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Exponential Dichotomy and Stability of Neutral Type Equations

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A linear functional differential equation of neutral type with unbounded delay

$$\mathscr{L}x \equiv (d/ds)\, Dx + Bx = 0,$$

where D and B are linear bounded retarded operators with exponentially fading memory, is considered. It is shown that if operator $\mathscr{L}$ is interpreted as operator from the space C into the special space C^{-1} of distributions, then its invertibility is equivalent to the presence of exponential dichotomy of the solutions of this equation. As applications, we prove the theorems on stability and instability in the first approximation for neutral functional differential equations of a general form.

1. INTRODUCTION

The theorems on dichotomy of solutions have their origin in a paper of Perron [47]. Subsequently, this problem was studied for ordinary differential equations by many authors. As far as the problems of ODE theory are beyond the area of our concern in this paper, we restrict ourselves to the remark that one can find these results and the history of the problem in the books of Massera and Schäffer [41], Hartman [27], Daleckij and Krejn [19], Krasnosel'skij, Burd and Kolesov [32], and Coppel [14].

The first results on dichotomy for differential equations with delayed argument are due to Burd and Kolesov [12] (for the proofs, see [30, 31]), Pecelli [45, 46], Coffman and Schäffer [13], Schäffer [48], and Corduneanu [15]. A special case of the theorem on dichotomy of solutions,

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when a subspace of exponentially increasing solutions is trivial, was considered in Halanay [20]. The analogue of this result is Theorem 3 below. Note that in the papers cited above only the case of bounded delay was discussed.

The dichotomy problems for a special case of equations with constant coefficients were discussed by Hale and Meyer [26], Henry [28], Birkgan [9, 11], Naito [43, 44], Kurbatov and Frolov [40], and Staffans [51], where the case of neutral type equations with bounded delay was investigated in [26, 28, 9, 11], the case of retarded equations with unbounded delay was investigated in [43, 44] and, finally, the case of neutral type equations with unbounded delay was investigated in [40, 51].

The dichotomy problem for the neutral type equations with variable coefficients was considered in Kurbatov [33, 34] and Birkgan [10]. These papers are devoted to the equations with a bounded delay. In [33, 34] a proof was given of equivalence of the exponential dichotomy of solutions for the neutral type equation with "internal" differentiation

$$D \frac{d}{ds} x + Bx = 0,$$

where D and B are retarded operators, in the phase space $C^1[-h, 0]$ and invertibility of the operator $\mathscr{L} = D(d/ds) + B$ as an operator from C^1 into C (where C is the space of continuous bounded functions on $\mathbb{R}$, and C^1 consists of functions that lie in C along with the first derivative). In [10] the equation with "external" differentiation

$$\frac{d}{ds} Dx + Bx = 0$$

was considered, but only half of the classical theorem on dichotomy has been proved: namely it is proved that if D has smooth coefficients and the operator $\mathscr{L} = (d/ds)D + B$: $C^1 \to C$ is invertible, then this equation possesses an exponential dichotomy in the phase space $C[-h, 0]$. Note that the notion of neutral type equations with external differentiation is due to Hale and Meyer [26] (see also Cruz and Hale [18], Hale [21]).

In this paper, the same problem is discussed for well-posed (see Definition 3 below) neutral functional differential equations with external differentiation and unbounded delay. The main result (Theorem 2) differs from the above results in the following points. First, we consider neutral type equations. Second, we only assume that the coefficients of the equations are continuous and bounded. Third, we admit unbounded delays; it should be noted that we consider only an exponentially fading memory. In contrast to [33, 34], we consider the equation with external rather than internal differentiation, and, which is most important, we consider the equation in a different phase space, in C and not in C^1. Finally, in comparison with [10], we also prove the inverse assertion, i.e., the implication "exponential dichotomy" $\Rightarrow$ "invertibility."

The main idea of the present paper consists in the examination of the operator $\mathscr{L}$ that corresponds to the equation as an operator from the space C into a special space C^{-1} of distributions. The invertibility of $\mathscr{L}$ just in this pair of spaces turns out to be equivalent to the presence of exponential dichotomy. The space C^{-1} was earlier used for the investigation of the neutral type equations in Kurbatov [38, 39]. In Section 2 the space C^{-1} and its properties are described.

In Section 3 we introduce the spaces $C_{-\gamma}$ of functions increasing on $\pm\infty$ as $O(e^{-\gamma s})$. The basic advantage of the space $C_{-\gamma}$ in comparison with C in the capacity of the phase space can be reduced to the following: the translation operator along the trajectories of the equation is contractive if the equation is stable. As a consequence, we have to consider the differential operator $\mathscr{L} = (d/ds)D + B$ not only as an operator from C into C^{-1} but also as an operator from $C_{-\gamma}$ into $C_{-\gamma}^{-1}$ for $\gamma > 0$. Due to this, we have to discuss the dependence of the properties of the operator $\mathscr{L}\colon C_{-\gamma} \to C_{-\gamma}^{-1}$ with respect to γ. We do this in Section 3. The main result of this section (Theorem 1) asserts that the invertibility of $\mathscr{L}$ does not depend on γ for sufficiently small γ.

The main result of the paper is Theorem 2 on the dichotomy of solutions. It is discussed and proved in Section 4. A significant special case of this theorem (Theorem 3) is devoted to the situation in which the dichotomy decomposition consists of the whole space and the zero-subspace. An essentially analogous result was considered in Kurbatov [38].

As an application of the theorem on dichotomy we prove generalizations of Lyapunov theorems on stability and isntability in the first approximation for neutral functional differential equations with external differentiation and unbounded delay

$$\frac{d}{ds}[Dx + d(x)] + Bx + b(x) = 0,$$

where D and B are linear retarded operators, and d and b are nonlinear retarded operators with zero derivative at zero. The theorem on the exponential dichotomy allows us to extract stable and unstable invariant manifolds for the translation operator along the trajectories of the linear part of this equation and to exploit the abstract theorems (see Theorems 4 and 5 in Section 5) on stability and instability in the first approximation for nonautonomous dynamical systems from Akhmerov and Kamenskij [3] (for the proofs, see [4]). The analogous of these theorems for the case of the autonomous linear part can be found in the books of Daleckij and Krejn [19] and Henry [29].

Theorems on stability in the first approximation for neutral type equations with internal differentiation were proved by Bellman and Cooke [8], Misnik [42], and (under weaker conditions) Akhmerov and Kamenskij [3, 4] (in the latter papers the theorem on instability also has been proved). Similar results in a simpler case of equations with periodic coefficients are due to Akhmerov and Kamenskij [1, 2] (see also [5, 6, 7]).

Theorems on stability in the first approximation for the neutral type equations with external differentiation were first obtained by Cruz and Hale [17] (see also Hale and Ize [23], Hale and Martinez-Amores [25], and the book of Hale [22]). In comparison with these results, we, first, consider the equations with unbounded delay. Second, we prove not only the theorem on stability but also the theorem on instability in the first approximation. Third, the scheme of the proof suggested below allows us to avoid the employment of the variation of constants formula and, therefore, to extend the scope of our theorems. Finally, we do not assume the time independence of the linear part of the equation.

2. The Space C^{-1}

Suppose $\mathbb{R}=(-\infty,+\infty)$, E is an arbitrary real or complex Banach space with norm $|\cdot|$, C is the space of all continuous bounded functions $x\colon \mathbb{R}\to E$ with the norm $\|x\|=\sup\{|x(s)|: s\in\mathbb{R}\}$. The spaces $C(-\infty, a]$ and $C[a, b]$ are defined similarly.

Let $\mathscr{D}$ be the space of infinitely differentiable functions $\psi\colon \mathbb{R}\to\mathbb{R}$ with compact support, and $\mathscr{D}'$ the space of $\mathbb{R}$-linear continuous vector-functionals $f\colon \mathscr{D}\to E$. For more details, see [50]. The elements of the space $\mathscr{D}'$ are called distributions. Evidently, one can imbed C into $\mathscr{D}'$.

Let us denote the space of distributions $f\in\mathscr{D}'$ representable as $f=u'+v$ with $u, v\in C$ by C^{-1}. The representation $f=u'+v$ is, of course, not unique. Designate the norm of an element f in C^{-1} by $\|f\|=\inf\{\|u\|_C+\|v\|_C\colon u, v\in C, f=u'+v\}$. We note that the space C^{-1} is isomorphic to the factor-space $(C\oplus C)/R$, where R is the subspace of $C\oplus C$ consisting of pairs (u, v) such that $u'+v=0$. By virtue of the theorem on completeness of factor-space, this implies the completeness of C^{-1}.

The space C^{-1} in connection with functional differential equations was discussed in [38, 39].

Any function $f\in C^{-1}$ has a primitive F in $\mathscr{D}'$ (it is defined by the relation $F'=f$). Clearly, the primitive F is a continuous but, generally speaking, unbounded function. It is defined to within a constant.

PROPOSITION 1 [39]. *The continuous function F is a primitive for some function $f \in C^{-1}$ if and only if F is bounded with respect to the semi-norm*

$$p(F) = \sup\{|F(t) - F(s)| : |t - s| \leqslant 1\}.$$

PROPOSITION 2 [38]. *The map $Ux = x' + x$ represents isomorphism between C and C^{-1}.*

For any pair of functions $f, g \in \mathscr{D}'$ the notion of equality on the interval (a, b) is defined. In the case of $f, g \in C^{-1}$ this notion can be understood in the following way: f and g concide on (a, b) if their primitives (they are continuous functions) differ on (a, b) by a constant.

PROPOSITION 3 [38]. *Let $a < b < c$ and functions $f, g \in C^{-1}$ coincide both on (a, b) and on (b, c). Then f and g coincide on (a, c).*

We define the projector $P(a, b)$ in C^{-1} for arbitrary $-\infty \leqslant a < b \leqslant +\infty$ in the following way. Suppose $f \in C^{-1}$ and F is the primitive of f. Let

$$[p(a, b) f](s) = \begin{cases} F(a) & \text{for } s \leqslant a, \\ F(s) & \text{for } a \leqslant s \leqslant b, \\ F(b) & \text{for } b \leqslant s \end{cases}$$

and

$$P(a, b) f = [p(a, b) f]'.$$

It is clear that $P(a, b) f$ coincides with f on (a, b) and is equal to zero on $(-\infty, a)$ and $(b, +\infty)$. These properties, by virtue of Proposition 3, define the function $P(a, b) f$ uniquely. It is also evident that the projectors $P(-\infty, t)$ and $P(t, +\infty)$ are mutually complementary.

3. OPERATORS WITH EXPONENTIALLY FADING MEMORY

For any $\gamma \in \mathbb{R}$ we denote the space of continuous functions $x: \mathbb{R} \to E$ bounded with respect to the norm

$$\|x\|_{-\gamma} = \sup\{|e^{\gamma s} x(s)| : s \in \mathbb{R}\}$$

by $C_{-\gamma}$. We define the spaces $C_{-\gamma}(-\infty, t]$ analogously, but let the norm in $C_{-\gamma}(-\infty, t]$ be set by the formula

$$\|x\|_{-\gamma, t} = \sup\{|e^{\gamma(s-t)} x(s)| : s \leqslant t\}. \tag{1}$$

This norm is more convenient because for $\nu < \gamma$

$$\|x\|_{-\gamma,t} \leqslant \|x\|_{-\nu,t}. \tag{2}$$

DEFINITION 1. An operator D is said to be a *linear operator with exponentially fading memory of order* $\gamma > 0$ if it is a linear bounded operator acting in $C_{-\gamma}$.

Such operators were discussed in [35, 36, 37] in somewhat different terms.

DEFINITION 2. An operator $D: C_{-\gamma} \to C_{-\gamma}$ is said to be *retarded* if

$$\forall (t \in \mathbb{R})\, \forall (x, y \in C_{-\gamma})[(x(s) = y(s) \text{ for } s < t)$$
$$\Rightarrow ((Dx)(s) = (Dy)(s) \text{ for } s < t)]. \tag{3}$$

In this section the symbol D designates an arbitrary linear retarded operator with exponentially fading memory of order $\gamma > 0$.

Condition (3) allows us to apply such an operator D not only to functions $x \in C_{-\gamma}$ but also to functions $x \in C_{-\gamma}(-\infty, t]$. Indeed, let $x \in C_{-\gamma}(-\infty, t]$ and y be some continuation of x to a function in $C_{-\gamma}$. Let for $s \leqslant t$

$$(Dx)(s) = (Dy)(s).$$

By the same token Dx is defined as an element of $C_{-\gamma}(-\infty, t]$. Condition (3) implies that Dx does not depend on the choice of continuation y.

Similarly, if $x: \mathbb{R} \to E$ is such a function that its restriction on $(-\infty, t]$ for any $t \in \mathbb{R}$ lies in $C_{-\gamma}(-\infty, t]$, then Dx makes sense. In particular, Dx makes sense if $x \in C_{-\nu}$ for $\nu < \gamma$.

PROPOSITION 4. *For any* $x \in C_{-\gamma}(-\infty, t]$

$$|(Dx)(t)| \leqslant \|D\|_{-\gamma} \cdot \|x\|_{-\gamma,t},$$

where $\|D\|_{-\gamma}$ *is the norm of* D *in* $C_{-\gamma}$.

Proof. Suppose a continuous function $y: \mathbb{R} \to E$ coincides with x on $(-\infty, t]$ and satisfies the condition

$$\sup\{|e^{\gamma(s-t)}y(s)| : s \in \mathbb{R}\} = \|x\|_{-\gamma,t}.$$

Condition (3) for D implies the equality

$$(Dy)(t) = (Dx)(t).$$

But $\|Dy\|_{-\gamma} \leqslant \|D\|_{-\gamma} \cdot \|y\|_{-\gamma}$. Notice that

$$\|y\|_{-\gamma} = \|x\|_{-\gamma, t} \cdot e^{-\gamma t}$$

and

$$\|Dy\|_{-\gamma} = \sup\{|e^{\gamma s}(Dy)(s)| : s \in \mathbb{R}\} \geqslant e^{\gamma t} |(Dy)(t)|.$$

Hence

$$|(Dx)(t)| = |(Dy)(t)| \leqslant \|D\|_{-\gamma} \cdot \|x\|_{-\gamma, t}. \quad \blacksquare$$

COROLLARY 1. *For any* $t \in \mathbb{R}$

$$\|D\|_{-\gamma, t} \leqslant \|D\|_{-\gamma},$$

where $\|D\|_{-\gamma, t}$ *is the norm of* D *in* $C_{-\gamma}(-\infty, t]$.

COROLLARY 2. *Let* $\nu < \gamma$ *and* $t \in \mathbb{R}$. *Then* D *acts in* $C_{-\nu}(-\infty, t]$ *and* $C_{-\nu}$. *Moreover,*

$$\|D\|_{-\nu, t} \leqslant \|D\|_{-\gamma, t}, \qquad \|D\|_{-\nu} \leqslant \|D\|_{-\gamma}.$$

Let now

$$(Tx)(s) = sx(s), \qquad s \in \mathbb{R}.$$

This operator acts, for example, from $C_{-\nu}(-\infty, t]$ into $C_{-\nu-\varepsilon}(-\infty, t]$ for all $\nu \in \mathbb{R}$ and $\varepsilon > 0$.

PROPOSITION 5. *For any* $\nu < \gamma$ *the operator* $DT - TD$ *acts in* $C_{-\nu}$ *and is bounded.*

Proof. Let $x \in C_{-\nu}$ and $\|x\|_{-\nu} \leqslant 1$, i.e., $|x(s)| \leqslant e^{-\nu s}$, and let $t \in \mathbb{R}$. Then

$$((DT - TD)x)(t) = (Dy)(t),$$

where

$$y(s) = sx(s) - tx(s) = (s - t)\, x(s).$$

Further,

$$\begin{aligned} \|y\|_{-\gamma, t} &= \sup\{|e^{\gamma(s-t)}(s-t)\, x(s)| : s \leqslant t\} \\ &\leqslant \sup\{|e^{\gamma(s-t)}(s-t)\, e^{-\nu s}| : s \leqslant t\} = Me^{-\nu t}, \end{aligned}$$

where M depends on v only. Proposition 4 implies the needed inequality

$$|((DT-TD)x)(t)| \leqslant M \cdot \|D\|_{-\gamma} e^{-vt}. \quad \blacksquare$$

Suppose provisionally that the space E is complex and for any $\lambda \in \mathbb{C}$ let

$$(\Psi_\lambda x)(s) = e^{\lambda s} x(s).$$

If $\lambda = v + i\omega$, then it is evident that Ψ_λ isomorphically maps C onto C_v.
Let us consider the operator

$$D_\lambda = \Psi_\lambda^{-1} D \Psi_\lambda.$$

By operator D on the right-hand side of this formula is meant the operator acting in C_v, where $v = \operatorname{Re} \lambda$. Thus, the operator-function

$$\lambda \mapsto D_\lambda : C \to C$$

is defined for $\operatorname{Re} \lambda \geqslant -\gamma$.

PROPOSITION 6. *The operator-function $\lambda \mapsto D_\lambda$ is analytic for* $\operatorname{Re} \lambda > -\gamma$. *Its derivative at the point λ is equal to $\Psi_\lambda^{-1}(DT-TD)\,\Psi_\lambda$.*

Proof. Let $x \in C$ and $\|x\|_C \leqslant 1$, i.e., $|x(s)| \leqslant 1$, and let $t \in \mathbb{R}$. A few simple calculations yield

$$\begin{aligned}&[((\Psi_{\lambda+\Delta\lambda}^{-1} D\Psi_{\lambda+\Delta\lambda} - \Psi_\lambda^{-1} D\Psi_\lambda)/\Delta\lambda - (DT-TD))x](t)\\ &\quad = (Dy)(t),\end{aligned}$$

where

$$y(s) = [(e^{(\lambda+\Delta\lambda)(s-t)} - e^{\lambda(s-t)})/\Delta\lambda - e^{\lambda(s-t)}(s-t)]\, x(s).$$

Further, we have

$$\|y\|_{-\gamma,t} \leqslant \sup\{|e^{(\gamma+\lambda)(s-t)}(e^{\Delta\lambda(s-t)} - 1 - \Delta\lambda(s-t))/\Delta\lambda| : s \leqslant t\}.$$

If $\operatorname{Re} \lambda > -\gamma$, then, evidently, the right-hand side of this inequality tends to zero as $\Delta\lambda \to 0$ uniformly in t. Using Proposition 4, the end of the proof is now supplied in the obvious way. $\blacksquare$

PROPOSITION 7. *If $D: C \to C$ is invertible, then there exists $v \in (0, \gamma)$ such that for any $\lambda \in [-v, v]$ the operator D is invertible as an operator from C_λ into C_λ. Moreover, the inverse operator does not depend on λ on the space $C_v \cap C_{-v}$.*

Remark 1. Some variants of this proposition were given in [35, 36, 37].

Proof. Without loss of generality one can assume that E is complex. The operator-function $\lambda \mapsto D_\lambda$ is analytic for $\operatorname{Re} \lambda > -\gamma$ and invertible at the point $\lambda = 0$. Hence, the operators D_λ are invertible for all λ in some neighborhood of zero. Choose $v \in (0, \gamma)$ in such a way that the operators D_λ are invertible for all $\lambda \in [-v, v]$. In this case it is evident that the operators

$$D = \Psi_\lambda D_\lambda \Psi_\lambda^{-1} \colon C_\lambda \to C_\lambda \quad (\lambda \in [-v, v])$$

are invertible and the first part of the proposition is proved.

We prove now that $D_\lambda^{-1} \colon C_\lambda \to C_\lambda$ does not depend on $\lambda \in [-v, v]$ on the space $C_v \cap C_{-v}$. Let $f \in C_v \cap C_{-v}$. The function $\lambda \mapsto \Psi_\lambda^{-1} f$ (with values in C) is analytic on $(-v, v)$ and continuous on $[-v, v]$ in the topology of uniform convergence on each compact subset of $\mathbb{R}$. Therefore, the function $g(\lambda) = D_\lambda^{-1} \Psi_\lambda^{-1} f$ will be the same as $\lambda \mapsto \Psi_\lambda^{-1} f$.

Note that for $\lambda = i\omega$

$$D_\lambda^{-1} = \Psi_\lambda^{-1} D^{-1} \Psi_\lambda$$

(here $D^{-1} \colon C \to C$) and hence for such λ

$$g(\lambda) = \Psi_\lambda^{-1} D^{-1} f. \tag{4}$$

We prove that (4) holds for $\lambda \in [-v, v]$. In fact, let $[a, b]$ be a bounded segment, and $Q[a, b] \colon C \to C[a, b]$ the canonical restriction. Then the operator-function $\lambda \mapsto Q[a, b]\ \Psi_\lambda^{-1} \colon C \to C[a, b]$ is analytic and hence the identity

$$Q[a, b]\ g(\lambda) = Q[a, b]\ \Psi_\lambda^{-1} D^{-1} f$$

holds for $\lambda \in (-v, v)$ because it holds for $\lambda = i\omega$ and by virtue of the uniqueness of analytic continuation. From continuity reasons, it follows that it also holds for $\lambda = \pm v$. Thus, (4) holds for all $\lambda \in [-v, v]$.

But $g(v)$, $g(-v) \in C$. Consequently, by the representation (4), $x = D^{-1} f \in C_v \cap C_{-v}$. It remains to note that by the definition of the operators $D \colon C_\lambda \to C_\lambda$ they map a function $x \in C_v \cap C_{-v}$ into one and the same function. ∎

For $\gamma \in \mathbb{R}$ we denote by $C_{-\gamma}^{-1}$ the space of distributions $f \in \mathscr{D}'$ such that the function $g(s) = e^{\gamma s} f(s)$ belongs to C^{-1}, equipped with the norm $\|f\| = \|g\|_{C^{-1}}$.

PROPOSITION 8. *A function $f \in \mathscr{D}'$ belongs to $C_{-\gamma}^{-1}$ if and only if f is representable in the form $f = u' + v$ with $u, v \in C_{-\gamma}$.*

Proof. This is a direct application of the definitions of C^{-1}, $C_{-\gamma}$, and $C_{-\gamma}^{-1}$. ∎

PROPOSITION 9. *The map* $Ux = x' + \gamma x$ $(\gamma > 0)$ *is an isomorphism between* $C_{-\nu}$ *and* $C_{-\nu}^{-1}$ *for all* $\nu < \gamma$. *The map* U^{-1} *is given by the formula*

$$(U^{-1}f)(t) = u(t) + \int_0^{+\infty} e^{-\gamma s}[v(t-s) + \gamma u(t-s)]\, ds$$

for $f = u' + v$.

Just in the same way as in C^{-1} (see Section 2), one can define in the spaces $C_{-\gamma}^{-1}$ the projectors $P(a, b)$.

Suppose now that D and B are linear retarded operators in $C_{-\gamma}$ for some $\gamma > 0$. Consider the operator

$$\mathscr{L}x = \frac{d}{ds} Dx + Bx.$$

It is clear that $\mathscr{L}$ maps $C_{-\gamma}$ into $C_{-\gamma}^{-1}$. But according to the above, $\mathscr{L}$ maps $C_{-\nu}$ into $C_{-\nu}^{-1}$ for all $\nu < \gamma$. It is also obvious that $\mathscr{L}$ is a retarded operator.

THEOREM 1. *Suppose* $\mathscr{L}: C \to C^{-1}$ *is invertible. Then there exists* $\nu \in (0, \gamma)$ *such that the operators* $\mathscr{L}: C_\lambda \to C_\lambda^{-1}$ *are invertible for any* $\lambda \in [-\nu, \nu]$ *and, moreover, the inverse operators do not depend on* λ *on the space* $C_\nu^{-1} \cap C_{-\nu}^{-1}$.

Proof. Consider the operator

$$R = U^{-1}\mathscr{L},$$

where $Ux = x' + 2\gamma x$. The operator R acts in $C_{-\gamma}$, it is a retarded operator invertible as an operator in C. Proposition 7 implies that for some $\nu \in (0, \gamma)$ the operator R is invertible on C_λ if $\lambda \in [-\nu, \nu]$. In addition, the inverse operator does not depend on λ on $C_\nu \cap C_{-\nu}$. Hence, the operators $\mathscr{L} = UR: C_\lambda \to C_\lambda^{-1}$ are invertible for the same λ. Moreover, the inverse operators $\mathscr{L}^{-1} = R^{-1}U^{-1}$ coincide on $C_\nu^{-1} \cap C_{-\nu}^{-1}$ into $C_\nu \cap C_{-\nu}$ where R^{-1} does not depend on λ. ▮

4. THEOREM ON DICHOTOMY

In this section we consider the linear neutral functional differential equation

$$\mathscr{L}x \equiv \frac{d}{ds} Dx + Bx = 0, \tag{5}$$

where D and B are linear retarded operators with exponentially fading memory of order $\gamma > 0$.

DEFINITION 3. We say that Eq. (5) (or the operator $\mathscr{L}$) is *well-posed* if for any $t \in \mathbb{R}$, $\tau \in (t, +\infty)$, $\varphi \in C_{-\gamma}(-\infty, t]$, and $f \in C^{-1}$ the initial value problem

$$(\mathscr{L}x)(s) = f(s) \qquad \text{for } t < s < \tau, \tag{6}$$

$$x(s) = \varphi(s) \qquad \text{for } s \leqslant t \tag{7}$$

has a unique solution $x \in C_{-\gamma}(-\infty, \tau]$, and, moreover,

$$\|x\|_{-\gamma, \tau} \leqslant M(\|f\|_{C^{-1}} + \|\varphi\|_{-\gamma, t}),$$

where M depends on $\tau - t$ only. It is easy to see that in this definition the number γ can be replaced by arbitrary $\nu < \gamma$.

The initial value problem (6)–(7) was considered in detail in [38] (see also [18, 22, 26]). We note only that (6) means the equality between two distributions and therefore it is meaningful only on the open interval (t, τ).

In what follows we shall assume that Eq. (5) is well-posed.

DEFINITION 4. Equation (5) is said *to possess a dichotomy in* $C_{-\nu}$ if for every $t \in \mathbb{R}$ the following four conditions are fulfilled.

1D. *The space* $C_{-\nu}(-\infty, t]$ *is decomposable into a direct sum*

$$C_{-\nu}(-\infty, t] = C^{+}_{-\nu}(-\infty, t] \oplus C^{-}_{-\nu}(-\infty, t].$$

For $\varphi \in C_{-\nu}(-\infty, t]$ we subsequently denote the projections of φ into subspaces $C^{+}_{-\nu}(-\infty, t]$ and $C^{-}_{-\nu}(-\infty, t]$ by φ^{+} and φ^{-} and the solutions of the initial value problems

$$\begin{aligned} (\mathscr{L}x)(s) &= 0 \qquad && \text{for } s > t, \\ x(s) &= \varphi^{\pm}(s) \qquad && \text{for } s \leqslant t \end{aligned} \tag{8}$$

by x^{+} and x^{-}. Here and below by a solution of Eq. (8) on $\mathbb{R}$ is meant a function $x: \mathbb{R} \to E$ such that the restriction of x onto $(-\infty, \tau]$ with $\tau > t$ is the solution of problem (8).

2D. *For* $\tau > t$ *the restrictions of the solutions* $x^{\pm}$ *onto* $(-\infty, \tau]$ *lie in* $C^{\pm}_{-\nu}(-\infty, \tau]$, *respectively.*

3D. *Function* φ^{-} *satisfies the equation*

$$(\mathscr{L}x)(s) = 0 \qquad \text{for } s < t$$

and for $\tau < t$ *the restriction of* φ^{-} $(-\infty, \tau]$ *lies in* $C_{-\nu}(-\infty, \tau]$.

4D. *There exists* $N<\infty$ *such that*

$$|x^+(s)| \leqslant Ne^{-\nu(s-t)}\,\|\varphi\|_{-\nu,t} \qquad \textit{for } s \geqslant t,$$
$$|\varphi^-(s)| \leqslant Ne^{\nu(s-t)}\,\|\varphi\|_{-\nu,t} \qquad \textit{for } s \leqslant t,$$

and, moreover, N *does not depend on* t.

The dichotomy is said to be *exponential* if 4D is replaced by the stronger condition:

5D. *There exist* $\varepsilon>0$ *and* $N<\infty$ *such that*

$$|x^+(s)| \leqslant Ne^{-(\nu+\varepsilon)(s-t)}\,\|\varphi\|_{-\nu,t} \qquad \textit{for } s \geqslant t,$$
$$|\varphi^-(s)| \leqslant Ne^{(\nu+\varepsilon)(s-t)}\,\|\varphi\|_{-\nu,t} \qquad \textit{for } s \leqslant t$$

(ε *and* N *do not depend on* t).

Remark 2. By virtue of inequality (2) the dichotomy in $C_{-\nu}$ implies the exponential dichotomy in $C_{-\lambda}$ for $\lambda<\nu$.

Remark 3. The exponential dichotomy is actually interesting only for $\nu=0$. The need for dichotomy in $C_{-\nu}$ for $\nu>0$ becomes apparent in applications to nonlinear equations (see the proofs of Theorems 6 and 7).

Remark 4. Let us define for $-\infty<t<\tau<+\infty$ the operator $V(t,\tau)$: $C_{-\nu}(-\infty,t]\to C_{-\nu}(-\infty,\tau]$ of translation along the trajectories of Eq. (5) by the formula $V(t,\tau)\varphi=x$, where x is the solution of the initial value problem

$$\begin{aligned}(\mathscr{L}x)(s)&=0 \qquad \text{for } t<s<\tau,\\ x(s)&=\varphi(s) \qquad \text{for } s\leqslant t.\end{aligned} \tag{9}$$

Condition 2D in terms of $V(t,\tau)$ means that $V(t,\tau)$ maps the subspaces $C^{\pm}_{-\nu}(-\infty,t]$ into $C^{\pm}_{-\nu}(-\infty,\tau]$, respectively. And condition 3D implies that the operator $V(\tau,t)$ for $\tau<t$ is invertible on the subspace $C^{-}_{-\nu}(-\infty,t]$. Indeed, let $\varphi^-\in C^{-}_{-\nu}(-\infty,t]$ and denote the restriction of φ^- onto $(-\infty,\tau]$ by φ^-_τ. Then $V(\tau,t)\,\varphi^-_\tau=\varphi^-$ as far as φ^- satisfies the homogeneous equation. Thus, $[V(\tau,t)]^{-1}$ on $C_{-\nu}(-\infty,t]$ is the operator of restriction onto $(-\infty,\tau]$.

Remark 5. The bound for φ^- in 4D can be rewritten in the following form

$$\|\varphi^-\|_{\nu,t} \leqslant N\,\|\varphi\|_{-\nu,t},$$

that together with (2) implies

$$\|\varphi^-\|_{-\nu,t} \leqslant N\,\|\varphi\|_{-\nu,t}.$$

The latter inequality means that the projectors onto the subspaces $C^{\pm}_{-\nu}(-\infty, t]$ are bounded uniformly with respect to t.

THEOREM 2. *Suppose D and B are linear retarded operators with exponentially fading memory of order $\gamma > 0$ and the operator $\mathscr{L} = (d/ds)D + B$ is well-posed. Then the following statements are equivalent*:

(i) *Operator $\mathscr{L}: C \to C^{-1}$ is invertible.*

(ii) *Equation* (5) *possesses a dichotomy in $C_{-\nu}$ for sufficiently small $\nu > 0$.*

(iii) *Equation* (5) *possesses an exponential dichotomy in $C_{-\nu}$ for sufficiently small $\nu \geqslant 0$.*

Proof. (ii) $\Rightarrow$ (iii) is obvious.

(i) $\Rightarrow$ (ii). By virtue of Theorem 1 we choose $\nu \in (0, \gamma)$ so that the operators $\mathscr{L}: C_\lambda \to C^{-1}_\lambda$ are invertible for all $\lambda \in [-\nu, \nu]$.

Let $\varphi \in C_{-\nu}(-\infty, t]$ and ψ be an arbitrary continuation of φ to a function in $C_{-\nu}$. Let

$$f = \mathscr{L}\psi, \qquad f^+ = P(-\infty, t) f, \qquad f^- = P(t, +\infty) f,$$
$$x^+ = \mathscr{L}^{-1} f^+, \qquad y^- = \mathscr{L}^{-1} f^-, \tag{10}$$
$$\varphi^+ = x^+ |_{(-\infty, t]}, \qquad \varphi^- = y^- |_{(-\infty, t]}.$$

Since $x^+ + y^- = \psi$, and ψ and φ coincide on $(-\infty, t]$, we have

$$\varphi = \varphi^+ + \varphi^-. \tag{11}$$

We show that the decomposition (11) does not depend on the choice of continuation ψ. Suppose ψ_1 and ψ_2 are two distinct continuations. Then the functions $f_1 = \mathscr{L}\psi_1$ and $f_2 = \mathscr{L}\psi_2$ coincide on $(-\infty, t]$ since $\mathscr{L}$ is a retarded operator. Hence $f_1^+ = f_2^+$, $x_1^+ = x_2^+$ and $\varphi_1^+ = \varphi_2^+$.

Thus we have constructed the decomposition into a direct sum as in 1D.

We note that the function x^+ is a solution of (8) since $\mathscr{L}x^+ = f^+$ and $f^+(s) = 0$ on $(t, +\infty)$. At the same time, y^- is not at all a solution through φ^-.

We now verify condition 2D, i.e., we show that the solutions $x^\pm$ of problem (8) remain in the subspaces $C^{\pm}_{-\nu}(-\infty, \tau]$ for all $\tau > t$. Define the function $\psi \in C_{-\nu}$ on $(-\infty, \tau]$ as the solution of initial value problem (9) and to the right of τ in an arbitrary way. Denote the restriction of ψ onto $(-\infty, \tau]$ by $\varphi_1 (\in C_{-\nu}(-\infty, \tau])$. We show that the restrictions of the solutions $x^\pm$ of (8) to $(-\infty, \tau]$ coincide with $\varphi_1^\pm$. Indeed, without loss of generality, it can be assumed that $\psi = \psi_1$ where ψ_1 is a continuation of φ_1. Since $(\mathscr{L}\psi)(s) = 0$ for $s \in (t, \tau)$, the functions f^+, f^-, x^+, and y^-, and f_1^+,

f_1^-, x_1^+, and y_1^- (constructed for the function φ_1 and initial moment τ according to the scheme (10)) coincide, respectively. And, also, y_1^- is a solution of the homogeneous equation on (t, τ), i.e., it coincides with x^- on $[t, \tau]$. It remains to note that $\varphi^\pm$ and $\varphi_1^\pm$ are the restrictions of the functions x^+, y^- and x_1^+, y_1^- to the corresponding half-line.

We now prove that condition 3D is satisfied. The definition of φ^- implies that φ^- satisfies the homogeneous equation. To prove the second part of 3D, suppose $\varphi \in C^-_{-\nu}(-\infty, t]$ and $\tau < t$. Let φ_1 be the restriction of φ to $(-\infty, \tau]$. Using (10) we construct the function φ^- and φ_1^- for φ, t and φ_1, τ. It is clear that $\varphi^- = \varphi$. Further, one can consider that $\psi = \psi_1$. Note that the function $f = \mathscr{L}\psi$ is equal to zero on $(-\infty, t)$. Otherwise we would have $f^+ \neq 0$ and $x^+ \neq 0$. But x^+ is the solution of problem (8) and, hence, $\varphi^+ \neq 0$. This contradicts the equality $\varphi^- = \varphi$. It is clear now that $f^- = f_1^-$ and, hence, $y^- = y_1^-$. And then φ_1^- coincides with the restriction of φ^- to $(-\infty, \tau]$. Therefore, this restriction lies in $C^-_{-\nu}(-\infty, \tau]$.

To prove condition 4D we choose a continuation ψ of $\varphi \in C_{-\nu}(-\infty, t]$ in such a way that

$$\sup\{|e^{\nu(s-t)}\psi(s)| : s \in \mathbb{R}\} = \|\varphi\|_{-\nu, t}.$$

In this case $\psi \in C_{-\nu}$, $f \in C^{-1}_{-\nu}$, $f^+ \in C^{-1}_{-\nu}$, and $x^+ \in C_{-\nu}$. The latter relation is precisely the first inequality in 4D. Further, $f^- \in C_\nu^{-1} \cap C^{-1}_{-\nu}$ and, hence, y^- satisfies the second inequality. The independence of the constant N can be verified directly.

(iii) $\Rightarrow$ (i). Suppose Eq. (5) possesses an exponential dichotomy in C. Let at first the support of a function $f \in C^{-1}$ be contained in a bounded segment $[t, \tau]$. Our objective is to construct $\mathscr{L}^{-1}f \in C$.

We define the function $\varphi \in C(-\infty, \tau]$ as the solution of the initial value problem

$$(\mathscr{L}x)(s) = f(s) \qquad \text{for } t < s < \tau,$$
$$x(s) = 0 \qquad \text{for } s \leqslant t.$$

According to 1D let $\varphi = \varphi^+ + \varphi^-$, where $\varphi^\pm \in C^\pm_{-\nu}(-\infty, \tau]$ and let x^+ be the solution of the initial value problem

$$(\mathscr{L}x)(s) = 0 \qquad \text{for } s > \tau,$$
$$x(s) = \varphi^+(s) \qquad \text{for } s \leqslant \tau.$$

Recall that x^+ is exponentially decreasing at $+\infty$. Further, by virtue of equality

$$x^+(s) = \varphi(s) - \varphi^-(s) \qquad \text{for } s \leqslant \tau$$

the function x^+ is also exponentially decreasing at $-\infty$. Thus,

$$|x^+(s)| \leqslant K e^{-\varepsilon|s-t|} \|f\|, \tag{12}$$

where $K < \infty$ depends only on $\tau - t$. It remains to note that

$$\mathscr{L} x^+ = f.$$

Indeed, for $s > \tau$ the left-hand and right-hand sides of this equality are equal to zero, and for $s < \tau$

$$(\mathscr{L} x^+)(s) = (\mathscr{L}\varphi)(s) - (\mathscr{L}\varphi^-)(s) = f(s) - 0 = f(s).$$

So, for any $f \in C^{-1}$ with compact support we construct a pre-image x^+ which satisfies the bound (12).

Let now $f \in C^{-1}$ be an arbitary function. We represent it as the sum of the series of functions $f_k = P(k, k+1)f$ and take the pre-images x_k term by term. The exponential bound (12) guarantees the uniform convergence of the series $\sum_{k=-\infty}^{\infty} x_k$ on each compact subset of $\mathbb{R}$ to some function $x \in C$. Evidently, $\mathscr{L} x = f$.

Thus, the image of $\mathscr{L}$ coincides with C^{-1}. We prove now that the kernel of $\mathscr{L}$ consists only of zero. Let $\mathscr{L} x = 0$ and $t \in \mathbb{R}$ be arbitrary. We denote the restriction of x to $(-\infty, t]$ by φ. The projections of φ into $C^{\pm}(-\infty, t]$ are denoted by $\varphi^{\pm}$. And, finally, we denote the solutions of problem (8) by $x^{\pm}$.

Suppose $\tau > t$ and $\psi^{\pm}$ are the restrictions of $x^{\pm}$ to $(-\infty, \tau]$. By virtue of condition 2D the functions $\psi^{\pm}$ lie in $C^{\pm}(-\infty, \tau]$. But then the function $\psi = \psi^+ + \psi^-$ is the solution of (9) since $\mathscr{L} x = 0$ and, hence, ψ coincides with the restriction of x to $(-\infty, \tau]$. These arguments show that $x^{\pm}$ do not depend on the choice of t.

We rewrite the bounds in 5D in the form

$$\sup\{|e^{\varepsilon(s-t)} x^+(s)| : s \geqslant t\} \leqslant N \|x\|_C,$$
$$\sup\{|e^{-\varepsilon(s-t)} x^-(s)| : s \leqslant t\} \leqslant N \|x\|_C.$$

They imply that $x^+ = 0$ and $x^- = 0$ since t is arbitrary. But $x^+ + x^- = x$ and, hence, $x = 0$. Thus, the kernel of $\mathscr{L}$ consists only of zero. This completes the proof. ∎

DEFINITION 5. Equation (5) is said to be *stable in* $C_{-\nu}$ if there exists $N < \infty$ such that for any $t \in \mathbb{R}$ and $\varphi \in C_{-\nu}(-\infty, t]$ the solution of the problem

$$\begin{aligned} (\mathscr{L} x)(s) &= 0 \qquad \text{for } s > t, \\ x(s) &= \varphi(s) \qquad \text{for } s \leqslant t \end{aligned} \tag{13}$$

satisfies the estimate

$$|x(s)| \leqslant Ne^{-\nu(s-t)} \|\varphi\|_{-\nu,t} \qquad \text{for } s \geqslant t.$$

This equation is said to be *exponentially stable in* $C_{-\nu}$ if for some $\varepsilon > 0$

$$|x(s)| \leqslant Ne^{-(\nu+\varepsilon)(s-t)} \|\varphi\|_{-\nu,t} \qquad \text{for } s \geqslant t.$$

By virtue of (2) stability in $C_{-\nu}$ implies exponential stability in $C_{-\lambda}$ for $\lambda < \nu$.

THEOREM 3. *Suppose the conditions of Theorem 2 are satisfied. Then exponential stability of Eq. (5) in C (i.e., in C_0) implies exponential stability in $C_{-\nu}$ for sufficiently small ν.*

Proof. Obviously, exponential stability in C_0 is dichotomy in $C_{-\nu}$ for $\nu = 0$ with $C^{-}_{-\nu}(-\infty, t] = \{0\}$. Therefore, exponential stability implies invertibility of the operator $\mathscr{L}: C \to C^{-1}$, which in turn implies dichotomy in $C_{-\nu}$ for some $\nu > 0$. But if in this dichotomy the subspace $C^{-}_{-\nu}(-\infty, t]$ is not zero, then the homogeneous equation has by virtue of the second estimate of 4D and condition 3D, an exponentially increasing solution. This is a contradiction. ▮

5. NONAUTONOMOUS DYNAMICAL SYSTEMS

In this section we formulate the theorems on stability and instability in the first approximation for operator equations in a Banach space. These theorems are elementary corollaries of Theorems 3 and 4 from [4].

So, suppose $\{E_i\}_{i=1}^{\infty}$ is an arbitrary sequence of Banach spaces and $f_i: E_i \to E_{i+1}$ is an arbitrary mapping for each $i = 1, 2, \ldots$. Let $F_k = f_k \circ f_{k-1} \circ \cdots \circ f_1$. It is clear that F_k acts from E_1 into E_{k+1}. We shall call the sequence $\{F_k\}$ the *nonautonomous dynamical system* (*NDS*) *generated by the sequence of operators* $\{f_i\}$. Further, assume that $f_i(0) = 0$ for every i. Then also $F_k(0) = 0$ for every k.

DEFINITION 6. Zero is said to be *exponentially stable with respect to the NDS* $\{F_k\}$ if there exist $q < 1$ and $r_0 > 0$ such that

$$\|F_k(x)\| \leqslant q^k \|x\|$$

for all k, provided $\|x\| \leqslant r_0$.

DEFINITION 7. Zero is said to be *unstable with respect to the NDS* $\{F_k\}$

if there exist a constant $L>0$, a sequence $\{x_m\} \subset E_1$, and a sequence of natural numbers $\{k_m\}$ such that

$$\lim_{m\to\infty} x_m = 0 \qquad \text{and} \qquad \|F_{k_m}(x_m)\| \geqslant L.$$

We formulate tests for exponential stability and instability expressed in terms of Fréchet derivatives $f_i'(0)$ of the operators f_i at zero. Therefore, it is assumed that the operators f_i are Fréchet differentiable at zero in the following uniform sence: for all natural numbers i and for all $x \in E_i$ with $\|x\| \leqslant r_0$

$$\|f_i(x) - f_i'(0)x\| \leqslant p(\|x\|)\,\|x\|,$$

where $p(r) \to 0$ as $r \to 0$. Evidently, one can assume without loss of generality that the function p is nondecreasing.

THEOREM 4. *If $\|f_i'(0)\| \leqslant q_0 < 1$ for all natural i, then zero is exponentially stable with respect to the NDS $\{F_k\}$ and, moreover, the numbers q and r_0 from Definition 6 depend on q_0 and the function p only.*

THEOREM 5. *Suppose that, for each i, the space E_i is decomposable into a direct sum of subspaces E_i^+ and E_i^- such that the following conditions are satisfied:*

(1) *$E_i^- \neq \{0\}$ and the corresponding projectors are uniformly bounded with respect to i.*

(2) $f_i'(0)\,E_i^+ \subseteq E_{i+1}^+$, $f_i'(0)\,E_i^- \subseteq E_{i+1}^-$ $(i = 1, 2, ...)$.

(3) *There exist constants $q_1 < 1$ and $q_2 < 1$ such that the following inequalities hold for every natural number i: $\|f_i'(0)x\| \leqslant q_1\|x\|$ for all $x \in E_i^+$, and $\|f_i'(0)x\| \geqslant q_2\|x\|$ for all $x \in E_i^-$.*

Then zero is unstable with respect to the NDS $\{F_k\}$, and moreover, one can choose a sequence $\{x_m\}$ from Definition 7 belonging to E_1^-.

6. Stability in the First Approximation for Neutral Type Equations

In this section we consider the following neutral functional differential equation with unbounded delay

$$\frac{d}{ds}\,[Dx + d(x)] + Bx + b(x) = 0. \tag{14}$$

We assume that the linear part of Eq. (14)

$$\mathscr{L}x \equiv \frac{d}{ds} Dx + Bx = 0$$

satisfies the conditions of Theorem 3, and that the nonlinear mappings d and b satisfy the following four conditions.

1N. *The operators d and b are operators with exponentially fading memory of order γ, i.e., they map $C_{-\gamma}$ into $C_{-\gamma}$ and they are continuous and bounded.* Boundedness means the existence of $M < \infty$ such that the inequalities $\|d(x)\|_{-\gamma} \leqslant M \|x\|_{-\gamma}$, $\|b(x)\|_{-\gamma} \leqslant M \|x\|_{-\gamma}$ hold for every $x \in C_{-\gamma}$.

In particular, $d(0) = b(0) = 0$.

2N. *The operators d and b are retarded in the sense of* (3).

These two conditions allow us to construct, as in the linear case, corresponding operators d and b that act in $C_{-\gamma}(-\infty, t]$, $C_{-\nu}(-\infty, t]$, and $C_{-\nu}$ for all $\nu < \gamma$.

Let

$$\mathfrak{L}x = \frac{d}{ds} [Dx + d(x)] + Bx + b(x).$$

By virtue of the previous paragraph the operator $\mathfrak{L}$ is defined both on functions in $C_{-\gamma}$ and on functions in $C_{-\nu}$ for $\nu < \gamma$. The values of $\mathfrak{L}$ lie in $C_{-\gamma}^{-1}$ and $C_{-\nu}^{-1}$, respectively.

3N. *For every $t \in \mathbb{R}$ and $\varphi \in C_{-\gamma}(-\infty, t]$ the initial value problem*

$$\begin{aligned} (\mathfrak{L}x)(s) &= 0 \qquad \textit{for } s > t, \\ x(s) &= \varphi(s) \qquad \textit{for } s \leqslant t \end{aligned} \tag{15}$$

has a unique solution. By a solution of problem (15) is meant, as in the linear case, a function $x\colon \mathbb{R} \to E$ whose restriction to any interval $(-\infty, \tau]$ with $\tau > t$ lies in $C_{-\gamma}(-\infty, \tau]$ and that satisfies (15).

We denote the solution of problem (15) by $x(\cdot, t, \varphi)$ and the solution of problem (13) by $x^{\mathscr{L}}(\cdot, t, \varphi)$.

4N. *The derivatives of d and b at zero are zero operators in the following sense*:

$$|x(s, t, \varphi) - x^{\mathscr{L}}(s, t, \varphi)| \leqslant \varepsilon(s - t, \|\varphi\|_{-\gamma, t}) \, \|\varphi\|_{-\gamma, t},$$

where $\varepsilon(\cdot, r) \to 0$ as $r \to 0$ uniformly on every bounded segment.

It is easy to see that the function ε in 4N may be assumed nondecreasing with respect to both arguments.

By virtue of (2), it is clear that in condition 4N $\|\cdot\|_{-\gamma,t}$ can be replaced by $\|\cdot\|_{-\nu,t}$ for any $\nu<\gamma$. In what follows we assume that $\nu<\gamma$.

Remark 6. Conditions 3N and 4N represent conditions of global existence, uniqueness, and differentiability of the solutions of problem (15) with respect to initial data. References to various concrete and general realizations of these conditions can be found in the book of Hale [22] and in the survey of Corduneanu and Lakshmikantham [10].

Let us define the operators $W(t,\tau)$, $V(t,\tau)$: $C_{-\gamma}(-\infty,t]\to C_{-\gamma}(-\infty,\tau]$ $(\tau>t)$ of translation along the trajectories of (14) and (5) by the formulae

$$[W(t,\tau)(\varphi)](s)=x(s,t,\varphi) \qquad (s\in(-\infty,\tau]),$$

$$[V(t,\tau)\varphi](s)=x^{\mathscr{L}}(s,t,\varphi) \qquad (s\in(-\infty,\tau]),$$

respectively. Clearly, $W(t,\tau)$ and $V(t,\tau)$ map $C_{-\nu}(-\infty,t]$ into $C_{-\nu}(-\infty,\tau]$ too.

Note that condition 4N implies uniform differentiability of $W(t,\tau)$ at zero:

$$\|W(t,\tau)(\varphi)-V(t,\tau)\varphi\|_{-\gamma,\tau}\leqslant\varepsilon(s-t,\|\varphi\|_{-\gamma,t})\,\|\varphi\|_{-\gamma,t};$$

thus, $[W(t,\tau)]'(0)=V(t,\tau)$. One can replace $\|\cdot\|_{-\gamma,t}$ by $\|\cdot\|_{-\nu,t}$ in this inequality.

DEFINITION 8. Zero of Eq. (14) is said to be *stable in* $C_{-\nu}$ if there exist $N<\infty$ and $r>0$ such that for every $t\in\mathbb{R}$, $s\geqslant t$, and $\varphi\in C_{-\nu}(-\infty,t]$

$$|x(s,t,\varphi)|<Ne^{-\nu(s-t)}\,\|\varphi\|_{-\nu,t},$$

provided $\|\varphi\|_{-\nu,t}\leqslant r$. Zero is called *exponentially stable in* $C_{-\nu}$ if this estimate is replaced by

$$|x(s,t,\varphi)|\leqslant Ne^{-(\nu+\varepsilon)(s-t)}\,\|\varphi\|_{-\nu,t}$$

for some $\varepsilon>0$.

It is easy to see that stability in $C_{-\nu}$ implies exponential stability in $C_{-\lambda}$ for $\lambda<\nu$.

THEOREM 6. *Assume that the operator* $\mathscr{L}$ *satisfies conditions of Theorem* 2, *the conditions* 1N–4N *are fulfilled, and Eq.* (5) *is exponentially stable in C. Then the trivial solution of Eq.* (14) *is exponentially stable in* $C_{-\nu}$ *for some* $\nu>0$.

Proof. According to Theorem 3 there exists $\nu_0 > 0$ such that the estimate

$$|x^{\mathscr{L}}(s, t, \varphi)| \leqslant N_\nu e^{-\nu(s-t)} \|\varphi\|_{-\nu, t} \qquad (s \geqslant t)$$

holds for all $\nu \in [0, \nu_0]$. Besides, since $\varphi \in C_{-\nu}(-\infty, t]$, one can assume that the same estimate holds on the left half-line:

$$|x^{\mathscr{L}}(s, t, \varphi)| \leqslant N_\nu e^{-\nu(s-t)} \|\varphi\|_{-\nu, t} \qquad (s \leqslant t).$$

Fix now $\varepsilon > 0$ and choose positive numbers ν, l, q, T such that $\nu < \nu_0$, $l < q < 1$, $N_\nu e^{-\nu T} < l$, and $-\log q \geqslant 2T(\nu + \varepsilon)$.

The above estimates imply the bound

$$|x^{\mathscr{L}}(s, t, \varphi)| \leqslant l e^{-\nu[s-(t+T)]} \|\varphi\|_{-\nu, t} \qquad (s \in \mathbb{R}).$$

In terms of the translation operator $V(t, \tau)$ this inequality means that

$$\|V(t, t+T)\| \leqslant l$$

for all $t \in \mathbb{R}$.

We fix $t \in \mathbb{R}$ and let $t_i = t + (i-1)T$, $E_i = C_{-\nu}(-\infty, t_i]$. Let $f_i\colon E_i \to E_{i+1}$ be defined by the formula

$$f_i = W(t_i, t_{i+1}) \qquad (i = 1, 2, ...).$$

It is easy to see that the thus defined NDS satisfies conditions of Theorem 4. This theorem asserts the existence of $r_0 > 0$ such that

$$\|W(t, t_{k+1})(\varphi)\|_{-\nu, t_k} \leqslant q^k \|\varphi\|_{-\nu, t} \tag{16}$$

for every $\varphi \in C_{-\nu}(-\infty, t]$, $\|\varphi\|_{-\nu, t} \leqslant r_0$.

Further, condition 4N and well-posedness of problem (6)–(7) imply the estimate

$$\|W(\xi, \eta)(\varphi)\|_{-\nu, \eta} \leqslant [M + \varepsilon(T, r_0)] \|\varphi\|_{-\nu, \xi} \equiv K \|\varphi\|_{-\nu, \xi} \tag{17}$$

for all $\xi \leqslant \eta \leqslant \xi + T$, $\varphi \in C_{-\nu}(-\infty, \xi]$, $\|\varphi\|_{-\nu, \xi} \leqslant r_0$. Fix an arbitrary $s \geqslant t$ and denote $[(s-t)/T]$ by k, where $[\cdot]$ denotes the greatest-integer function. Inequalities (16) and (17) imply that

$$\begin{aligned} \|W(t, s)(\varphi)\|_{-\nu, s} &= \|W(t_{k+1}, s) \circ W(t, t_{k+1})(\varphi)\|_{-\nu, s} \\ &\leqslant K \|W(t, t_{k+1})(\varphi)\|_{-\nu, t_{k+1}} \leqslant K q^k \|\varphi\|_{-\nu, t}. \end{aligned}$$

It remains to note that

$$Kq^k \leqslant Ne^{-(\nu+\varepsilon)(s-t)}$$

and the theorem is proved. ∎

DEFINITION 9. We shall call the trivial solution of Eq. (14) *unstable in* $C_{-\nu}$ if for every $t \in \mathbb{R}$ there exist a constant $L > 0$, a sequence $\{\varphi_m\} \subset C_{-\nu}(-\infty, t]$, and a sequence $\{s_m\} \subset [t, +\infty)$ such that

$$\lim_{m \to \infty} \|\varphi_m\|_{-\nu, t} = 0$$

and

$$|x(s_m, t, \varphi_m)| \geqslant L.$$

It is easy to see that instability in $C_{-\lambda}$ implies instability in $C_{-\nu}$ for $\lambda < \nu$.

THEOREM 7. *Assume that the operator* $\mathscr{L}$ *satisfies the conditions of Theorem* 2, *conditions* 1N–4N *are fulfilled, the operator* $\mathscr{L}: C \to C^{-1}$ *is invertible, and Eq.* (5) *is unstable in C. Then the trivial solution of Eq.* (14) *is unstable in* C_ν *for some* $\nu > 0$.

It should be emphasized that instability is asserted in C_ν and not in $C_{-\nu}$.

Proof. According to the assumptions Eq. (5) possesses an exponential dichotomy in $C_{-\nu}$ for some $\nu > 0$. Note that the subspaces $C^-_{-\nu}(-\infty, t]$ are nontrivial since otherwise Eq. (5) would be stable.

Let us choose $T > 0$ such that

$$Ne^{-\nu T} = q_1 < 1 \qquad \text{and} \qquad N^{-1}e^{\nu T} = q_2 > 1.$$

Just as in the proof of Theorem 6 the first inequality implies the estimate

$$\|V(t, t+T)\,\varphi^+\|_{-\nu, t+T} \leqslant q_1 \|\varphi^+\|_{-\nu, t}$$

for all $t \in \mathbb{R}$ and $\varphi \in C_{-\nu}(-\infty, t]$. We prove now that the second inequality implies the estimate

$$\|V(t, t+T)\,\varphi^-\|_{-\nu, t+T} \geqslant q_2 \|\varphi^-\|_{-\nu, t}$$

for all $t \in \mathbb{R}$ and $\varphi \in C_{-\nu}(-\infty, t]$.

Let $x = x^{\mathscr{L}}(\cdot, t, \varphi)$ and substitute x^- for φ and $t + T$ for t in the second inequality in 4D,

$$|x^-(s)| \leqslant Ne^{\nu[s-(t+T)]} \|x^-\|_{-\nu, t+T}$$

or

$$|e^{-\nu(s-t)}x^{-}(s)| \leqslant Ne^{-\nu T}\|x^{-}\|_{-\nu,t+T},$$

and thus

$$\begin{aligned} q_2\|\varphi^{-}\|_{-\nu,t} &\leqslant q_2\|\varphi^{-}\|_{\nu,t} \\ &\leqslant \|x^{-}\|_{-\nu,t+T} = \|V(t,t+T)\,\varphi^{-}\|_{-\nu,t+T}. \end{aligned}$$

Fix $t \in \mathbb{R}$ and as above set $t_i = t + (i-1)T$, $E_i = C_{-\nu}(-\infty, t_i]$, $E_i^{\pm} = C_{-\nu}^{\pm}(-\infty, t_i]$, and

$$f_i = W(t_i, t_{i+1}).$$

It is evident that the NDS defined by the sequence $\{f_i\}$ satisfies the conditions of Theorem 5, and, therefore, there exist a constant $L > 0$, a sequence $\{\varphi_m\} \subset C_{-\nu}(-\infty, t]$, and a sequence of natural numbers $\{k_m\}$ such that

$$\lim_{m\to\infty} \|\varphi_m\|_{-\nu,t} = 0 \tag{18}$$

and

$$\|W(t, t_{k_m})(\varphi_m)\|_{-\nu,t_{k_m}} \geqslant L.$$

Now, in the first place, the second inequality in 4D implies that $\varphi_m \in C_\nu(-\infty, t]$ and

$$\lim_{m\to\infty} \|\varphi_m\|_{\nu,t} = 0.$$

And, in the second place (we denote t_{k_m} by $t(m)$),

$$\begin{aligned} L &\leqslant \|W(t, t(m))(\varphi_m)\|_{-\nu,t(m)} \\ &= \sup\{|e^{\nu[s-t(m)]}x(s,t,\varphi_m)| : s \leqslant t(m)\} \\ &= \max\{\sup\{|e^{\nu[s-t(m)]}x(s,t,\varphi_m)| : s \leqslant t\}, \\ &\quad \sup\{|e^{\nu[s-t(m)]}x(s,t,\varphi_m)| : t \leqslant s \leqslant t(m)\}\}. \end{aligned}$$

The first term on the right-hand side under the max sign tends to zero as $m \to \infty$ because of (18), and, hence, without loss of generality one can assume that

$$\sup\{|e^{\nu[s-t(m)]}x(s,t,\varphi_m)| : t \leqslant s \leqslant t(m)\} \geqslant L.$$

This implies the existence of $s_m \geqslant t$ such that

$$|x(s_m, t, \varphi_m)| \geqslant L$$

and the proof is complete. ▮

Remark 7. Some conditions for the invertibility of the operator $\mathscr{L}$ can be obtained in the case of constant coefficients with the help of the results from [28, 40, 51].

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