Cell complexes, oriented matroids and digital geometry

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Abstract

Abstract cell complexes (ACCs) were introduced by Kovalevsky as a means of solving certain connectivity paradoxes in graph-theoretic digital topology, and to this extent provide an improved theoretical basis for image analysis. In this work we argue that ACCs are a very natural setting for digital geometry, to the extent that their use permits simple, almost trivial formulations of major convexity results, including Caratheodory’s, Helly’s and Radon’s theorems. We also discuss the relevance of oriented matroids to digital geometry.

1. Introduction

Why not work with discrete geometrical objects and discrete operations only? That is, to develop a purely discrete modelling methodology? ... This work is, nowadays, not done; but I consider it can be done. ... I consider as a fundamental trend of discrete (or digital) geometry to produce the concept of a full discrete computer imagery. J. Françon [3].

Abstract cell complexes (ACCs) were introduced by Kovalevsky [6] as a means of solving certain connectivity paradoxes in graph-theoretic digital topology, and to this extent provide an improved theoretical basis for image analysis. As a datatype for digital geometry, however, it seems that ACCs have hardly been considered, apart from by Kovalevsky [7]. In this work we argue that ACCs are a very natural setting for...
digital geometry, to the extent that their use permits simple, almost trivial formulations of major convexity results, including Caratheodory’s, Helly’s and Radon’s theorems.

The following example indicates the advantage of ACCs over graphs with regard to Helly’s theorem.¹

**Example 1.** In graph-theoretic digital topology the digital plane is composed entirely of pixels. A pixel is in effect two-dimensional, and for digitization purposes is regarded as a square in the Euclidean plane. The diagram on the left shows the digital image² of the filled-in Euclidean triangle with vertices \((0,0),(0,4),(4,0)\). This image is a digital convex set according to any of the standard definitions discussed, for example, in [4]. Rotate the image through 90° successively to get four digital convex sets: the intersection of any three of these is non-empty, but no pixel lies in the intersection of all four, which violates Helly’s theorem.

In cellular digital topology the digital plane is composed of elements of differing dimension, namely pixels (2-cells), edges (1-cells) and points (0-cells) which, for digitization purposes, may be interpreted as a partition of the Euclidean plane as indicated in the diagram on the right. This diagram shows the digital image of the same filled-in triangle, and the intersection of the four rotations contains the middle point.

1.1. Axiomatic digital geometry

Our ultimate goal is an axiomatic digital geometry. We believe that digital space can and should be considered as a model of a set of “Euclidean” axioms that are the foundations of a rich geometry. This is the view of Smyth [13], and Knuth [5] has developed an axiomatic foundation of computational geometry. Knuth’s CC-systems are a special class of oriented matroids, which form the basis of the geometry to be considered here.

¹ Helly’s theorem states that, for any collection of convex sets in \(\mathbb{R}^n\), if the intersection of any sub-collection of \(\leq n + 2\) of these sets is non-empty, then the intersection of the whole collection is non-empty.

² The digital image being the set of all squares (pixels) whose interior intersects the triangle. There are of course several different “digitization” definitions in the literature, but the point is that the digital triangle is uncontroversially a convex set in graph-theoretic digital topology.
We will now discuss briefly how axiomatic considerations lead to ACCs. Digital topology seeks to establish a “discrete Euclidean topology” on $\mathbb{Z}^n$ that models the Euclidean topology on $\mathbb{R}^n$. How may this methodology be extended to geometry? Consider, for example, what are to be the straight lines in $\mathbb{Z}^2$? Clearly, the horizontal and vertical lines ought to be straight, but given this natural and very weak constraint one very quickly sees that two fundamental properties are mutually exclusive. If straight lines are to be connected (according to any of the standard definitions in digital topology) then the Line Axiom\(^3\) cannot be satisfied. If, on the other hand, straight lines may be disconnected, as in the “vector” approaches in Geographical Information Systems (e.g. [12]), the Line Axiom can be satisfied, but necessarily at the expense of allowing that lines may cross without intersecting.

Example 2. The diagram on the left shows a digital straight line segment, according to [10], which is connected with respect to “8-connectivity”. The vertical line passing through the points $a_1, a_2$ is also straight, which violates the Line Axiom. The diagram on the right shows two disconnected lines that cross without intersecting.

![Diagram showing digital straight lines](image.png)

The Line Axiom is essentially a point axiom and so cannot be satisfied by a space composed entirely of pixels. Digital space composed entirely of points, however, has insufficiently many points to witness all intersection of non-parallel lines. ACCs are digital spaces populated by both points and pixels, and so might allow versions of both properties we seek. For example, we say that the set of four points outlined on the right is a pixel at which the point lines “intersect”. We will consider two types of geometry on $\mathbb{Z}^n$, namely a “point geometry” and a geometry of cells, which we will call a “digital geometry”. These geometries capture different aspects of classical Euclidean geometry, and it is in their interaction that digital versions of classical theorems find their expression.

\(^3\) The Line Axiom states that for any two distinct points there is a unique straight line passing through them.
2. Digital geometry

The digital spaces we consider here are the Cartesian ACCs introduced by Kova-
levsky in [7]. An ACC is an abstract, finite version of the classical notion of a cell
complex in combinatorial topology. The following definition is in fact that of a specific,
convenient model of Kovalevsky’s axioms: the set $\mathcal{D}$ of digital cells in a Cartesian
ACC, ordered by subset inclusion, is a partial order $(\mathcal{D}, \subseteq)$. This partial order together
with the dimension function $\mathcal{D} \rightarrow \mathbb{Z}$ is precisely an ACC as in [6], and, as the product
of one-dimensional ACCs, is a Cartesian ACC as in [7].

For any integer $k \geq 1$, let $I_k$ denote the set $\{0, 1, \ldots, k\}$. A 0-dimensional cell in
$I_k$ is any singleton subset, and a one-dimensional cell is any subset of the form $\{j, j+1\}$,
for $0 \leq j < k$.

Definition 3. An $n$-dimensional Cartesian ACC consists of a point set $I_{k_1} \times \cdots \times I_{k_n}$
together with all subsets, called digital cells, of the form $D = D_1 \times \cdots \times D_n$, where
each $D_i$ is a cell in $I_{k_i}$. The dimension of the cell $D$ is the sum of the dimensions
of the cells $D_i$. A cell of dimension $m$ is called an $m$-cell.

From now on we will refer to a Cartesian ACC as a digital space.

In digital topology it is standard to consider a digital space explicitly in relation to
Euclidean space, and this is the approach we take here. The points of an $n$-dimensional
digital space are considered as points of $\mathbb{R}^n$ in the obvious way. The digital cells are
then to be considered as subsets of $\mathbb{R}^n$ thus: we say that the continuous part of a
digital cell $D$ is the relative interior of the polytope in $\mathbb{R}^n$ that has $D$ as its set of
vertices. (For the basic theory of relative interiors, polytopes, and for basic Euclidean
affine geometry we refer to [14]). The proof of the following result is straightforward.

Proposition 4. Let $S$ be the point set of an $n$-dimensional digital space. The collection
of continuous parts of digital cells is a partition of the convex hull of $S$ in $\mathbb{R}^n$.

Definition 5. The digital image in an $n$-dimensional digital space of any $P \subseteq \mathbb{R}^n$ is the
set of all digital cells whose continuous part intersects $P$.

Example 6. A two-dimensional digital space, or digital plane, consists of a point set
$I_n \times I_m$ ($n, m \geq 1$; see above) together with the digital cells:

- 0-cells are the singleton subsets;
- 1-cells are the subsets $\{(x, y), (x, y + 1)\}$ and $\{(x + 1, y)\};$
- 2-cells are the subsets $\{(x, y), (x, y + 1), (x + 1, y), (x + 1, y + 1)\}$.

The corresponding continuous parts are:

- The continuous part of the 0-cell $\{p\}$ is $\{p\}$
- The continuous part of the 1-cell $\{p, q\}$ is the open interval in $\mathbb{R}^2$ between $p, q$;
- The continuous part of the 2-cell $\{p, q, r, s\}$ is the interior in $\mathbb{R}^2$ of the filled-in
  square with vertices $p, q, r, s$.

The diagram on the right shows the digital image of the shaded Euclidean set on the
left. The diagram on the right is supposed to represent a collection of digital cells,
but it is far clearer to draw a digital cell as its continuous part, and this convention is assumed throughout.

2.1. Point geometry: oriented matroids

The geometry we consider on the point set of a digital space is that of an oriented matroid. Oriented matroids are combinatorial structures that capture an astounding amount of Euclidean geometry, but to our knowledge have not yet been considered in digital geometry, so we think it appropriate to give the basic definitions and geometric intuitions here. Standard references for matroids and oriented matroids are [9,2].

Definition 7. A matroid is a finite set $S$ together with a collection $\mathcal{I}$ of subsets of $S$, called independent sets, satisfying:

1. $\emptyset \in \mathcal{I}$;
2. If $J \in \mathcal{I}$ and $I \subseteq J$ then $I \in \mathcal{I}$;
3. If $I, J \in \mathcal{I}$ and $|J| > |I|$ then there exists some $x \in J \setminus I$ such that $I \cup \{x\} \in \mathcal{I}$.

Matroids can be considered as combinatorial abstractions of linear or affine independence in vector spaces. A basis of any $T \subseteq S$ is any maximal independent subset of $T$. It is a basic, fundamental result that any set has a basis and that any two bases of the same set have the same cardinality, which is called its rank; the rank of $T$ is denoted $r(T)$. A set $T$ is closed if, for any independent $I \subseteq T$ and any $x$, if $I \cup \{x\}$ is not independent then $x \in T$. The closure of $T$, denoted $\langle T \rangle$, is the smallest closed set that contains $T$.

Matroids have a substantial structure, which can be considered as an affine geometry: closed sets are affine sets, rank is affine dimension. A hyperplane, for example, is any closed set $H$ such that $r(H) = r(S) - 1$. A matroid has no convex geometry, however, as there is no means of saying whether or not two points lie on opposite sides of a hyperplane.

An oriented matroid is a matroid in which each hyperplane $H$ is assigned two sets (one of which might be empty), called open half-spaces, that partition its complement. The half-spaces are labelled $H^+, H^-$ according to a given context, although often it does not matter which is labelled which, in which case an arbitrary labelling is assumed.\footnote{The standard approach is to use two opposite but fixed labellings, or “signatures”, but we think our approach is simpler in an elementary geometric introduction.}
Definition 8. An oriented matroid is a matroid together with open half-spaces that satisfy the weak elimination axiom:

Let $H, K$ be any two distinct hyperplanes and let $x \in H^- \cap K^+$. Then there exists a hyperplane $L$ containing $x$ such that one of the open half-spaces determined by $L$ is a subset of $H^- \cup K^-$ and the other is a subset of $H^+ \cup K^+$.

Examples relevant to digital geometry are oriented matroids that are realizable in Euclidean space. Any finite $S \subseteq \mathbb{R}^n$ inherits the following oriented matroid structure. The independent sets in $S$ are those of its subsets that are affine independent in $\mathbb{R}^n$, which gives a matroid. For any $T \subseteq S$, let $\langle T \rangle_R$ and $\text{dim}(T)$ denote, respectively, the Euclidean affine hull and affine dimension of $T$. The following result is almost true by definition.

Proposition 9. For any $T \subseteq S$:

1. $\langle T \rangle = \langle T \rangle_R \cap S$;
2. $r(T) = \text{dim}(T)$.

To get an oriented matroid, suppose that $S$ contains an affine basis of $\mathbb{R}^n$. By the result, if $H$ is a hyperplane in $S$ then $\langle H \rangle_R$ is a hyperplane in $\mathbb{R}^n$. The open half-spaces determined by $H$ are then defined as the intersection with $S$ of the open half-spaces determined by $\langle H \rangle_R$. It is a standard result that this gives an oriented matroid.

The convex sets in an oriented matroid are obtained as follows. A closed half-space is the union of a hyperplane with either of its open half-spaces; the closed half-space $H \cup H^+$ is denoted $H^0_+$. The convex hull of a set is then the intersection of all the closed half-spaces that contain it; the convex hull of $T$ is denoted $[T]$. It is a standard result that this gives an oriented matroid.

The point geometry of an $n$-dimensional digital space is the oriented matroid its point set inherits from $\mathbb{R}^n$. In this section we develop the digital geometry, or geometry of digital cells, entirely in terms of the point geometry. To distinguish clearly between the two types of geometry we will often speak of point convex sets, point half-spaces etc.

The 0-cells in a digital space are precisely its singleton subsets, and by abuse of terminology we will identify these with points. That a point set $T$ is a subset of a digital set (set of digital cells) $\mathcal{D}$ means that if $x \in T$ then $\{x\} \in \mathcal{D}$. The following two definitions are different to those in [7].

Definition 11. The digital closed half-space determined by a point closed half-space $H^{0+}$ is the set of all digital cells $D$ such that $D \subseteq H$ or $D \cap H^+ \neq \emptyset$. 
Definition 12. The digital convex hull of a point set $T$ is the intersection of all the digital closed half-spaces that contain $T$.

The use of the word “closed” in “digital closed half-space” is here used to distinguish between these and “open” half-spaces, rather than (and in contrast to [7]) in any topological sense. Indeed, as in the following example, digital closed half-spaces are not necessarily closed in the natural topology on the set of cells (where $(\mathcal{D}, \subseteq)$ is the partial order of the set of cells, the natural topology on $\mathcal{D}$ is that for which $\subseteq$ is the specialization order).

Example 13. The diagram on the left shows a point hyperplane (the set of crosses) and one of its open half-spaces. The diagram on the right shows the determined digital closed half-space.

The next diagram shows the intersection of three digital closed half-spaces, which is the digital convex hull of the point set $\{a, b, c\}$ together with the 2-cell $D$. This cell must be separated from the digital convex hull using a further closed half-space.

The point of the example is to illustrate the sort of problem that may arise in developing a theory of digital convexity. That the intersection of convex sets is a convex set is a fundamental property in axiomatic convexity theory (see [2], for example), but the intersection of the three digital half-spaces in the example is, arguably, not what we would want to count as a digital convex set—at least, it is not the digital image of any Euclidean convex set. The “digital convex hull of a point set” is perhaps unintuitive at first, but it is one of the main arguments of this work that a theory of digital convexity must involve explicit interaction between point sets and digital sets.

Proposition 14. The digital convex hull of a point set $T$ is the digital image of $[T]_\mathbb{R}$.

Proof. Let $D$ be a digital cell and let $\text{cont}(D)$ denote its continuous part. We want to show that $\text{cont}(D)$ and $[T]_\mathbb{R}$ are disjoint if and only if $D$ is not in the digital
convex hull of $T$. If $H$ is a hyperplane such that $T \subseteq H^0_0$ and $-(D \subseteq H)$ then $[T]_R \subseteq \langle H \rangle^0_0$ and, by Lemma 21, $\text{cont}(D) \subseteq \langle H \rangle^+_R$. For the converse, let $S$ denote the point set of the digital space in question. By Proposition 23, if $\text{cont}(D)$ and $[T]_R$ are disjoint then there exists some $S$-hyperplane $J$ such that $[T]_R \subseteq J^0_0$ and $\text{cont}(D) \subseteq J^-$. By Proposition 9, $H = J \cap S$ is a hyperplane in the digital space, and then $T \subseteq H^0_0$, $D \subseteq H^0_0$ and $-(D \subseteq H)$. (The proof of Proposition 10 is then obtained by the identification of points with 0-cells.)

Classical convexity theorems may now be translated into digital convexity theorems almost trivially. Carathéodory’s theorem has both a “point” and a “digital” version and does not rely on the interaction between the two types of geometry. Helly’s theorem and Radon’s theorem, on the other hand, do rely on this interaction. The following three results hold in any $n$-dimensional digital space.

**Proposition 15** (Carathéodory’s theorem). For any point set $T$, $[T]$ is the union of the convex hulls of the subsets of $T$ having cardinality $\leq n + 1$, and the digital convex hull of $T$ is the union of digital convex hulls of such subsets.

**Proof.** Carathéodory’s theorem states that $[T]_R$ is the union of the Euclidean convex hulls of those subsets of $T$ having cardinality $\leq n + 1$. The first part of the result then follows easily from Proposition 10. For the second part, if a digital cell $D$ is in the digital convex hull of $T$ then, by Proposition 14 and Carathéodory’s theorem, there is some $T' \subseteq T$ with cardinality $\leq n + 1$ such that $D$ is in the digital convex hull of $T'$.

**Proposition 16** (Radon’s theorem). Any point set $T$ that has cardinality $\geq n + 2$ admits a partition into two sets whose respective digital convex hulls intersect.

**Proof.** Radon’s theorem states that $T$ can be partitioned into two sets $T_1, T_2$ whose respective Euclidean convex hulls intersect. For any point $x$ that lies in this intersection let $D$ be the digital cell whose continuous part contains $x$. Then $D$ lies in the digital convex hulls of both $T_1, T_2$.

**Proposition 17** (Helly’s theorem). Let $\mathcal{T}$ be any collection of point convex sets. If the intersection of any $n + 1$ members of $\mathcal{T}$ is non-empty then the intersection of all the digital convex hulls of members of $\mathcal{T}$ is non-empty.

**Proof.** Helly’s theorem gives that the intersection of the collection of Euclidean convex hulls of members of $\mathcal{T}$ is non-empty. For any point $x$ that lies in this intersection let $D$ be the digital cell whose continuous part contains $x$. Then $D$ lies in the intersection of all the digital convex hulls of members of $\mathcal{T}$.

An extreme point of a subset $T$ of an oriented matroid is any $x \in T$ such that $x \notin [T \setminus x]$. The set of all extreme points of $T$ is denoted $E(T)$. Oriented matroids admit a version of the Krein–Milman theorem, namely that $[T] = [E(T)]$; see [1]. A digital version of this result is:
Proposition 18 (Krein–Milman theorem). For any point set $T$, the digital convex hull of $T$ is the digital convex hull of $E(T)$.

**Proof.** From Proposition 10 we have that $E(T)$ is the set of extreme points of $T$ in $\mathbb{R}^n$. The Krein–Milman theorem states that $[T]_\mathbb{R} = [E(T)]_\mathbb{R}$.

2.3. Resolution

In this section we show briefly that the digital geometry of a digital space captures Euclidean geometry to the extent considered so far regardless of which resolution is considered. Let $S$ denote the point set of an $n$-dimensional digital space. We have so far only considered the embedding $x \mapsto x$ of $S$ into Euclidean space. But different resolutions of digital space correspond to different embeddings. For example, the embedding $x \mapsto x/2$ corresponds to a resolution twice as high as that considered so far.

The embeddings well-behaved enough for our purposes are as follows. We say that a linear basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ determines an embedding $e : S \to \mathbb{R}^n$, $(m_1, \ldots, m_n) \mapsto m_1 b_1 + \cdots + m_n b_n$. (The embedding considered so far is obviously that determined by the basis $c_1 = (1, 0, \ldots, 0), \ldots, c_n = (0, \ldots, 0, 1)$.) With respect to this embedding, we say that the continuous part of a digital cell $D$ is the relative interior of the polytope that has $e(D)$ as its set of vertices. The digital image of a Euclidean set is then defined accordingly—see Definition 5.

For any polytope $P$ in $\mathbb{R}^n$, let $ri(P)$ denote its relative interior.

Proposition 19. For any point set $T$, the digital convex hull of $T$ is the digital image of $[e(T)]_\mathbb{R}$.

**Proof.** Let $i$ be the embedding determined by the linear basis $c_1, \ldots, c_n$. Then the mapping $k : \mathbb{R}^n \to \mathbb{R}^n, \sum \lambda_i c_i \mapsto \sum \lambda_i b_i$ is a linear isomorphism such that $e = k \circ i$. We want to show that, for any digital cell $D$, $ri([i(D)]_\mathbb{R})$ intersects $[e(T)]_\mathbb{R}$ if and only if $D$ is in the digital convex hull of $T$. From Proposition 14 we know that $D$ is in the digital convex hull of $T$ if and only if $ri([i(D)]_\mathbb{R})$ intersects $[i(T)]_\mathbb{R}$. Linear isomorphisms preserve both the convex hull and relative interior operations, therefore $ri([e(D)]_\mathbb{R}) = k(ri([i(D)]_\mathbb{R}))$ and $[e(T)]_\mathbb{R} = k([i(T)]_\mathbb{R})$. The result is then given by the fact that linear isomorphisms are bijections.

3. Further work

We have tried to indicate some of the elements of an axiomatic digital geometry. We have shown that ACCs allow simple formulations of classical convexity results, and allow the introduction of fundamental combinatorial structures into digital geometry.

The theory given here is not fully axiomatic as, for example, we have not axiomatized the particular class of oriented matroids considered. Moreover, although a standard technique in digital topology, we feel it a theoretical weakness to have to embed
digital spaces into Euclidean space in order to understand them. A fully self-contained
digital geometry would not rely on classical geometry in this way. This indeed is one
of the achievements, in computational geometry, of Knuth, whose CC-systems are not
even necessarily embeddable in $\mathbb{R}^n$ but still support algorithms directly applicable
to classical computational geometry. A similar comment applies to oriented matroids in
general, which have purely combinatorial versions of all the convexity results given
here.

A more general and more axiomatic approach would be to consider cell complexes
on arbitrary oriented matroids. Triangulations of oriented matroids are discussed in
[1], and a recent, substantial theory of such is [11]. One main aim would then be to
obtain, in this context, digital versions of classical convexity results. The convexity
theory for oriented matroids mentioned above is, to a large extent, couched in terms
of “point extensions”: a “point of intersection of convex sets in an oriented matroid
$\mathcal{M}$ is obtained by embedding $\mathcal{M}$ into a richer oriented matroid $\mathcal{M}'$ that contains this
point. This contrasts with our view that cells in the original oriented matroid $\mathcal{M}$ should
witness intersection.

Perhaps the notion of an ACC can be refined further, taking into account the theory
of abstract polytopes [8], which is still under active development. An abstract polytope
is a poset that satisfies certain simple combinatorial conditions that capture the prop-
erties of the poset of faces of a Euclidean polytope. Presumably, an ACC should be
a collection of abstract polytopes that fit together in a way that abstracts the classical
definition of a cell complex in combinatorial topology.

4. Appendix A. Separation of polytopes in $\mathbb{R}^n$

There are several very standard results on separation of convex polytopes by
hyperplanes in $\mathbb{R}^n$—for the basic Euclidean geometry assumed in this section we
refer to [14]. All polytopes here are assumed to be non-empty. In this work we
are concerned with the subgeometry on some $S \subseteq \mathbb{R}^n$, and are therefore interested
in when two polytopes with vertices in $S$, which we call $S$-polytopes, can be sep-
arated by a hyperplane that is generated by points of $S$, which we will call an
$S$-hyperplane.

Recall that a hyperplane $H$ separates two sets if they lie in different closed half-
spaces, and separates the sets properly if, in addition, not both are subsets of $H$. When the two sets in question are polytopes, proper separation has several equivalent
formulations according to the following standard result.

For any polytope $P$, its relative interior is denoted $ri(P)$ and its set of vertices is
denoted $V(P)$.

**Lemma 20.** For any hyperplane $H$ and any polytope $P$ the following are equivalent:

1. $P \subseteq H^0^+$, $\neg(P \subseteq H)$;
2. $V(P) \subseteq H^0^+$, $\neg(V(P) \subseteq H)$;
3. $ri(P) \subseteq H^+$. 


By a hyperplane \textit{in} an affine set \( H \) we mean any affine \( K \subseteq H \) such that \( \dim(K) = \dim(H) - 1 \), where \( \dim \) denotes affine dimension.

**Lemma 21.** Let \( H \) be a hyperplane that separates the polytopes \( P, Q \) properly. If there exists a hyperplane \( K \) in \( H \) that contains \( (V(P) \cup V(Q)) \cap H \), then there exists a hyperplane \( L \) containing \( K \) that separates \( P, Q \) and contains a point of \( (V(P) \cup V(Q)) \backslash H \).

**Proof** (Outline). Consider the set of all hyperplanes \( L \) such that \( L \cap H = K \). Choose any \( x \in H \setminus K \) and, for each hyperplane \( L \), label the open half-spaces \( L^+, L^- \) such that \( x \in L^+ \). Fix a labelling of the open half-spaces of \( H \): the relation \( L \leq M \) if \( L^+ \cap H^+ \subseteq M^+ \) is a total partial order the set of hyperplanes. The least element of this order that contains a point of \((V(P) \cup V(Q)) \backslash H\) separates \( P, Q \).

The hyperplane obtained in the proof is said to be the result of \textit{rotating} \( H \) \textit{around} \( K \) \textit{through} \( H^+ \) \textit{away from} \( x \) until it hits \((P \cup Q) \backslash H\).

**Proposition 22.** Let \( S \) be any subset of \( \mathbb{R}^n \) that contains an affine basis, and let \( P, Q \) be \( S \)-polytopes whose respective relative interiors are disjoint. There exists an \( S \)-hyperplane that separates \( P, Q \) properly.

**Proof.**\footnote{This proof was communicated to me by Günter Ziegler.} We will first prove the result for when \( S = V(P) \cup V(Q) \). It is a standard result that there exists at least one proper separating hyperplane; let \( H \) be such a hyperplane with the property that, for any other proper separating hyperplane \( J \), \( \dim(H \cap S) \geq \dim(J \cap S) \). If \( \dim(H \cap S) = n \) we are done. If not, there exists a hyperplane \( K \) in \( H \) that contains \( H \cap S \). Rotate \( H \) around \( K \) to obtain a separating hyperplane \( L \) that contains a point of \( S \setminus H \). Then \( \dim(L \cap S) > \dim(H \cap S) \) so \( L \) cannot be a proper separating hyperplane. But then \( S \subseteq L \), which contradicts that \( S \) contains an affine basis. For the general result, we have just proved, in effect, that there exists a proper separating hyperplane \( K \) in the affine hull \( \langle P \cup Q \rangle \) that is generated by points of \( V(P) \cup V(Q) \). Then any extension of \( K \) to an \( S \)-hyperplane in \( \mathbb{R}^n \) separates \( P, Q \) properly.

We actually require a slightly stronger result that states which polytope is not contained in a properly separating hyperplane. We say that a hyperplane \( H \) separates a polytope \( P \) \textit{from} a polytope \( Q \) if it separates them and if \( \neg(P \subseteq H) \).

**Proposition 23.** Let \( S \) be any subset of \( \mathbb{R}^n \) that contains an affine basis, and let \( P, Q \) be \( S \)-polytopes such that \( \text{ri}(P) \cap Q = \emptyset \). There exists an \( S \)-hyperplane that separates \( P \) from \( Q \).

**Proof.** We will first prove, by induction on \( n \), that the result holds when \( S = V(P) \cup V(Q) \). When \( n = 1 \) the result is very simple, so assume that \( n > 1 \). Let \( H \) be a proper separating \( S \)-hyperplane. If \( \neg(P \subseteq H) \) we are done, so assume that \( P \subseteq H \), in which case \( \neg(Q \subseteq H) \). Let \( F \) denote the polytope with vertices \( V(Q) \cap H \). Then \( F \) is a face \( \subseteq H \).
of \( Q \), so \( ri(P) \cap F = \emptyset \). By the induction hypothesis there is a hyperplane \( K \) in \( H \) that is generated by points of \( S \) such that \( ri(P) \subseteq K^+ \) and \( F \subseteq K^{0-} \). Choose any \( x \in ri(P) \), let \( Q \subseteq H^{0+} \) and rotate \( H \) around \( K \) through \( H^+ \) away from \( x \) until it hits \( Q \setminus H \): the resultant \( S \)-hyperplane separates \( P \) from \( Q \). For the general result, we have just proved, in effect, that there exists a hyperplane \( K \) in the affine hull \( \langle P \cup Q \rangle \) that is generated by points of \( V(P) \cup V(Q) \) such that \( ri(P) \subseteq K^+ \) and \( Q \subseteq K^{0-} \). Then extend \( K \) to an \( S \)-hyperplane.

References