MATHEMATICS

THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF CERTAIN DIFFERENCE-DIFFERENTIAL EQUATIONS

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ABSTRACT

The paper settles the asymptotic behaviour of the solutions of the difference-differential equation (1.1) in a case that was left open in N. G. DE BRUIJN [1]. Theorem 3.1 states that in this case all solutions have limits for $x \to \infty$. A result about the speed of convergence is given by theorem 4.1.

1. INTRODUCTION

In this paper we shall study the asymptotic behaviour of the solutions of equations of the following kind:

$$w(x)f'(x) = (1+q(x))f(x-1) - (1+p(x))f(x) + r(x), \tag{1.1}$$

where w is a positive continuous function on $[1, \infty)$ and p, q, r are continuous real functions on $[1, \infty)$ satisfying the inequalities $p(x) > -1$, $q(x) > -1$, $|p(x)| < \varphi(x)$, $|q(x)| < \varphi(x)$, $|r(x)| < \varphi(x)$, where φ is a decreasing positive function on $[1, \infty)$ with convergent integral $\int_1^\infty \varphi(x)dx$.

A function f is called a solution of (1.1) if f is defined, real-valued and continuous on $[0, \infty)$, differentiable on $(1, \infty)$ and satisfying (1.1) on $(1, \infty)$. Every continuous function f on $[0, 1]$ can be extended to a solution by means of solving the differential equation $w(x)y'(x) + (1+p(x))y(x) = (1+q(x))f(x-1) + r(x)$ for $1 \leqslant x \leqslant 2$ (with the initial condition $y(1) = f(1)$), and so on.

N. G. DE BRUIJN ([1], [2]) solved the question of the asymptotic behaviour of the solutions with relatively mild restrictions on the coefficients of the equation. In order to give an idea, we explain the situation in the case $w(x) = x^{-\alpha}$, α constant. For $\alpha \leqslant 0$ all solutions have limits if $x \to \infty$ ([2]); for $\alpha > 1$ all solutions are asymptotically periodic ([1]); for $\frac{1}{2} < \alpha \leqslant 1$ they are asymptotically periodic in a modified sense (argument $x - \int_1^x t^{-\alpha}dt$ instead of x) ([1]). For $0 < \alpha \leqslant \frac{1}{2}$ the question was left open ([1]). In this paper we shall prove that the behaviour in the case $0 < \alpha \leqslant \frac{1}{2}$ is the same as in the case $\alpha \leqslant 0$ i.e. that every solution has a limit. Moreover, in the special case $w(x) = x^{-1/2}$ we shall prove a result about the speed of convergence.

2. Preliminaries

We use the following notation:

$$M(x)=\max_{x\leqslant t\leqslant x+1} f(t),\ m(x)=\min_{x\leqslant t\leqslant x+1} f(t),\ \delta(x)=M(x)-m(x),$$

where f is a solution of (1.1).

Furthermore we use formulas like $g(x)\leqslant O(h(x))$; this one denotes that there exist numbers C_1, C_2 such that $g(x)\leqslant C_1h(x)$ for all $x\geqslant C_2$.

Lemma 2.1. Every solution f of (1.1) is bounded. Moreover, the limits of $M(x)$, $m(x)$ and $\delta(x)$ for $x\to\infty$ exist.

Proof. Let $x\geqslant 0$. Let x_0 be a number such that $M(x+1)=f(x_0)$, $x+1\leqslant x_0\leqslant x+2$. If $x_0=x+1$ then trivially $M(x+1)\leqslant M(x)$. If $x+1<$ $<x_0\leqslant x+2$ then $f'(x_0)\geqslant 0$ and then we have, by (1.1),

$$(1+p(x_0))\,M(x+1)\leqslant(1+q(x_0))\,M(x)+r(x_0),$$

hence

$$M(x+1)\leqslant M(x)+(1+A(x))O(\varphi(x+1)), \tag{2.1}$$

where $A(x)=\max\{|M(x)|,\,|m(x)|\}$. In a similar manner we obtain

$$m(x+1)\geqslant m(x)+(1+A(x))O(\varphi(x+1)). \tag{2.2}$$

It follows that

$$A(x+1)\leqslant(1+O(\varphi(x+1))A(x)+O(\varphi(x+1)),$$

from which we infer that A is a bounded function. Using this in (2.1) and (2.2) we conclude that $M(x)$ and $m(x)$ have limits for $x\to\infty$.

Corollary 2.1. If $\lim\limits_{x\to\infty}\delta(x)=0$ then $\lim\limits_{x\to\infty} f(x)$ exists.

Lemma 2.2. If p, q, r are identically zero then M and δ are monotonically non-increasing and m is monotonically non-decreasing.

Proof. Arguing in a similar manner as in the proof of lemma 2.1 we find that $M(x+1)\leqslant M(x)$. Suppose that $M(x)$ is not monotonically non-increasing. Then there are numbers x_1 and ε, $x_1\geqslant 0$, $0<\varepsilon<1$, such that $M(x_1)<M(x_1+\varepsilon)$.

Let the maximum $M(x_1+\varepsilon)$ be attained at x_2, i.e. $M(x_1+\varepsilon)=f(x_2)$. Then $x_1+1<x_2\leqslant x_1+\varepsilon+1$, since $x_1+\varepsilon\leqslant x_2\leqslant x_1+1$ would imply $M(x_1+\varepsilon)=$ $=f(x_2)\leqslant M(x_1)$. Hence $f(x_2)\leqslant M(x_1+1)$. But then we have a contradiction: $M(x_1)<M(x_1+\varepsilon)=f(x_2)\leqslant M(x_1+1)\leqslant M(x_1)$. We can treat $m(x)$ in a similar manner.

3. The asymptotic behaviour

For convenience we introduce the abbreviations V and W: if w is of bounded variation on every interval $[1, x]$, $x \geqslant 1$, then $V(x)$ will denote the total variation on this interval

$$V(x) = \int_1^x |dw(x)|;$$

further we introduce a sequence $W(n)$, $n=2, 3, \ldots$, defined by

$$W(n) = \sum_{k=1}^{n-1} \Big[\int_k^{k+1} (w(x))^{-2}\, dx\Big]^{-1}.$$

Theorem 3.1. If w satisfies the following additional conditions

i) w is of bounded variation on every interval $[1, x]$, $x \geqslant 1$,

ii) $W(n) \to \infty$ if $n \to \infty$,

iii) $\liminf_{n\to\infty} V(n)/W(n) < 1$,

then every solution $f(x)$ of (1.1) has a limit for $x \to \infty$.

Proof. Squaring $w(x)f'(x)+f(x)$, using (1.1) and integrating from 1 to n, $n=2, 3, \ldots$, we obtain

$$(3.1)\quad \left\{ \begin{aligned} &\int_1^n [w(x)f'(x)]^2\, dx + 2\int_1^n w(x)f'(x)f(x)\, dx = \\ &\qquad = \int_0^1 (f(x))^2\, dx - \int_{n-1}^n (f(x))^2\, dx + R(n), \end{aligned} \right.$$

where $R(n)$ is a bounded function of n (since f is bounded).

For our purpose we need an upper bound for the right-hand side of (3.1) and lower bounds for the integrals on the left-hand side of (3.1). The right-hand side of (3.1) is bounded above since f is bounded.

We treat the second integral on the left-hand side of (3.1) as follows: the function g defined by $g(x) = \int_1^x w(t)f'(t)\, dt$ is bounded, for, by (1.1), we have $g(x) = \int_0^1 f(t)\, dt - \int_{x-1}^x f(t)\, dt + \int_1^x (g(t)\, f(t-1) + p(t)\, f(t) + r(t))\, dt$ is bounded. Integrating by parts we obtain

$$2\int_1^n w(x)f'(x)f(x)\, dx = 2\int_1^n w(x)f'(x)(f(x)-f(n))\, dx + 2f(n)\, g(n) =$$

$$= 2f(n)\, g(n) - (f(1)-f(n))^2\, w(1) - \int_1^n (f(x)-f(n))^2\, dw(x).$$

Hence

$$(3.2)\qquad 2\int_1^n w(x)f'(x)f(x)\, dx \geqslant O(1) - \int_1^n (f(x)-f(n))^2\, dV(x).$$

If V is bounded then the right-hand side of (3.2) is bounded below. If V is unbounded we proceed as follows: Let M_1, m_1, δ_1 be the limits of M, m and δ, respectively. We assume $\delta_1 > 0$. For every ε, $0 < \varepsilon < \delta_1$, there is a natural number $N(\varepsilon)$ such that $|M(x)-M_1|<\varepsilon/2$, $|m(x)-m_1|<\varepsilon/2$, $|\delta(x)-\delta_1|<\varepsilon/2$ if $x\geqslant N(\varepsilon)$. Then $(f(x)-f(n))^2<(\delta_1+\varepsilon)^2$ if $x\geqslant N(\varepsilon)$. Hence, for $n>N(\varepsilon)$,

$$\int_1^n (f(x)-f(n))^2\,dV(x)\leqslant \int_1^{N(\varepsilon)} (f(x)-f(n))^2\,dV(x)+(\delta_1+\varepsilon)^2\,V(n).$$

It follows that in all cases we have

$$2\int_1^n w(x)f'(x)f(x)\,dx\geqslant \mathrm{O}(1)-(\delta_1+\varepsilon)^2\,V(n), \tag{3.3}$$

if ε is fixed.

Next we deal with the first integral on the left-hand side of (3.1). We have

$$\delta(k)\leqslant \int_k^{k+1} |f'(x)|dx=\int_k^{k+1} w(x)f'(x)\frac{dx}{w(x)}.$$

Application of Schwarz's inequality gives

$$(\delta(k))^2\leqslant \int_k^{k+1} [w(x)f'(x)]^2\,dx \int_k^{k+1} (w(x))^{-2}\,dx.$$

Dividing by the second integral on the right-hand side, summing over k from 1 to $n-1$, we obtain

$$\int_1^n [w(x)f'(x)]^2\,dx\geqslant \sum_{k=1}^{n-1} [\int_k^{k+1} (w(x))^2\,dx]^{-1}(\delta(k))^2. \tag{3.4}$$

Furthermore,

$$\sum_{k=1}^{n-1} [\int_k^{k+1} (w(x))^{-2}\,dx]^{-1}(\delta(k))^2\geqslant \sum_{k=N(\varepsilon)+1}^{n-1} \geqslant(\delta_1-\varepsilon/2)^2\,W(n)+\mathrm{O}(1) \tag{3.5}$$

if ε is fixed. Combining (3.1), (3.3) and (3.5) we obtain

$$(\delta_1+\varepsilon/2)^2\,W(n)-(\delta_1+\varepsilon)^2\,V(n)\leqslant \mathrm{O}(1),$$

if ε is fixed. Dividing both sides by $W(n)$ and letting $n\to\infty$ through values of n such that $V(n)/W(n)\to\alpha=\liminf V(n)/W(n)$, we obtain $(\delta_1-\varepsilon/2)^2\leqslant(\delta_1+\varepsilon)^2\alpha$. Making $\varepsilon\to 0$ we get a contradiction. Hence $\delta_1=0$. Application of corollary 2.1 completes the proof.

Remark 1. Theorem 3.1 applies if w is a monotonically non-increasing function with divergent integral $\int_1^\infty (w(x))^2\,dx$, in particular if $w(x)=x^{-\alpha}$, $0\leqslant\alpha\leqslant\frac{1}{2}$. The theorem also applies if $w(x)=x^{-\alpha}$, $\alpha\leqslant 0$.

Remark 2. If $w(x)=x^{-\alpha}$, $0<\alpha\leqslant\frac{1}{2}$, $p(x)=q(x)=r(x)=0$, then it is easily proved that every solution f has n continuous derivatives for $x>n$, $n=1, 2, \ldots$. Moreover, $f^{(k)}(x+k)$, $k\geqslant 1$, is a solution of an equation of type (1.1). (See [1]). Hence, by theorem 3.1, all derivatives have limits for $x \to \infty$. Of course, $\lim f^{(k)}(x)=0$ if $k>0$, since f is bounded.

REMARK 3. If we examine the proof of theorem 3.1 then several generalizations come into mind. For instance, we can replace $f(x-1)$ in the right-hand side of (1.1) by a linear convex sum in which f appears with different retardations of the argument. We can also drop the linearity of the right-hand side of (1.1). These considerations suggest a generalization to cases where a Stieltjes integral appears in the right-hand side of (1.1). We give theorem 3.2 as an example, although further generalizations are still possible.

THEOREM 3.2. Let g be a non-decreasing function on $[0, 1]$ with the properties: $g(1)-g(0)=1$, $g(1)-g(\frac{1}{2})=\beta>0$. Let h be an increasing locally Lipschitzian function on $(-\infty, \infty)$ (i.e. $|h(x)-h(y)|/|x-y|$ is bounded if x and y are bounded) with the properties $h(0)=0$, $|h(x)| \to \infty$ if $|x| \to \infty$. Let w, p, q and r be as in theorem 3.1 except for condition iii), which is replaced by the stronger condition $\liminf V(n)/W(n)=0$. Then every solution $f(x)$ of

$$w(x)f'(x)=(1+q(x))\int_0^1 h(f(x-t))\,dg(t)-(1+p(x))\,h(f(x))+r(x) \tag{3.6}$$

tends to a constant if $x \to \infty$. (Our Stieltjes integrals $\int_0^1$ are taken over the closed interval $[0, 1]$; in other words they are what is often denoted by $\int_{0-}^{1+}$).

REMARK. It can be proved that there exists precisely one solution f which equals a prescribed continuous function on $[0, 1]$ (For example see [4]).

PROOF OF THEOREM 3.2. First we prove the analog of lemma 2.1 for solutions of (3.6). Let $x \geqslant 0$. There is a number x_0, $x+1 \leqslant x_0 \leqslant x+2$, such that $M(x+1)=f(x_0)$. If $x+1 \leqslant x_0 \leqslant x+\frac{3}{2}$ then trivially $M(x+1) \leqslant M(x+\frac{1}{2})$, $h(M(x+1) \leqslant h(M(x+\frac{1}{2}))$. If $x+\frac{3}{2}<x_0 \leqslant x+2$ then $f'(x_0) \geqslant 0$. In this case we have, by (3.3),

$$(1+p(x_0))\,h(M(x+1)) \leqslant (1+q(x_0))\int_0^1 h(f(x_0-t))\,dg(t)+r(x_0) \leqslant$$

$$\leqslant \alpha_1(1+q(x_0))\,h(M(x+1))+\beta_1(1+q(x_0))\,h(M(x+\tfrac{1}{2}))+r(x_0),$$

where

$$\beta_1=\int_{x_0-x-3/2}^{1} dg(t)>\beta,\ \alpha_1=1-\beta_1.$$

It follows that

$$h(M(x+1)) \leqslant h(M(x+\tfrac{1}{2}))+(1+B(x+\tfrac{1}{2}))\mathrm{O}(\varphi(x+1)),$$

where $B(x) = \max\{|h(M(x))|, |h(m(x))|\}$. In a similar manner we obtain

$$h(m(x+1)) \geqslant h(m(x+\tfrac{1}{2})) + (1+B(x+\tfrac{1}{2}))O(\varphi(x+1)).$$

It results that

$$B(x+1) \leqslant (1+O(\varphi(x+1)))B(x+\tfrac{1}{2}) + O(\varphi(x+1)),$$

from which it follows that B is a bounded function. Using this result in the inequalities for M and m we conclude that $M(x)$ and $m(x)$ have limits for $x \to \infty$.

Proceeding in the same way as in the proof of theorem 3.1 (squaring integrating) we obtain

$$(3.7)\quad \left\{\begin{aligned} &\int_1^n [w(x)f'(x)]^2\,dx + 2\int_1^n w(x)f'(x)\,h(f(x))\,dx = \\ &\qquad = \int_1^n \{[\int_0^1 h(f(x-t))\,dg(t)]^2 - [h(f(x))]^2\}\,dx + R(n). \end{aligned}\right.$$

Put $H(x) = \int_0^x h(s)\,ds$. Obviously $H(x) \geqslant 0$. Integrating by parts we obtain

$$(3.8)\quad \left\{\begin{aligned} &2\int_1^n w(x)f'(x)\,h(f(x))\,dx = 2w(n)\,H(f(n)) - 2w(1)\,H(f(1)) - \\ &\qquad - 2\int_1^n H(f(x))\,dw(x) \geqslant O(1+V(n)). \end{aligned}\right.$$

In order to prove that the right-hand side of (3.7) is bounded above we proceed as follows:

$$\int_1^n \{[\int_0^1 h(f(x-t))\,dg(t)]^2 - [h(f(x))]^2\}\,dx =$$

$$= \int_1^n dx \int_0^1 \int_0^1 \{h(f(x-t))\,h(f(x-s)) - [h(f(x))]^2\}dg(t)\,dg(s) =$$

$$= \int_0^1 \int_0^1 dg(t)\,dg(s) \int_1^n \{h(f(x-t))\,h(f(x-s)) - [h(f(x))]^2\}\,dx =$$

$$= \int_0^1 \int_0^1 dg(t)\,dg(s) \int_1^n \{-\tfrac{1}{2}[h(f(x-t)) - h(f(x-s))]^2 +$$

$$+ \tfrac{1}{2}[h(f(x-t))]^2 + \tfrac{1}{2}[h(f(x-s))]^2 - [h(f(x))]^2\}\,dx \leqslant$$

$$\leqslant \int_0^1 dg(t) \int_1^n \{[h(f(x-t))]^2 - [h(f(x))]^2\}\,dx =$$

$$= \int_0^1 dg(t)\{\int_{1-t}^1 [h(f(x))]^2\,dx - \int_{n-t}^n [h(f(x))]^2\,dx\} \leqslant \int_0^1 [h(f(x))]^2\,dx.$$

The proof of theorem 3.2 is completed as follows. Let lim $\delta(x)=\delta_1$. Then the right-hand side of (3.4) (obviously (3.4) is valid) is $\geqslant\frac{1}{2}\delta_1^2W(n)$ for n sufficiently large, since $W(n)\to\infty$ if $n\to\infty$. Using this result and combining (3.7), (3.4) and (3.8) we obtain

$$\tfrac{1}{2}\delta_1^2W(n)\leqslant O(1+V(n)).$$

Dividing by $W(n)$ and letting $n\to\infty$ through values of n such that $V(n)/W(n)\to 0$ we obtain $\delta_1=0$.

4. The speed of convergence

We shall restrict ourselves to the equation

$$x^{-\alpha}f'(x)=f(x-1)-f(x), \tag{4.1}$$

with $\alpha=\frac{1}{2}$, although the method can be used if $0<\alpha\leqslant\frac{1}{2}$.

If f is a solution of (4.1), we already know that

$$f(x)=C+o(1)\quad(x\to\infty).$$

The question arises what more can be said. We shall make the agreement that in this section f will denote a solution of (4.1) with the property that $f(x)\to 0$ if $x\to\infty$. This is possible since (4.1) is linear and constants are solutions.

Lemma 4.1. There exists a sequence $\{x_k\}_{k=1}^{\infty}$ with the properties: $f(x_k)=0$, $0<x_{k+1}-x_k\leqslant 1$ for $k=1, 2 \ldots, x_k\to\infty$ if $k\to\infty$.

Proof. Application of lemma 2.2 yields $M(x)\geqslant 0$, $m(x)\leqslant 0$ for $x\geqslant 0$. Hence there is at least one zero of f in every closed interval of length one. For $n\geqslant 1$ let u_n be the largest zero with $u_n\leqslant n$, and v_n the smallest with $v_n\geqslant n$. Hence $n-1\leqslant u_n\leqslant v_n\leqslant n+1$, $v_n-u_n\leqslant 1$, $v_n\leqslant u_{n+1}$, $u_{n+1}-v_n\leqslant 1$. Omitting duplications from the sequence $u_1, v_1, u_2, v_2, \ldots$, we obtain a sequence with the required properties.

For later reference we quote the following well-known inequality of Poincaré ([3], theorem 257):

Lemma 4.2. If $g\in C^1([a, b])$, $b>a$ and if $g(a)=g(b)=0$ then

$$\int_a^b (g'(x))^2\,dx\geqslant M^2(b-a)^{-2}\int_a^b (g(x))^2\,dx.$$

Theorem 4.1. If $f(x)$ is a solution of (4.1) which tends to zero for $x\to\infty$, then

$$f(x)=O(x^{-\pi^2/2})\quad(x\to\infty).$$

PROOF. Squaring $x^{-1/2}f'(x)+f(x)$, using (4.1), integrating from n to m, $1\leqslant n<m$ natural numbers and letting $m\to\infty$ we obtain

$$\int_n^\infty (f'(x))^2x^{-1}\,dx=\int_{n-1}^n (f(x))^2\,dx+(f(n))^2n^{-1/2}-\tfrac{1}{2}\int_n^\infty (f(x))^2x^{-3/2}\,dx. \tag{4.2}$$

Put

$$u(x)=\int_{x-1}^x (f(t))^2\,dt,\quad v(x)=u(x)+(f(x))^2x^{-1/2}.$$

We have $v'(x)=-(f(x)-f(x-1))^2-\frac{1}{2}(f(x))^2x^{-3/2}$, whence v is non-increasing. We note that

$$\int_n^\infty (f'(x))^2x^{-1}\,dx\leqslant v(n). \tag{4.3}$$

Using lemmas 4.1 and 4.2 we obtain the following chain of inequalities:

$$\left\{\begin{aligned}&\int_n^\infty (f'(t))^2t^{-1}\,dt\geqslant\sum_{x_k\geqslant n} x_{k+1}^{-1}\int_{x_k}^{x_{k+1}} (f'(t))^2\,dt\geqslant\\ &\geqslant\sum_{x_k\geqslant n}\pi^2x_{k+1}^{-1}(x_{k+1}-x_k)^{-2}\int_{x_k}^{x_{k+1}} (f(t))^2\,dt\geqslant\\ &\geqslant\pi^2\sum_{x_k\geqslant n}\int_{x_k}^{x_{k+1}} (f(t))^2(t+1)^{-1}\,dt\geqslant\pi^2\int_{n+1}^\infty (f(t))^2(t+1)^{-1}\,dt\geqslant\\ &\geqslant\pi^2\sum_{k=n+1}^\infty (k+2)^{-1}\int_k^{k+1} (f(t))^2\,dt=\pi^2\sum_{k=n}^\infty (k+3)^{-1}u(k+2).\end{aligned}\right. \tag{4.4}$$

Further we have the inequality

$$\int_n^\infty (f'(t))^2t^{-1}\,dt\geqslant\sum_{k=n}^\infty (k+\tfrac{1}{2})^{-1}(\delta(k))^2, \tag{4.5}$$

which can be derived in a similar way as (3.4). From lemma 2.2 and the fact that $f(x)\to 0$ if $x\to\infty$ it follows that $m(k)\leqslant 0\leqslant M(k)$, whence $(f(k+2))^2\leqslant(\delta(k+2))^2\leqslant(\delta(k))^2$. Using this in (4.5) we derive

$$\pi^2(n+2)^{-1/2}\int_n^\infty (f'(t))^2t^{-1}\,dt\geqslant\pi^2\sum_{k=n}^\infty (k+3)^{-1}(f(k+2))^2(k+2)^{-1/2}. \tag{4.6}$$

Adding (4.4) and (4.6), using (4.3), we infer

$$\pi^2\sum_{k=n}^\infty (k+3)^{-1}v(k+2)\leqslant(1+\pi^2(n+2)^{-1/2})\,v(n),$$

or, since v is non-increasing

$$\pi^2\sum_{k=n}^\infty (k+2)^{-1}v(k+1)\leqslant(1+\pi^2(n+2)^{-1/2}+\pi^2(n+2)^{-1})\,v(n). \tag{4.7}$$

Putting

$$S_n = \sum_{k=n}^{\infty} (k+2)^{-1} v(k+1)$$

we obtain from (4.7)

$$\left(1+\frac{\pi^2}{n+1}\right) S_n < S_{n-1},$$

from which it follows that

$$S_n = O(n^{-\pi^2}).$$

So we have

$$\sum_{k=n}^{\infty} (k+2)^{-1} v(k+1) = O(n^{-\pi^2}).$$

Since v is monotonically non-increasing we have

$$v(n) \sum_{k \geqslant n/2}^{n} (k+2)^{-1} = O(n^{-\pi^2}),$$

whence

$$v(n) = O(n^{-\pi^2}).$$

From (4.3) and (4.5) it follows that

$$\sum_{k=n}^{\infty} (k+\tfrac{1}{2})^{-1} (\delta(k))^2 = O(n^{-\pi^2}),$$

hence, by the same device (since $(\delta(k))^2$ is also monotonically non-increasing),

$$(\delta(n))^2 = O(n^{-\pi^2}),$$

which completes the proof.

Remark. There are reasons to believe that the result of theorem 4.1 is not the best possible one.

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