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The Limiting Equations of Nonautonomous Ordinary Differential Equations

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1. INTRODUCTION

The limiting equations of the nonautonomous ordinary differentiable equation $\dot{x} = f(x, s)$ are limit points, as $|t| \to \infty$, of the translated equations $\dot{x} = f^t(x, s)$, where f^t is defined by $f^t(x, s) = f(x, t + s)$, i.e., a translation in the time variable. The limit is taken in a prespecified space, and has to obey certain properties which will be discussed later.

The general motivation for introducing the limiting equations is that there is a tight connection between the asymptotic behavior of the solutions of the original equation and the solutions of the limiting equations. This was demonstrated already by Markus [13] who considered an autonomous equation $\dot{x} = f(x)$ as the unique limiting equation of a nonautonomous perturbation $\dot{x} = f(x) + g(x, s)$, where g^t vanishes (in a certain sense) as $t \to \infty$. Miller [14] used the dynamics generated by the translations f^t and the limiting equations to investigate almost periodic systems. Sell [17] established the fundamental theory of the limiting equations within the framework of topological dynamics. For a more complete description of the evolution of the concept and the various applications see Sell [18], Miller and Sell [15], and LaSalle [10].

The usual approach was to embed the translates f^t of the function f in a certain function space, and then to identify a suitable convergence. For instance, uniform convergence on compact sets was used by Miller [14] and Sell [17]; a certain weak convergence was used by Wakeman [21]. In all these cases the limiting equations were always ordinary differential equations. It was observed in [3] that the consequences and applications of the theory hold even if we allow the limiting equations to be "unordinary," i.e., not ordinary, equations.

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The idea is to embed the translates $\dot{x} = f^t(x, s)$ in a space of equations, not necessarily ode's, and then under the appropriate convergence to show that the asymptotic behavior of solutions of the equation $\dot{x} = f(x, s)$ is governed by the limiting equations. This was done in [3] where under certain conditions a space of Kurzweil equations served the embedding.

The purpose of the present paper is to find a general form of limiting equations of nonautonomous ordinary differential equations. The assumptions on the equation will be weak enough to include many examples of ordinary and unordinary equations, but strong enough so that the dynamics of the translates and the solutions can be developed.

The assumptions placed on the nonautonomous ordinary differential equation are given in Section 2. The general form of the limiting equations will be an ordinary integral-like operator equation, an object which will be defined and discussed in Section 3. Two special cases of such equations will be mentioned in the examples of Section 10. The basic property that the limiting equation has to satisfy is the continuous dependence of the solutions when this limit is taken. Sections 4 and 5 are devoted to the continuous dependence. In Section 6 we formally give the definitions of the translated equations, limiting equations, and the hull of the equation, together with some explanatory remarks.

Section 7 presents one relation between the solutions of the original ordinary differential equation and those of the limiting equations. It is the invariance property. We give one application of it to stability theory. In Section 8 we give sufficient conditions for the precompactness of the set of translates, i.e., for the property that each sequence of translates has a subsequence which converges to a limiting equation.

The limiting equations are not in general ode's. But we shall find that if the equation is asymptotically autonomous, the unique limiting equation is necessarily an autonomous ordinary differential equation, and the original equation has the form $\dot{x} = h(x) + g(x, s)$, where g^t converges to zero. This is the main result of Section 9. (Asymptotically autonomous equations were discussed in [13, 20].) This conclusion is already untrue if the equation is asymptotically periodic, as the example in Section 10 shows.

In the last section we turn to the formal construction of a local flow, as was done by Sell [17, 19], show how to build it under our conditions and point out some differences.

2. The Assumptions

Let $f(x, s)$ be the right-hand side of $\dot{x} = f(x, s)$. We assume that f is continuous in $x \in R^n$ and measurable in $s \in R$, and satisfies locally the Carathéodory condition (i.e., for x in bounded sets $|f(x, s)| \leqslant m(s)$ with m locally integrable). We shall make an assumption on the equicontinuity of the primitive of f.

For this let μ: $[0, \infty) \to [0, \infty]$ be nondecreasing, continuous at 0 with $\mu(0) = 0$. We say that the function γ: $(a, b) \to R^n$ admits μ as a modulus of continuity if $|\gamma(s) - \gamma(t)| \leqslant \mu(|s - t|)$ for every s, t in $[a, b]$ (see [4, p. 72]).

ASSUMPTION (A). For every compact $K \subset R^n$ there is a nondecreasing function μ_K: $[0, \infty) \to [0, \infty]$, continuous at 0 with $\mu_K(0) = 0$, so that whenever u: $[a, b] \to K$ is continuous the primitive $F(t) = \int_a^t f(u(s), s)\, ds$ is defined and admits μ_K as a modulus of continuity.

The assumption is not as restrictive as its length might suggest. If $f(x, s)$ is bounded for x in compact sets then f certainly satisfies Assumption (A). A sufficient condition then is that $|f(x, s)| \leqslant m_K(s)$ for $x \in K$ compact, where m_K is locally integrable and has a uniformly continuous primitive. If $f(x, s)$ oscillates in s fast enough then the assumption might be satisfied while $|f(x, s)| \to \infty$ when $s \to \infty$, as the following example (borrowed from [20]) shows: $f(x, \tau) = (\tau \sin \tau^3, \tau \cos \tau^3)$. The reason for this type of hypothesis is basically to obtain the last conclusion in the following result.

PROPOSITION 2.1. *Under Assumption* (A), *for each* $(t_0, x_0) \in R \times R^n$ *there is a solution* $\phi(s)$ *of the initial-value problem* $\dot{x} = f(x, s)$, $x(t_0) = x_0$, *which is defined on a maximal interval* (α, ω). *Also* $|\phi(s)| \to \infty$ *as* $s \to \alpha^+$ (*resp.* $s \to \omega^-$) *if* $\alpha > -\infty$ (*resp. if* $\omega < \infty$). *Moreover, every solution* ϕ *of* $\dot{x} = f(x, s)$ *admits* μ_K *as a modulus of continuity on any interval* $[a, b]$ *where* ϕ: $[a, b] \to K$.

Proof. For a proof of the first two statements compare [6, Theorems 5.1, 5.2]. The last statement follows immediately from the observation that if ϕ is a solution then $\phi(t) = \phi(a) + \int_a^t f(\phi(s), s)\, ds$.

3. ORDINARY INTEGRAL-LIKE OPERATOR EQUATIONS

In [16] Neustadt developed the fundamental theory of integral-like operator equations. We shall be interested in a special class of these equations, namely, those which might arise as limiting equations of ordinary differential equations. We shall give the formal definition of our ordinary integral-like operator equations below, but the motivation is fairly simple and will be described now. We recall that the ode $\dot{x} = h(x, s)$ is equivalent to the integral equation $x(t) = x(a) + \int_a^t h(x(s), s)\, ds$. The integral part can be viewed as an operator H which maps the function ϕ into the function $H_a\phi(t) = \int_a^t h(\phi(s), s)\, ds$, and the problem is to solve the functional equation $u = u(a) + H_a u$. The ordinary integral-like operator equations will have the same structure as these functional equations but without assuming the representation as integrals with a kernel.

DEFINITION 3.1. An ordinary integral-like operator H is a mapping which associates with each R^n-valued continuous function ϕ and a in the domain of ϕ, a continuous function $H_a\phi$ so that (1) H_a: $C[a, b] \to C[a, b]$ is continuous for each $[a, b]$; (2) $H_a\phi(t) = H_a\phi(s) + H_s\phi(t)$ for all a, s, t in the domain of ϕ.

We say that H is consistent with Assumption (A) if whenever ϕ: $[a, b] \to K \subset R^n$, K compact, then $H_a\phi$ admits μ_K as a modulus of continuity.

It is quite clear what equation and concept of a solution should be associated with the ordinary integral-like operator H. We shall denote the equation by $u = Hu$ (notice that Hu is not defined, only $H_a u$ is defined). A solution is a continuous function ϕ such that $\phi = \phi(b) + H_b\phi$ for any b in its domain. A *maximally defined* solution is one which cannot be extended to a solution on a strictly larger domain. The initial value problem $u = Hu$, $u(b) = z$ can be equivalently written as $u = z + H_b u$. The following result is analogous to Proposition 2.1.

PROPOSITION 3.2. *Let H be an ordinary integral-like operator which is consistent with Assumption* (A). *Then for each* $(t_0, x_0) \in R \times R^n$ *there is a maximally defined solution ϕ, defined on (α, ω) of the initial value problem* $u = x_0 + H_{t_0}u$. *Also* $|\phi(s)| \to \infty$ *as* $s \to \alpha^+$ (*resp.* $s \to \omega^-$) *if* $\alpha > -\infty$ (*resp.* $\omega < \infty$). *Moreover any solution ϕ of $u = Hu$ admits μ_K as a modulus of continuity on every interval* $[a, b]$ *where* ϕ: $[a, b] \to K$.

Proof. We again refer the reader to [6, Chap. 5] where the result is proved for ordinary equations but by using only properties of the corresponding integral equation. The last statement follows immediately from the equality $\phi = \phi(a) + H_a\phi$ and that H is consistent with Assumption (A).

It should be clear that the concept of ordinary integral-like operator equations is an extension of an ordinary differential equation. Every ode can be viewed as an operator equation but not every ordinary integral-like operator equation can be represented as an ode. We shall encounter some examples in Section 10 below, but it is worth noting that any continuous function ψ is a solution of a certain ordinary integral-like operator equation. For instance the equation determined by the constant valued operator defined by $H_a\phi(t) = \psi(t) - \psi(a)$.

4. A PREPARATION FOR THE CONTINUOUS DEPENDENCE

Our assumptions allow escape of the solution in a finite time, i.e., maximally defined solutions with a bounded domain. Different solutions might have different domains. We shall need to consider the convergence of a sequence of solutions in order to establish the continuous dependence, and it will be convenient to embed all the solutions in one metric space.

The family of functions which we define below will contain all the candidates for solutions of $u = x + H_0 u$ where H is consistent with Assumption (A), including the solutions of the original ordinary differential equation. Compare with Propositions 2.1 and 3.2. The modulus of continuity μ used in the following definition is the one provided by Assumption (A).

DEFINITION 4.1. Let Γ be the collection of all continuous noncontinuable functions $\gamma: (\alpha(\gamma), \omega(\gamma)) \to R^n$, so that $0 \in (\alpha(\gamma), \omega(\gamma))$ and such that γ admits μ_K as a modulus of continuity on each interval $[a, b]$ where $\gamma: [a, b] \to K$. In particular this implies that $|\gamma(s)| \to \infty$ as $s \to \alpha(\gamma)^+$ (resp. $s \to \omega(\gamma)^-$) if $\alpha(\gamma) > -\infty$ (resp. $\omega(\gamma) < \infty$). A convergence is defined on Γ as follows. The sequence γ_k converges to γ_0 if $\gamma_k(t) \to \gamma_0(t)$ uniformly on each compact subinterval of $(\alpha(\gamma_0), \omega(\gamma_0))$. In particular, $\alpha(\gamma_0) \geqslant \limsup \alpha(\gamma_k)$ and $\omega(\gamma_0) \leqslant \liminf \omega(\gamma_k)$.

Remark. The type of convergence defined above is the one used in the continuous dependence results needed in the applications of topological dynamics. See [3, 10, 15, 18] and also Section 11 below.

PROPOSITION 4.2. *The space Γ can be made into a complete metric space so that the convergence in the metric coincides with the convergence given in Definition 4.1. Also if $K \subset R^n$ is compact then the set $\{\gamma \in \Gamma: \gamma(0) \in K\}$ is compact in Γ.*

Proof. Notice that the convergence is actually the compact open convergence and the only modification that has to be made is taking into account the different domains. Let m be a positive integer. For each $\gamma \in \Gamma$ let $t_m(\gamma) = \sup\{s < 0: |\gamma(s)| > m\}$ and $T_m(\gamma) = \inf\{s > 0: |\gamma(s)| > m\}$, ($\sup \phi = -\infty$ and $\inf \phi = \infty$). Let γ_m be defined on all R by $\gamma_m(s) = \gamma(s)$ if $s \in [t_m(\gamma), T_m(\gamma)]$, $\gamma_m(s) = \gamma(t_m(\gamma))$ if $s < t_m(\gamma)$ and $\gamma_m(s) = \gamma(T_m(\gamma))$ if $s > T_m(\gamma)$. Let d_m be the semimetric

$$d_m(\gamma, \delta) = \sup\{|\gamma_m(s) - \delta_m(s)|: -\infty < s < \infty\}.$$

Notice that being close in d_m means that γ and δ are close at least until one gets out of the ball with radius m. Define now a metric by

$$d(\gamma, \delta) = \sum_{m=1}^{\infty} (1/m2^m)\, d_m(\gamma, \delta).$$

Then d is the desired metric. The completeness and compactness are immediate consequences of the equicontinuity hypothesis.

5. CONVERGENCE OF EQUATIONS AND CONTINUOUS DEPENDENCE

All the ordinary integral-like operators are assumed to be consistent with Assumption (A); i.e., if $\phi: [a, b] \to K \subset R^n$ then $H_a\phi$ admits μ_K as a modulus of continuity where μ_K is provided by Assumption (A). We denote the collection of all these operators by $\mathscr{H}$.

DEFINITION 5.1. The sequence $H^{(1)}, H^{(2)},\ldots,$ of ordinary integral-like operators converges to H if whenever $\phi_k: [a, b] \to R^n$ is a sequence of continuous functions which converge uniformly to ϕ then $H_a^{(j)}\phi_j$ converge uniformly to $H_a\phi$. (The equicontinuity assumption implies that pointwise convergence $H_a^{(j)}\phi(t) \to H_a\phi(t)$ implies uniform convergence.)

Remark 5.2. The converging sequences in the space coincide with the converging *sequences* of the compact–open topology on $\mathscr{H}$. The convergence of Definition 5.1 can be easily generalized to convergence of nets (generalized sequences) but then the convergence would not be the convergence in the compact–open topology, and as a matter of fact will not be generated by a topology at all. See [1, Appendix B]. However, with the convergence given in Definition 5.1 $\mathscr{H}$ becomes a convergence space ($\mathscr{L}^*$ space in the terminology of [7]) and the convergence is the continuous convergence, see [7, p. 197].

THEOREM 5.3. *Suppose that $H^{(1)}, H^{(2)},\ldots,$ converge to H, and let $z_k \to z$ in R^n. Let γ_k be a maximally defined solution of $u = z_k + H_0^{(k)}u$. Then a subsequence of γ_k exists which converges in Γ to a solution of $u = z + H_0u$.*

Proof. The sequence $\gamma_k(0) = z_k$ is bounded. By the compactness property of Γ (see Proposition 4.2) a converging subsequence, say $\gamma_m \to \gamma$, exists. The convergence $\gamma_m \to \gamma$ is uniform on compact subintervals $[0, t]$ or $[t, 0]$ of the domain of γ. On these intervals $H_0^{(m)}\gamma_m$ converge to $H_0\gamma$. But the equality $\gamma_m - z_m = H_0^{(m)}\gamma_m$ shows that the latter converge also to $\gamma - z$. So $H_0\gamma = \gamma - z$ and γ is the desired solution.

Remarks. The continuous dependence could be stated more elegantly in terms of the set $s(z, H)$ of solutions of $u = z + H_0u$. The results simply say that this set-valued function is upper semicontinuous (see [2]). It is also easy to extend Theorem 5.2 and to prove also continuous dependence with respect to a in the equation $u = z + H_au$ (in the theorem $a = 0$ was fixed). The type of continuous dependence presented in Theorem 5.3 is analogous to the Kamke lemma for ode's (see [17]).

The convergence given in Definition 5.1 is quite relaxed, but still not a necessary condition for the continuous dependence. For instance, let h_k be defined on $R \times R$ by $h_k(x, s) = 0$ if $| x - s - 1 | > 1/k$; $h_k(x, x - 1) = \frac{1}{2}$, and let $0 \leqslant h_k(x, s) \leqslant \frac{1}{2}$ be a continuous extension on the rest of $R \times R$. Then any sequence of solutions of $\dot{x} = h_k(x, s)$, $x(0) = z_k$, where $z_k \to z$. converges to the constant function $z(s) = z$. However, the sequence h_k does not integrally converge to $h = 0$. The convergence is relaxed enough to be a necessary condition for the continuous dependence of solutions of $u(t) =$

$z(t) + H_0u(t)$, where we demand the dependence also on the continuous forcing term $z(t)$. Indeed, if $\phi_k \to \phi$ as in Definition 5.1 we define $z_k = \phi_k - H_0^{(k)}\phi_k$. Since the collection $H_0^{(k)}\phi_k$ is precompact there is a converging subsequence $H_0^{(m)}\phi_m$ and then $z_m(t)$ converges, say to $z(t)$. Now clearly ϕ_m is a solution of $u = z_m(t) + H_0^{(m)}u$ and if ϕ is a solution of $u = z(t) + H_0u$ it follows that $H_0^{(k)}\phi_k \to H_0^{(k)}\phi$, and $H^{(k)}$ therefore converges to H. Compare [1, 2].

6. Translates, Limiting Equations, and the Hull

Let $\dot{x} = f(x, s)$ be the ordinary differential equation which satisfies Assumption (A). For $t \in R$ we define the *translate* f^t of f by

$$f^t(x, s) = f(x, t + s).$$

We denote by tran(f) the collection of all translates f^t, $t \in R$.

Definition 6.1. The ordinary integral-like operator equation $u = Hu$ is a *limiting equation* of $\dot{x} = f(x, s)$ if there is a sequence of times $t_1, t_2, \ldots$, $|t_k| \to \infty$ so that f^{t_j} integrally converges to H (see Definition 5.1); i.e., whenever $\phi_j\colon [a, b] \to R^n$ converges uniformly to ϕ then $\int_a^b f(\phi_j(s), t_j + s)\, ds$ converges to $H_a\phi(b)$. The *positive* (*negative*) *limiting equations* are those where the sequence $t_j \to \infty$ (resp. $t_j \to -\infty$).

Remark 6.2. It is clear that if f^{t_j} integrally converges to H then for a fixed τ the sequence $f^{t_j+\tau}$ also integrally converges, and to the translate H^τ of H, where H^τ is defined in a natural way. A formal way of defining H^τ is as follows. For a function ϕ define ϕ^τ by $\phi^\tau(s) = \phi(\tau + s)$. Then H^τ is given by

$$(H^\tau)_a\, \phi(b) = H_{\tau+a}\phi^\tau(b + \tau).$$

Remark 6.3. Notice that the limiting equations are defined with respect to the particular convergence given in Definition 5.1. A different notion of convergence will produce a different class of limiting equations. Also, being an $\mathscr{L}^*$ space, or a convergence space, the closure of a set in $\mathscr{H}$ is naturally defined to be the collection of all limits of sequences in the set, and a set is closed if it contains all these limits. Notice that the closure of a set is not necessarily closed.

Definition 6.4. The hull of f, denoted by hull(f), is the closure of tran(f).

Remark. Obviously $t_k \to t$ implies $f^{t_k} \to f^t$. Therefore, hull(f) is the union of tran(f) and the limiting operators of f.

COROLLARY 6.5. *The hull of f is translation invariant, i.e.,* $H \in \text{hull}(f)$ *implies* $H^\tau \in \text{hull}(f)$ *for all* τ *(see Remark 6.2).*

The following result is the analog of the completeness of the space.

PROPOSITION 6.6. *Let $t_1, t_2, \ldots, |t_k| \to \infty$ be a sequence of times. Suppose that whenever $\phi_k: [a, b] \to R^n$ converge uniformly then $\int_a^b f(\phi_k(s), t_k + s)\, ds$ is a Cauchy sequence in R^n. Then the operator H defined by $H_a\phi(b) = \lim \int_a^b f(\phi(s), t_k + s)\, ds$ is an ordinary integral-like operator, consistent with Assumption* (A), *and consequently $u = Hu$ is a limiting equation of $\dot{x} = f(x, s)$.*

Proof. We have to check the conditions in Definition 3.1. Assumption (A) implies that the limit $H_a\phi$ of the integrals admits also μ_K as a modulus of continuity if $\phi: [a, b] \to K$, so H is consistent. This also implies that $H_a: C[a, b] \to C[a, b]$ will be continuous if $H_a\phi(t)$ is pointwise continuous. Let us show the pointwise continuity. Let $\phi_j \to \phi$. For each j there is a time t_j with $|t_j|$ large enough so that $H_a\phi_j(b)$ is close to $\int_a^b f(\phi_j(s), t_j + s)\, ds$. The latter converges as $j \to \infty$ to $H_a\phi(b)$, so $H_a\phi_j(b) \to H_a\phi(b)$. This proves that condition (1) in Definition 3.1 is satisfied. Condition (2) is obvious.

We shall need the concept of a compact and a precompact set, and naturally, a set is *precompact* if any sequence in it has a converging subsequence; the set is *compact* if it is precompact and closed. But notice that the closure of a precompact set is not necessarily compact.

DEFINITION 6.7. The set tran(f) (of all translates f^t of f) is positively (negatively) precompact if $\{f^t: t \geqslant 0\}$ (resp. $\{f^t: t \leqslant 0\}$) is precompact.

7. INVARIANCE

The ω-limit set of a function $\phi(t)$, denoted by $\Omega(\phi)$, is the set of all limit points of $\phi(t)$ as $t \to \infty$. If ϕ is a solution of the autonomous ordinary differential equation $\dot{x} = f(x)$ and if the solution through any initial value is unique, then $\Omega(\phi)$ is invariant in the sense that the solution through a point of $\Omega(\phi)$ stays in $\Omega(\phi)$ on its entire domain. This well-known observation was found to be very fruitful in many areas. In connection with stability theory it is the basis of the LaSalle invariance principle which combines the invariance and Lyapunov functions to form a powerful tool in detecting stability properties of the system. See [10, 15] and the references therein.

If $\dot{x} = f(x, s)$ is nonautonomous a problem arises, namely, with respect to what equation will the ω-limit set be invariant? Since f varies in time we cannot expect that solutions of f itself will not leave $\Omega(\phi)$. The answer was given by constructing the limiting equations. Indeed, under appropriate conditions the ω-limit set is invariant under the right limiting equation. This was established by Markus [13] when the only limiting equation is autonomous (see also [20]), and was generalized by Dafermos, LaSalle, Miller, Sell, Wakeman, and others. See [10, 15] for historical notes and references.

We shall establish invariance under the weak concept of convergence given in Definition 5.1. Here the limiting equations can be unordinary differential equations. The formulation and definition of the invariance follow the treatment given by Strauss and Yorke [20] for asymptotically autonomous systems.

DEFINITION 7.1. The set $B \subset R^n$ is *semiinvariant* with respect to the equation $u = u(0) + H_0 u$ if for each $z \in B$ there is a maximally defined solution ϕ of $u = z + H_0 u$ so that $\phi(t) \in B$ for all t in the domain of ϕ. The set $B \subset R^n$ is *semiinvariant with respect to a class of equations* $\{u = u(0) + H_0 u\colon H \in \mathscr{H}^1\}$ if for each $z \in B$ there is an $H \in \mathscr{H}^1$ and a maximally defined solution ϕ of $u = z + H_0 u$ so that $\phi(t) \in B$ for all t in its domain.

If the uniqueness of the solution of $u = z + H_0 u$ holds then semiinvariance is reduced to the usual invariance, as described at the beginning of the section. However, the quantifier in the second part of Definition 7.1 justifies the use of the "semi" before the invariance.

DEFINITION 7.2 (Compare [20, Definition 2.2]). Let Φ be a family of functions. The generalized ω-limit set $\Omega(\Phi)$ of Φ consists of those points $z \in R^n$ for which there exist sequences $t_j \to \infty$ and ϕ_j in Φ so that $\phi_j(t_j) \to z$ as $j \to \infty$. If Φ consists of one element ϕ we write $\Omega(\phi)$ for $\Omega(\{\phi\})$.

The generalized ω-limit set is obviously closed. The following theorem is the generalization of Strauss and Yorke [20, Theorem 2.4] to the non-autonomous case. Strauss and Yorke [20, Sect. 2] have shown how this type of invariance applies to a variety of situations. For the definition of tran(f), and the precompactness see the previous section.

THEOREM 7.3. *Suppose that* tran(f) *is positively precompact. Let Φ be a collection of maximally defined solutions of $\dot{x} = f(x, t)$. Then $\Omega(\Phi)$ is semiinvariant with respect to $\{u = u(0) + H_0 u\colon H \in \mathscr{H}_p\}$ where $\mathscr{H}_p$ is the family of the positive limiting equations.*

Proof. Let $z \in \Omega(\Phi)$. Then $z = \lim z_k$ with $z_k = \phi_k(t_k)$, $t_k \to \infty$ and $\phi_k \in \Phi$. The function $\gamma_k(s) = \phi_k(t_k + s)$ is a maximally defined solution of $\dot{x} = f^{t_k}(x, s)$, $x(0) = z_k$. By the positive precompactness of tran(f) a subsequence, say f^{t_j}, integrally converges to a positive limiting integral-like operator H. By Theorem 5.3 a subsequence γ_i of γ_j exists which converges in Γ to a solution γ of $u = z + H_0 u$. We shall show that $\gamma(t) \in \Omega(\Phi)$ for all t in the domain of γ. The convergence in Γ (see Definition 4.1) implies that $\gamma(t)$ is the limit of $\gamma_i(t) = \phi_i(t_i + t)$. But also, $t_i + t \to \infty$, so $\gamma(t) \in \Omega(\Phi)$. This completes the proof.

The theorem does not guarantee a solution which is defined for all $t \in R$. In fact, escape in a finite time might occur even if the original equation is autonomous and Φ contains only one solution. However, the following holds.

PROPOSITION 7.4. *If $\Omega(\Phi)$ in Theorem 7.4 is bounded then for each $z \in \Omega(\Phi)$ there is a solution ϕ of a positive limiting equation $u = z + H_0 u$ so that $\phi(t)$ is defined and belongs to $\Omega(\Phi)$, for all $t \in R$.*

Proof. Immediate from Theorem 7.3 and Proposition 3.2.

We shall demonstrate now how the invariance can be used in establishing stability. The basic idea is due to LaSalle, and was used later by several authors (see [10]). In order to isolate the role of the invariance we shall make some assumptions that usually are deduced as properties from the structure of the system. See the remark following the example.

EXAMPLE 7.5. Consider the xy system (x and y are vectors)

$$\dot{x} = g(x, y, t),$$
$$\dot{y} = h(x, y, t)y + p(x, y).$$

Assume that: (1) $p(x, y) \neq 0$ if $x \neq 0$; (2) every solution $(x(t), y(t))$ is positively bounded and satisfies $y(t) \to 0$ as $t \to \infty$; (3) Assumption (A) is satisfied and tran(f), where f represents the right-hand side of the system, is positively precompact. Then each solution $(x(t), y(t))$ converges to $(0, 0)$ as $t \to \infty$.

Proof. By (2) the ω-limit of the solution $(x(t), y(t))$ is not empty and contained in $\{(x, y): y = 0\}$. It is also semiinvariant as Theorem 7.3 assures. We shall show that the only semiinvariant subset of $\{(x, y): y = 0\}$ with respect to hull(f) is the origin. Indeed, if $\phi = (\xi(t), 0)$ is a function and H is any limiting equation, then the y coordinates of $H_a\phi$ equal to $\int_a^t p(\xi(s), 0)\, ds$, and by (1) it is not identically zero if $\xi(s)$ is not identically zero. So a function $(\xi(s), 0)$ cannot be a solution of *any* limiting equation unless $\xi(s) = 0$, and $\{(0, 0)\}$ is the only candidate for a semiinvariant set. This completes the proof.

Remarks. The point in Example 7.5 is that we do *not* have to actually compute the limiting equations, which might be fairly complicated. The structure of the equation yields the necessary information on the limiting equations.

Property (2) in the example is quite strong and usually is deduced from the structure of the equation with the aid of Lyapunov functions. The particular scalar example.

$$\ddot{x} + h(x, \dot{x}, t)\dot{x} + p(x) = 0 \tag{7.6}$$

which is equivalent to $\dot{x} = y$, $\dot{y} = -h(x, y, t)y - p(x)$ arises in reactor dynamics and was first studied by Levin and Nohel [12] where (under appropriate assumptions) a Lyapunov function is used to establish property (2). Wakeman [21] gave a proof of the global asymptotic stability of the origin of (7.6) under assumptions similar to [12], and by using the same invariance argument which we use in the present paper, but his limiting equations are always ode's. A relaxation of the conditions was obtained in [3] by allowing more general limiting equations. Property (3) above together with the precompactness criterion in Theorem 8.1 below give a further relaxation of the conditions.

A more general system, where x and y might be vectors, was treated by Levin [11]. The form of the equations in [11] is more general than our example, but for the particular form of Example 7.5 our conditions are weaker than those of [11].

8. The Precompactness of tran(f)

The precompactness of tran(f) plays a significant role in establishing the semiinvariance (Theorem 7.3) and its applications (Example 7.5). It is important in other applications too. We shall give here a sufficient condition for the precompactness.

The idea is simple. If $f(x, s)$ satisfies Assumption (A) then for a fixed continuous ϕ: $[a, b] \to R^n$ any sequence $\int_a^b f(\phi(s), t_j + s)\, ds$ where $|t_j| \to \infty$ has a convergent subsequence. A diagonal process will give a subsequence of t_j for which the integral converges for every ϕ in a dense sequence of $C[a, b]$. An equicontinuity property of $f(x, s)$ in x will guarantee the joint convergence required in Definition 5.1. Our purpose is to find an appropriate equicontinuity assumption.

Recall that we assumed that $f(x, s)$ is continuous in x, and therefore uniformly continuous on compact subsets of R^n. See Section 2 for a definition of a modulus of continuity.

Theorem 8.1. *Suppose that for every fixed s and a fixed compact set $K \subset R^n$ the function $f(\cdot, s)$ admits $\nu_K(\cdot, s)$ as a modulus of continuity, where $\nu_K(\delta, s)$ is integrable in s for a fixed δ and $\int_s^{s+1} \nu_K(\delta, \tau)\, d\tau \leqslant N_K(\delta)$ for every s with $N_K(\delta) \to 0$ as $\delta \to 0$. Then* tran(f) *is precompact. If the conditions are satisfied only for s in a set $[s_0, \infty)$ then* tran(f) *is positively precompact.*

Proof. From the equicontinuity assumption in the theorem we easily get the following estimation. If ϕ, ψ: $[a, b] \to K$ are continuous and $\sup |\phi(s) - \psi(s)| \leqslant \delta$ then for every $t \in R$

$$\left| \int_a^b (f(\phi(s), t + s) - f(\psi(s), t + s))\, ds \right| \leqslant (|b - a| + 1)\, N_K(\delta). \quad (8.2)$$

Let t_k be a sequence of times. For a fixed interval $[a, b]$ let ϕ_j be a dense sequence in $C[a, b]$. For $j = 1$ the sequence $\int_a^b f(\phi_1(s), t_k + s)\, ds$ has a convergent subsequence, and denote the corresponding sequence of times by $t_{k,1}$. For $j = 2$ the sequence $\int_a^b f(\phi_2(s), t_{k,1} + s)\, ds$ has a convergent subsequence and denote the corresponding sequence of times by $t_{k,2}$, and so on. The diagonal sequence $\tau_l = t_{l,l}$ has the property that $\int_a^b f(\phi_j(s), \tau_l + s)\, ds$ converges for any fixed j.

We now use the estimation (8.2) to show that whenever $\psi_l \to \psi$ in $C[a, b]$ the sequence $\int_a^b f(\psi_l(s), \tau_l + s)\, ds$ is a Cauchy sequence in R^n. Indeed,

$$\begin{aligned} &\left| \int_a^b (f(\psi_l(s), \tau_l + s) - f(\psi_m(s), \tau_m + s))\, ds \right| \\ &\quad \leqslant \left| \int_a^b (f(\psi_l(s), \tau_l + s) - f(\phi(s), \tau_l + s))\, ds \right| \\ &\qquad + \left| \int_a^b (f(\phi(s), \tau_l + s) - f(\phi(s), \tau_m + s))\, ds \right| \\ &\qquad + \left| \int_a^b (f(\phi(s), \tau_m + s) - f(\psi_m(s), \tau_m + s)\, ds \right| \\ &\quad = I_1 + I_2 + I_3, \end{aligned}$$

where ϕ is fixed and belongs to the dense sequence ϕ_j above. Since ψ_l converges it follows that for every $\delta > 0$ a ϕ can be found so that if l is large enough $\sup |\phi(s) - \psi_l(s)| \leqslant \delta$. By using (8.2) we deduce that I_1 and I_3 are less than $(|b - a| + 1)\, N_K(\delta)$ where K is a compact set containing all $\psi_l(s)$. Since $N_K(\delta)$ is small when δ is small it follows that I_1 and I_2 are small for l, m large enough. The quantity I_2 is small for l, m large since ϕ belongs to the dense set chosen before.

So far, for a fixed interval $[a, b]$ we found a subsequence τ_l such that if $\psi_l \to \psi$ in $C[a, b]$ the sequence $\int_a^b f(\psi_l(s), \tau_l + s)\, ds$ converges. We can find a subsequence of it, $\tau_{l,1}$, so that the integrals on another interval, say $[a_1, b_1]$ will converge, and a subsubsequence $\tau_{l,2}$ for $[a_2, b_2]$ and so on for a dense collection of subintervals. The diagonal sequence $t_l = \tau_{l,l}$ has the property that f^{t_l} integrally converges. Indeed the conditions of Proposition 6.6 are fulfilled for a dense collection of intervals $[a_i, b_i]$, but the equicontinuity in Assumption (A) implies that they hold for every $[a, b]$. This completes the proof of the main statement. In view of this the last statement is obvious.

Remark. The sufficient condition in Theorem 8.1 is given in terms of the function $f(x, s)$. It is clear that the validity of an estimation of the type of (8.2) is what is really needed for the proof. It would be interesting to establish precompactness without equicontinuity. In general tran(f) is not precompact even if $f(x, s)$ is bounded, (see for instance the example in Remark 9.2 below).

9. Asymptotically Autonomous Equations

Intuitively an equation $\dot{x} = f(x, s)$ is asymptotically autonomous if as time progresses, $s \to \infty$, the solutions behave like solutions of an autonomous system. An example is a perturbation $\dot{x} = h(x) + g(x, s)$ of the autonomous system $\dot{x} = h(x)$, where g converges to zero as $s \to \infty$. Here, for s large the solutions are close to solutions of the autonomous system $\dot{x} = h(x)$ which is the unique limiting equation. It is reasonable to define "asymptotically autonomous" by requiring that tran(f) will be positively precompact and that a unique positive limiting equation exists. In this case, this limiting equation is invariant under translates (Remark 6.2) and is thus autonomous. Compare Dafermos [5, Definition 4.4 and the observation which follows it]. The main result of this section is to show that if the equation $\dot{x} = f(x, s)$ is asymptotically autonomous in this sense then the unique limiting equation is necessarily an autonomous ordinary differential equation, i.e., $f(x, s) = h(x) + g(x, s)$ where g^t integrally converges to zero.

Perturbed autonomous equations $\dot{x} = h(x) + g(x, s)$, where the perturbation $g(x, s)$ becomes small in some sense when $s \to \infty$, were studied by Markus [13], and the conditions were eased by Strauss and Yorke [20]. (The conditions used in [13] or [20] imply that Assumption (A) is satisfied.) The convergence given in Definition 5.1 provides a "sense" for the convergence of g to zero so that the autonomous unperturbed equation $\dot{x} = h(x)$ will govern the behavior of the solutions for large s. Notably this convergence is weaker than the "mostly converges to zero" introduced by Strauss and Yorke [20]. While we demand that $\int_a^b g(\phi_k(s), t_k + s)\, ds \to 0$ if $\phi_k(s) \to \phi(s)$ uniformly and $t_k \to \infty$, in [20] it is required that the integral will converge to zero for every bounded sequence ϕ_k. The sequence $g_{1,m}$ in the example constructed in [1, Appendix B] integrally converges to 0 but not mostly converges to zero.

Theorem 9.1. *Suppose that* tran(f) *is positively precompact and that there is only one limiting equation when* $t \to \infty$. *Then this limiting equation is an autonomous ordinary differential equation.*

Proof. (The positive precompactness is not needed in the proof. See the remark below.) We know that the limiting equation is an ordinary integral-like operator equation $u = Hu$ (Definitions 3.1, 6.1). We have to show the existence of a continuous function $h(x)$ so that $H_a\phi(b) = \int_a^b h(\phi(s))\, ds$ for every $\phi: [a, b] \to R^n$ continuous.

Consider the constant function $z(t) = z$, and define $e_z(b) = H_0 z(b)$. Since any translate H^τ of H is equal to H it follows that the continuous function $e_z(b)$ has the property $e_z(2b) = 2e_z(b)$. It is then an easy exercise to show that $e_z(b) = be_z(1)$ or if we denote $e_z(1) = h(z)$ then $H_0 z(t) = h(z)t$. The continuity of H_0 on $C[0, 1]$ implies that $h(z)$ is continuous.

Let ϕ: $[a, b] \to R^n$ be continuous. We construct a piecewise constant function $\psi_k(s)$ on $[a, b]$ as follows. Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a uniform partition, i.e., $t_{j+1} - t_j = \Delta_j$ is constant. Define $\psi_k(s) = \phi(t_j)$ if $t_j \leqslant s < t_{j+1}$ this for $j = 1,..., k$. For τ_k large enough the integral $\int_a^b f(\psi_k(s), s)\, ds$ is close to $\sum_{j=1}^k h(\phi(t_j))\Delta_j$. If we change the values of ψ_k on a small portion of $[a, b]$ and the change is not big, it will cause only a small change in the value of the integral. So let us change ψ_k on the intervals $[t_j - \epsilon, t_j]$ for ϵ small, and make it a continuous function ϕ_k (say, by changing the constant to a linear function). Now $\phi_k \to \phi$ in $C[a, b]$ as $k \to \infty$, and therefore $\int_a^b f(\phi_k(s), \tau_k + s)\, ds \to H_a\phi(b)$. On the other hand the integral converges as $k \to \infty$ to the Riemann integral

$$\int_a^b h(\phi(s))\, ds = \lim_{k\to\infty} \sum_{j=1}^{k} h(\phi(t_j))\, \Delta_j$$

(here $t_j = t_j(k)$ and $\Delta_j = \Delta_j(k)$). This completes the proof.

Remark 9.2. As was mentioned above, it is enough to assume that there is a unique positive limiting equation in order to conclude that it is an autonomous ordinary differential equation. (And actually the proof can be modified in order to show that an autonomous ordinary integral-like operator equation which is consistent with Assumption (A) is already an ordinary differential equation.) The positive precompactness of tran(f) was added because uniqueness of the limiting equation alone does not provide the asymptotic results in which we are interested, and the equation is not asymptotically autonomous.

Consider the following example (based on [21, (20)]). The function $f(x, s)$ is defined on $R \times R$. We set $f(x, s) = 1$ if $s - x = 2^n$ for a certain $n = 1, 2,...$. Also $f(x, s) = 0$ if $| s - x - 2^n | \geqslant 2^{-n}$ for every $n = 1, 2,...$. We now extend f continuously to all (x, s) but so that $0 \leqslant f(x, s) \leqslant 1$. There is only one limiting equation, namely, $\dot{x} = 0$. But for $t_n = 2^n$ the sequence f^{t_n} does not have a converging subsequence, and moreover, the continuous dependence does not hold, namely, the unique solution of $\dot{x} = f(x, 2^n + s)$, $x(0) = 0$ is $x(s) = s$ and it does not converge to the zero solution.

Remark. Theorem 9.1 does not rule out the possibility that all the limiting equations will be autonomous equations, but there will be more than one limiting equation. For instance if $f(x, s) = \sin \ln(| s | + 1)$, then tran($f$) is precompact and the limiting equations (positive or negative) are exactly the autonomous equations $\dot{x} = \alpha$ for $-1 \leqslant \alpha \leqslant 1$.

10. Some Examples

We start with an example where the limiting equations consist of one periodic orbit under translations, but yet no limiting equation is an ordinary differential equation. This shows that the uniqueness in Theorem 9.1 cannot be replaced by periodicity.

EXAMPLE 10.1. Let $\eta(t)$ be a continuous nondecreasing function on $[0, 2\pi]$ with $\eta(0) = 0$, $\eta(2\pi) = 2\pi$ and such that η is *not* absolutely continuous with respect to the Lebesgue measure (for instance the Cantor function). Extend η to the whole line by letting $\eta(t) = 2\pi k + \eta(t - 2\pi k)$ if $2\pi k \leqslant t \leqslant 2\pi(k+1)$. Let $F(t)$ be a C^1 function, nondecreasing so that $|F(t) - \eta(t)| \to 0$ as $t \to \infty$, i.e., for large $|t|$, $F(t)$ is a smooth approximation of $\eta(t)$. Let $f(t) = F'(t)$ be the derivative of F. Consider now the two-dimensional equation

$$\begin{aligned} \dot{x} &= f(t)y, \\ \dot{y} &= -f(t)x. \end{aligned} \tag{10.2}$$

The general solution of (10.2) is easily computed and has the form $(x(t), y(t)) = (A\sin(F(t) - \alpha), A\cos(F(t) - \alpha))$. Such a solution clearly converges to the periodic solution $(A\sin(\eta(t) - \alpha), A\cos(\eta(t) - \alpha))$. But the latter is not absolutely continuous so it is not a solution of any ordinary differential equation. We shall find now the limiting equations that govern the limiting behavior of our system. The ode (10.2) is equivalent to

$$(x(t), y(t)) = (x(a), y(a)) + \int_a^t (y(s), -x(s))\, dF(s), \tag{10.3}$$

keeping in mind that $f(s)\, ds = dF(s)$, and when the integral is the Lebesgue–Stieltjes integral. The translation of the equation by τ has the form

$$(x(t), y(t)) = (x(a), y(a)) + \int_a^t (y(s), -x(s))\, dF(s - \tau). \tag{10.4}$$

If $|\tau_k| \to \infty$ and $\tau_k (\mathrm{mod}\ 2\pi) \to \sigma$ then the sequence of translates converges to

$$(x(t), y(t)) = (x(a), y(a)) + \int_a^t (y(s), -x(s))\, d\eta(s - \sigma). \tag{10.5}$$

The construction of $d\eta$ as a periodic measure which is not absolutely continuous with respect to the Lebesgue measure dt completes the argument.

The example above is a particular case of the class of equations

$$x(t) = x(a) + \int_a^t h(x(s), s)\, d\eta(s), \tag{10.6}$$

where $x \in R^m$ and η is a function with bounded variation. These equations might arise as limiting equations of nonautonomous ordinary differential equations, but are not ordinary differential equations, unless η is absolutely continuous. If (10.6) is consistent with Assumption (A) then η has to be continuous, so a jump might not occur in the context of this paper.

Another observation is that (10.6) cannot be autonomous unless $d\eta(s)$ is a constant multiplication of dt. Indeed, the only translation invariant measure of the line is (up to a multiplication) the Lebesgue measure. This of course agrees with Remark 9.2.

Another typical example of unordinary equations which might appear as limiting equations of nonautonomous ordinary differential equations are the Kurzweil equations. Kurzweil [8, 9] developed a theory of generalized ordinary differential equations. In [3] a special class of Kurzweil equations was investigated in connection with the subject of the present paper. Conditions were given under which every limiting equation is a Kurzweil equation. Let us mention that Kurzweil equations which do not satisfy the conditions in [3] might also appear as limiting equations.

11. Nonautonomous Equations as Local Flows

The basic construction of Sell [17] enables us to view the dynamical process generated by a nonautonomous equation as an autonomous system, a flow, on an appropriate phase space. Thus we can apply the abstract theory of flows to deduce properties of the nonautonomous system. This fruitful technique was demonstrated in many occasions, see the recent survey by Miller and Sell [15]. The invariance result treated in Section 7 above is one consequence of the theory, compare LaSalle [10], but there are more.

We want to show that the basic flow can be built under the assumptions adopted in this paper. We shall follow the construction in [15, 17, 18], with only some changes which we list now. The first change is that the phase space contains unordinary equations, namely, ordinary integral-like operator equations. We also follow Sell [19] and construct the local flow without the uniqueness assumption. For this purpose we add the space Γ defined in Section 4 to the phase space. Finally, we construct the flow on a convergence space ($\mathscr{L}^*$ space) and not on a topological space. This was already done by LaSalle [10].

We first define the phase space, then construct the candidate for local flow, and only later mention the definition of a local flow and make some comments.

Let $\dot{x} = f(x, s)$ be an ordinary differential equation satisfying Assumption (A). Let $\mathscr{H}$ be the collection of all ordinary integral-like operator equations which are consistent with Assumption (A) (see Definition 3.1). For each $H \in \mathscr{H}$ let $\Gamma(H)$ be the collection of the maximally defined solutions $\gamma \in \Gamma$ of $u = u(0) + H_0 u$ (see Definition 4.1 and Proposition 4.2). In Remark 6.2 we defined the translation H^τ of H. For $\gamma \in \Gamma$ we let $\gamma^\tau(s) = \gamma(\tau + s)$. If γ is a solution of $u = Hu$ then clearly γ^τ is a solution of $u = H^\tau u$. Let

$$X = \{(\gamma, H) \in \Gamma \times \mathscr{H} : \gamma \in \Gamma(H)\}.$$

Then X is translation invariant. For t in the domain of $\gamma \in \Gamma(H)$ define

$$\pi(t, \gamma, H) = (\gamma^t, H^t).$$

The space Γ is a complete metric space (Proposition 4.2). The space $\mathscr{H}$ is a convergence space (Definition 5.1, Remark 5.2). Therefore, the natural product convergence is defined on $\Gamma \times \mathscr{H}$ which makes it a convergence space. Theorem 5.3 implies that X is a closed subset of $\Gamma \times \mathscr{H}$. (The space X is even complete when the completeness in the $\mathscr{H}$ coordinate is along the lines of Proposition 6.6 but with respect to a sequence of operators.)

The mapping π has the following properties. Here p stands for a typical element of X, and I_p is the domain of $\pi(t, p)$, (i.e., the domain of γ if $p = (\gamma, H)$).

(1) $\pi(0, p) = p$.

(2) $t \in I_p$ and $s \in I_{\pi(t,p)}$ implies $t + s \in I_p$ and $\pi(s, \pi(t, p)) = \pi(t + s, p)$.

(3) Each $I_p = (\alpha_p, \omega_p)$ is maximal in the sense that either $\omega_p = \infty$ ($\alpha_p = -\infty$) or $\{\pi(t, p): t \geqslant 0\}$ (resp. $\{\pi(t, p): t \leqslant 0\}$) is not precompact.

(4) $\pi: R \times X \to X$ is continuous when defined, i.e., continuous on $\{(t, p): t \in I_p\}$.

(5) $p_k \to p$ implies $I_p \subset \liminf I_{p_k}$.

Conditions (1) and (2) are easily checked. Condition (3) is a consequence of Proposition 3.2. Condition (4) is implied by Definitions 4.1, 5.1, and Theorem 5.3. Condition (5) follows from Definition 4.1.

Properties (1)–(5) are the requirements from a local flow (compare [10, 15, 17]).

For most purposes it is enough to consider the restriction of π to the smallest closed set $\mathscr{H}^1$ which contains the hull of f. (Although hull(f) is the closure of tran(f), it might not be closed since we work in a convergence space.) If the solution $\gamma = \gamma(t, z, H)$ of $u = z + H_0 u$ for $H \in \mathscr{H}^1$ is always unique, then the local flow could be constructed on $R^n \times \mathscr{H}^1$ (rather than on $\Gamma \times \mathscr{H}^1$) by

$$\pi(t, z, H) = (\gamma(t, z, H), H^t).$$

Then Theorem 5.3 guarantees that conditions (4) and (5) hold, while Proposition 3.2 implies condition (3).

We constructed the local flow on a convergence space. Sometimes one needs a topological space, even with a uniformity structure (see [18]). It is clear that if we topologize X, and the topology is determined by convergence of sequences, then π will still be a local flow provided the convergence in the topology implies the convergence in X as above. If condition (4) is reformulated and the requirement is sequential continuity then the compact–open topology on $\mathscr{H}$ is an appropriate one (see Remark 5.2). Notice, however, that some ingredients of the structure are not preserved by changing the convergence structure or the topology. For instance, the hull of f as the closure of tran(f) might be different. We also might loose the precompactness of tran(f) by taking a topology which is too strong.

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