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Graded rings associated with contracted ideals

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1. Introduction

The study of the ideals in a regular local ring $(R, \mathfrak{m})$ of dimension 2 has a long and important tradition dating back to the fundamental work of Zariski [ZS]. More recent contributions are due to several authors including Cutkosky, Huneke, Lipman, Rees, Sally, and Teissier among others, see [C1,C2,H,HS,L,LT,R]. One of the main result in this setting is the unique factorization theorem for complete (i.e., integrally closed) ideals proved originally by Zariski [ZS, Theorem 3, Appendix 5]. It asserts that any complete ideal can be factorized as a product of simple complete ideals in a unique way (up to the order of the factors). By definition, an ideal is simple if it cannot be written as a product of two proper ideals. Another important property of a complete ideal I is that its reduction number is 1 which in turns implies that the associated graded ring $\mathrm{gr}_I(R)$ is Cohen–Macaulay and its Hilbert series is well-understood; this is due to Lipman and Teissier [LT], see also [HS].

The class of contracted ideals plays an important role in the original work of Zariski as well as in the work of Huneke [H]. An ideal I of R is contracted if $I = R \cap IR[\mathfrak{m}/x]$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Any complete ideal is contracted but not the other way round. The associated graded ring $\mathrm{gr}_I(R)$ of a contracted ideal I need not be Cohen–Macaulay and its Hilbert series can be very complicated.

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Our goal is to study depth, Hilbert function, and defining equations of the various graded rings (Rees algebra, associated graded ring and fiber cone) of homogeneous $\mathfrak{m}$-primary contracted ideals in the polynomial ring $R = k[x, y]$ over an algebraically closed field k of characteristic 0.

In Section 3 we present several equivalent characterizations of contracted ideals in the graded and local case. The main result of this section is Theorem 3.12. It asserts that the depth of $\operatorname{gr}_I(R)$ is equal to the minimum of depth $\operatorname{gr}_{I'S_N}(S_N)$, where $S = R[\mathfrak{m}/x]$, I' is the transform of I and N varies in the set of maximal ideals of S containing I'.

An important invariant of a contracted ideal I of order (i.e., initial degree) d is the so-called characteristic form. In the graded setting the characteristic form of I is $\mathrm{GCD}(I_d)$, where I_d denotes the homogeneous component of degree d of I. The more general contracted ideals are those with a square-free (i.e., no multiple factors) characteristic form. On the other hand, the more special contracted ideals are those whose characteristic form is a power of a linear form; these ideals are the so-called lex-segment ideals. The lex-segment ideals are in bijective correspondence with the Hilbert functions (in the graded sense) of graded ideals so that to specify a lex-segment ideal is equivalent to specify a Hilbert function.

In the graded setting Zariski's factorization theorem for contracted ideals ([ZS, Theorem 1, Appendix 5] or Theorem 3.8) says that any contracted ideal I can be written as a product of lex-segment ideals $I = \mathfrak{m}^c L_1 \cdots L_k$. Here each L_i is a lex-segment monomial ideal with respect to an appropriate system of coordinates x_i, y_i which depends on i. Furthermore L_i has exactly one generator in its initial degree which is a power of x_i.

As a consequence of Theorem 3.12 we have that the depth of $\operatorname{gr}_I(R)$ is equal to the minimum of the depth of $\operatorname{gr}_{L_i}(R)$ (see Corollary 3.14). We can also express the Hilbert series of I in terms of the Hilbert series of the L_i's and of the characteristic form of I (see Proposition 3.10).

For a contracted ideal with a square-free characteristic form we show in Theorem 3.17 that the Rees algebra $\mathcal{R}(I)$, the associated graded ring $\operatorname{gr}_I(R)$ and the fiber cone $\mathrm{F}(I)$ are all Cohen–Macaulay with expected defining equations in the sense of [Vas] and [MU]. Furthermore $\mathcal{R}(I)$ is normal, the fiber cone $\mathrm{F}(I)$ is reduced and we determine explicitly the Hilbert function of $\operatorname{gr}_I(R)$.

Section 3 ends with a statement and a conjecture on the coefficients of the h-vector of a contracted ideal. Denote by $h_i(I)$ the ith coefficient of the h-vector of I and by $\mu(I)$ the minimal number of generators of I. We show that for any contracted (or monomial) ideal I one has $h_1(I) \geqslant (\mu(I) + 1)\mu(I)/2$ (Proposition 3.18) and we conjecture that $h_2(I) \geqslant 0$.

Sections 4 and 5 are devoted to the study of the lex-segment ideals. This class is important since, as we said above, information about the associated graded ring of a contracted ideal I can be derived from information about the associated graded rings of the lex-segment ideals appearing in the Zariski's factorization of I. Another motivation for studying the associated graded rings of lex-segment ideals comes from Section 2. There it is proved that if I is any ideal and $\operatorname{in}(I)$ is its initial ideal with respect to some term order then $H_I(n) \leqslant H_{\operatorname{in}(I)}(n)$ for all n provided depth $\operatorname{gr}_{\operatorname{in}(I)}(R) > 0$. In two variables initial ideals in generic coordinates are lex-segment ideals. We detect several classes of lex-segment ideals for which the associated graded ring is Cohen–Macaulay or at least has positive depth.

In particular, consider the lex-segment ideal L associated with the Hilbert function of an ideal generated by generic forms $f_1, \dots, f_s$; equivalently, set $L = \mathrm{in}((f_1, \dots, f_s))$, where the forms f_i are generic and in generic coordinates. We show that $\operatorname{depth} \operatorname{gr}_L(R) > 0$ (see Theorem 5.3). Furthermore $\operatorname{gr}_L(R)$ is Cohen–Macaulay if the forms f_i have all the same degree. In Section 6 we describe the defining equations of the Rees algebra of various classes of lex-segment ideals.

Some of the results and the examples presented in this paper have been inspired and suggested by computations performed by the computer algebra system CoCoA [Co]. In particular, we have made extensive use of the local algebra package.

2. Initial ideals and associated graded rings

Let R be a regular local ring of dimension d, with maximal ideal $\mathfrak{m}$ and residue field k, or, alternatively, let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k, and $\mathfrak{m} = (x_1, \dots, x_d)$. Throughout the paper we assume that $k = R/\mathfrak{m}$ is algebraically closed of characteristic 0. Moreover, let I be an $\mathfrak{m}$-primary ideal. For every integer n, the length $\lambda(R/I^{n+1})$ of R/I^{n+1} as R-module is finite. For $n \gg 0$, $\lambda(R/I^{n+1})$ is a polynomial $\mathrm{HP}_I(n)$ of degree d in n. The polynomial $\mathrm{HP}_I(n)$ is called the Hilbert–Samuel polynomial of I and one has

$$\mathrm{HP}_I(n) = \frac{e(I)}{d!} n^d + \text{lower degree terms.}$$

In particular, $e(I)$ is the ordinary multiplicity of the associated graded ring $\operatorname{gr}_I(R)$ to I,

$$\operatorname{gr}_I(R) = \bigoplus_{n \geqslant 0} I^n / I^{n+1}.$$

The Hilbert function $\mathrm{HF}_I(n)$ of I is defined as

$$\mathrm{HF}_I(n) = \lambda\big(I^n / I^{n+1}\big)$$

and it is by definition the Hilbert function of $\operatorname{gr}_I(R)$. The Hilbert series of I is

$$\mathrm{HS}_I(z) = \sum_{n \geqslant 0} \mathrm{HF}_I(n) z^n.$$

It is well known that the Hilbert series is of the form

$$\mathrm{HS}_I(z) = \frac{h_0(I) + h_1(I)z + \cdots + h_s(I)z^s}{(1-z)^d},$$

with $h_i(I) \in \mathbb{Z}$ for every i, $h_0(I) = \lambda(R/I)$ and $\sum_{i=0}^{s} h_i(I) = e(I)$.

In the local case, most important tools for studying the associated graded ring are minimal reductions and superficial elements. Those tools are not available in the non-local case, so we need to pass to the localization.

Lemma 2.1. *Let S be a flat extension of a ring R and let $I \subset R$ be an ideal. Suppose that $S/IS \simeq R/I$ as R-modules. Then*

$$\mathrm{gr}_I(R) \simeq \mathrm{gr}_{IS}(S).$$

Proof. It is enough to prove that $I^n/I^{n+1} \simeq I^nS/I^{n+1}S$ as R-modules. Since S is a flat extension of R, one has

$$\begin{aligned} I^nS/I^{n+1}S &\simeq I^nS \otimes_S S/IS \simeq I^n \otimes_R S \otimes_S S/IS \simeq I^n \otimes_R S/IS \\ &\simeq I^n \otimes_R R/I \simeq I^n/I^{n+1}, \end{aligned}$$

and the multiplicative structure is the same. □

Remark 2.2. We apply the above lemma to our setting. The localization $R_\mathfrak{m}$ is a flat extension of R and $R/I \simeq R_\mathfrak{m}/I_\mathfrak{m}$. By the above lemma, we get

$$\mathrm{gr}_I(R) \simeq \mathrm{gr}_{I_\mathfrak{m}}(R_\mathfrak{m}).$$

In particular, one has $\mathrm{HF}_{I_\mathfrak{m}}(n) = \mathrm{HF}_I(n)$ and hence $\mathrm{HS}_{I_\mathfrak{m}}(z) = \mathrm{HS}_I(z)$.

We are interested in studying the behaviour of $\mathrm{HF}_I(n)$ under Gröbner deformation. In the following we denote by I an $\mathfrak{m}$-primary ideal of $R = k[x_1, \dots, x_d]$ with $\mathfrak{m} = (x_1, \dots, x_d)$. We fix a term order on R and consider the initial ideal $\mathrm{in}(I)$ of I. Recall that $\lambda(R/I) = \lambda(R/\mathrm{in}(I))$.

We want to compare $\mathrm{HF}_I(n)$ and $\mathrm{HF}_{\mathrm{in}(I)}(n)$. First note that

$$e(I) \leqslant e\big(\mathrm{in}(I)\big).$$

This inequality follows easily from the fact that the multiplicities can be read from the leading coefficients of the Hilbert–Samuel polynomials of I and of $\mathrm{in}(I)$. In fact, since $\mathrm{in}(I)^n \subseteq \mathrm{in}(I^n)$, we have

$$\lambda\big(R/I^n\big) = \lambda\big(R/\mathrm{in}\big(I^n\big)\big) \leqslant \lambda\big(R/\mathrm{in}(I)^n\big) \tag{1}$$

and hence for $n \gg 0$ we get the required inequality on the multiplicities. Notice that in [DTVVW, 4.3] equality has been characterized.

As a consequence of the next lemma one has that the same inequality holds for the Hilbert function asymptotically, in a more general setting. Moreover, under some assumption, the inequality holds from the beginning.

Lemma 2.3. *Let J be an $\mathfrak{m}$-primary ideal in $R = k[x_1, \dots, x_d]$ and let $\mathcal{F} = \{\mathcal{F}_n\}_{n \geqslant 0}$ be a filtration of ideals, such that $J\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $J^n \subseteq \mathcal{F}_n$ for every $n \geqslant 0$. Then*

(1) $\lambda(\mathcal{F}_n/\mathcal{F}_{n+1}) \leqslant \lambda(J^n/J^{n+1})$ *for $n \gg 0$;*
(2) *if* $\operatorname{depth} \mathrm{gr}_J(R) > 0$, *then* $\lambda(\mathcal{F}_n/\mathcal{F}_{n+1}) \leqslant \lambda(J^n/J^{n+1})$ *for every $n \geqslant 0$.*

Proof. (1) Since for every $n \geqslant 0$ we have $J^n \subseteq \mathcal{F}_n$, it is equivalent to prove

$$\lambda\big(\mathcal{F}_n/J^n\big) \leqslant \lambda\big(\mathcal{F}_{n+1}/J^{n+1}\big).$$

By Remark 2.2, and since $\lambda(\mathcal{F}_n/\mathcal{F}_{n+1}) = \lambda(\mathcal{F}_n R_\mathfrak{m}/\mathcal{F}_{n+1} R_\mathfrak{m})$, we may transfer the problem to the local ring $R_\mathfrak{m}$, identifying J with $JR_\mathfrak{m}$ and $\mathcal{F}_n$ with $\mathcal{F}_n R_\mathfrak{m}$. Let a be a superficial element for J and consider the following exact sequence induced by the multiplication by a:

$$0 \to \big[\big(J^{n+1} : a\big) \cap \mathcal{F}_n\big]/J^n \to \mathcal{F}_n/J^n \xrightarrow{\cdot a} \mathcal{F}_{n+1}/J^{n+1} \to \mathcal{F}_{n+1}/a\mathcal{F}_n + J^{n+1} \to 0.$$

Since a is superficial and regular, one has $J^{n+1} : a = J^n$ for $n \gg 0$, and this proves (1).

(2) If depth $\mathrm{gr}_J(R) > 0$, then $\bar{a} \in J/J^2$ is regular (see [HM1, 2.1]) and $J^{n+1} : a = J^n$ for every n. This forces the required inequality and concludes the proof. □

As a consequence of the above lemma one has:

Theorem 2.4. *Fix any term order on $R = k[x_1, \dots, x_d]$, and let I be an $\mathfrak{m}$-primary ideal in R. The following facts hold*:

(1) $\mathrm{HF}_I(n) \leqslant \mathrm{HF}_{\mathrm{in}(I)}(n)$ *for* $n \gg 0$;
(2) *if* $\operatorname{depth} \mathrm{gr}_{\mathrm{in}(I)}(R) > 0$, *then* $\mathrm{HF}_I(n) \leqslant \mathrm{HF}_{\mathrm{in}(I)}(n)$ *for every* $n \geqslant 0$.

Proof. Since $\lambda(R/I^n) = \lambda(R/\operatorname{in}(I^n))$ for every n, one has

$$\mathrm{HF}_I(n) = \lambda\big(I^n/I^{n+1}\big) = \lambda\big(\operatorname{in}\big(I^n\big)/\operatorname{in}\big(I^{n+1}\big)\big).$$

Now the results follow by applying Lemma 2.3 with $J = \operatorname{in}(I)$ and $\mathcal{F}_n = \operatorname{in}(I^n)$. Note that part (1) can also be proved directly from Eq. (1). □

A lex-segment ideal is a monomial ideal L such that whenever n, m are monomials of the same degree with $n > m$ in the lexicographical order then $m \in L$ implies $n \in L$. Macaulay's theorem on Hilbert function implies that for every homogeneous ideal I there is a unique lex-segment ideal L with $\dim I_s = \dim L_s$ for all s. We will denote this ideal by $\mathrm{Lex}(I)$. Note however that $\mathrm{Lex}(I)$ depends only on the Hilbert function of I.

The following examples show that the conclusion of part (2) in Theorem 2.4 does not hold if the depth of $\mathrm{gr}_{\mathrm{in}(I)}(R)$ is 0.

Example 2.5. (a) Consider $R = \mathbb{Q}[x, y]$ equipped with the lexicographic order, with $x > y$. If $I = (x^5, x^4y^2, x^2y^5(x + y), xy^8, y^{10})$, then $\operatorname{in}(I) = (x^5, x^4y^2, x^3y^5, x^2y^7, xy^8, y^{10})$. In this case the associated graded ring to $\operatorname{in}(I)$ has depth zero and one has

$$\mathrm{HS}_I(z) = \frac{32 + 14z + 6z^2 - 2z^3}{(1 - z)^2}, \qquad \mathrm{HS}_{\mathrm{in}(I)}(z) = \frac{32 + 16z + 4z^2 - 2z^3}{(1 - z)^2}.$$

Thus $e(I) = e(\mathrm{in}(I))$, and $\mathrm{HF}_I(n) = \mathrm{HF}_{\mathrm{in}(I)}(n)$ for $n \geqslant 3$, but $\mathrm{HF}_I(2) = 130 > 128 = \mathrm{HF}_{\mathrm{in}(I)}(2)$. Note that $\mathrm{in}(I)$ is a lex-segment ideal, thus in particular, it is also the generic initial ideal of I.

(b) Let

$$I = \left(x^9, x^7y, x^6y^3, x^5y^5, x^4y^{12}, x^3y^{13}, x^2y^{14}, xy^{17}, y^{19}\right) \subseteq \mathbb{Q}[x, y].$$

Its generic initial ideal is the lex-segment ideal

$$L = \left(x^8, x^7y^2, x^6y^3, x^5y^5, x^4y^{12}, x^3y^{13}, x^2y^{14}, xy^{17}, y^{19}\right).$$

Notice that I is contracted, since it has the same number of generators as L (contracted ideals will be defined and studied in Section 3). In this case as well, depth $\mathrm{gr}_L(R) = 0$, and the conclusion of Theorem 2.4(2) fails. In fact, one has

$$\mathrm{HS}_I(z) = \frac{85 + 42z + 10z^2 - 3z^3}{(1-z)^2} \quad \text{and} \quad \mathrm{HS}_L(z) = \frac{85 + 43z + 7z^2 - z^3}{(1-z)^2},$$

thus $\mathrm{HF}_I(2) = 349 > 348 = \mathrm{HF}_L(2)$.

By Theorem 2.4, in $k[x, y]$, one has

$$\lambda\left(I^n/I^{n+1}\right) \leqslant \lambda\left(\mathrm{Lex}(I)^n/\mathrm{Lex}(I)^{n+1}\right) \quad \text{for every } n \gg 0.$$

Such inequality does not hold in 3 or more variables, see the next example.

Example 2.6. Let $I = (x^2, y^2, xy, xz^2, yz^2, z^4) \subset \mathbb{Q}[x, y, z]$. One has $\mathrm{Lex}(I) = (xz, xy, x^2, yz^2, y^2z, y^3, z^4)$, and

$$\mathrm{HS}_I(z) = \frac{8 + 8z}{(1-z)^3}, \qquad \mathrm{HS}_{\mathrm{Lex}(I)}(z) = \frac{8 + 7z}{(1-z)^3}.$$

Thus $\mathrm{HF}_I(n) \geqslant \mathrm{HF}_{\mathrm{Lex}(I)}(n)$ for every $n \geqslant 1$, and also $e(I) > e(\mathrm{Lex}(I))$.

In the last part of the section we restrict ourselves to the case of dimension two, and we collect some Cohen–Macaulayness criteria for the associated graded ring. We recall the following important result.

Proposition 2.7 [HM, Theorem A, Proposition 2.6]. *Let I be an $\mathfrak{m}$-primary ideal of a regular local ring $(R, \mathfrak{m})$ of dimension two and J a minimal reduction of I. Then $\mathrm{gr}_I(R)$ is Cohen–Macaulay if and only if $I^2 = JI$.*

It follows from [LT, 5.5] and [HS, 3.1] that:

Proposition 2.8. *Let I be an $\mathfrak{m}$-primary ideal in a regular local ring $(R, \mathfrak{m})$ of dimension two. If I is integrally closed, then $\mathrm{gr}_I(R)$ is Cohen–Macaulay.*

Proposition 2.7 does not have a corresponding version in the graded case since minimal reductions need not exist in that setting. But the next corollary holds both in the graded and in the local setting.

Proposition 2.9. *Let I be an $\mathfrak{m}$-primary ideal in a regular ring R of dimension two. Then $\mathrm{gr}_I(R)$ is Cohen–Macaulay if and only if*

$$\mathrm{HS}_I(z) = \frac{h_0(I) + h_1(I)z}{(1-z)^2}.$$

Proof. This is a simple consequence of the fact that $\mathrm{gr}_I(R) \simeq \mathrm{gr}_{I_\mathfrak{m}}(R_\mathfrak{m})$. In fact, by Proposition 2.7, $\mathrm{gr}_{I_\mathfrak{m}}(R_\mathfrak{m})$ is Cohen–Macaulay if and only if $I_\mathfrak{m}^2 = JI_\mathfrak{m}$ for some reduction J of $I_\mathfrak{m}$. Since $R_\mathfrak{m}$ is a local Cohen–Macaulay ring, this is equivalent to

$$\mathrm{HS}_{I_\mathfrak{m}}(z) = \frac{h_0(I) + h_1(I)z}{(1-z)^2}$$

(see, for example, [GR, 2.5]). The conclusion follows by Remark 2.2. $\square$

We will apply the above criteria for proving the Cohen–Macaulayness of the associated graded rings of certain classes of monomial ideals.

Let $R = k[x, y]$ and denote by $\mathfrak{m}$ the ideal (x, y). There are many ways of encoding an $\mathfrak{m}$-primary monomial ideal I. Among them we choose the following.

Set $d = \min\{j\colon x^j \in I\}$. Then for $i = 0, \dots, d$ we set $a_i(I) = \min\{j\colon x^{d-i}y^j \in I\}$ and

$$a(I) = \big(a_0(I), a_1(I), \dots, a_d(I)\big).$$

The sequence $a(I)$ is said to be the column sequence of I. By the very definition we have that $a_0(I) = 0$ and $1 \leqslant a_1(I) \leqslant a_2(I) \leqslant \cdots \leqslant a_d(I)$. Conversely, any sequence satisfying these conditions corresponds to a monomial ideal. For example,

$$I = \big(x^3, xy^3, y^5\big) \quad \longleftrightarrow \quad a(I) = (0, 3, 3, 5),$$

$$I = \big(x^4, x^3y, x^2y^4, xy^7, y^9\big) \quad \longleftrightarrow \quad a(I) = (0, 1, 4, 7, 9).$$

It is easy to see that

$$\lambda(R/I) = \big|a(I)\big| = \sum_i a_i(I).$$

Note also that the minimal generators of I are the monomials $x^{d-i}y^{a_i(I)}$ with $a_i(I) < a_{i+1}(I)$ or $i = d$.

Remark 2.10. In two variables, the $\mathfrak{m}$-primary lex-segment ideals correspond exactly to strictly increasing column sequences. In other words, any $\mathfrak{m}$-primary lex-segment ideal L can be written as

$$L = \left(x^d, x^{d-1}y^{a_1}, \ldots, xy^{a_{d-1}}, y^{a_d}\right),$$

where $0 = a_0 < a_1 < \cdots < a_d$. In particular, the minimal number of generators of L is exactly one more than the initial degree. Ideals with this property are called contracted, see Section 3. Furthermore, in characteristic 0, the lex-segment ideals are exactly the Borel fixed ideals and this implies that the generic initial ideal $\mathrm{gin}(I)$ of I is equal to $\mathrm{Lex}(I)$.

The b-sequence (or differences sequence) of I is denoted by $b_1(I), \ldots, b_d(I)$ and defined as

$$b_i(I) = a_i(I) - a_{i-1}(I).$$

We will use a_i for $a_i(I)$ and b_i for $b_i(I)$ if there is no confusion.

If I and J are monomial $\mathfrak{m}$-primary ideals, then the column sequence of the product IJ is given by

$$a_i(IJ) = \min\left\{a_j(I) + a_k(J)\colon\ j + k = i\right\}.$$

In particular, the column sequence of I^n is given by

$$a_i\left(I^n\right) = \min\left\{a_{j_1}(I) + \cdots + a_{j_n}(I)\colon\ j_1 + \cdots + j_n = i\right\}.$$

Example 2.11. Let I be a monomial ideal with b-sequence $b_1, \ldots, b_d$.

(a) Assume $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_d$. Then it is easy to see that for every $n \in \mathbf{N}$ one has $a_i(I^n) = (n-r)a_q(I) + ra_{q+1}(I)$, where $i = qn + r$ with $0 \leqslant r < n$. Summing up, we have $|a(I^n)| = n^2|a(I)| - \binom{n}{2}a_d(I)$. It follows that

$$\mathrm{HS}_I(z) = \frac{\lambda(R/I) + (\lambda(R/I) - a_d(I))z}{(1-z)^2}.$$

Note that by [E, Example 4.22] the ideal I is integrally closed.

(b) Assume $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_d$. Then it is easy to see that for every $n \in \mathbf{N}$ one has $a_i(I^n) = qa_d(I) + a_r(I)$, where $i = qd + r$ with $0 \leqslant r < d$. Summing up, we have $|a(I^n)| = n|a(I)| + \binom{n}{2}da_d(I)$. It follows that

$$\mathrm{HS}_I(z) = \frac{\lambda(R/I) + (da_d(I) - \lambda(R/I))z}{(1-z)^2}.$$

Moreover, the ideal $J = (x^d, y^{a_d})$ is a minimal reduction of I, and $I^2 = JI$.

In both cases the associated graded ring of I is Cohen–Macaulay by 2.9.

3. Contracted ideals

Let R be either a polynomial ring over a field or a regular local ring. Assume that $\dim R = 2$. Denote by $\mathfrak{m}$ the (homogeneous) maximal ideal of R. Most of the results of this section hold for any infinite base field. But, to avoid endless repetitions, we assume k is algebraically closed of characteristic 0.

Let $I \subset R$ be an ideal, homogeneous in the graded case. The ideals we are going to study were introduced by Zariski [ZS]:

Definition 3.1. An ideal $I \subset R$ is said to be contracted if there exists $\ell \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $I = IS \cap R$, where $S = R[\mathfrak{m}/\ell]$.

From now on we assume that I is $\mathfrak{m}$-primary. For any non-zero a in $\mathfrak{m}$, the order $o(a)$ of a is the $\mathfrak{m}$-adic valuation of a, that is, the greatest integer n such that $a \in \mathfrak{m}^n$. If $o(a) = r$, then we denote by a^* the initial form of a in $\mathrm{gr}_{\mathfrak{m}}(A)$, that is $a^* = \bar{a} \in \mathfrak{m}^r/\mathfrak{m}^{r+1}$. Denote by $\mu(I)$ the minimum number of generators of I and by $o(I)$ the order of I, that is, the largest h such that $I \subseteq \mathfrak{m}^h$. In the graded case $o(I)$ is simply the least degree of non-zero elements in I. In the local case, if I^* is the homogeneous ideal of $\mathrm{gr}_{\mathfrak{m}}(R)$ generated by the initial forms of the elements of I, then $o(I)$ is the least degree of an element in I^*. As in [ZS], we call *characteristic form* the GCD of the elements of degree $o(I)$ in I^*. In the graded case the characteristic form is just the GCD of the elements of degree $o(I)$ in I.

By the Hilbert–Burch theorem, I is generated by the maximal minors of a $(t-1) \times t$ matrix, say X, where $t = \mu(I)$. It follows that $\mu(I) \leqslant o(I) + 1$.

Remark 3.2. In the graded setting, if $g_1 \leqslant \cdots \leqslant g_t$ are the degrees of the generators of I and $z_1 \leqslant \cdots \leqslant z_{t-1}$ the degrees of the syzygies, then the ij-entry of X has degree $z_i - g_j$. Here we use the convention that 0 has any degree. The matrix (u_{ij}), $u_{ij} = z_i - g_j$, is called the degree matrix of I. It is easy to see that u_{ij} must be positive for all i, j with $j - i \leqslant 1$ and that $o(I) = \sum_{i=1}^{t-1} u_{i,i+1}$.

We have the following proposition.

Proposition 3.3. *The following conditions are equivalent*:

(1) *I is contracted,*
(2) *there exists $\ell \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $I : (\ell) = I : \mathfrak{m}$,*
(3) $\mu(I) = o(I) + 1$.

Furthermore, in the graded case, (1)–(3) *are equivalent to any of the following conditions*:

(4) *I is Gotzmann, i.e., I and* $\mathrm{Lex}(I)$ *have the same number of generators,*
(5) *I is componentwise linear, i.e., the ideal generated by every homogeneous component of I has a linear resolution,*

(6) *for every* $h \geqslant o(I)$, *the degree* h *component of* I *has the form* $I_h = f_h R_{h-s_h}$, *where* f_h *is a homogeneous polynomial of degree* s_h,
(7) $u_{i,i+1} = 1$ *for* $i = 1, \dots, \mu(I) - 1$.

Proof. The equivalence between (1)–(3) is proved in [H, 2.1, 2.3] in the local case and the arguments work also in the graded case. In the graded case the equivalence between (3) and (7) follows from Remark 3.2. The equivalence of (3) and (4) holds because, obviously, $o(I) = o(\mathrm{Lex}(I))$ and any lex-segment ideal in 2 variables satisfies (3), see Remark 2.10. That (4) implies (5) is a general fact [HH, Example 1.1b] while that (5) is equivalent to (6) follows from the fact that in two variables the only ideals with linear resolution have the form $f\mathfrak{m}^u$, where f is a form. To conclude, it suffices to show that (6) implies (2), where ℓ is any linear form not dividing the characteristic form of I and this is an easy check. □

Definition 3.4. An ideal $I \subset R$ is said to be contracted from $S = R[\mathfrak{m}/\ell]$ if $I = IS \cap R$. Moreover, we say that ℓ is coprime for I if its initial form in $\mathrm{gr}_{\mathfrak{m}}(R)$ does not divide the characteristic form of I.

Note that in the graded case, a coprime element for I is just a linear form ℓ not dividing the GCD of the elements of degree $o(I)$ of I.

If I is contracted, then conditions (2) and (3) hold true for any ℓ coprime for I. Since k is infinite, coprime elements for I exist. More generally, given a finite number of ideals one can always find an element which is coprime for any ideal.

By Remark 2.10, the minimum number of generators of any lex-segment ideal L is exactly one more than the initial degree; hence L is contracted. Moreover, $y + ax$ is coprime for L for all $a \in k$.

Remark 3.5. The homogeneous component I_h of a homogeneous contracted ideal I has the form $I_h = f_h R_{h-s_h}$ for all $h \geqslant o(I)$. The element f_h is the GCD of the elements in I_h. Furthermore, it divides f_{h-1} for all $h > o(I)$. Here $s_h = \deg f_h$ is also the dimension of R/I in degree h. So we have

$$\lambda(R/I) = \binom{\mu(I)}{2} + \sum_{h \geqslant o(I)} s_h.$$

The number of generators of I in degree $h > o(I)$ is $s_{h-1} - s_h$. The lex-segment ideal $L = \mathrm{Lex}(I)$ associated with I has the following form $L_h = x^{s_h} R_{h-s_h}$.

Next we give a characterization of contracted ideals in terms of the Hilbert–Burch matrix:

Proposition 3.6. *Let $R = k[x, y]$ and let d be a positive integer. Let $\alpha_1, \ldots, \alpha_d$ be elements of the base field k and $b_1, \ldots, b_d$ positive integers. Then the ideal generated by the d-minors of the $d \times (d+1)$ matrix*

$$\begin{pmatrix} y^{b_1} & x+\alpha_1 y & 0 & 0 & \cdots & \cdots & 0 \\ 0 & y^{b_2} & x+\alpha_2 y & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & y^{b_d} & x+\alpha_d y \end{pmatrix}$$

is contracted of order d, and y is coprime for I. Conversely, every contracted ideal I of order d can be realized, after a change of coordinates, in this way.

Proof. That the ideal of minors of such a matrix is contracted follows directly from the definition. Conversely, assume that I is contracted and let L be its associated lex-segment ideal. Say $a = (a_0, \ldots, a_d)$ is the column sequence of L. The matrix above with $b_i = a_i - a_{i-1} > 0$ and all the $\alpha_i = 0$ defines L. On the other hand, for every choice of the α_i we get a contracted ideal with the Hilbert function of I. We will show that a particular choice of the α_i will define the ideal I. By definition, I is determined by d and by the form $f_h = \mathrm{GCD}(I_h)$ for $h \geqslant d$. Since we assume that k is algebraically closed, every f_h is a product of linear forms. Since f_{h+1} divides f_h, we may find linear forms $\ell_{d-s_h+1}, \ldots, \ell_d$ so that $f_h = \prod_{j=d+1-s_h}^{d} \ell_j$, where $s_h = \deg f_h$. Take linear forms $\ell_1, \ldots, \ell_{d-s_h}$ in any way. Then take a system of coordinates x, y so that y does not divide any of the ℓ_i. In other words, up to irrelevant scalars, $\ell_i = x + \alpha_i y$ for all $i = 1, \ldots, d$. We claim that this choice of the α_i works. The degree of f_h is determined by the Hilbert function. So, it is enough to show that f_h divides every d-minor of degree $\leqslant h$ of the matrix. This is easy to check. □

An important property of contracted ideals is the following:

Proposition 3.7. *The product of contracted ideals is contracted.*

Proof. The proof given in the local case in [H, 2.6] works also in the graded setting. □

Next we recall Zariski's factorization theorem for contracted ideals [ZS, Theorem 1, Appendix 5]:

Theorem 3.8. *Let I be a contracted ideal of order d and characteristic form g of degree s. Let $g = g_1^{\beta_1} \cdots g_k^{\beta_k}$ be the factorization of g, where the g_i are distinct irreducible forms. Then I has a unique factorization as*

$$I = \mathfrak{m}^{d-s} L_1 L_2 \cdots L_k,$$

where the L_i are contracted ideals with characteristic form $g_i^{\beta_i}$.

Since we assume that k is algebraically closed the g_i are indeed linear forms. In the graded setting it follows that the L_i are lex-segment ideals in a system of coordinates with g_i as first coordinate. We deduce the following:

Lemma 3.9. *In the graded setting and with the notation of Theorem* 3.8 *we have*

$$\lambda(R/I) = \sum_{j=1}^{k} \lambda(R/L_j) + \binom{d+1}{2} - \sum_{j=1}^{k} \binom{\beta_j + 1}{2}.$$

Proof. The ideals L_i can be described quite explicitly in terms of the data of I: if g_i appears in the GCD of I_{d+j} with exponent β, then the GCD of L_i in degree $\beta_i + j$ is exactly g_i^{β}. Then using Remark 3.5, one gets the formula. $\square$

From the factorization of I given in Theorem 3.8 immediately follows that

$$I^n = \mathfrak{m}^{n(d-s)} L_1^n L_2^n \cdots L_k^n$$

is the analogous factorization for I^n. Applying Lemma 3.9 to I^n, summing up, and using the formula

$$\sum_{n=0}^{\infty} \binom{(n+1)\gamma + 1}{2} z^n = \frac{\gamma(1-z) + \gamma^2(1+z)}{2(1-z)^3},$$

we obtain:

Proposition 3.10. *In the graded setting and with the notation of Theorem* 3.8 *we have*

$$\mathrm{HS}_I(z) = \sum_{j=1}^{k} \mathrm{HS}_{L_j}(z) + \frac{\binom{d+1}{2} + \binom{d}{2} z - \sum_{j=1}^{k} \left[\binom{\beta_j+1}{2} + \binom{\beta_j}{2} z \right]}{(1-z)^2},$$

and in particular,

$$e(I) = \sum_{j=1}^{k} e(L_j) + d^2 - \sum_{j=1}^{k} \beta_j^2.$$

Similarly one can write all the coefficients of the Hilbert–Samuel polynomial of I in terms of those of the L_i.

In this part of the section $(R, \mathfrak{m})$ will denote a regular local ring of dimension two and I an $\mathfrak{m}$-primary ideal. We consider a coprime element ℓ for I, and we fix a minimal system of generators of $\mathfrak{m} = (x, \ell)$.

We define now the *transform* of an ideal I (not necessarily contracted) in $S = R[\mathfrak{m}/\ell]$. If a is in I and $d = o(I)$ is the order of I, then a/ℓ^d is in S and we may write

$$IS = \ell^d I',$$

where I' is an ideal of S. Such an ideal I' is called the transform of I in S.

Notice that if $I = (f_1, \dots, f_t)$, then I' is generated (not minimally in general) by $f_1/\ell^d, \dots, f_t/\ell^d$. If I and J are two ideals of R, then $(IJ)' = I'J'$. In fact, if $d = o(I)$ and $s = o(J)$, then $(IJ)S = (IS)(JS) = \ell^d \ell^s I'J'$. Therefore $(IJ)S = \ell^{d+s}(IJ)'$, so the conclusion follows. In particular, $(I^n)' = (I')^n$ for any integer n.

If $I = \mathfrak{m}^d$, then $I' = S$. In particular, if $I = \mathfrak{m}^s J$, then $I' = J'$ in S.

In the following we always denote by I' the transform of I in $S = R[\mathfrak{m}/\ell]$.

We note that $S = R[\mathfrak{m}/\ell] = R[x/\ell]$ is isomorphic to the ring $R[z]/(x - z\ell)$. The ring S is not local and its maximal ideals N which contain $\mathfrak{m}$ are in one-to-one correspondence with the irreducible polynomials g in $k[z]$.

We denote by T the localization of S at one of its maximal ideal N. Then T is a 2-dimensional regular local ring called the *first quadratic transform* of R. In algebraic geometry this construction is the well-known "locally quadratic" transformation of an algebraic surface, with center at a given simple point P of the surface.

If I' is the transform of I in S, then $(I')_N = I'T$ is the transform of I in T and $IT = \ell^d (I')_N$ if $d = o(I)$. We remark that ℓ is a regular element both in S and T. It is known that if I is primary for $\mathfrak{m}$ and N is a maximal ideal which contains I', then $(I')_N$ is primary for N or is a unit ideal. If I is contracted, then $(I')_N$ is a unit ideal if and only if $I = \mathfrak{m}^d$ (see [ZS, Proposition 2 and Corollary, Appendix 5]).

In the following we assume I is not a power of the maximal ideal. Then I' is a zero-dimensional ideal of S which is not necessarily primary. We will denote by $\operatorname{Max}(I')$ the set of the maximal ideals associated to I'. The maximal ideals in $\operatorname{Max}(I')$ depend on the characteristic form of I and on the field k.

Denote by T any localization of S at a maximal ideal $N \in \operatorname{Max}(I')$. The following easy facts will be useful in the proof of Theorem 3.12.

Remark 3.11. Let $d = o(I)$, ℓ be coprime for I, J be a minimal reduction of I and $S = R[\mathfrak{m}/\ell]$. The following facts hold:

(1) ℓ is coprime for I^n and for $\mathfrak{m}^s I^n$ for every positive integers n and s. In particular, if I is contracted from S, then I^n and $\mathfrak{m}^s I^n$ are contracted from S.
(2) ℓ is coprime for J. In fact, there exists n such that $I^{n+1} = JI^n$ and since $o(J) = d$, to conclude it is enough to look at the minimal degree $nd + d$ part of the corresponding ideals of the initial forms.
(3) If I is contracted, then JI is contracted [H, proof of Theorem 5.1], and by (2) it is contracted from S.
(4) If $J = (a, b)$, then $J' = (a/\ell^d, b/\ell^d)$ is a minimal reduction of I' both in S and T. In fact, if $I^{n+1} = JI^n$, then $I^{n+1}S = JI^nS$. Since $o(I^{n+1}) = o(JI^n)$, we have

$$(I')^{n+1} = J'(I')^n.$$

In particular, from the last equality it follows easily that $\operatorname{Max}(I') = \operatorname{Max}(J'I')$.

We are ready to state the main result of this section.

Theorem 3.12. *Let I be a contracted ideal of a local regular ring $(R, \mathfrak{m})$ of dimension two. With the above notation we have*

$$\operatorname{depth} \operatorname{gr}_I(R) = \min\{\operatorname{depth} \operatorname{gr}_{I'_N}(S_N):\ N \in \operatorname{Max}(I')\}.$$

Proof. First we prove that $\operatorname{gr}_I(R)$ is Cohen–Macaulay if and only if $\operatorname{gr}_{I'_N}(S_N)$ is Cohen–Macaulay for every $N \in \operatorname{Max}(I')$.

Assume that $\operatorname{gr}_I(R)$ is Cohen–Macaulay and let (a, b) be a minimal reduction of I. By Proposition 2.7, we have $I^2 = (a, b)I$, and in particular, $I^2S = (a, b)IS$. By definition of the transform of an ideal one has that $\ell^{2d}(I')^2 = \ell^{2d}(a, b)'I' = \ell^{2d}(a', b')I'$. Since ℓ is regular in S, it follows that $(I')^2 = (a', b')I'$ in S, hence $(I')^2 = (a', b')I'$ in T. Thus $\operatorname{gr}_{I'_N}(S_N)$ is Cohen–Macaulay for every $N \in \operatorname{Max}(I')$.

For the converse let (a, b) be a minimal reduction of I. Since $\{a' = a/\ell^d,\ b' = b/\ell^d\}$ generates a minimal reduction of I' in T, by Proposition 2.7 we get $(I')^2T = (a', b')I'T$. Now, by Remark 3.11(4), we note that $\operatorname{Max}(I') = \operatorname{Max}((a', b')I')$. Hence the equality is local on all maximal ideals N which contain $(a', b')I'$, thus in S

$$(I')^2 = (a', b')I'.$$

It follows that $\ell^{2d}(I')^2 = \ell^{2d}(a', b')I' = \ell^d(a', b')\ell^d I'$, that is,

$$I^2S = (a, b)IS.$$

By Remark 3.11(1) and (3), both I^2 and $(a, b)I$ are contracted from S, hence $I^2 = I^2S \cap R = (a, b)IS \cap R = (a, b)I$. Therefore $I^2 = (a, b)I$ and thus $\operatorname{gr}_I(R)$ is Cohen–Macaulay. This concludes the proof of the first part of the theorem.

Now it is enough to prove that $\operatorname{depth} \operatorname{gr}_I(R) > 0$ if and only if $\operatorname{depth} \operatorname{gr}_{I'_N}(S_N) > 0$ for every $N \in \operatorname{Max}(I')$. Assume $\operatorname{depth} \operatorname{gr}_I(R) > 0$. In particular, one has that $I^{n+1} :_R a = I^n$ for every $n \geqslant 0$ with a superficial for I. Let $a' = a/\ell^d$, it is enough to prove that $(I')^{n+1} :_S a' = (I')^n$ for every n. In fact, from this it follows that a' is regular in $\operatorname{gr}_{I'_N}(S_N)$ for every localization $T = S_N$ because $(I')^{n+1}T :_T a' = ((I')^{n+1} :_S a')_N = (I')^n_N = (I')^nT$ for every n.

Let c/ℓ^s be any element of S, with $c \in \mathfrak{m}^s$. Suppose c/ℓ^s is in $(I')^{n+1} :_S a'$, that is, $\frac{c}{\ell^s}\frac{a}{\ell^d} \in (I')^{n+1}$. To prove that $c/\ell^s \in (I')^n$ we distinguish two cases.

If $s \leqslant dn$, then

$$\ell^{d(n+1)}\frac{ca}{\ell^{s+d}} \in \ell^{d(n+1)}(I')^{n+1} = I^{n+1}S,$$

that is, $\ell^{dn-s}ca \in I^{n+1}S \cap R = I^{n+1}$. Thus $\ell^{dn-s}c \in I^n$, and

$$\ell^{dn-s}c = \ell^{dn}\frac{c}{\ell^s} \in I^nS = \ell^{dn}(I')^n.$$

Since ℓ is regular in S, it follows that $c/\ell^s \in (I')^n$, and this concludes this case.

Assume now that $s > dn$ and let $\mathfrak{m} = (x, \ell)$. Since ℓ is coprime for $\mathfrak{m}^{s-dn} I^n$, there exists $f \in \mathfrak{m}^{s-dn} I^n$ such that $f = x^s - p\ell$ with $p \in \mathfrak{m}^{s-1}$. Hence we get that

$$\left(\frac{x}{\ell}\right)^s = \left(\frac{p}{\ell^{s-1}}\right) + \frac{f}{\ell^s} \quad \text{with } \frac{f}{\ell^s} \in (I')^n.$$

Since $c \in \mathfrak{m}^s$, we may write $c = ux^s + q\ell$ with $u \in R$, $q \in \mathfrak{m}^{s-1}$. Now

$$\frac{c}{\ell^s} = u\left(\frac{x}{\ell}\right)^s + \frac{q}{\ell^{s-1}} = \frac{q + pu}{\ell^{s-1}} + \frac{f}{\ell^s}.$$

Hence $\frac{q+pu}{\ell^{s-1}} \in (I')^{n+1} : a'$ and we have to prove that $\frac{q+pu}{\ell^{s-1}} \in (I')^n$. By repeating this argument, after $s - dn$ steps we are in the already discussed case $s < dn$.

We now assume that $\operatorname{depth} \operatorname{gr}_{I'_N}(S_N) > 0$ for every $N \in \operatorname{Max}(I')$ and we have to prove that $\operatorname{depth} \operatorname{gr}_I(R) > 0$. We recall that if $J = (a, b)$ is a minimal reduction of I, then $J' = (a/\ell^d, b/\ell^d)$ is a minimal reduction of I' in $T = S_N$ and, by the assumption, we may suppose that $a' = a/\ell^d$ is regular in $\operatorname{gr}_{I'_N}(T)$.

Thus $(I')_N^{n+1} :_T a' = (I')_N^n$ for every $n \geqslant 0$ and for every $N \in \operatorname{Max}(I')$ which implies $(I')^{n+1} :_S a' = (I')^n$ because it is a local fact on the maximal ideals $N \in \operatorname{Max}(I')$.

We conclude if we prove that $I^{n+1} : a = I^n$ for every $n \geqslant 0$.

Let $b \in I^{n+1} :_R a$, that is, $ba \in I^{n+1}$. Since $o(I^{n+1}) = d(n+1)$ and $o(a) = d$, one has $o(b) \geqslant dn$. Then

$$ba = \ell^{d(n+1)} \frac{ba}{\ell^{d(n+1)}} = \ell^{d(n+1)} \frac{b}{\ell^{dn}} \frac{a}{\ell^d} \in I^{n+1} = \ell^{d(n+1)} (I')^{n+1},$$

and since ℓ is regular in S, $\frac{b}{\ell^{dn}} \frac{a}{\ell^d} \in (I')^{n+1}$, thus $\frac{b}{\ell^{dn}} \in (I')^n$. It follows that $b = \ell^{dn} \frac{b}{\ell^{dn}} \in \ell^{dn} (I')^n = I^n S$, and then $b \in I^n S \cap R = I^n$, since I^n is contracted from S. $\square$

Theorem 3.12 can be applied also in the graded setting by localizing. We present now some corollaries which hold both in the local and in the graded case.

Corollary 3.13. *Let I and J be contracted ideals with coprime characteristic forms. Then*

$$\operatorname{depth} \operatorname{gr}_{IJ}(R) = \min\bigl\{\operatorname{depth} \operatorname{gr}_I(R), \operatorname{depth} \operatorname{gr}_J(R)\bigr\}.$$

Proof. Note that if $g = g_1^{\beta_1} \dots g_k^{\beta_k}$ is a factorization in irreducible factors of the characteristic form g of an ideal I, then

$$\operatorname{Max}(I') = \bigl\{(g_i/\ell, \ell)\colon i = 1, \dots, k\bigr\},$$

where ℓ is a coprime element for I. Since the characteristic forms of I and J are coprime, $\operatorname{Max}(I') \cap \operatorname{Max}(J') = \emptyset$. Thus $(IJ)'_N = I'_N$ for every $N \in \operatorname{Max}(I')$ and $(IJ)'_M = J'_M$ for every $M \in \operatorname{Max}(J')$. By using twice Theorem 3.12, we have

$$\begin{aligned}\operatorname{depth}\operatorname{gr}_{IJ}(R) &= \min\bigl\{\operatorname{depth}\operatorname{gr}_{I'_N}(S_N), \operatorname{depth}\operatorname{gr}_{J'_M}(S_M)\colon\ N \in \operatorname{Max}(I'), M \in \operatorname{Max}(J')\bigr\}\\ &= \min\bigl\{\operatorname{depth}\operatorname{gr}_I(R), \operatorname{depth}\operatorname{gr}_J(R)\bigr\}. \qquad \square\end{aligned}$$

In particular:

Corollary 3.14. *Consider the factorization of a contracted ideal I as in Theorem* 3.8. *Then*

$$\operatorname{depth}\operatorname{gr}_I(R) = \min\bigl\{\operatorname{depth}\operatorname{gr}_{L_i}(R)\colon\ i = 1, \dots, k\bigr\}.$$

The above result leads to study the depth of the associated graded ring of a lex-segment ideal. This will be the topic of the next section.

Remark 3.15. Trung and Hoa gave in [TH] a combinatorial characterization of the Cohen–Macaulayness of semigroup rings which can be applied to the study of the Cohen–Macaulay property of the Rees algebra of monomial ideals. In principle their result in connection with Corollary 3.14 can be used to give combinatorial description of the Cohen–Macaulayness of the associated graded rings of contracted ideals. In practice, however, we have not been able to obtain such a characterization.

By Corollary 3.14 and Example 2.11, we have

Corollary 3.16. *Consider the factorization of a contracted ideal I as in Theorem* 3.8. *If $o(L_i) \leqslant 2$ for every $i = 1, \dots, k$, then $\operatorname{gr}_I(R)$ is Cohen–Macaulay.*

Consider the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\in\mathbf{N}} I^n$ of I, and the fiber cone $\mathrm{F}(I) = \operatorname{gr}_I(R) \otimes R/\mathfrak{m}$ of I. In the special case $o(L_i) = 1$, one has the following theorem.

Theorem 3.17. *Let $I \subset R = k[x, y]$ be a homogeneous contracted ideal with $o(I) = d$. Assume that the characteristic form is a square-free polynomial* (*it has no multiple factors*). *Then:*

(1) *The Rees algebra $\mathcal{R}(I)$ is a Cohen–Macaulay normal domain and the defining ideal J of $\mathcal{R}(I)$ has the expected form in the sense of* [Vas, §8.2] *and* [MU, 1.2], *that is, J is the ideal of* 2*-minors of a* $2 \times (d + 1)$ *matrix H.*
(2) *The associated graded ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay with Hilbert series*

$$\operatorname{HS}_I(z) = \frac{\lambda(R/I) + \binom{d}{2}z}{(1-z)^2}.$$

(3) *The fiber cone $F(I)$ is a Cohen–Macaulay reduced ring defined by the* 2*-minors of a* $2 \times d$ *matrix of linear forms. Furthermore, $F(I)$ is a domain if and only if $I = \mathfrak{m}^d$.*

Proof. By Corollary 3.16, $\operatorname{gr}_I(R)$ is Cohen–Macaulay. Hence $\mathcal{R}(I)$ is also Cohen–Macaulay. Since $\operatorname{GCD}(I_d)$ is square free, I is given by the d-minors of the matrix ϕ of

Proposition 3.6 with $\alpha_i \neq \alpha_j$ if $i \neq j$. It follows immediately that the ideal of the entries of ϕ is $\mathfrak{m}$ and that the ideal of the $(d-1)$-minors of ϕ is $\mathfrak{m}^{d-1}$. By [MU, 1.2], we conclude that J has the expected form, that is, it is given by the 2-minors of a certain matrix H. We can write down explicitly the matrix. If we present $\mathcal{R}(I)$ as $R[t_0, \dots, t_d]/J$ by sending t_i to $(-1)^i$ times the d-minor of ϕ obtained by deleting the $(i+1)$th column, then J is generated by the 2-minors of the matrix:

$$H = \begin{pmatrix} x & \alpha_1 t_1 + y^{b_1-1}t_0 & \alpha_2 t_2 + y^{b_2-1}t_1 & \dots & \alpha_d t_d + y^{b_d-1}t_{d-1} \\ -y & t_1 & t_2 & \dots & t_d \end{pmatrix}$$

By Theorem 3.8, I is a product of complete intersections of order 1. Thus I is integrally closed. In a two-dimensional regular ring, this is equivalent to the normality of $\mathcal{R}(I)$. This conclude the proof of part (1). For part (3) one notes that the defining equation of $F(I)$ are the 2-minors of the matrix obtained from H by replacing x and y with 0. The dimension of $F(I)$ is 2 and the codimension of $F(I)$ is $\mu(I) - 2$, i.e., $d - 1$. So $F(I)$ is defined by a determinantal ideal with the expected codimension, thus it is Cohen–Macaulay (see [BV]). That $F(I)$ is reduced follows by the fact that one of the initial ideals of its defining ideal is $(t_i t_j : 1 \leqslant i < j \leqslant d)$. Finally, if I is not $\mathfrak{m}^d$, then at least one of the b_i, say b_k, is > 1 and then some of the generators of the defining ideal of $F(I)$ have t_k as a factor. Therefore $F(I)$ is not a domain. Now, if $I = \mathfrak{m}^d$, then $F(I)$ is the dth Veronese algebra of R, hence a domain. It remains to prove the assertion on $\mathrm{HS}_I(z)$. Since $\mathrm{gr}_I(R)$ is Cohen–Macaulay, its h-vector has length $\leqslant 1$. Obviously $h_0(I) = \lambda(R/I)$. Since the L_i are complete intersections and $\beta_i = 1$, from Proposition 3.10 it follows that $h_1(I) = \binom{d}{2}$. $\square$

In the above theorem it is proved that $h_1(I) = \binom{d}{2}$ for contracted ideal with square free characteristic form. In general the following inequalities hold:

Proposition 3.18. *Let $I \subset R = k[x, y]$ be an $\mathfrak{m}$-primary homogeneous ideal. If I is monomial or contracted, then $h_1(I) \geqslant \binom{\mu(I)-1}{2}$ and $e(I) \geqslant \lambda(R/I) + \binom{\mu(I)-1}{2}$.*

Proof. In general, one knows that $e(I) \geqslant h_0(I) + h_1(I)$, see [V1, Lemma 1]. So it is enough to prove the first inequality. By Proposition 3.10, if the inequality holds for lex-segment ideals, then it holds for contracted ideals. Thus to conclude it is enough to prove the first inequality for monomial ideals.

Let I be a monomial ideal, say with associated column sequence $a = (a_0, \dots, a_d)$ and differences sequence $b = (b_1, \dots, b_d)$. Now one has $\mu(I) = |\{i\colon b_i > 0\}| + 1$. Suppose that one of the b_i is > 1, say $b_k > 1$. Set $c = (c_1, \dots, c_d)$ with $c_i = b_i$ if $i \neq k$ and $c_k = b_k - 1$. Denote by f the sequence whose differences sequence is c, i.e., $f_0 = 0$ and $f_i = \sum_{j=1}^{i} c_j$, and by J the corresponding monomial ideal. In other words, $f_j = a_j$ if $j < k$ and $f_j = a_j - 1$ if $j \geqslant k$. We claim that

$$h_1(I) \geqslant h_1(J). \tag{2}$$

To prove this note first that

$$\lambda(R/I) - \lambda(R/J) = \sum_i a_i - \sum_i f_i = d - k + 1.$$

Therefore

$$h_1(I) - h_1(J) = \lambda\big(R/I^2\big) - \lambda\big(R/J^2\big) - 3(d - k + 1)$$

and hence (2) is equivalent to:

$$\lambda\big(R/I^2\big) - \lambda\big(R/J^2\big) \geqslant 3(d - k + 1).$$

Denote by $a^{(2)}$ and $f^{(2)}$ the column sequences associated with I^2 and J^2, respectively. Note that $a_i^{(2)} = a_j + a_h$ for some j and h with $0 \leqslant j, h \leqslant d$ and $j + h = i$. If $i \geqslant 2k - 1$, then at least one among j, h is $\geqslant k$ and if $i \geqslant k + d$, then both j, h are $\geqslant k$. It follows that

$$a_i^{(2)} \geqslant \begin{cases} f_i^{(2)}, & \text{if } i = 0, \dots, 2k - 2, \\ f_i^{(2)} + 1, & \text{if } i = 2k - 1, \dots, k + d - 1, \\ f_i^{(2)} + 2, & \text{if } i = k + d, \dots, 2d. \end{cases}$$

We may conclude that

$$\lambda\big(R/I^2\big) - \lambda\big(R/J^2\big) = \sum_{i=0}^{2d} a_i^{(2)} - \sum_{i=0}^{2d} f_i^{(2)} \geqslant 3(d - k + 1)$$

as desired. Since the number of generators of I and J is, by construction, the same, it is now enough to prove the assertion for J. Repeating the argument it is enough to prove the statement for a monomial ideal H whose differences sequence consists only of 0 and 1. Such an ideal has $\alpha + 1$ generators and one generator, namely y^α, of degree α. In particular, it is a lex-segment ideal with respect to y, whose differences sequence does not contain 0. After exchanging x and y and by applying the same procedure as above to H, one ends up with a power of the maximal ideal, for which it is easy to see that the inequality holds. $\square$

One may wonder whether the inequality $h_1(I) \geqslant \binom{\mu(I)-1}{2}$ holds more generally for every $\mathfrak{m}$-primary ideal I. We believe that this is indeed the case.

In general, $h_2(I)$ need not be non-negative for an $\mathfrak{m}$-primary ideal I. The ideal I generated by 4 generic polynomials of degree 7 and one generic polynomial of degree 8 (take for example $x^7, y^7, x^3y^4, x^6y - xy^6, x^2y^6 - x^5y^3$) has $h_2(I) = -1$. On the other hand, there is some computational evidence that

Conjecture 3.19. *For a contracted ideal I one has $h_2(I) \geqslant 0$.*

Note that, in view of Proposition 3.10, to prove the conjecture one may assume right away that I is a lex-segment ideal.

4. Lex-segment ideals and depth of the associated graded ring

In this section we study the depth of the associated graded ring of a lex-segment ideal in $k[x, y]$. This is strongly motivated by Corollary 3.14 which moves the computation of the depth of the associated graded ring from contracted ideals to lex-segment ideals.

We start by giving classes of lex-segment ideals whose associated graded rings have positive depth or are Cohen–Macaulay. Notice that we can apply Theorem 2.4 to such classes since, in our setting, $\mathrm{gin}(I)$ is a lex-segment ideal. In the second part of the section we find new classes of lex-segment ideals whose associated graded ring is Cohen–Macaulay by interpreting Theorem 3.12 in the case of lex-segment ideals.

Let L be a lex-segment ideal in $R = k[x, y]$. As we have already seen, one has

$$L = \left(x^d, x^{d-1}y^{a_1}, x^{d-2}y^{a_2}, \dots, y^{a_d}\right),$$

with $0 = a_0 < a_1 < a_2 < \cdots < a_d$. The sequence $(b_1, \dots, b_d)$, with $b_i = a_i - a_{i-1}$, is the differences sequence of L. From now on we may assume $a_d > d$, otherwise $L = \mathfrak{m}^d$ and its associated graded ring is Cohen–Macaulay.

By Proposition 2.7, if I is an $\mathfrak{m}$-primary ideal in R with $I^2 = JI$ for a minimal reduction J of I, then $\mathrm{gr}_I(R)$ is Cohen–Macaulay. We show now that in the class of lex-segment ideals, $L^2 = JL$ for certain kind of (non-minimal) reduction J, will yield positive depth for the associated graded ring.

Proposition 4.1. *Let $L = (x^d, \dots, y^{a_d})$ be a lex-segment ideal. If $L^2 = (x^d, x^{d-i}y^{a_i}, y^{a_d})L$ for some $i = 0, \dots, d$, then* $\operatorname{depth} \mathrm{gr}_L(R) > 0$.

Proof. We show that $L^n : (x^d, y^{a_d}) = L^{n-1}$ for all $n \geqslant 1$. For $n = 1$, it is obvious. Let $n > 1$ and assume that the result is true for $n - 1$. Let $h \in L^n : (x^d, y^{a_d})$. Without loss of generality we may assume that h is a monomial. Since $L^n = J^{n-1}L$, we may write

$$hx^d = \left(x^d\right)^{r_1}\left(x^{d-i}y^{a_i}\right)^{r_2}\left(y^{a_d}\right)^{r_3}g_1, \tag{3}$$

$$hy^{a_d} = \left(x^d\right)^{s_1}\left(x^{d-i}y^{a_i}\right)^{s_2}\left(y^{a_d}\right)^{s_3}g_2 \tag{4}$$

for some $g_1, g_2 \in L$ and $\sum_i r_i = n - 1 = \sum_j s_j$. We need to show that $h \in L^{n-1}$. If $r_1 > 0$ or $s_3 > 0$, then clearly $h \in L^{n-1}$. Suppose $r_1 = s_3 = 0$. If $s_2 = 0$, then $s_1 = n-1$. Therefore, $x-\deg h \geqslant (n-1)d$ so that $h \in L^{n-1}$. Similarly, if $r_2 = 0$, then $r_3 = n - 1$. Hence $y-\deg h \geqslant (n-1)a_d$ so that $h \in L^{n-1}$. Suppose $r_2 \geqslant 1$ and $s_2 \geqslant 1$. Then from (3) it follows that $y-\deg h \geqslant a_{d-i}$ and from (4) it follows that $x-\deg h \geqslant d-i$. Therefore, $x^{d-i}y^{a_i}$ divides h. Write $h = x^{d-i}y^{a_i}h_1$. Then we have

$$h_1x^d = \left(x^d\right)^{r_1}\left(x^{d-i}y^{a_i}\right)^{r_2-1}\left(y^{a_d}\right)^{r_3}g_1,$$
$$h_1y^{a_d} = \left(x^d\right)^{s_1}\left(x^{d-i}y^{a_i}\right)^{s_2-1}\left(y^{a_d}\right)^{s_3}g_2.$$

Therefore $h_1 \in L^{n-1} : (x^d, y^{a_d}) = L^{n-2}$, by induction hypothesis. Hence $h = x^{d-i}y^{a_i}h_1 \in L^{n-1}$. Therefore $L^n : (x^d, y^{a_d}) = L^{n-1}$ for all $n \geqslant 1$ and hence $\operatorname{depth}\operatorname{gr}_L(R) > 0$. $\square$

The following proposition gives a class of lex-segment ideals to which one can apply Proposition 4.1:

Proposition 4.2. *Let $L = (x^d, \dots, y^{a_d})$ be a lex-segment ideal such that $b_2 \geqslant b_3 \geqslant \dots \geqslant b_d$. Then $L^2 = (x^d, x^{d-1}y^{a_1}, y^{a_d})L$. In particular,* $\operatorname{depth}\operatorname{gr}_L(R) > 0$.

Proof. Set $J = (x^d, x^{d-1}y^{a_1}, y^{a_d})$. We need to show that for all $0 \leqslant i \leqslant j \leqslant d$, $x^{d-i}y^{a_i}x^{d-j}y^{a_j} \in JL$.

We split the proof into two cases:

Case I. If $i + j - 1 \leqslant d$, then we show that $x^{2d-i-j}y^{a_i+a_j} = x^{d-1}y^{a_1} \cdot x^{d-i-j+1}y^{a_{i+j-1}} \cdot m$ for some monomial m.

Consider the following equations:

(1) $a_{i+j-1} - a_j = b_{i+j-1} + b_{i+j-2} + \dots + b_{j+1}$,
(2) $a_i - a_1 = b_i + b_{i-1} + \dots + b_2$.

Since $b_2 \geqslant b_3 \geqslant \dots \geqslant b_d$, $a_i - a_1 \geqslant a_{i+j-1} - a_j$. Therefore, $a_i + a_j \geqslant a_{i+j-1} + a_1$. Hence, we may write $x^{2d-i-j}y^{a_i+a_j} = x^{d-1}y^{a_1} \cdot x^{d-i-j+1}y^{a_{i+j-1}} \cdot m$ for some monomial m, so that $x^{2d-i-j}y^{a_i+a_j} \in JL$.

Case II. If $i + j - 1 > d$, then $x^{2d-i-j}y^{a_i+a_j} = x^{d-k-1}y^{a_{k+1}} \cdot y^{a_d} \cdot m'$ for some monomial m', where $k = i + j - 1 - d$.

As in Case I, write $a_d - a_i$ and $a_j - a_{k+1}$ as sum of b_l and conclude that $a_i + a_j \geqslant a_d + a_{k+1}$. Therefore $x^{2d-i-j}y^{a_i+a_j} = x^{d-k-1}y^{a_{k+1}} \cdot y^{a_d} \cdot m'$, so that $x^{2d-i-j}y^{a_i+a_j} \in JL$.

Therefore, for all $0 \leqslant i \leqslant j \leqslant d$, $x^{2d-i-j}y^{a_i+a_j} \in JL$ and hence $L^2 = JL$. Now using Proposition 4.2, we may conclude that $\operatorname{depth}\operatorname{gr}_L(R) > 0$. $\square$

We apply now the theory developed in Section 3 to lex-segment ideals. Recall that a lex-segment ideal $L = (x^d, x^{d-1}y^{a_1}, \dots, y^{a_d})$ is contracted from $S = R[\mathfrak{m}/y] = R[x/y]$.

In particular, $LS \cap R = L$ and $LS = y^dL'$, where L' is the monomial ideal of S generated by the elements $(\frac{x}{y})^{d-i}y^{a_i-i}$ for every $i = 0, \dots, d$.

It will be useful to consider $\varphi : S = R[x/y] = k[x, y, z]/(x - yz) \to P = k[y, z]$ the natural ring homomorphism defined by sending the class of $f(x, y, z)$ to $f(yz, y, z)$ for every $f(x, y, z) \in k[x, y, z]$. It is easy to see that φ is an isomorphism. We set $T(L) =$

$\varphi(L')$, so that $T(L)$ could be identified with a monomial ideal in $k[x,y]$. Moreover, one has

$$\mathrm{gr}_{T(L)}(P) \simeq \mathrm{gr}_{L'}(S).$$

These facts hold for every contracted ideal, not only for lex-segment ideals. In practice, the ideal $T(L)$ can be obtained from L by substituting x with yz and dividing any generator by y^d, where $d = o(L)$. In the following examples we explain in details the procedure.

Example 4.3. Let $L = (x^4, x^3y, x^2y^3, xy^4, y^{10}) \subset R = k[x,y]$. Note that $o(L) = 4$. The transform L' of L is defined to be the ideal of $S = R[z]/(x - zy)$ such that $LS = y^4L'$. Thus one has

$$LS = \left(y^4z^4, y^4z^3, y^5z^2, y^5z, y^{10}\right)S = y^4\left(z^4, z^3, yz^2, yz, y^6\right)S = y^4\left(z^3, yz, y^6\right)S,$$

and via the isomorphism φ one gets $T(L) = (z^3, yz, y^6)$ in $P = k[y,z]$.

We remark that in the particular case of a lex-segment ideal L, its transform L' is a primary ideal for $N = (y, x/y)$ or equivalently $T(L)$ is a primary ideal for (y,z). Hence, by Remark 2.2,

$$\mathrm{gr}_{L'}(S) \simeq \mathrm{gr}_{L'_N}(S_N).$$

Now we may rephrase Theorem 3.12 in the case of a lex-segment ideal. As a consequence one has that to compute the depth of the associated graded ring of L one can pass to the transform $T(L)$ of L, which is in general easier to study. In particular, $\mu(T(I)) \leqslant \mu(I)$ and $e(T(I)) < e(I)$, see [H, 3.6].

Theorem 4.4. *Let L be a lex-segment ideal in $R = k[x,y]$. With the above notation we have*

$$\operatorname{depth} \mathrm{gr}_L(R) = \operatorname{depth} \mathrm{gr}_{T(L)}(P).$$

Proof. Since $\mathrm{gr}_{T(L)}(P) \simeq \mathrm{gr}_{L'}(S)$, it suffices to prove that $\operatorname{depth} \mathrm{gr}_L(R) = \operatorname{depth} \mathrm{gr}_{L'}(S)$. The ideal L is primary for $\mathfrak{m} = (x,y)$ and L' is primary for the maximal ideal $N = \mathfrak{m} + (x/y)$, hence by Remark 2.2, $\mathrm{gr}_L(R) \simeq \mathrm{gr}_{L_\mathfrak{m}}(R_\mathfrak{m})$ and $\mathrm{gr}_{L'}(S) \simeq \mathrm{gr}_{L'_N}(S_N)$. Now the result follows by using Theorem 3.12. □

As an immediate application of the theorem, we find classes of lex-segment ideals, whose associated graded ring is Cohen–Macaulay.

First, we want to give an explicit description of the ideal $T(L)$, where L is a lex-segment ideal. Let $L = (B_1, \dots, B_{s+1})$ be the decomposition of the minimal set of generators of L in subsets of elements of the same degree, that is, B_i is the block of the elements of degree $d + i - 1$. Assume $B_{s+1} \neq \emptyset$. In Example 4.3 one has $B_1 = \{x^4, x^3y\}$, $B_2 = \{x^2y^3, xy^4\}$, $B_3 = \cdots = B_6 = \emptyset$, $B_7 = \{y^{10}\}$.

Let

$$|B_1| = p_1 + 1, \qquad |B_i| = p_i \quad \text{for } i = 2, \dots, s+1.$$

Proposition 4.5. *With the above notation one has*

$$T(L) = \left(z^{d-p_1}\right) + \left(z^{d-(p_1+\cdots+p_i)}y^{i-1}\colon\ i \geqslant 2,\ p_i \neq 0\right).$$

Proof. By definition one has $B_1 = \{x^d, \dots, x^{d-p_1}y^{p_1}\}$ and

$$B_i = \left\{x^{d-(p_1+\cdots+p_{i-1}+1)}y^{p_1+\cdots+p_{i-1}+i}, \dots, x^{d-(p_1+\cdots+p_i)}y^{p_1+\cdots+p_i+i-1}\right\}$$

for $i \geqslant 2$. Applying the transform to the elements of B_i, one obtains the set

$$T(B_i) = \left\{z^{d-(p_1+\cdots+p_{i-1}+1)}y^{i-1}, \dots, z^{d-(p_1+\cdots+p_i)}y^{i-1}\right\},$$

hence the ideal

$$\left(T(B_i)\right) = \left(z^{d-(p_1+\cdots+p_i)}y^{i-1}\right)$$

and this concludes the proof. $\square$

It is natural to ask under which conditions is $T(L)$ a lex-segment ideal. As an easy consequence of Proposition 4.5, one gets a characterization:

Lemma 4.6. *Let L be a lex-segment ideal. Then $T(L)$ is a lex-segment ideal if and only if one of the following holds*:

(1) $p_i \leqslant 1$ *for every* $i \geqslant 2$;
(2) $p_i \neq 0$ *for every* $i \geqslant 2$.

Moreover, in case (2) $T(L)$ *is a lex-segment ideal with respect to* y *and its differences sequence is* $(p_{s+1}, p_s, \dots, p_3, p_2)$.

Proof. By Proposition 4.5, one has

$$T(L) = \left(z^{d-p_1}, z^{d-(p_1+p_2)}y, z^{d-(p_1+p_2+p_3)}y^2, \dots, z^{d-(p_1+\cdots+p_s)}y^{s-1}, y^s\right).$$

It is clear that $T(L)$ is a lex-segment ideal with respect to z if and only if $p_i \leqslant 1$ for every $i \geqslant 2$. Note that by definition $\sum_{i=1}^{s+1} p_i = d$ and $s + 1 = a_d - d + 1$, and let rewrite $T(L)$ as

$$T(L) = \left(y^{a_d-d}, y^{a_d-d-1}z^{p_{s+1}}, \dots, y^2z^{p_4+\cdots+p_{s+1}}, yz^{p_3+\cdots+p_{s+1}}, z^{p_2+\cdots+p_{s+1}}\right).$$

It follows that $T(L)$ is a lex-segment ideal with respect to y if and only if $p_i \neq 0$ for every $i \geqslant 2$. When this is the case the differences sequence is $(p_{s+1}, p_s, \dots, p_3, p_2)$. $\square$

Remark 4.7. Recall that the differences sequence of L is $(b_1, \dots, b_d)$, with $b_i = a_i - a_{i-1}$. Note that the difference between the degree of $x^{d-i}y^{a_i}$ and the degree of $x^{d-i+1}y^{a_{i-1}}$ is $b_i - 1$. Thus the conditions of the lemma above can be written in terms of the b_i. In fact, condition (1) is equivalent to $b_i \geqslant 2$ if $b_{i-1} \geqslant 2$, that is, the generators of L of degree $> d$ have all different degrees. Condition (2) holds if and only if $b_i \in \{1, 2\}$ for every i, that is, there are generators in every degree between d and a_d.

By the above lemma, Example 2.11, and Theorem 4.4, we get new classes of lex-segment ideals with Cohen–Macaulay associated graded ring:

Proposition 4.8. *Let L be a lex-segment ideal. Assume that one of the following holds*:

(1) $0 < p_2 \leqslant p_3 \leqslant \cdots \leqslant p_{s+1}$, *or*
(2) $p_2 \geqslant p_3 \geqslant \cdots \geqslant p_{s+1}$.

Then $\mathrm{gr}_L(R)$ *is Cohen–Macaulay.*

5. Generic forms and lex-segment ideals

Theorem 2.4 points to a question: "find classes of ideals in R such that the associated graded ring of its initial ideal has positive depth." It is known that if char $k = 0$ and I is an ideal in $R = k[x, y]$, then the generic initial ideal $\mathrm{gin}(I)$ is a lex-segment ideal. We say that an ideal I is a generic ideal if it is generated by generic forms of given degrees and a lex-segment ideal L is generic, if it is the lex-segment ideal of a generic ideal. It is not always true that the associated graded ring of a lex-segment ideal has positive depth, see Example 2.5(a). In this section we produce a sub-class of the lex-segment ideals, namely lex-segment ideals of generic $\mathfrak{m}$-primary ideals, with positive depth associated graded ring. Let I be a generic $\mathfrak{m}$-primary ideal in R. We begin with a lemma which will help us in identifying the structure of a generic lex-segment ideal in R. For a polynomial $f(z) = \sum_i a_i z^i \in \mathbb{Z}[z]$, we let $|f(z)| = \sum_i b_i z^i$ with $b_i = a_i$ if $a_0, \dots, a_i > 0$ and $b_i = 0$ if $a_j \leqslant 0$ for some $j \leqslant i$, and let $\Delta f(z) = \sum_i (a_i - a_{i-1}) z^i$.

Proposition 5.1. *Let $H(z) \in \mathbb{Z}[z]$. Then*

$$H(z) = \left| \frac{\prod_{i=1}^{r+1}(1 - z^{d_i})}{(1-z)^2} \right|$$

for some integers $d_1, \dots, d_{r+1}$, $r \geqslant 1$ if and only if

$$\Delta H(z) = 1 + z + \cdots + z^{d_1 - 1} - p_1 z^{d_1} - p_2 z^{d_1+1} - \cdots - p_s z^{d_1+s-1} - c z^{d_1+s},$$

where $0 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_s$, $0 \leqslant c < p_s$ and $\sum_{i=1}^{s} p_s + c = d_1$.

Proof. Assume that $H(z)$ has the given form. We induct on r. Let $r = 1$. Then

$$
\begin{aligned}
H(z) &= \left|\frac{(1-z^{d_1})(1-z^{d_2})}{(1-z)^2}\right| = \left|\left(1+z+\cdots+z^{d_1-1}\right)\left(1+z+\cdots+z^{d_2-1}\right)\right| \\
&= 1+2z+\cdots+d_1z^{d_1-1}+\cdots+d_1z^{d_2-1}+(d_1-1)z^{d_2}+\cdots+z^{d_1+d_2-2}.
\end{aligned}
$$

Therefore

$$\Delta H(z) = 1+z+\cdots+z^{d_1-1}-z^{d_2}-z^{d_2+1}-\cdots-z^{d_1+d_2-1}.$$

Then

$$0 = p_1 = \cdots = p_{d_2-d_1} < 1 = p_{d_2-d_1+1} = \cdots = p_{d_2-1}.$$

Also

$$\sum_i p_i = (d_1+d_2-1)-(d_2-1) = d_1.$$

Hence the assertion follows.

Now assume that $r > 1$ and that the assertion is true for all $l < r$. Let

$$H'(z) = \left|\frac{\prod_{i=1}^r (1-z_i^d)}{(1-z)^2}\right|.$$

Then by inductive hypothesis, there exist $p_1, \ldots, p_s, c$ such that $0 \leqslant p_1 \leqslant \cdots \leqslant p_s$; $0 \leqslant c < p_s$; $\sum_i p_i + c = d_1$ and $\Delta H'(z) = 1+z+\cdots+z^{d_1-1} - p_1z^{d_1} - \cdots - p_sz^{d_1+s-1} - cz^{d_1+s}$. Therefore, if we write $H'(z) = \sum_i a_iz^i$, then

$$a_i = \begin{cases} i+1, & \text{if } i = 0,1,\ldots,d_1-1, \\ d_1 - \sum_{j=1}^{i-d_1+1} p_j, & \text{if } d_1 \leqslant i \leqslant d_1+s-1, \\ 0, & \text{if } i \geqslant d_1+s. \end{cases} \tag{5}$$

We have

$$H(z) = \left|\prod_{i=1}^r \frac{(1-z^{d_i})}{(1-z)^2}\left(1-z^{d_{r+1}}\right)\right| = \left|H'(z)\left(1-z^{d_{r+1}}\right)\right|.$$

If $d_{r+1} > \deg H'(z) = d_1+s-1$, then $|H'(z)(1-z^{d_{r+1}})| = |H'(z)| = H'(z)$. Therefore assume that $d_{r+1} \leqslant d_1+s-1$. If we set $H'(z)(1-z^{d_{r+1}}) = \sum_i b_iz^i$, then

$$b_i = \begin{cases} a_i, & \text{if } 0 \leqslant i \leqslant d_{r+1}-1, \\ a_i-(j+1), & \text{if } i = d_{r+1}+j. \end{cases}$$

Set $h = \max\{i \geqslant d_{r+1}\colon b_i > 0\}$. Therefore,

$$\left|H'(z)\left(1-z^{d_{r+1}}\right)\right| = \left|\sum_i b_iz^i\right| = \sum_{i=0}^h b_iz^i =: P(z).$$

We need to prove that $\Delta P(z)$ has the required properties. Denote by $\Delta P(z)_i$, the coefficient of $\Delta P(z)$ in degree i. Then

$$\Delta P(z)_i = \begin{cases} \Delta H'(z)_i, & \text{if } i \leqslant d_{r+1} - 1, \\ \Delta H'(z)_i - 1, & \text{if } d_{r+1} \leqslant i \leqslant h, \end{cases}$$

and $\Delta P(z)_{h+1} = -b_h$. Then

$$\begin{aligned} \sum_{i=d_1}^{h} \Delta P(z)_i + b_h &= \sum_{i=1}^{d_{r+1}-d_1} p_i + \sum_{i=d_{r+1}-d_1+1}^{h-d_1+1} [p_i + 1] + (a_h - h + d_{r+1} - 1) \\ &= \sum_{i=1}^{h-d_1+1} p_i + (h - d_{r+1} + 1) + a_h - (h - d_{r+1} + 1) \\ &= \sum_{i=1}^{h-d_1+1} p_i + a_h = d_1 \end{aligned}$$

because by Eq. (5) one has $a_h = d_1 - \sum_{j=1}^{h-d_1+1} p_j$. Therefore $\Delta P(z) = \Delta H(z)$ satisfies the required properties.

Let

$$\Delta H(z) = 1 + z + \cdots + z^{d_1-1} - p_1 z^{d_1} - \cdots - p_s z^{d_1+s-1} - c z^{d_1+s}$$

with p_i and c satisfying the given properties. We prove by induction on p_s. Suppose $p_s = 1$. Then $c = 0$ and $[p_1, \ldots, p_s] = [0, \ldots, 0, 1, \ldots, 1]$ for certain number of 0's, say l, and 1's, say m. Set $d_2 = d_1 + l$. Since $l + m = s$, we have

$\deg n$	0	1	...	$d_1 - 1$	d_1	...	d_2	...	$d_1 + d_2 - 1$
$\Delta H(z)_n$	1	2	...	1	0	...	-1	...	-1

Therefore,

$$H(z) = \left| \frac{(1 - z^{d_1})(1 - z^{d_2})}{(1 - z)^2} \right|.$$

Now assume that $p_s > 1$ and set $j = \max\{n\colon\ p_n > p_{n-1}\}$. Then we have, $1 \leqslant j \leqslant s$ and $p_j = \cdots = p_s$. Since $p_s - 1 > 0$, there exist non-negative integers q, r such that $c + s - j + 1 = (p_s - 1)q + r$ with $0 \leqslant r < p_s - 1$.

Define a polynomial $H'(z) \in \mathbb{Z}[z]$ such that

$$\Delta H'(z)_i = \begin{cases} 1, & \text{if } 0 \leqslant i \leqslant d_1 - 1, \\ -p_{i+1}, & \text{if } d_1 \leqslant i \leqslant d_1 + j - 2, \\ -p_j + 1, & \text{if } d_1 + j - 1 \leqslant i \leqslant d_1 + s - 1 + q, \\ -r, & \text{if } i = d_1 + s + q, \\ 0, & \text{if } i > d_1 + s + q. \end{cases}$$

Then

$$0 \leqslant p_1 \leqslant \cdots \leqslant p_{j-1} \leqslant p_j - 1 = \cdots = p_j - 1;\ 0 \leqslant r < p_j - 1 = p_s - 1$$

and

$$\begin{aligned}\sum_{i=1}^{j-1} p_i + (s-j+1+q)(p_j-1) + r &= \sum_{i=1}^{s} p_i - (s-j+1) + q(p_s-1) + r\\ &= \sum_{i=1}^{s} p_i + c = d_1.\end{aligned}$$

Therefore, by induction there exist $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$ such that

$$H'(z) = \left|\frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^2}\right|.$$

We show that $H(z) = |H'(z)(1-z^{d_1+j-1})|$. Note that $H'(z)_i = H(z)_i$ for $i \leqslant d_1 + j - 2$ and $H'(z)_i = H(z)_i + i - (d_1 + j - 2)$. Therefore $[H'(z)(1-z^{d_1+j-1})]_i = H(z)_i$ for $i \leqslant d_1+s-1$ and $H(z)_i = 0$ for $i > d_1+s-1$. To complete the proof, we need to show that $H'(z)_{d_1+s-1} - (p_j-1) - (s-j+2) \leqslant 0$. Since $H'(z)_{d_1+s-1} = H(z)_{d_1+s-1} + s - j + 1 = c + s - j + 1$, we have $H'(z)_{d_1+s-1} - (p_j-1) - (s-j+2) = c - p_s < 0$. Therefore

$$H(z) = \bigl|H'(z)\bigl(1-z^{d_1+j-1}\bigr)\bigr|. \qquad \square$$

Using the above proposition, we describe the structure of generic lex-segment ideals in R. We recall that, given a homogeneous ideal $I \subset R$, the Hilbert series $\mathrm{HS}_{R/I}(z)$ of R/I is defined to be $\sum_{t\geqslant 0}\mathrm{HF}_{R/I}(t)z^t$, where $\mathrm{HF}_{R/I}(t) = \dim_k(R/I)_t$ is the Hilbert function of R/I. If $\dim R/I = 0$, then $\mathrm{HS}_{R/I}(z)$ is a polynomial.

Proposition 5.2. *Let $I \subseteq R$ be an ideal generated by $r \geqslant 2$ generic forms of degrees $d_1, \ldots, d_r$ respectively. Let $d = \min\{d_i\}$. Then*

(1) *the Hilbert series of R/I is such that*

$$\Delta\,\mathrm{HS}_{R/I}(z) = 1 + z + \cdots + z^{d-1} - p_1 z^d - \cdots - p_s z^{d+s-1} - c z^{d+s},$$

with $0 \leqslant p_1 \leqslant \cdots \leqslant p_s$, $0 \leqslant c < p_s$ and $\sum_i p_i + c = d$.

(2) $\mathrm{Lex}(I) = (x^d, x^d y^{a_1}, \ldots, y^{a_d})$ *such that there are $p_1 + 1$ elements in degree d and p_i elements in degree $d + i - 1$ for $i = 2, \ldots, s$ and c elements in degree $d + s$.*

Proof. (1) The Hilbert series of R/I is given by

$$\mathrm{HS}_{R/I}(z) = \left|\frac{(1-z^{d_1})\cdots(1-z^{d_r})}{(1-z)^2}\right|,$$

for a simple proof of this fact, see [V2, 4.3]. Now the assertion follows directly from Lemma 5.1.

(2) Since the $\mathrm{Lex}(I)$ and I have same Hilbert function, the assertion follows from (1). $\square$

We set the notation for the rest of the section. Let $L = (x^d, x^{d-1}y^{a_1}, \ldots, y^{a_d})$ be a lex-segment ideal in $R = k[x, y]$, where $a_i + 1 \leqslant a_{i+1}$. Recall the notation set up in Section 4: let $L = (B_1, \ldots, B_{s+1})$ be the block decomposition of L such that $|B_1| = p_1 + 1$, $|B_i| = p_i$ for $i = 2, \ldots, s+1$. For the rest of the paper, we set

$$c = p_{s+1}.$$

From Proposition 5.2, we have $0 \leqslant p_1 \leqslant \cdots \leqslant p_s$ and $0 \leqslant c < p_s$.

Now we proceed to prove that the associated graded rings of generic lex-segment ideals have positive depth. Recall that there are lex-segment ideals whose associated graded rings have depth zero, see Example 2.5.

Theorem 5.3. *Let L be a generic lex-segment ideal in R. Then* $\operatorname{depth} \operatorname{gr}_L(R) > 0$.

Proof. We split the proof into two cases, namely $p_2 = 0$ and $p_2 > 0$.

Let $p_2 = 0$. Note that, in this case, in degree d, the lex-segment ideal L has only one generator, namely x^d. By Proposition 4.1, it is enough to prove that for some i, $L^2 = (x^d, x^{d-i}y^{a_i}, y^d)L$. Following the notation set up above, for $i = 1, \ldots, s+1$, let B_i denote the ith block of elements of degree $d + i - 1$, of the minimal generating set of L. Let $x^p y^q$ be the last element in the block B_s. Then

Claim. $L^2 = (x^d, x^p y^q, y^{a_d})L$.

To prove the claim, we need to show that, for any $1 \leqslant i \leqslant j \leqslant d$, $x^{2d-i-j}y^{a_i+a_j} \in JL$, where $J = (x^d, x^p y^q, y^{a_d})$. As in the proof of Proposition 4.2, we split the proof of the claim into different cases.

We first show that if $i + j \leqslant d$, then $x^{2d-i-j}y^{a_i+a_j} = x^d \cdot x^{d-i-j}y^{a_{i+j}} \cdot m$ for some monomial m. It is enough to prove that $a_i + a_j \geqslant a_{i+j}$. Let $b_i = a_i - a_{i-1}$. Consider the following equations:

- $a_{i+j} - a_j = b_{i+j} + b_{i+j-1} + \cdots + b_{j+1}$.
- $a_i = b_i + b_{i-1} + \cdots + b_1$.

Since $p_2 = 0$, $a_1 \geqslant 2$. Also note that since $p_2 \leqslant p_3 \leqslant \cdots \leqslant p_s$, the number of 2's appearing in $\{b_{i+j}, \ldots, b_{j+1}\}$ is at most the number of 2's appearing in $\{b_i, \ldots, b_1\}$. Hence $a_i \geqslant a_{i+j} - a_j$. Therefore $a_i + a_j \geqslant a_{i+j}$ and hence $x^{2d-i-j}y^{a_i+a_j} = x^d \cdot x^{d-i-j}y^{a_{i+j}} \cdot m$, for some monomial m.

Using similar arguments, we can show that

- if $d < i + j \leqslant d + t$, where $t = d - p$, then $x^{2d-i-j}y^{a_i+a_j} = x^p y^q \cdot x^{i+j-t}y^{a_{d-i-j+t}} \cdot m$ for some monomial m, and

- if $i+j>d+t$, then $x^{2d-i-j}y^{a_i+a_j}=x^{2d-i-j}y^{a_{i+j-d}}\cdot y^{a_d}\cdot n$ for some monomial n.

Therefore $x^{2d-i-j}y^{a_i+a_j}\in JL$ for all $0\leqslant i\leqslant j\leqslant d$, so that $L^2=JL$. Hence, by Proposition 4.1, $\operatorname{depth}\operatorname{gr}_L(R)>0$.

Now let $p_2\geqslant 1$. Note that in this case, there are minimal generators of L in all degrees from d to a_d. By Lemma 4.6, $T(L)$ is a lex-segment ideal with respect to y in $P=k[z,y]$. Write

$$T(L)=\left(y^{a_d-d},\, y^{a_d-d-1}z^c,\, y^{a_d-d-2}z^{c+p_s},\, \ldots,\, yz^{c+p_s+\cdots+p_3},\, z^{c+p_s+\cdots+p_2}\right).$$

Then $b_1=c$ and $b_i=p_{s-i+2}$, for $i=2,\ldots,s$. Hence we have $b_2\geqslant b_3\geqslant\cdots\geqslant b_s$. Therefore,

$$\left(T(L)\right)^2=\left(y^{s-1},\, y^{s-2}z^c,\, z^{c+p_s+\cdots+p_2}\right)T(L),$$

by Proposition 4.2. Therefore by Proposition 4.1, $\operatorname{depth}\operatorname{gr}_{T(L)}(P)>0$ and hence by Theorem 4.4 $\operatorname{depth}\operatorname{gr}_L(R)>0$. □

Example 5.4. (a) Let $I=(f_1,f_2,f_3)$ be a generic ideal such that $\deg f_1=5$, $\deg f_2=7$, $\deg f_3=8$. Then, a computation as in the proof of Proposition 5.1, will give that $\Delta H=[1,1,1,1,1,0,0,-1,-2,-2]$. Therefore the corresponding lex-segment ideal is $L=(x^5,x^4y^3,\ x^3y^5,x^2y^6,xy^8,y^9)$. It can be seen that $L^2=(x^5,y^9)L$ and hence $\operatorname{gr}_L(R)$ is Cohen–Macaulay. In Theorem 6.4 we actually prove that the Rees algebra of such ideals are normal.

(b) Let $I=(f_1,f_2,f_3,f_4,f_5)$ be a generic ideal such that $\deg f_1=10$, $\deg f_2=12$, $\deg f_3=13$, $\deg f_4=15$, $\deg f_5=15$. The corresponding lex-segment ideal is

$$L=\left(x^{10},x^9y^3,x^8y^5,x^7y^6,x^6y^8,x^5y^9,x^4y^{11},x^3y^{12},x^2y^{13},xy^{14},y^{16}\right)$$

and its Hilbert series is

$$\operatorname{HS}_L(z)=\frac{97+58z+z^3}{(1-z)^2}.$$

By Proposition 2.9, one has that $\operatorname{depth}\operatorname{gr}_L(R)=1$.

For a generic lex-segment ideal L, we have seen that $|B_1|-1\leqslant|B_2|\leqslant\cdots\leqslant|B_s|$ and $|B_{s+1}|<|B_s|$, where $L=(B_1,\ldots,B_{s+1})$ is a block decomposition of L. Therefore, in terms of the number of generators in each degree, there can be an "irregularity" in the last block of elements. Since we have shown that the associated graded ring of generic lex-segment ideals have positive depth, it is natural to ask, whether the associated graded ring is Cohen–Macaulay when this "irregularity" is removed. In the following theorem, we answer this question affirmatively.

Theorem 5.5. *Let L be a generic lex-segment ideal in R such that $c = 0$. Let $J = (x^{d-p_1}y^{a_{p_1}}, x^d + y^{a_d})$. Then $L^2 = JL$ and L is integrally closed. In particular,* $\mathrm{gr}_L(R)$ *is Cohen–Macaulay.*

Proof. Let $L = (x^d, x^{d-1}y^{a_1}, \dots, xy^{a_{d-1}}, y^{a_d})$. Following our previous notation, let $p_1 + 1 = |B_1|$ and $p_i = |B_i|$ for $i = 2, \dots, s$. First we prove that $L^2 = JL$ for $J = (x^{d-p_1}y^{a_{p_1}}, x^d + y^{a_d})$. We show that for $1 \leqslant i \leqslant j \leqslant d$, $x^{2d-i-j}y^{a_i+a_j} \in JL$. We split the proof into three cases:

Case I. $i + j < p_1$. Write

$$x^{2d-i-j}y^{a_i+a_j} = \left(x^d + y^{a_d}\right)\left(x^{d-i-j}y^{a_i+a_j}\right) - x^{d-i-j}y^{a_i+a_j+a_d}.$$

Note that for $i \leqslant p_1$, $a_{i-1} + 1 = a_i$. Therefore, $a_i + a_j = a_{i+j}$, if $i + j < p_1$. Hence $(x^d + y^{a_d})x^{d-i-j}y^{a_i+a_j} \in JL$. Now,

$$x^{d-i-j}y^{a_i+a_j+a_d} = \left(x^{d-p_1}y^{a_{p_1}}\right)\left(x^{d-(d-p_1+i+j)}y^{a_i+a_j+a_d-a_{p_1}}\right).$$

Since the number of minimal generators of L in each degree in increasing, as argued in the proof of Theorem 5.3, we can show that $a_i + a_j + a_d \geqslant a_{d-p_1+i+j} + a_{p_1}$. Therefore $(x^{d-(d-p_1+i+j)}y^{a_i+a_j+a_d-a_{p_1}}) \in L$ so that $x^{2d-i-j}y^{a_i+a_j} \in JL$.

Case II. $p_1 \leqslant i + j \leqslant d + p_1$. Writing $a_j - a_{p_1}$ and $a_{i+j-p_1} - a_i$ as in the proof of Theorem 5.3, one can easily see that, in this case $a_i + a_j \geqslant a_{p_1} + a_{i+j-p_1}$. Hence $x^{2d-i-j}y^{a_i+a_j} \in JL$.

Case III. $d + p_1 < i + j$. Then $2d - i - j = d - p_1 - k$ for some $k \geqslant 1$. Therefore we can write

$$x^{2d-i-j}y^{a_i+a_j} = \left(x^d + y^{a_d}\right)\left(x^{d-p_1-k}y^{a_i+a_j-a_d}\right) - x^{2d-p_1-k}y^{a_i+a_j-a_d}.$$

Arguments similar to that of in the proof of Case I will show that $x^{d-p_1-k}y^{a_i+a_j-a_d} \in L$ and $x^{2d-p_1-k}y^{a_i+a_j-a_d} \in JL$. Hence $x^{2d-i-j}y^{a_i+a_j} \in JL$. Therefore $L^2 = JL$.

Now we proceed to prove that L is integrally closed. From Corollary 3.14, it follows that if L has r generators in the initial degree, then $L = \mathfrak{m}^r N$ for a lex-segment ideal N. It can easily be seen that if L is generic, then so is N. Note also that there is only one generator in the initial degree (i.e., $p_1 = 0$) and $c = 0$ for N. We have considered such ideals in the next section. In Theorem 6.4 we have proved that lex-segment ideals with $p_1 = c = 0$ are integrally closed. Therefore N is integrally closed. Since L is a product of power of the maximal ideal (which is integrally closed) and N, L is integrally closed. Hence $\mathrm{gr}_L(R)$ is Cohen–Macaulay. □

We end the section with another class of lex-segment ideals whose associated graded rings are Cohen–Macaulay, namely lex-segment ideals corresponding to ideals generated by generic forms of equal degree.

Proposition 5.6. *Let I be an $\mathfrak{m}$-primary ideal generated by generic forms of equal degree. Let L be the lex-segment ideal corresponding to I. Then $\mathrm{gr}_L(R)$ is Cohen–Macaulay.*

Proof. Let I be generated by r forms of degree d. Then the Hilbert series of R/I is

$$\mathrm{HS}_{R/I}(z) = \left|\frac{(1-z^d)^r}{(1-z)^2}\right|.$$

A direct computation shows that $\mathrm{HF}_{R/I}(n) = n+1$ for $n = 0, \dots, d-1$, and for $i \geqslant 0$

$$\mathrm{HF}_{R/I}(d+i) = \begin{cases} d-(r-1)(i+1), & \text{if } d-(r-1)(i+1) \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\Delta H = [1, 1, \dots, 1, -r+1, -r+1, \dots, -r+1, -c],$$

where 1 is repeated $d-1$ times, $-r+1$ is repeated $[d/(r-1)]$ times, where $[d/(r-1)]$ denotes the largest integer smaller or equal to $d/(r-1)$, and $0 \leqslant c < r-1$. Hence the corresponding lex-segment ideal L have r generators in degree d, $r-1$ generators in degree $d+j$ for $j = 1, \dots, d+[d/(r-1)]-1$ and c generators in degree $d+[d/(r-1)]$. Thus, by Proposition 4.8, $\mathrm{gr}_L(R)$ is Cohen–Macaulay. □

6. Rees algebras of lex-segment ideals

In this section we study the Rees algebras of lex-segment ideals. For an ideal I in a ring R, the Rees algebra $\mathcal{R}(I)$ is defined to be the R-graded algebra $\bigoplus_{n\geqslant 0} I^n$. It can be identified with the R-subalgebra, $R[It]$ of $R[t]$ generated by It, where t is an indeterminate over R.

Let $I = (x^d, x^{d-1}y^{a_1}, \dots, y^{a_d})$ be a lex-segment ideal in $R = k[x, y]$. Consider the epimorphism of R-graded algebras

$$\psi : R[T_0, \dots, T_d] \longrightarrow \mathcal{R}(I)$$

defined by setting $\psi(T_i) = x^{d-i}y^{a_i}t$ and let $H = \ker\psi$ be the ideal of the presentation of $\mathcal{R}(I)$. The goal of this section is to describe explicitly a Gröbner basis of H, for some of the classes of lex-segment ideals we have considered. We begin by describing a set of binomials which are not in $\ker\psi$.

Lemma 6.1. *Let a, b, c, f, g be integers bigger than or equal to 0. The ideal H does not contain non-zero elements of the following forms*:

$$T_i^a T_{i+1}^b - y^c T_j^f T_{j+1}^g \quad \textit{with } 1 \leqslant i, j \leqslant d-1,$$

$$y^a T_i^b T_{i+1}^c - y^f T_j^g \quad \textit{with } 0 \leqslant i, j \leqslant d-1,$$

$$T_0^a T_d^b T_j^l - y^c T_0^f T_d^g T_k^h \quad \textit{with } 1 \leqslant j \neq k \leqslant d-1, 0 \leqslant l, h \leqslant 1.$$

Proof. Suppose that $m_i - m_j = T_i^a T_{i+1}^b - y^c T_j^f T_{j+1}^g$ is in H; then $\psi(T_i^a T_{i+1}^b) = \psi(y^c T_j^f T_{j+1}^g)$, and by comparing the degrees respectively of t, x one has

$$\begin{cases} a+b = f+g, \\ a(d-i) + b(d-i-1) = f(d-j) + g(d-j-1). \end{cases} \tag{6}$$

Since $a+b = f+g$, $ad+bd = fd+gd$ and thus from (6), it follows that $ai+bi+b = fj+gj+g$. Therefore $g-b = (f+g)(i-j)$.

If $i > j$, then, since $f+g > 0$, $g-b > 0$. Again from (6), we get that $g-b \geqslant f+g$, i.e., $-b \geqslant f$. Therefore, the only possibility is that $f = b = 0$ and hence $i = j+1$. This implies that $a = g$ and $c = 0$. Hence $m_i = m_j$.

If $i < j$, then one concludes in the same way as before, since one has $b-g = (a+b) \times (j-i)$.

If $i = j$, then by (6) one has $b = g$, and therefore $a = f$ and $c = 0$. Again we have $m_i = m_j$.

Identical arguments will show that a non-zero equation of the form $y^a T_i^b T_{i+1}^c - y^f T_j^g$, with $0 \leqslant i, j \leqslant d-1$, is not in H.

Suppose now that an element of the form $m_j - m_k = T_0^a T_d^b T_j^l - y^c T_0^f T_d^g T_k^h$ is in H. By a degree comparison this implies

$$\begin{cases} a+b+l = f+g+h, \\ ad + l(d-j) = df + h(d-k), \\ ba_d + la_j = c + ga_d + ha_k. \end{cases}$$

We distinguish different cases.

If $l = h = 0$, then one has $a = f$, $b = g$, and $c = 0$. Thus $m_j = m_k$.

If $l = 0$ and $h = 1$, then it follows that $ad = fd + d - k$, that is, $(a-f)d = d-k$. This is a contradiction since d cannot divide $d-k$. If $l = 1$ and $h = 0$, one concludes as in the previous case that $m_j = m_k$.

If $l = h = 1$, then $ad + d - j = fd + d - k$, that is, $(a-f)d = j-k$. This implies that d divides $j-k$ and this is a contradiction, since $1 \leqslant j \neq k \leqslant d-1$. □

In the following two propositions, we describe explicitly a Gröbner basis for the presentation ideal of Rees algebras of lex-segment ideals with increasing and decreasing differences sequence, already considered in Example 2.11.

Proposition 6.2. *Let I be a lex-segment ideal in R. Suppose that its differences sequence is such that $b_i \leqslant b_{i+1}$ for $i = 1, \ldots, d-1$. Then the set of elements*

$$\left\{ \begin{array}{l} xT_i - y^{b_i} T_{i-1},\ i = 1, \ldots, d, \\ T_i T_{j-1} - y^{b_i - b_j} T_{i-1} T_j,\ i, j \in \{1, \ldots, d\},\ d \geqslant i > j \geqslant 1 \end{array} \right\}$$

form a Gröbner basis of H with respect to any term order such that the initial term of any of the elements above is the term on the left side. Also, the Rees algebra $\mathcal{R}(L)$ is normal.

Proof. Let Q be the ideal generated by $xT_i,\ i=1,\dots,d$, and $T_iT_{j-1},\ i>j\geqslant 1$. Since the binomial relations form a universal Gröbner basis of H, to prove that $Q=\mathrm{in}(H)$ it suffices to prove that there are no relations in H involving only monomials which are not in Q. Note that such monomials are of the form $x^aT_0^b,\ y^aT_i^bT_{i+1}^c,\ y^aT_i^b$ for some a,b,c. Since $\psi(x^aT_0^b)$ involves only x, it cannot be the term of an element in H. Moreover, if $y^aT_i^b-y^cT_j^e$ is in H, then it is easy to see that $b=e$ and $i=j$. From Lemma 6.1 it follows that the given relations are a Gröbner basis of H. Note that the initial terms of the elements of the Gröbner basis are square-free, thus by [St, 13.15] the Rees algebra $\mathcal{R}(L)$ is normal. $\square$

Proposition 6.3. *Let L be a lex-segment ideal in R. Suppose that its differences sequence is such that $b_i\geqslant b_{i+1}$ for $i=1,\dots,d-1$. Then the set of elements*

$$\left\{\begin{array}{l} xT_i-y^{b_i}T_{i-1},\ i=1,\dots,d,\\ T_iT_{j-1}-y^{b_i-b_j}T_{i-1}T_j,\ i,j\in\{1,\dots,d\},\ 1\leqslant i<j\leqslant d\end{array}\right\}$$

form a Gröbner basis of H with respect to any term order such that the initial term of any of the elements above is the term on the left.

Proof. Let Q be the ideal generated by $xT_i,\ i=1,\dots,d$, and $T_iT_{j-1},\ 1\leqslant i<j\leqslant d$. In particular, $T_i^2\in Q$ for $i=1,\dots,d-1$. Thus the monomials which are not in Q are the ones of the form $x^aT_0^b,\ y^aT_0^bT_d^cT_j^e$, with $0<j<d$, $0\leqslant e\leqslant 1$, and some a,b,c. Using Proposition 6.1 and arguing as in Proposition 6.2, one concludes that $Q=\mathrm{in}(H)$. $\square$

It is easy to see that in general a lex-segment ideal L as in the proposition above is not integrally closed, thus $\mathcal{R}(L)$ is not normal.

In the following theorem, we obtain the Gröbner basis for the presentation ideal of the Rees algebra of another sub-class of generic lex-segment ideal and then use it to produce another class of normal Rees algebras.

Theorem 6.4. *Let L be a generic lex-segment ideal in R such that $c=p_1=0$. Then the set of elements*

$$\left\{\begin{array}{l} xT_i-y^{b_i}T_{i-1},\ i=1,\dots,d,\\ T_iT_j-y^{\alpha}T_0T_{i+j},\ 1\leqslant i\leqslant j<d,\ 1\leqslant i+j\leqslant d \text{ and } \alpha=a_i+a_j-a_{i+j},\\ T_iT_j-y^{\beta}T_{i+j-d}T_d,\ 1\leqslant i\leqslant j<d,\ d<i+j \text{ and } \beta=a_i+a_j-(a_{i+j-d}+a_d)\end{array}\right\}$$

form a Gröbner basis for H with respect to any term order such that the initial term of any of the elements in the above set is the term on the left. Also, the Rees algebra $\mathcal{R}(L)$ is normal.

Proof. Since $c = p_1 = 0$, it follows from Theorem 5.5 that $L^2 = JL$ for $J = (x^d, y^{a_d})$. Let $\mathcal{B}$ denote the set of elements given in the statement of the theorem. We first show that $\mathcal{B}$ is a Gröbner basis for H.

Let Q be the ideal generated by $xT_i,\ i = 1, \ldots, d$ and $T_iT_j,\ 1 \leqslant i \leqslant j < d$. The monomials which are not in Q are of the form $x^a y^b T_0^c,\ y^a T_0^b T_d^c T_j^e,\ y^a T_i^b$ with $0 < j < d,\ 0 \leqslant i \leqslant d$ and some non-negative integers a, b, c, e. Then, as in the proof of Proposition 6.2, it can easily be seen that $\operatorname{in}(H) = Q$ and hence $\mathcal{B}$ is a Gröbner basis for H.

Now we prove that $\mathcal{R}(I)$ is normal. Since $\mathcal{R}(I)$ is a semi-group ring, it is enough to prove that for any three monomials $f, g, h \in \mathcal{R}(I)$, if for some integer p, $f^p = g^p h$, then $h = h_1^p$ for some monomial $h_1 \in \mathcal{R}(I)$ (see [BH, 6.1.4]). Let $S = R[T_0, \ldots, T_d]$. Then $\mathcal{R}(I) \cong S/H$. Since $\mathcal{B}$ is a Gröbner basis for H and $\operatorname{in}(H) = Q$, the set

$$Q' = \left\{ \begin{array}{l} x^a y^b T_0^c,\ a, b, c \geqslant 0, \\ y^a T_0^b T_d^c T_j^e,\ 0 < j < d;\ a, b, c, e \geqslant 0, \\ y^a T_i^b,\ a, b \geqslant 0;\ 0 \leqslant i \leqslant d \end{array} \right\}$$

form a monomial basis for S/H. Note that any monomial in S/H will be a power of either of the above forms. Let $f, g, h \in S/H$ be monomials such that

$$f^p = g^p h \quad \text{for some } p \geqslant 0. \tag{7}$$

Let $f = x^a y^b T_0^c$ for some integers a, b, c. Then, from (7) and comparing the x and y degrees of $\psi(f), \psi(g)$ and $\psi(h)$, we can conclude that both g and h can not contain T_j for $j \neq 0$. Write $g = x^{a_1} y^{b_1} T_0^{c_1}$. From (7), it follows that $a_1 \leqslant a,\ b_1 \leqslant b$ and $c_1 \leqslant c$. Therefore, $h = (x^{a-a_1} y^{b-b_1} T_0^{c-c_1})^p$. Hence $h = h_1^p$, for $h_1 = x^{a-a_1} y^{b-b_1} T_0^{c-c_1}$.

Now assume that $f = y^a T_0^b T_d^c T_j^e$ for some $0 < j < d$ and non-negative integers a, b, c, e. If $g = x^{a_1} y^{b_1} T_0^{c_1}$. Then comparing the x-degrees, we get that $a_1 = 0$. Therefore $g = y^{b_1} T_0^{c_1}$, such that $b_1 \leqslant a$ and $c_1 \leqslant b$. Therefore $h = (y^{a-b_1} T_0^{b-c_1} T_d^c T_j^e)^p$. Hence $h = h_1^p$ for $h_1 = y^{a-b_1} T_0^{b-c_1} T_d^c T_j^e$. Suppose $g = y^{a_1} T_0^{b_1} T_d^{c_1} T_i^{e_1}$ for some i. Then again from (7), it follows that $i = j$ and $h = h_1^p$ for $h_1 = y^{a-a_1} T_0^{b_1} T_d^{c_1} T_j^{e-e_1}$.

Let $f = y^a T_i^b$ for some $a, b \geqslant 0$ and $0 \leqslant i \leqslant d$. Then it is obvious from (7) that g and h have to be of the same form. Thus, as in the previous cases, we conclude that $h = h_1^p$ for some h_1.

Therefore S/H is normal and hence $\mathcal{R}(L)$ is normal. □

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