcal{T}$ has demand $d_i \in \mathbf{Z}$. Additionally, we have a set $\mathcal{E}_X$ of unordered pairs (t_i, t_j) of terminals in *different* sectors that must not use the same frequency due to possible interference. We can represent this setting by a *weighted interference graph* $(G, d) = (V, E, d)$, where the node set V stands for the customer terminals, the edge set E for pairs of interfering customers, and the node weights d for the communication demands. Throughout this paper we denote by $n = |V|$ resp. $m = |E|$ the number of nodes resp. edges of G.

In base stations, oscillators provide the different frequencies — with a possible difference Δ to the required frequency. Thus, between the frequency intervals of possibly interfering links $(t_i, t_j) \in \mathcal{E}_X$ in different sectors, a guard distance $g = 2\Delta$ has to be obeyed. Finally, we have the available radio frequency spectrum $[0, s]$, with $s \in \mathbf{Z}_+$, where all the frequency intervals have to be placed in. Thus, the problem input consists in the quadruple (G, d, s, g).

The desired output is an assignment of an interval $I(i) = [l_i, r_i] \subseteq [0, s]$, with $l_i, r_i \in \mathbf{Z}_+$, to each customer $t_i \in \mathcal{T}$ such that $r_i - l_i \geq d_i$ for every $t_i \in \mathcal{T}$ and

$$\max\{l_i, l_j\} - \min\{r_i, r_j\} \geq \begin{cases} 0 & \text{if } t_i \text{ and } t_j \text{ belong to the same sector} \\ g & \text{if } (t_i, t_j) \in \mathcal{E}_X \end{cases}$$

for every pair of interfering customers $t_i, t_j \in \mathcal{T}$. For $g = 0$, the problem can be seen as a chromatic scheduling problem [3] or a consecutive coloring problem [4] on the weighted graph (G, d); the problem corresponds to the ordinary graph coloring problem if $d = \mathbf{1}$ holds in addition.

Note that the interval $I(i) = [l_i, r_i]$ assigned to customer i might oversatisfy the demand by $r_i - l_i > d_i$. Since the company operating the PMP-system must usually buy a license to use the whole spectrum $[0, s]$, such a bandwidth upgrade will result in a better service level at no additional cost, provided a feasible assignment is still possible within $[0, s]$. In fact, in some practical settings a set $M \subseteq V$ of *main customers* is given and the objective is to maximize $\sum_{i \in M}(r_i - l_i)$ due to the following reason, see [1].

The main customer in a sector stands typically for a group of individual customers with similar interference conditions, so maximizing the frequency interval allocated for this group opens the opportunity for the provider to keep the same frequency plan running even if new customers join the company (as long as they can be integrated in the main customer group). Hence, maximizing the frequency allocated for the main customer groups produces a frequency plan which is in some sense *robust* under extensions of the set of customers. This is an important issue for the provider as changing the frequency plan typically causes serious problems: if no incremental change from the old to the new frequency plan can be found, the system must be shut down to restart with the new plan from scratch, and no service is possible in the meantime.

Small instances of the bandwidth allocation problem could be handled by greedy-like heuristics [1], but in order to tackle problem sizes of real world applications, algorithms have to be designed that rely on a deeper insight of the problem structure. The polyhedral approach, consisting of an in-depth investigation of polytopes associated with a combinatorial structure and the application of linear programming based cutting plane techniques, has been very successful in recent years. To apply such methods to the bandwidth allocation problem, the convex hull of the incidence vectors of all feasible solutions has to be studied.

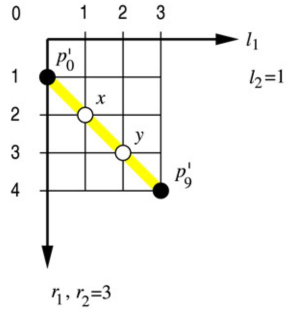


Fig. 2. Convex hull of two feasible (l, r)-vectors.

In order to represent a solution, we use a vector

$$(l, r, x)^{\mathrm{T}} = (l_1, \ldots, l_n, r_1, \ldots, r_n, x_1, \ldots, x_m)^{\mathrm{T}}$$

where, for all nodes $i \in V$, l_i and r_i stand for the interval bounds and, for all edges $ij \in E$, $i < j$, $x_{ij} \in \{0, 1\}$ represents an ordering variable with $x_{ij} = 1$ if and only if $r_i \leq l_j$. It is indeed necessary to introduce the latter variables as the convex hull of the solutions represented only by the interval bounds may contain infeasible integer points. For example, consider the instance $(K_2, d, 5, 0)$, with $d = (1, 2)$. The vectors $z = (0, 1, 1, 3, 1)$ and $z' = (3, 1, 4, 3, 0)$ represent feasible solutions, but dropping the information given by x_{12}, the convex hull of even these two points contains two infeasible but integral points, namely $(1, 1, 2, 3)$ and $(2, 1, 3, 3)$, see Fig. 2.

For $ij \in E$, define δ_{ij} to be $\delta_{ij} = 0$ if t_i and t_j belong to the same sector, and $\delta_{ij} = g$ otherwise. A feasible solution is an assignment of values to $l_i, r_i\ \forall i \in V$ and $x_{ij}\ \forall ij \in E, i < j$, such that the following constraints are satisfied:

$$d_i \leq r_i - l_i \quad \forall i \in V \tag{1}$$

$$0 \leq l_i \leq r_i \leq s \quad \forall i \in V \tag{2}$$

$$r_i + \delta_{ij} \leq l_j + s(1 - x_{ij}) \quad \forall ij \in E,\ i < j \tag{3}$$

$$r_j + \delta_{ij} \leq l_i + s x_{ij} \quad \forall ij \in E,\ i < j \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad \forall ij \in E,\ i < j \tag{5}$$

$$l_i, r_i \in \mathbf{Z} \quad \forall i \in V. \tag{6}$$

The *demand constraints* (1) and the *bound constraints* (2) assert that the interval $I(i) = [l_i, r_i]$ must satisfy the demand d_i and fit within the available frequency spectrum $[0, s]$. Inequalities (3) and (4) realize the *antiparallelity constraints*, which prevent interfering pairs of intervals from overlapping. Note that the intervals corresponding to pairs of customers located in the same sector must not overlap, and there must be a distance of at least g between the intervals corresponding to pairs of interfering customers in different sectors. Finally, the *integrality constraints* (5) resp. (6) force the x-variables to be binary resp. the interval bounds to be integral.

Definition 1 (*Chromatic Scheduling Polytope*). Let (G, d, s, g) be an instance of the bandwidth allocation problem in PMP-systems. We define the *chromatic scheduling polytope* $P(G, d, s, g) \subseteq \mathbf{R}^{2n+m}$ to be the convex hull of all feasible solutions $(l, r, x) \in \mathbf{Z}^{2n+m}$ satisfying constraints (1)–(6).

Chromatic scheduling polytopes are empty if the frequency span s is too small and pass through several stages as s increases: from a nonempty but low-dimensional stage to full-dimensionality and, finally, to a combinatorially steady state (where increasing s further does not change the structure of the faces of the polytope anymore), see [6,7] for more details.

We define $s_{\min}(G, d, g)$ to be the minimum frequency span s such that $P(G, d, s, g)$ is nonempty (i.e., $P(G, d, s, g) \neq \emptyset$ if and only if $s \geq s_{\min}(G, d, g)$), and $s_{\text{full}}(G, d, g)$ to be the minimum frequency span s such that $P(G, d, s, g)$ is full-dimensional. It is worth noting that there exist instances such that $s_{\min}(G, d, g) < s_{\text{full}}(G, d, g)$, especially when the customer demands d are not uniform. Conversely, the full-dimensionality threshold can be bounded by $s_{\text{full}}(G, d, g) \leq s_{\min}(G, d, g) + 2d_{\max} + g =: \alpha(G, d, g)$, where $d_{\max} = \max_{i \in V} d_i$ [6,7].

Chromatic scheduling polytopes admit very interesting symmetry properties [6,8]. In particular, every feasible solution $y = (l, r, x) \in P(G, d, s, g)$ has an associated symmetrical solution $y' = (s\,\mathbf{1} - r, s\,\mathbf{1} - l, \mathbf{1} - x) \in P(G, d, s, g)$.

This symmetry implies that every (facet-inducing) valid inequality $\pi y \leq \pi_0$ admits a symmetric (facet-inducing) valid inequality, obtained by replacing the above expression for y' in $\pi y \leq \pi_0$ For instance, the bounds $0 \leq l_i$ and $r_i \leq s$ are such a pair of symmetric inequalities.

The theoretical strength of the model constraints (1)–(5) was analyzed in [6,8], where it was shown that the interval bounds (2) and the antiparallelity constraints (3) and (4) do not define facets in general. A procedure to strengthen these families was presented, proving that the resulting inequalities are facet-defining for $P(G, d, s, g)$ if $s \geq s_{\min}(G, d, g) + O(1)$.

These so-called *covering-clique* and *double covering-clique inequalities*, respectively, are interesting and strong classes of valid inequalities, and this fact motivates the search for further classes based on similar concepts. In this paper we address this issue.

This paper is organized as follows. Section 2 recalls the definitions and main results on covering-clique and double covering-clique inequalities presented in [6,8], also providing a proof of the $\mathcal{NP}$-completeness of the associated separation problems. Sections 3 and 4 introduce two generalizations of covering-clique inequalities, and Section 5 presents three further classes of facet-inducing inequalities based on variations of double covering-clique inequalities. Section 6 reports the results of preliminary computational experiments with these classes of valid inequalities, and Section 7 closes the paper with some concluding remarks and open problems for further research. Most of the results in this paper were first presented in [6].

2. Covering-clique inequalities

For $ij \in E$, $i < j$, we introduce $x_{ji} = 1 - x_{ij}$ as a notational convenience. If $i \in V$, we denote by $N(i) = \{j \in V : ij \in E\}$ the set of neighbors of i in the interference graph. Let $A \subseteq V$ be a node subset, and let $K \subseteq A$ be a clique. We say that K *covers* A if every node $k \in A \setminus K$ satisfies $d_k \le \sum_{i \in K \setminus N(k)} d_i$. For $d = \mathbf{1}$, every maximal clique in A is a covering clique.

Every node subset $A \subseteq V$ admits a covering clique, and such a clique can be found in polynomial time [6,8]. To this end, let $i_1, \ldots, i_n$ be an ordering of the nodes in A such that $d_{i_k} \ge d_{i_{k+1}}$. Consider every node in this sequence and construct K iteratively as follows. At step k, we must decide whether i_k has to be inserted into K or not. If there is some $i_t \in K$ with $i_k i_t \notin E$, then do not insert i_k into K. Otherwise, insert i_k into K. Note that in both cases K remains a covering clique of $\{i_1, \ldots, i_k\}$ due to the ordering of the nodes, so upon termination of the algorithm K is a clique covering A.

Definition 2 (*Covering-clique Inequalities*). Let $i \in V$ be a node of G, and let $K \subseteq N(i)$ be a clique covering $N(i)$. We define

$$\sum_{k \in K} d_k\, x_{ki} \le l_i \tag{7}$$

to be the *covering-clique inequality* associated with i and K.

The inequalities (7) are valid for $P(G, d, s, g)$ even if K is a clique not covering $N(i)$, and define facets only if K is a covering clique and $s \ge s_{\min}(G, d, g) + 3(d_{\max} + g)$ [6,8]. The covering-clique inequality (7) is a direct strengthening of the interval bound $l_i \ge 0$. The opposite interval bound $r_i \le s$ can be strengthened in a similar way, generating the following symmetric inequality with identical properties:

$$r_i \le s - \sum_{k \in K} d_k\, x_{ki}.$$

As direct strengthenings of the antiparallelity constraints (3) we obtain:

Definition 3 (*Double Covering-clique Inequalities*). Let $ij \in E$ and let $K \subseteq N(i) \cap N(j)$ be a clique covering $N(i) \cap N(j)$. We define

$$r_i + \sum_{k \in K} d_k (x_{ik} - x_{jk}) \le l_j + (s - d(K)) x_{ji} \tag{8}$$

to be the *double covering-clique inequality* associated with ij and K, where $d(K) = \sum_{k \in K} d_k$.

Double covering-clique inequalities are valid for $P(G, d, s, g)$ and define facets if $s \ge s_{\min}(G, d, g) + 4(d_{\max} + g)$ [6].

To close this section, we address the computational complexity of the associated separation problems. In [6] it is shown that separating the covering-clique inequalities is $\mathcal{NP}$-complete. We provide here a similar proof for the double covering-clique inequalities. If $P_{LP}(G, d, s, g)$ denotes the *linear relaxation* of $P(G, d, s, g)$ (i.e., the polytope defined by the constraints (1)-(4) together with the bounds $0 \le x_{ij} \le 1$ for every $ij \in E$), then the separation problem for this class of inequalities can be defined as follows:

> Double covering-clique inequalities separation
>
> **Instance**: A point $y = (l, r, x) \in P_{LP}(G, d, s, g)$
>
> **Question**: Does y violate some double covering-clique inequality?

Theorem 1. Double covering-clique inequalities separation *is $\mathcal{NP}$-complete.*

Proof. We can easily check that this problem belongs to the class $\mathcal{NP}$, since we can nondeterministically generate an edge $ij \in E$ and a clique $K \subseteq N(i) \cap N(j)$ and verify in deterministic polynomial time whether K covers $N(i) \cap N(j)$ and the double covering-clique inequality associated with ij and K is violated by the point y or not. To complete the proof, we construct a polynomial reduction from Max-clique. An instance of the latter is given by a pair (H, p), where $H = (V_H, E_H)$ is a graph and $p \in \mathbf{Z}_+$ is an integer such that $1 \le p \le |V_H|$, and consists in deciding whether H has a clique of size at least p or not. Assume w.l.o.g. $|V_H| \geq 2$ and that H is noncomplete. To reduce this instance to an instance of Double covering-clique inequalities separation, we construct a graph $G = (V, E)$ from H by adding two universal nodes i and j, thus

$$V = V_H \cup \{i, j\}$$
$$E = E_H \cup \{ti, tj : t \in V_H\} \cup \{ij\}.$$

Set $d = \mathbf{1}$, $g = 0$ and $s = 2n$, where $n = |V|$. Finally, define a point $y = (l, r, x) \in \mathbf{R}^{2|V|+|E|}$ as follows:

$$y_{l_t} = \begin{cases} 0 & \text{if } t \neq j \\ \dfrac{p+1}{2} & \text{if } t = j \end{cases} \quad \forall t \in V$$
$$y_{r_t} = y_{l_t} + 1 \qquad \forall t \in V$$
$$y_{x_e} = \begin{cases} 1 & \text{if } e = tj \text{ for some } t \in V \\ \dfrac{1}{2} & \text{otherwise} \end{cases} \qquad \forall e \in E.$$

This construction is polynomial in the size of H. We now verify that $y \in P_{LP}(G, \mathbf{1}, 2n, 0)$ by checking that the point y satisfies all the constraints of this relaxed polytope. The demand constraints, the interval bounds and the relaxed constraints $0 \leq x_e \leq 1$ for every $e \in E$ are trivially satisfied by construction. So we are left to verify that the antiparallelity constraints $l_k + d_k \leq l_t + sx_{tk}$ for every $kt \in E$ are also satisfied. To this end, consider the following cases:

(1) If $k, t \neq j$, then $y_{x_{tk}} = 1/2$ and, therefore,

$$y_{l_k} + d_k = 1 \leq n = y_{l_t} + sy_{x_{tk}}.$$

(2) If $k = j$, then $y_{x_{tk}} = 1$ and we have that

$$y_{l_j} + d_j = \frac{p+1}{2} + 1 \leq 2n = y_{l_t} + sy_{x_{tj}}.$$

(3) If $t = j$, then $y_{x_{tk}} = 0$ and

$$y_{l_k} + d_k = 1 \leq \frac{p+1}{2} = y_{l_j} + sy_{x_{jk}}.$$

Therefore, $y \in P_{LP}(G, \mathbf{1}, 2n, 0)$. To complete the proof, we show that the prescribed transformation maps affirmative instances of Max-clique onto affirmative instances of Double covering-clique inequalities separation and conversely, i.e., $\omega(H) \geq p$ if and only if y violates some double covering-clique inequality.

($\Rightarrow$) Let $K \subseteq V_H$ be a maximal clique of H of size at least p. Since i and j are universal nodes, then $K \subseteq N_G(i) \cap N_G(j)$. Moreover, $d = \mathbf{1}$ implies K covers $N_G(i) \cap N_G(j) = V_H$. The following calculation shows that the double covering-clique inequality associated with $(K, V_H \setminus K)$ is violated by this point:

$$y_{l_i} + d_i + \sum_{k \in K} d_k (y_{x_{ik}} - y_{x_{jk}}) = 1 + \frac{d(K)}{2} > \frac{p+1}{2} = y_{l_j} + (s - d(K)) y_{x_{ji}}.$$

($\Leftarrow$) Conversely, suppose the double covering-clique inequality defined by the nodes k and t and the clique $K \subseteq N_G(k) \cap N_G(t)$ is violated, i.e.,

$$y_{l_k} + d_k + \sum_{l \in K} d_l (y_{x_{kl}} - y_{x_{tl}}) \; > \; y_{l_t} + (s - d(K)) y_{x_{tk}}. \tag{9}$$

We first show that $t = j$ holds. Suppose in contrary $t \neq j$ and consider two cases.

- If $k \neq j$, then $y_{x_{kl}} - y_{x_{tl}} = 0$ for every $l \in V \setminus \{k, t\}$, and therefore (9) has LHS $= 1$ and RHS $= \frac{1}{2}(s - d(K)) \geq \frac{1}{2}(2n - \omega(H)) \geq 1$. Hence (9) does not hold, a contradiction.
- On the other hand, if $k = j$ then LHS $= 1 + \frac{1}{2}(p + 1 - |K|)$ and RHS $= 2n - d(K)$. Again, we have LHS $\leq$ RHS, contradicting (9).

Thus we have indeed $t = j$. This implies that, in this setting, violated double covering-clique inequalities must have $I(j)$ as the right interval. Since $t = j$, then $y_{l_t} = \frac{p+1}{2}$ and $y_{x_{kl}} - y_{x_{tl}} = 1/2$ for every $l \in K$. Hence (9) reads $1 + \frac{|K|}{2} > \frac{p+1}{2}$, implying $|K| \geq p$. Therefore K is a clique of G with at least p nodes. Now, if $i \notin K$ then $K \subseteq V_H$ and $\omega(H) \geq p$. On the other hand, if $i \in K$ then $(K \setminus \{i\}) \cup \{k\}$ is a clique of H on p nodes, also implying $\omega(H) \geq p$.

Therefore, the transformation maps affirmative instances of Max-Clique onto affirmative instances of Double covering-clique inequalities separation and conversely, hence the latter is $\mathcal{NP}$-complete. $\square$

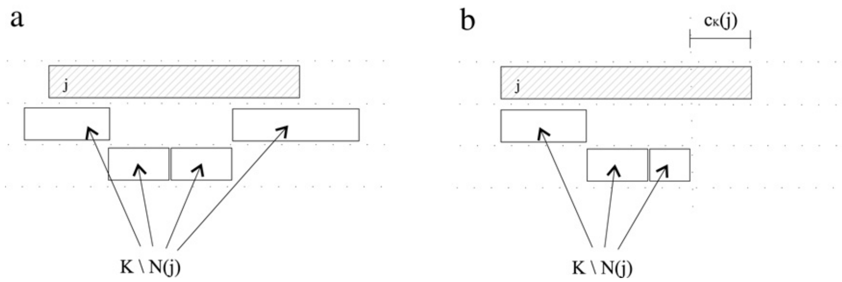


Fig. 3. (a) Example of $c_K(j) = 0$, and (b) example of $c_K(j) > 0$.

3. Reinforced covering-clique inequalities

In this section we present a class of valid inequalities for $P(G, d, s, g)$ generalizing the standard covering-clique inequalities, and we prove that this new class is facet-defining for $P(G, d, s, 0)$ when $s \geq s_{\min}(G, d, 0) + O(1)$. We also show how the ideas leading to this new family can also be applied to generalize the double covering-clique inequalities. If $K \subseteq V$ and $j \in V \setminus K$, we define $c_K(j) = \max\{0,\ d_j - \sum_{k \in K \setminus N(j)} d_k\}$ (see Fig. 3).

Definition 4 (*Reinforced Covering-clique Inequalities*). Let $i \in V$ be a node of G and fix a clique $K' \subseteq N(i)$. Furthermore, let K be a clique covering $N(i) \setminus K'$. We define

$$\sum_{k \in K} d_k x_{ki} + \sum_{k \in K'} c_K(k) x_{ki} \ \leq \ l_i \tag{10}$$

to be the *reinforced covering-clique inequality* associated with K and K'.

The standard covering-clique inequalities discussed in Section 2 can be obtained as a special case of these reinforced covering-clique inequalities by setting $K' = \emptyset$.

Proposition 1. *The reinforced covering-clique inequalities (10) are valid for the polytope $P(G, d, s, g)$.*

Proof. Let $y = (l, r, x)^{\mathrm{T}} \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a feasible solution, and define the node sets $A = \{k \in K' : y_{x_{ki}} = 1 \text{ and } c_K(k) > 0\}$ and $B = \{t \in K : y_{x_{ti}} = 1\}$. Since K resp. K' is a clique, the intervals corresponding to nodes in K resp. K' do not overlap. Moreover, define $Q = \{t \in K :\ tk \in E\ \forall k \in A\}$. Note that $A \cup Q$ is a clique, hence $A \cup (B \cap Q)$ is also a clique. The following calculation establishes the validity of (10):

$$\begin{aligned}
y_{l_i} &\geq \sum_{k \in A} d_k + \sum_{t \in B \cap Q} d_t \\
&= \sum_{k \in A} d_k + \sum_{t \in B \cap Q} d_t - \sum_{t \in B \setminus Q} d_t + \sum_{t \in B \setminus Q} d_t \\
&= \sum_{k \in A} d_k - \sum_{t \in B \setminus Q} d_t + \left(\sum_{t \in B \cap Q} d_t + \sum_{t \in B \setminus Q} d_t \right) \\
&\geq \sum_{k \in A} \left(d_k - \sum_{t \in K \setminus N(k)} d_t \right) + \sum_{t \in B} d_t \\
&= \sum_{k \in A} c_K(k) + \sum_{t \in B} d_t \\
&= \sum_{k \in K'} c_K(k)\, y_{x_{ki}} + \sum_{k \in K} d_k\, y_{x_{ki}}. \quad \square
\end{aligned}$$

Theorem 2. *The reinforced covering-clique inequalities (10) are facet-inducing for $P(G, d, s, 0)$ if $s \geq s_{\min}(G, d, 0) + 3d_{\max}$.*

Proof. Since $s \geq s_{\min}(G, d, 0) + 3d_{\max} \geq \alpha(G, d, 0)$, then $P(G, d, s, 0)$ is full-dimensional. Let F be the face of $P(G, d, s, 0)$ defined by (10), and suppose $\lambda^{\mathrm{T}} z = \lambda_0$ for every feasible solution $z \in P(G, d, s, 0) \cap F$. We shall show that λ is a multiple of the coefficient vector of (10), hence proving that this inequality induces a facet of $P(G, d, s, 0)$.

Claim 1 : $\lambda_{l_j} = 0$ *for* $j \neq i$. Consider the feasible solutions z and z' presented in Fig. 4(a) and (b), respectively. Both solutions have $I(i) = [0, d_i]$ and $r_j = d_i + d_j + 1$, but differ in their l_j-coordinate, since $z_{l_j} = d_i + 1$ and $z'_{l_j} = d_i$. All the remaining intervals are assigned to the right of $I(j)$, which is possible since $s > s_{\min}(G, d, 0) + 2d_{\max}$. It is not difficult to verify that $z, z' \in F$ and, therefore, $\lambda^T z = \lambda_0 = \lambda^T z'$. Since these points only differ in their l_j-coordinate, $\lambda_{l_j} = 0$ follows. $\diamond$

Claim 2: $\lambda_{r_j} = 0$ *for every* $j \in V$. We provide a similar construction as in Claim 1. The feasible solutions presented in Fig. 4(c) and (d) satisfy (10) at equality, implying $\lambda_{r_j} = 0$ for $j \neq i$. A similar argument shows $\lambda_{r_i} = 0$. $\diamond$

Claim 3: $\lambda_{x_{jt}} = 0$ *for every* $jt \in E$ *with* $j, t \neq i$. Consider the feasible solution z presented in Fig. 4(e), having $I(i) = [0, d_i]$, $I[j] = [d_i, d_i + d_j]$, $I[t] = [d_i + d_j, d_i + d_j + d_t]$, and the remaining intervals located to the right of $I(t)$ (this construction is possible since $s \geq s_{\min}(G, d, 0) + 3d_{\max}$). Let z' be the solution obtained from z by swapping the intervals $I(j)$ and $I(t)$ (see Fig. 4(f)). We know from the previous claims that $\lambda_{l_j} = \lambda_{r_j} = 0$ and $\lambda_{l_t} = \lambda_{r_t} = 0$, thus $\lambda_{x_{jt}} = 0$. $\diamond$

Claim 4: $\lambda_{x_{ki}} = -d_k\lambda_{l_i}$ *for every* $k \in K$. Let z be a feasible solution with $I(i) = [0, d_i]$, $I(k) = [d_i, d_i + d_k]$ and the remaining intervals located to the right of $I(k)$ (see Fig. 4(g)). Let z' be the feasible solution obtained from z by swapping the intervals $I(i)$ and $I(k)$ (see Fig. 4(h)). These constructions are feasible since $s > s_{\min}(G, d, 0) + 2d_{\max}$. Both solutions satisfy (10) at equality, hence $\lambda_{x_{ki}} = -d_k\lambda_{l_i}$. $\diamond$

Claim 5: $\lambda_{x_{ki}} = -c_K(k)\lambda_{l_i}$ *for every* $k \in K'$ *with* $c_K(k) > 0$. Let $z \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution with $z_{l_i} = 0$. Now construct a feasible solution $z' \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_k} = 0$ and $z_{l_i} = d_k$, and assigning every interval $I(t)$, for $t \in K \setminus N(k)$, to the left of the interval $I(i)$ (see Fig. 4(i)). These two feasible solutions satisfy (10) at equality and, therefore, $\lambda_{x_{ki}} = -c_K(k)\lambda_{l_i}$. $\diamond$

Claim 6: $\lambda_{x_{ki}} = 0$ *for every* $k \in K'$ *with* $c_K(k) = 0$ *and every* $k \in N(i) \setminus (K \cup K')$. Again, let $z \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution with $z_{l_i} = 0$, and construct a feasible solution $z' \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_k} = 0$, $z_{l_i} = \sum_{l \in K \setminus N(k)} d_l$, and assigning every interval $I(t)$, for $t \in K \setminus N(k)$, to the left of the interval $I(i)$ (see Fig. 4(j)). These two points satisfy (10) at equality, implying $\lambda_{x_{ki}} = 0$. $\diamond$

Hence we verify that λ is a multiple of the coefficient vector of (10) and thus this inequality induces a facet of $P(G, d, s, 0)$. $\square$

We have proved the facetness of (10) for $s \geq s_{\min}(G, d, 0) + 3d_{\max}$. Note that this bound ensures that the constructions in the proof of Theorem 2 are feasible, and guarantees the full-dimensionality of $P(G, d, s, 0)$. It is interesting to note that the full dimension of $P(G, d, s, 0)$ is not sufficient to prove the facetness of the reinforced covering-clique inequalities, since these inequalities generalize the covering-clique inequalities, which may not define facets for $s = s_{\text{full}}(G, d, g)$ [8].

The inequality symmetric to (10) is

$$r_i \leq s - \sum_{k \in K} d_k x_{ik} - \sum_{k \in K'} c_K(k) x_{ik}, \tag{11}$$

which is valid and facet-inducing for $P(G, d, s, 0)$ under the same setting as in Theorem 2. Note that whereas the reinforced covering-clique inequality (10) describes the interaction between the left bound of the interval $I(i)$ and the left bound of the frequency spectrum $[0, s]$, the symmetric inequalities (11) describe the interaction between the right bound of the interval $I(i)$ and the right bound of the frequency spectrum $[0, s]$.

We can also reinforce the double covering-clique inequalities as follows:

Definition 5 (*Reinforced Double Covering-clique Inequalities*). Let $ij \in E$ be an edge of G and fix a clique $K' \subseteq N(i) \cap N(j)$. Furthermore, let K be a clique covering $[N(i) \cap N(j)] \setminus K'$. We call

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) + \sum_{k \in K'} c_K(k)(x_{ik} - x_{jk}) \leq l_j + \left(s - \sum_{k \in K} d_k - \sum_{k \in K'} c_K(k)\right) x_{ij} \tag{12}$$

the *reinforced double covering-clique inequality* associated with K and K'.

The proofs of validity and facetness for the reinforced double covering-clique inequalities are similar to the proofs of Proposition 1 and Theorem 2. Thus, we obtain:

Theorem 3. *The reinforced double covering-clique inequalities are valid for* $P(G, d, s, g)$*, and define facets of* $P(G, d, s, 0)$ *if* $s \geq s_{\min}(G, d, 0) + 4d_{\max}$.

4. Replicated covering-clique inequalities

We now introduce a second generalization of covering-clique inequalities, based on the replication of a subset of the associated covering clique. We prove that these new valid inequalities are facet-inducing if $s \geq s_{\min}(G, d, g) + O(1)$, and we analyze the corresponding symmetric family.

Definition 6 (*Replicated Covering-clique Inequalities*). Fix a node $i \in V$ and let K be a clique covering $N(i)$. Consider a subset $K' = \{k_1, \ldots, k_t\} \subseteq K$ and a clique $Q = \{p_{k_1}, \ldots, p_{k_t}\} \subseteq V \setminus N(i)$ such that $p_k k \in E$ for every $k \in K'$ (see Fig. 5). We define

$$\sum_{k \in K} d_k x_{ki} + \sum_{k \in K'} c_K(p_k)(x_{p_k k} - x_{ik}) \leq l_i \tag{13}$$

to be the *replicated covering-clique inequality* associated with the node i and the cliques K and Q.

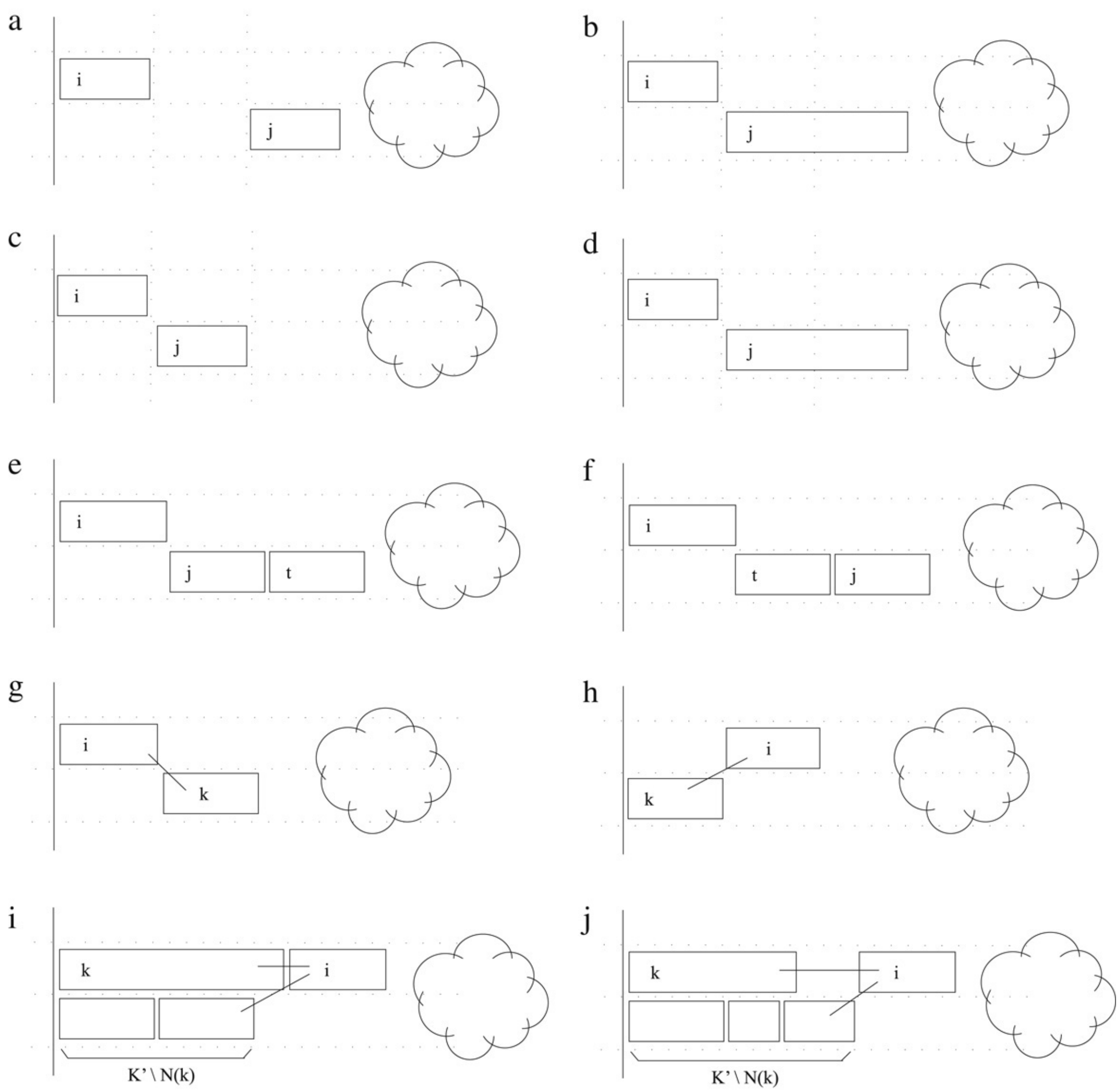


Fig. 4. Constructions for the proof of Theorem 2.

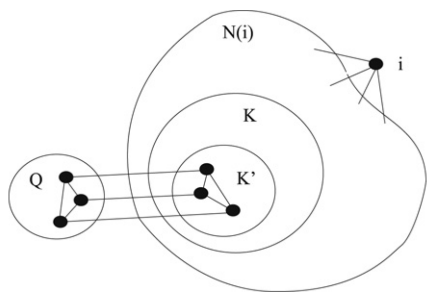


Fig. 5. Structure for replicated covering-clique inequalities.

Note that the definition of the replicated covering-clique inequalities allows edges between K and Q other than $p_k k$, for $k \in K'$. In the case $Q = \emptyset$, the replicated covering-clique inequality (13) is equivalent to the standard covering-clique inequality (7). Moreover, when both K and Q are singletons, these inequalities are equivalent to the *path inequalities* introduced in [5].

Proposition 2. *The replicated covering-clique inequalities (13) are valid for $P(G, d, s, g)$.*

Proof. Let $y = (l, r, x)^{\mathrm{T}} \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ denote an arbitrary integer solution, and define $A = \{k \in K : y_{x_{ki}} = 1\}$ and $B = \{k \in K' : y_{x_{p_k k}} = 1,\ y_{x_{ki}} = 1,\ c_K(p_k) > 0\}$. Also define $T = \{k \in K : kt \in E\ \forall t \in Q\}$, and note that $Q \cup T$ is a clique. The following calculation establishes the validity of (13):

$$\begin{aligned}
y_{l_i} &\geq \sum_{k\in B} d_{p_k} + \sum_{k\in T\cap A} d_k \\
&= \sum_{k\in B} d_{p_k} + \sum_{k\in T\cap A} d_k + \sum_{k\in A\setminus T} d_k - \sum_{k\in A\setminus T} d_k \\
&= \sum_{k\in B} d_{p_k} - \sum_{k\in A\setminus T} d_k + \left(\sum_{k\in T\cap A} d_k + \sum_{k\in A\setminus T} d_k\right) \\
&\geq \sum_{k\in B}\left(d_{p_k} - \sum_{t\in K\setminus N(p_k)} d_t\right) + \sum_{k\in A} d_k \\
&= \sum_{k\in B} c_K(p_k) + \sum_{k\in A} d_k \\
&\geq \sum_{k\in K'} c_K(p_k)(y_{x_{p_k k}} - y_{x_{ik}}) + \sum_{k\in K} d_k y_{x_{ki}}. \quad \square
\end{aligned}$$

Let $\gamma(G, d, g)$ be the minimum frequency span s such that $P(G, d, s, g)$ admits a feasible solution such that $I(k)$ is located after $I(p_k)$, for every $k \in K'$. Note that $\gamma(G, d, g) \geq s_{\min}(G, d, g)$.

Theorem 4. *If $s \geq \gamma(G, d, 0) + 3d_{\max}$, then the replicated covering-clique inequality (13) defines a facet of $P(G, d, s, 0)$.*

Proof. Since $s \geq s_{\min}(G, d, 0) + 3d_{\max} \geq \alpha(G, d, 0)$, then $P(G, d, s, 0)$ is full-dimensional. Let F be the face of $P(G, d, s, 0)$ defined by (13), and suppose every point $y \in F$ satisfies $\lambda^{\mathrm{T}} y = \lambda_0$. We will show that λ is a multiple of the coefficient vector of (13), implying that this inequality induces a facet.

Claim 1: *$\lambda_{l_j} = 0$ for $j \neq i$ and $\lambda_{r_j} = 0$ for $j \in V$.* We show first $\lambda_{l_j} = \lambda_{r_j} = 0$ with the help of the constructions illustrated in Fig. 6(a) and (b). Points y_1 and y_2 (Fig. 6(a) and (b), respectively) are constructed with $l_i = 0$, and thus $x_{ki} = 0$ for every $k \in K$. We also take care of assigning every $k \in K'$ after its associated node p_k, so that $x_{p_k k} - x_{ik} = 0$. This implies that y_1 and y_2 are in F, and thus $\lambda^{\mathrm{T}} y_1 = \lambda_0 = \lambda^{\mathrm{T}} y_2$. These points only differ in their l_j-coordinates, hence $\lambda_{l_j} = 0$ for $j \neq i$. A similar argument shows $\lambda_{r_j} = 0$ for every j (including node i). $\diamond$

Claim 2: *$\lambda_{x_{jl}} = 0$ for every $jl \in E$ such that $j, l \neq i$ and $jl \neq p_k k$ for every $k \in K'$.* We construct the points depicted in Fig. 6(c) and (d), again assigning every $k \in K'$ after p_k, so these points belong to F. Since $\lambda_{l_j} = \lambda_{r_j} = \lambda_{l_l} = \lambda_{r_l} = 0$, we have $\lambda_{x_{jl}} = 0$. $\diamond$

Claim 3: *$\lambda_{x_{p_k k}} = -c_K(p_k)\lambda_{l_i}$ for every $k \in K'$.* Suppose $K \setminus N(p_k) = \{k_1, \ldots, k_\ell\}$, so that $c_K(p_k) = d_{p_k} - \sum_{v=1}^{\ell} d_{k_v}$. Consider the pair of points depicted in Fig. 6(e) and (f), where every $t \in K'$, $t \neq k$, is assigned after p_k. Since both points belong to F they satisfy $\lambda^{\mathrm{T}} x = \lambda_0$ at equality, and we have

$$(d_k + d_{k_1} + \cdots + d_{k_\ell})\lambda_{l_i} = \lambda_{x_{p_k k}} + (d_{p_k} + d_k)\lambda_{l_i},$$

implying

$$\begin{aligned}
\lambda_{x_{p_k k}} &= (d_{k_1} + \cdots + d_{k_\ell} - d_{p_k})\lambda_{l_i} \\
&= -c_K(p_k)\lambda_{l_i}.
\end{aligned} \tag{14}$$

Claim 4: *$\lambda_{x_{ki}} = -d_k\lambda_{l_i}$ for every $k \in K \setminus K'$.* Let z be a feasible solution with $I(i) = [0, d_i]$, $I(k) = [d_i, d_i + d_k]$ and such that all the remaining intervals are located to the right of $I(k)$, taking care of assigning $I(p_t)$ before $I(t)$, for every $t \in K'$ (see Fig. 6(g)). Let z' be the solution obtained from z by swapping the intervals $I(i)$ and $I(k)$ (see Fig. 6(h)). It is not difficult to verify that both points satisfy (13) at equality, hence $\lambda_{x_{ki}} + d_k\lambda_{l_i} = 0$. $\diamond$

Claim 5: *$\lambda_{x_{ki}} = -(d_k + c_K(p_k))\lambda_{l_i}$ for every $k \in K'$.* The two points depicted in Fig. 6(i) and (j) satisfy (13). Since $\lambda_{l_j} = \lambda_{r_j} = 0$, we have $\lambda_{x_{p_k k}} = \lambda_{x_{ki}} + d_k\lambda_{l_i}$. From (14) we have $\lambda_{x_{p_k k}} = -c_K(p_k)\lambda_{l_i}$, implying $\lambda_{x_{ki}} = -(d_k + c_K(p_k))\lambda_{l_i}$. $\diamond$

Claim 6: *$\lambda_{x_{ij}} = 0$ for every $j \in N(i) \setminus K$.* Let $z \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution with $z_{l_i} = 0$, and construct a feasible solution $z' \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_j} = 0$, $z_{l_i} = \sum_{l\in K\setminus N(j)} d_l$, and assigning every interval $I(t)$, for $t \in K \setminus N(j)$, to the left of the interval $I(i)$. These two points belong to F, implying $\lambda_{x_{ij}} = 0$. $\diamond$

Therefore, we have $\lambda = -\lambda_{l_i}\pi$, where π denotes the coefficient vector of (13). Hence the replicated covering-clique inequality (13) defines a facet of $P(G, d, s, 0)$. $\square$

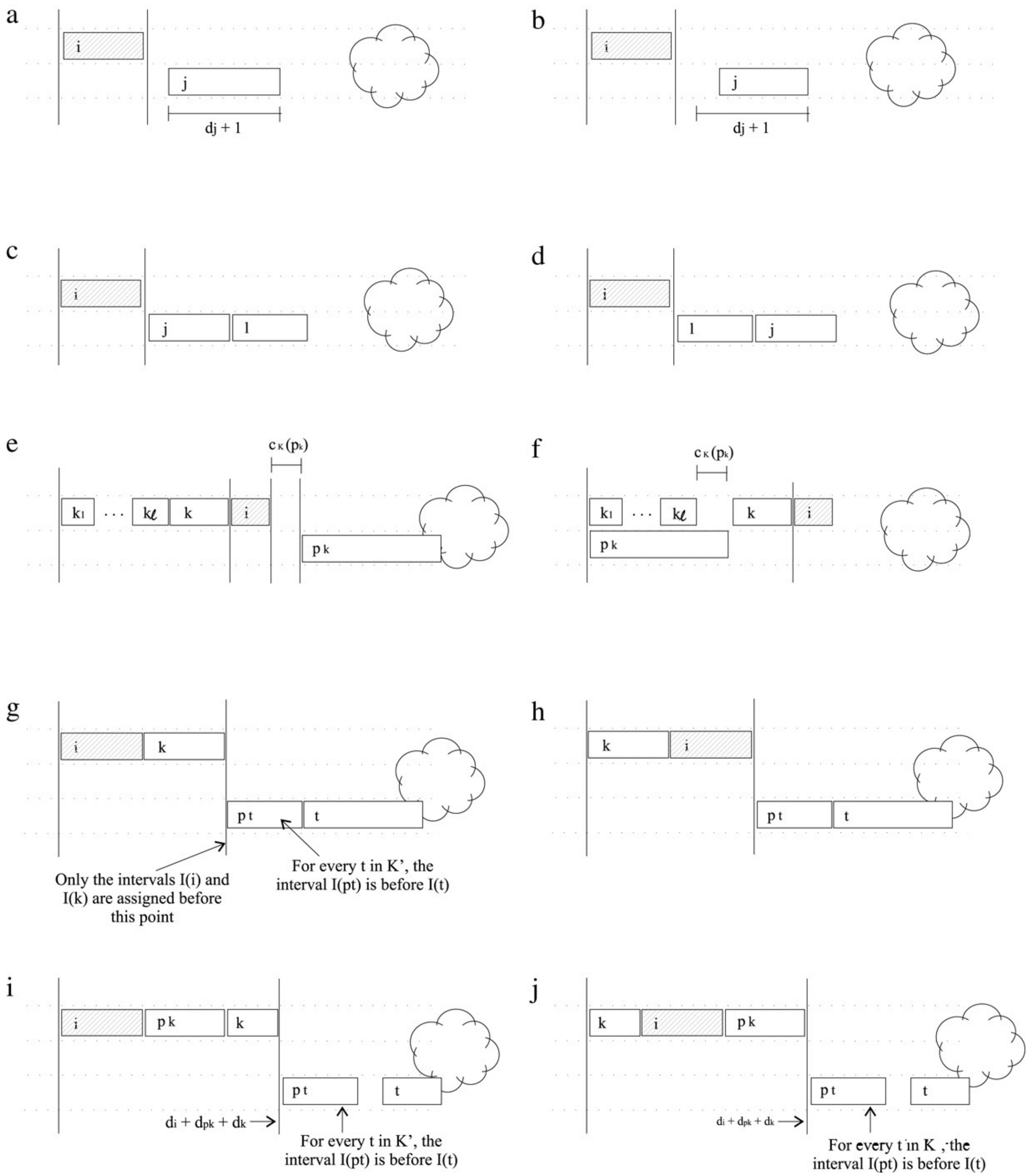


Fig. 6. Constructions for the proof of Theorem 4.

Note that the bound $s \geq \gamma(G, d, 0) + 3d_{\max}$ is necessary in order to ensure the feasibility of the constructions in the proof of Theorem 4, and to guarantee the full-dimensionality of $P(G, d, s, 0)$. It is interesting to note that the full-dimensionality alone is not sufficient to ensure the facetness of the replicated covering-clique inequalities, as this is not the case for the particular case of the covering-clique inequalities [8].

Again, the symmetric inequalities associated to the replicated covering-clique inequalities describe the interaction between the interval $I(i)$ and the cliques K and K' with the right bound of the frequency spectrum [0, s]. Under the same setting as in Theorem 4, the following symmetric inequality is valid and facet-inducing for $P(G, d, s, 0)$:

$$r_i \;\leq\; s - \sum_{k \in K} d_k x_{ik} + \sum_{k \in K'} c_K(p_k)(x_{kp_k} - x_{ki}).$$

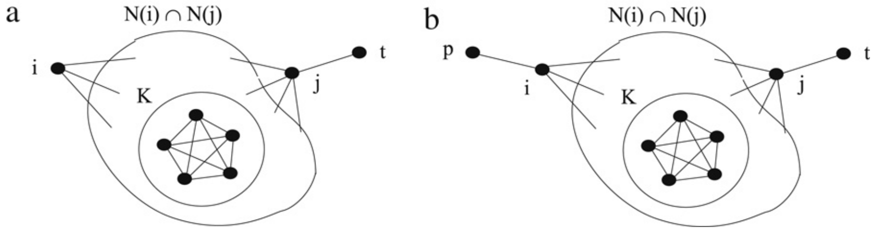


Fig. 7. Supports for extended double covering-clique inequalities.

5. Extensions of double covering-clique inequalities

The ideas involved in the development of double covering-clique inequalities do not restrict to that particular family of inequalities, but can be further exploited to find new classes of facet-inducing inequalities based on similar concepts. In this section we explore facet-defining valid inequalities over slightly different structures, analyzing the effect of these structure changes in the resulting inequalities.

Definition 7 (*Extended Double Covering-clique Inequalities*). Let $i, j \in V$ be two adjacent nodes, and let K be a clique covering $N(i) \cap N(j)$. Furthermore, fix some node $t \in N(j) \setminus N(i)$ (see Fig. 7(a)). We define

$$r_i + \sum_{k\in K} d_k(x_{ik} - x_{jk}) \ \le\ l_j + \varphi x_{ji} + c_K(t)x_{jt} \tag{15}$$

to be the *extended double covering-clique inequality* associated with K and t, where $\varphi = s - d(K) - c_K(t)$.

Proposition 3. *The extended double covering-clique inequalities (15) are valid for the polytope $P(G, d, s, g)$.*

Proof. Let $y = (l, r, x)^{\mathrm{T}} \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a feasible integer solution. If $y_{x_{ji}} = 0$, then the inequality (15) is dominated by the standard double covering-clique inequality (8), and thus is satisfied by y. On the other hand, if $y_{x_{ji}} = 1$ and $y_{x_{jt}} = 1$ then (15) reads as a standard double covering-clique inequality, and is therefore satisfied by y.

Hence the only nontrivial case for validity is $y_{x_{ji}} = 1$ and $y_{x_{jt}} = 0$. Assume this holds, so the interval $I(t)$ is located before $I(j)$, which in turn is located before $I(i)$. Define A, resp. B, resp. C to be the set of intervals from K located before $I(j)$ resp. between $I(j)$ and $I(i)$, resp. after $I(i)$. For $k \in K$, note that $y_{x_{ik}} - y_{x_{jk}} = -1$ if $k \in B$, and $y_{x_{ik}} - y_{x_{jk}} = 0$ otherwise.

Since $I(t)$ is located before $I(j)$ and A is a clique, then $y_{l_j} \ge d(A) + c_A(t)$ holds. Moreover, $A \subseteq K$ implies $c_K(t) \le c_A(t)$. Finally, since C is a clique, then $y_{r_i} \le s - d(C)$ clearly holds. Combining these observations, the following calculation establishes the validity of (15):

$$\begin{aligned} y_{r_i} + \sum_{k\in K} d_k(y_{x_{ik}} - y_{x_{jk}}) - y_{l_j} &\le \left(s - \sum_{k\in C} d_k\right) - \sum_{k\in B} d_k - \left(\sum_{k\in A} d_k + c_A(t)\right) \\ &\le s - \sum_{k\in K} d_k - c_K(t) \\ &= \varphi y_{x_{ji}} + c_K(t) y_{x_{jt}}. \quad \square \end{aligned}$$

The proofs of all the facetness results in this section go along the argumentation of the proof of facetness for the standard double covering-clique inequalities presented in [6,8].

Theorem 5. *If $s \ge s_{\min}(G, d, 0) + 4d_{\max}$, then the extended double covering-clique inequalities (15) induce facets of $P(G, d, s, 0)$.*

It is interesting to compare the standard double covering-clique inequalities (8) with the extended inequalities (15). The coefficient of x_{ji} is smaller in the extended inequality, which in turn has a new positive coefficient in the RHS, corresponding to x_{jt}. This reflects the fact that we cannot reinforce the original inequalities with a nonnegative coefficient in x_{jt} for free: when we force this variable to have a nonzero coefficient, the variable x_{ji} decreases its coefficient to maintain validity.

Moreover, it is worthwhile to compute the symmetric inequality of this new class. The symmetric of a double covering-clique inequality is again a double covering-clique inequality, but the symmetric of this extension is a new valid inequality:

$$r_j + \sum_{k\in K} d_k(x_{ik} - x_{jk}) \ \le\ l_i + \varphi x_{ij} + c_K(t)x_{tj}. \tag{16}$$

In this case, the inequality is reinforced by adding a coefficient associated with the edge $tj \in E$, but now the interval $I(j)$ is the left interval in the inequality.

Definition 8 (*2-extended Double Covering-clique Inequalities*). Let $i, j \in V$ be two adjacent nodes, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \setminus N(j)$ and $t \in N(j) \setminus N(i)$ (see Fig. 7(b)). We define

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\le\; l_j + \varphi' x_{ji} + c_K(t)x_{pi} + c_K(p)x_{jt} \tag{17}$$

to be the *2-extended double covering-clique inequality* associated with K and nodes t and p, where $\varphi' = s - d(K) - (c_K(t) + c_K(p))$.

Note that the 2-reinforced double covering-clique inequalities are obtained by "combining" inequalities (15) and (16) into a new valid one. Now we have two new nodes, namely p and t, adjacent to nodes i and j, respectively. The standard double covering-clique inequality is reinforced with nonzero coefficients associated with the variables x_{ip} and x_{jt}.

Theorem 6. *If $s \ge s_{\min}(G, d, 0) + 4d_{\max}$, then the 2-extended double covering-clique inequalities are facet-inducing for $P(G, d, s, 0)$.*

Definition 9 (*Closed Double Covering-clique Inequalities*). Let $i, j \in V$ be two adjacent nodes, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \setminus N(j)$ and $t \in N(j) \setminus N(i)$ such that $pt \in E$ and $pk, tk \in E$ for all $k \in K$. We define

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\le\; l_j + \varphi'' x_{ji} + \varphi_p x_{pi} + \varphi_t x_{jt} - \varphi_{pt} x_{pt} \tag{18}$$

to be the *closed double covering-clique inequality* associated with K and nodes t and p, where

$$\begin{aligned}
\varphi'' &= s - d(K) - (d_p + d_t)\\
\varphi_t &= d_t + \min\{d_p, d_t\}\\
\varphi_p &= d_p\\
\varphi_{pt} &= \min\{d_p, d_t\}.
\end{aligned}$$

By a similar argumentation as before, we can show:

Theorem 7. *If $s \ge s_{\min}(G, d, 0) + 4d_{\max}$, then the closed double covering-clique inequalities (18) induce facets of $P(G, d, s, 0)$.*

6. Computational experiments

This section reports the results of preliminary computational experiments performed in order to assess the contribution of the classes of valid inequalities presented in this paper to the practical solution of the bandwidth allocation problem in PMP-systems. Our goal is to evaluate the polyhedral strength of the studied valid inequalities w.r.t. their impact as user-defined cuts in a commercial IP solver.

The experiments are performed over random instances generated in the following way, in order to resemble the structure of the test instances reported in [1]. Given the number k of sectors, the geographical position of the corresponding base stations is randomly determined within the box $[0, 10] \times [0, 10]$ with uniform distribution. Each of the n customer terminals is randomly assigned to a base station and located within a random position at most at distance 3 from the base station. An inter-sector edge is added between every two customer terminals such that there is a distance of at most 2 between them. Each sector has a main customer with demand taken randomly from the interval $[0.7 \times s, 0.9 \times s]$, and the individual demands for the remaining customers are taken randomly from $[1, 4]$. The objective function asks to maximize $\sum_{i \in M}(r_i - l_i)$, where $M \subseteq V$ is the set of main customers. All the instances have $g = 0$.

Table 1 reports the improvement in the optimal value of the linear relaxation of $P(G, d, s, g)$ when all the inequalities from each of the classes discussed in this paper are added to the original formulation. The column LR represents the optimal value of the linear relaxation, and the following columns represent the percentage improvement for each class of valid inequalities. The last column represents the improvement when all the inequalities from all the considered classes are added to the linear relaxation. Blank entries correspond to infeasible linear relaxations.

As Table 1 shows, the covering-clique inequalities and their reinforcements do not improve the linear relaxation for these instances. On the other hand, the double covering-clique inequalities have a greater (although limited) impact in the linear relaxation, and their reinforcements and generalizations produce a marginal benefit only. A more interesting contribution is given by the fact that these inequalities allow to detect infeasible instances even in the linear relaxation, which is crucial in practical settings. It is important to note that this guarantee cannot be achieved by heuristic procedures.

In spite of the fact that the covering-clique inequalities and its reinforcements do not improve the linear relaxation, these inequalities can be very useful within a branch-and-bound procedure. Table 2 reports the total time of a branch-and-bound algorithm over the original integer programming formulation reinforced with all the inequalities from each class presented in this paper. The column "Time" contains the total running time in seconds to optimality of a branch-and-bound algorithm on the original integer programming formulation, whereas the following columns contain the total running time of the same algorithm on the formulation reinforced with all the inequalities of the corresponding class, expressed as a fraction of the original IP time. Finally, the last column represents the running time – as a fraction of the original IP time – when all the

Table 1
Improvement in the linear relaxation

Instance	n	k	s	LR	Clique	Double	Reinf	Dreinf	Repl	Ext	2-Ext	Closed	All
bsc.A.15.4.50.0	15	4	50	189	0.00		−0.10		0.00	−2.10	−2.12	0.00	
bsc.A.15.4.52.0	15	4	52	197	0.00		−0.09		0.00	−2.02	−2.03	0.00	
bsc.A.15.4.61.0	15	4	61	233	0.00	−6.44	−0.06	−3.00	0.00	−1.70	−1.72	0.00	−6.44
bsc.A.15.4.69.0	15	4	69	265	0.00	−5.66	−0.05	−2.64	0.00	−1.50	−1.51	0.00	−5.66
bsc.A.15.4.72.0	15	4	72	277	0.00	−5.42	−0.04	−2.53	0.00	−1.44	−1.44	0.00	−5.42
bsc.A.15.4.76.0	15	4	76	293	0.00	−5.12	−0.03	−2.39	0.00	−1.35	−1.37	0.00	−5.12
bsc.A.15.4.79.0	15	4	79	305	0.00	−4.92	−0.02	−2.30	0.00	−1.30	−1.31	0.00	−4.92
bsc.B.15.4.50.0	15	4	50	190	0.00		0.00	−1.55	0.00	0.00	0.00	0.00	
bsc.B.15.4.51.0	15	4	51	194	0.00	−8.76	0.00	−1.52	0.00	0.00	0.00	0.00	−8.76
bsc.B.15.4.53.0	15	4	53	202	0.00	−8.42	0.00	−1.46	0.00	0.00	0.00	0.00	−8.42
bsc.B.15.4.62.0	15	4	62	238	0.00	−7.14	0.00	−1.23	0.00	0.00	0.00	0.00	−7.14
bsc.B.15.4.66.0	15	4	66	254	0.00	−6.69	0.00	−1.15	0.00	0.00	0.00	0.00	−6.69
bsc.B.15.4.69.0	15	4	69	266	0.00	−6.39	0.00	−1.10	0.00	0.00	0.00	0.00	−6.39
bsc.B.15.4.71.0	15	4	71	274	0.00	−6.20	0.00	−1.07	0.00	0.00	0.00	0.00	−6.20
bsc.B.15.4.79.0	15	4	79	306	0.00	−5.56	0.00	−0.96	0.00	0.00	0.00	0.00	−5.56
bsc.B.15.4.80.0	15	4	80	310	0.00	−5.48	0.00	−0.95	0.00	0.00	0.00	0.00	−5.48
bsc.C.15.4.50.0	15	4	50	189	0.00		0.00	−3.17	0.00	0.00	0.00	0.00	
bsc.C.15.4.51.0	15	4	51	193	0.00		0.00	−3.11	0.00	0.00	0.00	0.00	
bsc.C.15.4.60.0	15	4	60	229	0.00	−5.68	0.00	−2.62	0.00	0.00	0.00	0.00	−5.68
bsc.C.15.4.75.0	15	4	75	289	0.00	−4.50	0.00	−2.08	0.00	0.00	0.00	0.00	−4.50
bsc.C.15.4.77.0	15	4	77	297	0.00	−4.38	0.00	−2.02	0.00	0.00	0.00	0.00	−4.38
bsc.C.15.4.78.0	15	4	78	301	0.00	−4.32	0.00	−1.99	0.00	0.00	0.00	0.00	−4.32
bsc.D.15.4.72.0	15	4	72	277	0.00	−5.05	−0.06	−2.29	0.00	−0.01	0.00	0.00	−5.05
bsc.E.15.4.61.0	15	4	61	234	0.00	−3.85	0.00	−0.85	0.00	0.00	0.00	0.00	−3.85
bsc.E.15.4.69.0	15	4	69	266	0.00	−3.38	0.00	−0.75	0.00	0.00	0.00	0.00	−3.38
bsc.E.15.4.75.0	15	4	75	290	0.00	−3.10	0.00	−0.69	0.00	0.00	0.00	0.00	−3.10
bsc.E.15.4.76.0	15	4	76	294	0.00	−3.06	0.00	−0.68	0.00	0.00	0.00	0.00	−3.06
bsc.E.15.4.79.0	15	4	79	306	0.00	−2.94	0.00	−0.65	0.00	0.00	0.00	0.00	−2.94
bsc.F.15.4.51.0	15	4	51	195	0.00	−6.15	0.00	−1.49	0.00	0.00	0.00	0.00	−6.15
bsc.F.15.4.52.0	15	4	52	199	0.00	−6.03	0.00	−1.46	0.00	0.00	0.00	0.00	−6.03
bsc.F.15.4.54.0	15	4	54	207	0.00	−5.80	0.00	−1.40	0.00	0.00	0.00	0.00	−5.80
bsc.F.15.4.56.0	15	4	56	215	0.00	−5.58	0.00	−1.35	0.00	0.00	0.00	0.00	−5.58
bsc.F.15.4.60.0	15	4	60	231	0.00	−5.19	0.00	−1.26	0.00	0.00	0.00	0.00	−5.19
bsc.F.15.4.65.0	15	4	65	251	0.00	−4.78	0.00	−1.16	0.00	0.00	0.00	0.00	−4.78
bsc.F.15.4.71.0	15	4	71	275	0.00	−4.36	0.00	−1.05	0.00	0.00	0.00	0.00	−4.36
bsc.A.20.4.173.0	20	4	17	432	0.00		0.00	−0.52	0.00			0.00	
bsc.A.20.4.175.0	20	4	17	438	0.00		0.00	−0.51	0.00			0.00	
bsc.A.20.4.176.0	20	4	17	441	0.00		0.00	−0.51	0.00			0.00	
bsc.B.20.4.50.0	20	4	50	188	0.00		0.00	−1.80	0.00	0.00	0.00	0.00	
bsc.A.22.5.94.0	22	5	94	367	0.00		0.00	−0.08	0.00			0.00	
bsc.B.22.5.88.0	22	5	88	261	0.00		0.00	−0.24	0.00			0.00	
bsc.B.22.5.89.0	22	5	89	264	0.00		0.00	−0.24	0.00			0.00	
bsc.D.22.5.100.0	22	5	10	344	0.00		0.00		0.00			0.00	
bsc.D.22.5.107.0	22	5	10	368.5	0.00		0.00		0.00			0.00	
bsc.D.22.5.91.0	22	5	91	312.5	0.00		0.00		0.00			0.00	
bsc.D.22.5.96.0	22	5	96	330	0.00		0.00		0.00			0.00	

The column LR represents the optimal value of the linear relaxation, and the following columns represent the percentage improvement for each class of valid inequalities. Blank entries correspond to infeasible linear relaxations.

inequalities from all the considered classes are added to the original formulation. The experiments were performed with Cplex 9 in a Pentium IV PC with a 2.4 GHz processor and 1.25 GB of RAM memory.

As Table 2 shows, the covering-clique inequalities and the double covering-clique inequalities have an important impact on the overall running times. Furthermore, the reinforcements of these two classes also produce an interesting impact. In spite of the fact that the four final valid inequalities in this table generate a limited contribution to the average running time, it is interesting to note that in some instances they allow to detect infeasibility in very short running times, which we recall is a fundamental issue in practical settings.

Finally, it is remarkable that the addition of all the inequalities from all the considered classes allows to solve the problem in extremely small running times, in spite of the fact that this reinforced formulation contains a larger number of constraints. Although the addition of these inequalities does not further improve the linear relaxation, it allows the branch-and-bound procedure to quickly find optimal and near-optimal feasible solutions, thus closing the duality gap in very short running times. These results suggest that the combined action of these inequalities may be crucial for the practical solution of this problem, in particular for finding feasible solutions.

It is interesting to observe that the overall performance of the branch-and-bound procedure was not worsened in our experiments by the addition of all the inequalities from each class. However, the addition of such numbers of valid inequalities to the IP formulation may not be practical for large instances, in particular for instances containing very large

Table 2
Running time in seconds of a branch-and-bound algorithm

Instance	n	k	s	Time (s)	Clique	Double	Reinf	Dreinf	Repl	Ext	2-Ext	Closed	All
bsc.A.15.4.50.0	15	4	50	0.44	0.70	0.68	0.77	0.70	0.95	0.93	0.82	0.93	0.03
bsc.A.15.4.52.0	15	4	52	0.78	0.42	0.38	0.54	0.40	0.62	0.60	0.53	0.96	0.05
bsc.A.15.4.61.0	15	4	61	10.45	0.15	0.03	0.15	0.91	0.86	0.15	0.23	1.00	0.05
bsc.A.15.4.69.0	15	4	69	10.20	0.21	0.03	0.17	0.96	0.65	0.08	0.15	1.00	0.05
bsc.A.15.4.72.0	15	4	72	13.91	0.11	0.02	0.13	0.69	0.68	0.04	0.06	0.99	0.06
bsc.A.15.4.76.0	15	4	76	12.69	0.13	0.03	0.13	0.83	0.36	0.10	0.05	1.00	0.06
bsc.A.15.4.79.0	15	4	79	12.30	0.25	0.03	0.13	0.67	0.28	0.10	0.11	1.00	0.05
bsc.B.15.4.50.0	15	4	50	0.69	0.57	0.45	0.48	0.90	0.97	0.97	0.97	0.97	0.03
bsc.B.15.4.51.0	15	4	51	11.23	0.16	0.03	0.27	0.33	1.00	1.00	1.00	1.00	0.03
bsc.B.15.4.53.0	15	4	53	11.45	0.29	0.03	0.23	0.34	1.00	1.00	1.00	1.00	0.03
bsc.B.15.4.62.0	15	4	62	12.70	0.21	0.03	0.25	0.40	1.00	1.00	1.00	1.00	0.03
bsc.B.15.4.66.0	15	4	66	7.92	0.28	0.04	0.31	0.64	1.00	1.00	1.00	1.00	0.05
bsc.B.15.4.69.0	15	4	69	10.47	0.31	0.03	0.38	0.40	1.00	1.00	1.00	1.00	0.05
bsc.B.15.4.71.0	15	4	71	7.03	0.38	0.05	0.43	0.53	1.00	1.00	0.99	1.00	0.05
bsc.B.15.4.79.0	15	4	79	9.34	0.51	0.04	0.40	0.50	1.00	1.00	1.00	1.00	0.05
bsc.B.15.4.80.0	15	4	80	11.56	0.25	0.03	0.34	0.42	1.00	1.00	1.00	1.00	0.05
bsc.C.15.4.50.0	15	4	50	0.69	0.49	0.45	0.49	0.68	0.93	0.96	0.96	0.93	0.05
bsc.C.15.4.51.0	15	4	51	0.62	0.61	0.50	0.58	0.73	1.00	1.00	1.00	1.00	0.05
bsc.C.15.4.60.0	15	4	60	17.94	0.19	0.02	0.13	0.46	1.00	1.00	1.00	0.99	0.05
bsc.C.15.4.75.0	15	4	75	15.08	0.11	0.02	0.15	0.55	1.00	1.00	1.00	1.00	0.05
bsc.C.15.4.77.0	15	4	77	12.86	0.15	0.03	0.28	0.67	1.00	1.00	1.00	1.00	0.05
bsc.C.15.4.78.0	15	4	78	9.33	0.22	0.03	0.40	0.92	0.99	1.00	0.99	0.99	0.05
bsc.D.15.4.72.0	15	4	72	8.77	0.55	0.04	0.23	0.81	0.25	0.28	0.78	1.00	0.06
bsc.E.15.4.61.0	15	4	61	2.19	0.68	0.15	0.56	0.59	0.99	0.99	0.99	0.99	0.05
bsc.E.15.4.69.0	15	4	69	2.12	0.85	0.15	0.68	0.50	0.97	1.00	1.00	0.99	0.05
bsc.E.15.4.75.0	15	4	75	2.08	0.40	0.15	0.75	0.50	1.00	0.98	1.00	0.98	0.06
bsc.E.15.4.76.0	15	4	76	2.28	0.55	0.14	0.49	0.51	0.98	1.00	1.00	0.97	0.05
bsc.E.15.4.79.0	15	4	79	2.00	0.48	0.16	0.52	0.75	0.98	0.98	0.98	0.98	0.05
bsc.F.15.4.51.0	15	4	51	65.59	0.21	0.00	0.32	0.28	1.00	1.00	1.00	1.00	0.05
bsc.F.15.4.52.0	15	4	52	52.86	0.32	0.01	0.30	0.42	1.00	1.00	1.00	1.00	0.05
bsc.F.15.4.54.0	15	4	54	60.09	0.18	0.01	0.23	0.42	1.00	1.00	1.00	1.00	0.05
bsc.F.15.4.56.0	15	4	56	48.95	0.31	0.01	0.37	0.52	1.00	1.00	1.00	1.00	0.05
bsc.F.15.4.60.0	15	4	60	73.00	0.17	0.00	0.30	0.25	0.99	1.00	1.00	1.00	0.06
bsc.F.15.4.65.0	15	4	65	71.97	0.18	0.00	0.22	0.34	0.99	0.99	0.99	0.99	0.05
bsc.F.15.4.71.0	15	4	71	65.59	0.24	0.00	0.29	0.29	1.00	1.00	1.00	1.00	0.03
bsc.A.20.4.173.0	20	4	17	0.31	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.05
bsc.A.20.4.175.0	20	4	17	7.75	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04	0.03
bsc.A.20.4.176.0	20	4	17	67.72	0.01	0.00	0.00	0.47	0.01	0.00	0.00	0.01	0.05
bsc.B.20.4.50.0	20	4	50	28.89	0.01	0.01	0.01	0.02	1.00	1.00	1.00	1.00	0.05
bsc.A.22.5.94.0	22	5	94	36.55	0.16	0.01	0.65	0.61	0.09	0.01	0.01	0.43	0.03
bsc.B.22.5.88.0	22	5	88	8.11	0.05	0.04	0.06	0.06	0.05	0.04	0.04	0.99	0.03
bsc.B.22.5.89.0	22	5	89	11.02	0.17	0.03	0.31	0.05	0.45	0.03	0.03	0.16	0.05
bsc.D.22.5.100.0	22	5	10	0.34	0.91	0.91	0.91	0.97	0.97	0.97	0.91	0.91	0.05
bsc.D.22.5.107.0	22	5	10	0.33	1.00	0.94	0.94	0.94	1.00	0.94	0.94	0.94	0.03
bsc.D.22.5.91.0	22	5	91	0.31	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.03
bsc.D.22.5.96.0	22	5	96	0.33	0.94	0.94	0.94	0.94	0.94	0.94	1.00	0.94	0.05
Average					0.37	0.19	0.40	0.56	0.83	0.76	0.77	0.91	0.05

The column "Time" contains the total running time of a branch-and-bound algorithm on the original integer programming formulation, whereas the following columns contain the total running time of the same algorithm on the formulation reinforced with all the inequalities of the corresponding class, expressed as a fraction of the original IP time.

numbers of cliques. In such a setting, solving the linear relaxation at each node of the branching tree may be a time-consuming step. This effect can be avoided by dynamically generating cuts from these classes of valid inequalities with suitable separation heuristics.

7. Concluding remarks

In this paper we explored classes of valid inequalities for chromatic scheduling polytopes based on covering cliques and related concepts. We introduced the reinforced covering-clique and the replicated covering-clique inequalities as direct generalizations of the standard covering-clique inequalities, and we presented three further classes based on slight variations on the original family.

It is worth comparing the inequalities from these families arising from the same graph structure. Suppose $N(i) \cap N(j) = \emptyset$ (so that $K = \emptyset$) and take $d = \mathbf{1}$. Moreover, set $s = 4$ and suppose $P(G, d, 4, 0)$ is nonempty. In this setting, the standard and the extended double covering-clique inequalities have the following form:

$$\text{standard} \rightarrow r_i \leq l_j + 4x_{ji}$$

$$\text{extended} \rightarrow r_i \leq l_j + 3x_{ji} + x_{jt}$$

extended (symm.) $\rightarrow\ r_i \leq l_j + 3x_{ji} + x_{pi}$

2-extended $\rightarrow\ r_i \leq l_j + 2x_{ji} + x_{jt} + x_{pi}$

closed $\rightarrow\ r_i \leq l_j + 2x_{ji} + 2x_{jt} + x_{pi} - x_{pt}$

closed (symm.) $\rightarrow\ r_i \leq l_j + 2x_{ji} + x_{jt} + 2x_{pi} - x_{pt}$.

These inequalities show an interesting interplay among the coefficients of the ordering variables involving the new nodes t and p. In the extended and 2-extended covering-clique inequalities, as we decrease the coefficient of the variable x_{ji}, the variables x_{jt} and x_{pi} receive nonzero coefficients in order to maintain validity while ensuring facetness. In the closed covering-clique inequalities, we have decreased the coefficient associated with the variable x_{pt}, and now the variables x_{jt} and x_{pi} must alternatively increase their coefficients.

It is remarkable that all these inequalities are facet-inducing, showing that the ideas leading to the covering-clique inequalities appear in many different facet-defining inequalities of chromatic scheduling polytopes. These results give another hint of the hardness of these polytopes, since so many variations of a same idea are present as facets. It is worth mentioning that all these inequalities are different even in the uniform case $d = \mathbf{1}$, i.e., for the graph coloring polytope associated to a variation of the orientation model for frequency assignment problems [2].

In this work we have studied the facetness properties for the case $g = 0$. It would be interesting to analyze the facetness properties of these classes of valid inequalities for the case $g > 0$. It would also be interesting to search for further variations of covering-clique inequalities involving more than two nodes outside the standard-clique structure.

Finally, some preliminary computational experiments were presented, showing the potential of these theoretical results to the practical solution of the bandwidth allocation problem in PMP-systems. The addition of all the inequalities from all the considered classes achieved a remarkable performance, and we plan to perform further experiments to assess the potential of the combined action of these classes.

These experiments were based on the addition of all the inequalities from each class to the original integer programming formulation, and it would be interesting to study the dynamic addition of these inequalities via separation procedures within a branch-and-cut environment. However, this task is not straightforward, as even the simplest covering-clique and double covering-clique inequalities have $\mathcal{NP}$-complete separation problems. We leave the exploration of separation heuristics for these classes of valid inequalities as an open problem for future research.

Acknowledgements

We thank the two anonymous referees for their valuable comments and suggestions which helped to improve the results and the final presentation of this paper. The first author was partially supported by Ubacyt Grant X212, Foncyt Grant 11-09112 (Argentina), and CNPq under Prosul project Proc. 490333/2004-4 (Brazil).

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