International Journal of Solids and Structures 48 (2011) 280–291

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Random homogenization analysis in linear elasticity based on analytical bounds and estimates

Juan Ma [a,b,*], Ilker Temizer [b], Peter Wriggers [b]

[a] School of Electronical and Mechanical Engineering, Xidian University, Xi'an 710071, PR China
[b] Faculty of Mechanical Engineering, Leibniz Universität Hannover, 30167 Hannover, Germany

ARTICLE INFO

Article history:
Received 23 June 2010
Received in revised form 16 September 2010
Available online 26 October 2010

Keywords:
Homogenization
Linear elasticity
Random effective properties
Random Factor Method
Monte-Carlo Method

ABSTRACT

In this work, random homogenization analysis of heterogeneous materials is addressed in the context of elasticity, where the randomness and correlation of components' properties are fully considered and random effective properties together with their correlation for the two-phase heterogeneous material are then sought. Based on the analytical results of homogenization in linear elasticity, when the randomness of bulk and shear moduli, the volume fraction of each constituent material and correlation among random variables are considered simultaneously, formulas of random mean values and mean square deviations of analytical bounds and estimates are derived from Random Factor Method. Results from the Random Factor Method and the Monte-Carlo Method are compared with each other through numerical examples, and impacts of randomness and correlation of random variables on the random homogenization results are inspected by two methods. Moreover, the correlation coefficients of random effective properties are obtained by the Monte-Carlo Method. The Random Factor Method is found to deliver rapid results with comparable accuracy to the Monte-Carlo approach.

1. Introduction

The homogenization method has been developed and extended to reduce the number of composite design parameters significantly by the introduction of effective characteristics using potential or complementary energy principles (Markovic and Ibrahimbegovic, 2006; Aboudi, 1991; Zohdi and Wriggers, 2005). The method relies on a statistically representative sample of material, referred to as a representative volume element (RVE). It is a finite sized sample from the heterogeneous material that characterizes its macroscopic behavior (Aboudi, 1991; Zohdi and Wriggers, 2005; Torquato, 2002). Although this technique, in its modern version, is more than 40 years old, there are many novel approaches and applications, such as in the food industry (Kanit, 2006), some composites made of wood (Lux, 2006), superconductors (Kaminski, 2005), even for time-dependent cases by "equation free" approach (Samaey et al., 2006); a variety of materially nonlinear multi-component composites can be homogenized as well (Idiart, 2006). Following numerous engineering applications, the strength of composites can also be estimated by the homogenization method (Steeves and Fleck, 2006).

Homogenization techniques deliver effective properties of heterogeneous materials. Exact computational approaches are summarized in Zohdi and Wriggers (2005). Here, the attention is focused to estimates and bounds. In this context, early approximations for the effective properties were first developed by Voigt (1889) and Reuss (1929). In 1957, Eshelby (1957) obtained a relatively compact solution that has been a basis for many approximation methods. Based on variational principles, Hashin and Shtrikman (1962) developed a model that improved solutions of the effective properties. Additional classical models have been proposed to estimate the effective properties, including the Self-Consistent method, the dilute distribution method, and the Mori and Tanaka (1973) method. Further approaches for estimating or bounding the effective responses of heterogeneous materials can be found for instance in Aboudi (1991), Mura (1987) and Nemat-Nasser and Hori (1999).

In recent years, a lot of attention is paid to random composites because of an uncertainty in reinforcement location/shape and/or pore spatial distribution in matrices, and randomness in components. Kaminski reported the perturbation-based homogenization analysis of two-phase composites (Kaminski and Kleiber, 2000) and the perturbation-based homogenization analysis for thermal conductivity of unidirectional fiber reinforced composites (Kaminski, 2001). Sakata obtained a macroscopic response by applying stochastic homogenization analysis for unidirectional fiber reinforced composites using the Monte-Carlo simulation (Sakata

* Corresponding author at: School of Electronical and Mechanical Engineering, Xidian University, Xi'an 710071, PR China.
E-mail addresses: juan@ikm.uni-hannover.de (J. Ma), temizer@ikm.uni-hannover.de (I. Temizer), wriggers@ikm.uni-hannover.de (P. Wriggers).

0020-7683/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2010.10.004

et al., 2008). Sakata also reported the three-dimensional results of perturbation analysis for the homogenized elastic tensor and the equivalent elastic properties (Sakata et al., 2008) or the second-order perturbation-based homogenization method (Sakata et al., 2008). Kaminski also developed a higher order perturbation-based analysis (Kaminski, 2007). Ostoja-Starzewski (2002) and Xu and Brady (2005) designed other approaches like the Fourier Galerkin method for random homogenization analysis. So far, most of the analytical models still consider the randomness of the geometric configuration like shape, size, location and distribution of particles, and the perturbation-based homogenization analysis is used as a main solution. For composites with different constituents, randomness of physical properties and volume fraction of the different constituents has an important effect on the effective properties after homogenization. Especially, the correlation among random variables should be fully considered, as should be the correlation among final random results. The perturbation method is based on the hypothesis that a random variable has a small perturbation about the mean value and subsequently Taylor series is used to describe a random variable as the sum of a determinate part plus a perturbation part that together transform the nonlinear equations into linear recursion formulas. For this reason, it is easy to get the first-order perturbation expansion, but a great deal of computation is needed to get the second-order or higher-order perturbation terms and the final results because of second-order or higher-order partial derivatives included. Moreover, this approach can quickly become numerically intractable when a large number of random variables are involved (Kaminski and Kleiber, 2000). Finally, due to the existence of secular terms, the accuracy and application of perturbation method is limited to some degree.

The goal of this work is to solve the random homogenization problem by two different methods while completely considering the randomness and correlation of the heterogeneous material. Based on the summary of the analytical results regarding the estimation of effective linear elasticity parameters, random effective properties of the two-phase heterogeneous materials are analyzed by the Random Factor Method (RFM) and the Monte-Carlo Method (MCM), in which the randomness of the bulk and shear moduli, volume fractions of the two constituents and the correlation among the random variables are considered fully. The numerical characteristics of effective properties after homogenization are derived by means of the random variable's moment method, and they are then compared with those obtained by MCM in order to verify the effectiveness of the method given in this paper. A future aim along this direction is to introduce the uncertainty to the finite element analysis of linear and nonlinear heterogeneous materials and multiscale engineering problems with heterogeneities distributed over multiple length scales.

2. Random analysis of the analytical bounds and estimates for the effective elasticity moduli

2.1. Monte-Carlo Method (MCM)

Monte-Carlo Method, the alternative to RFM, is used to solve the random problem by the test of random samples. According to the principle of MCM, samples associated with every random variable should be generated from their probabilistic distribution and correlation. Each sample realization is analyzed to obtain target quantities that display a statistical distribution. For normal distributions that are typically obtained in homogenization techniques, this statistical distribution is characterized by the mean value and the mean square deviation. In most Monte-Carlo simulations, different random variables are assumed to be independent of each other. However, this assumption does not hold in many engineering problems. Based on the Cholesky factorization of covariance matrix of random vector (Ali Touran, 1992), the simulation of the correlation among random variables is realized by MCM in this work. Some computational results and conclusions about correlation of random variables are given in Section 3.3.

2.2. Random Factor Method

The main ideas of Random Factor Method (RFM) (Ma et al., 2006; Gao et al., 2004) are as follows. A random variable y can be expressed as a random factor $\tilde{y}$ multiplied by its mean value $\mu_y : y = \tilde{y} \cdot \mu_y$. The random factor represents the randomness of the variable; its mean value is 1.0 and its mean square deviation is that of the random variable. $\tilde{y}$ obeys the same probabilistic distribution as y.

The main analyzing procedure by RFM in this paper is: firstly, the constituent's random variable is expressed as its random factor multiplied by its mean value; secondly, the material's effective properties are then written as random factors of constituents' random properties and volume fractions multiplied by their mean values respectively, that is, the material's effective properties are the functions of these random factors; finally, the mean values and mean square deviations of the effective properties can be obtained by using moment method of random variables.

RFM can directly and clearly reflect the influence of any random variable on the results. Additionally, based on the random variables' moment method, it is easy to consider the effect of correlation among random variables on homogenization results by RFM.

In the following, aiming at two-phase heterogeneous materials, the analytical bounds (Reuss–Voigt bounds (RV), Hashin–Shtrikman bounds (HS)) and estimates (Maxwell/Mori–Tanaka model (MW), Self-Consistent model (SC), Differential model (DF)) listed in Tables 1 and 2 (Torquato, 2002) will be considered. The mean values and mean square deviations of random effective properties will be derived by RFM.

In Tables 1 and 2, the volume average $\langle(\cdot)\rangle \overset{def}{=} V^{(1)}(\cdot)^{(1)} + V^{(2)}(\cdot)^{(2)}$, $\langle(\tilde{\cdot})\rangle \overset{def}{=} V^{(1)}(\cdot)^{(2)} + V^{(2)}(\cdot)^{(1)}$; $k^{(1)}, u^{(1)}, V^{(1)}$ are the bulk and shear moduli and the volume fractions of the first type of constituent; $k^{(2)}$, $u^{(2)}$ and $V^{(2)}$ are those of the second type of constituent; $f(x,y) = \frac{y[dx/2+(d+1)(d-2)y/d]}{x+2y}$, $g(x) = \frac{2(d-1)}{d}x$; for HS model, it is assumed that $k^{(2)} \geqslant k^{(1)}$, $u^{(2)} \geqslant u^{(1)}$.

Considering the randomness of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$ and $V^{(2)}$ simultaneously, they can be written as $k^{(1)} = \tilde{k}^{(1)} \cdot \mu_{k^{(1)}}, k^{(2)} = \tilde{k}^{(2)} \cdot \mu_{k^{(2)}}, u^{(1)} = \tilde{u}^{(1)} \cdot \mu_{u^{(1)}}, u^{(2)} = \tilde{u}^{(2)} \cdot \mu_{u^{(2)}}$ and $V^{(2)} = \tilde{V}^{(2)} \cdot \mu_{V^{(2)}}$ from RFM, where $\tilde{k}^{(1)}$, $\tilde{u}^{(1)}$, $\tilde{k}^{(2)}$, $\tilde{u}^{(2)}$, $\tilde{V}^{(2)}$ are the random factors of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ respectively, mean values of random factors are 1.0 and their mean square deviation are those $\sigma_{k^{(1)}}, \sigma_{u^{(1)}}, \sigma_{k^{(2)}}, \sigma_{u^{(2)}}, \sigma_{V^{(2)}}$ of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ respectively, $\mu_{k^{(1)}}, \mu_{u^{(1)}}, \mu_{k^{(2)}}, \mu_{u^{(2)}}, \mu_{V^{(2)}}$ are mean values of every random variable respectively.

For RV, HS and MW models in Tables 2 and 1, they are explicit and can be generally described as follows:

$$k^* = G\left(\tilde{k}^{(1)}\mu_{k^{(1)}}, \tilde{u}^{(1)}\mu_{u^{(1)}}, \tilde{k}^{(2)}\mu_{k^{(2)}}, \tilde{u}^{(2)}\mu_{u^{(2)}}, \tilde{V}^{(2)}\mu_{V^{(2)}}\right), \tag{1}$$

Table 1
A summary of analytical estimates for the effective elasticity moduli for spherical particles in d-dimensions.

Model	k^*	u^*
SC	$\sum_{i=1}^{2} V^{(i)} \frac{k^{(i)}-k^*}{k^{(i)}+g(u^*)} = 0$	$\sum_{i=1}^{2} V^{(i)} \frac{u^{(i)}-u^*}{u^{(i)}+f(k^*,u^*)} = 0$
DF	$\frac{dk^*}{dV^{(2)}} = \frac{1}{1-V^{(2)}}[k^* + g(u^*)]\frac{k^{(2)}-k^*}{k^{(2)}+g(u^*)}$	$\frac{du^*}{dV^{(2)}} = \frac{1}{1-V^{(2)}}[u^* + f(k^*,u^*)]\frac{u^{(2)}-u^*}{u^{(2)}+f(k^*,u^*)}$
MW	$\frac{k^*-k^{(1)}}{k^*+g(u^{(1)})} = V^{(2)}\left[\frac{k^{(2)}-k^{(1)}}{k^{(2)}+g(u^{(1)})}\right]$	$\frac{u^*-u^{(1)}}{u^*+f(k^{(1)},u^{(1)})} = V^{(2)}\left[\frac{u^{(2)}-u^{(1)}}{u^{(2)}+f(k^{(1)},u^{(1)})}\right]$

Here Maxwell/Mori–Tanaka model yields identical results to the Mori–Tanaka model, although the derivation procedures are slightly different (Torquato, 2002).

Table 2
A summary of analytical bounds for the effective elasticity moduli of macroscopically isotropic composites in d-dimensions in increasing order from left to right.

Matrix	RV^-	HS^-	Effective	HS^+	RV^+	Particle
$k^{(1)}$	$\langle k^{-1}\rangle^{-1}$	$\langle k\rangle - \frac{V^{(1)}V^{(2)}(k^{(2)}-k^{(1)})^2}{\langle k\rangle+g(u^{(1)})}$	k^*	$\langle k\rangle - \frac{V^{(1)}V^{(2)}(k^{(2)}-k^{(1)})^2}{\langle k\rangle+g(u^{(2)})}$	$\langle k\rangle$	$k^{(2)}$
$u^{(1)}$	$\langle u^{-1}\rangle^{-1}$	$\langle u\rangle - \frac{V^{(1)}V^{(2)}(u^{(2)}-u^{(1)})^2}{\langle u\rangle+f(k^{(1)},u^{(1)})}$	u^*	$\langle u\rangle - \frac{V^{(1)}V^{(2)}(u^{(2)}-u^{(1)})^2}{\langle u\rangle+f(k^{(2)},u^{(2)})}$	$\langle u\rangle$	$u^{(2)}$

$$u^* = F\left(\tilde{k}^{(1)}\mu_{k^{(1)}}, \tilde{u}^{(1)}\mu_{u^{(1)}}, \tilde{k}^{(2)}\mu_{k^{(2)}}, \tilde{u}^{(2)}\mu_{u^{(2)}}, \tilde{V}^{(2)}\mu_{V^{(2)}}\right). \quad (2)$$

According to the random variables' moment method (Gao et al., 2004), one can obtain the mean values of k^* and u^* by substituting the mean values $\mu_{k^{(1)}}, \mu_{u^{(1)}}, \mu_{k^{(2)}}, \mu_{u^{(2)}}, \mu_{V^{(2)}}$ into Eqs. (1), (2), and their mean square deviations can be computed by Eqs. (1) and (2) in Appendix, where $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ in $\frac{\partial k^*}{\partial k^{(1)}}, \frac{\partial u^*}{\partial k^{(1)}}, \frac{\partial k^*}{\partial k^{(2)}}, \frac{\partial u^*}{\partial k^{(2)}}, \frac{\partial k^*}{\partial u^{(1)}}, \frac{\partial u^*}{\partial u^{(1)}}, \frac{\partial k^*}{\partial u^{(2)}}, \frac{\partial u^*}{\partial u^{(2)}}, \frac{\partial k^*}{\partial V^{(2)}}, \frac{\partial u^*}{\partial V^{(2)}}$ are taken as their mean values; $\sigma_{u^{(1)}}, \sigma_{u^{(2)}}, \sigma_{k^{(1)}}, \sigma_{k^{(2)}}, \sigma_{V^{(2)}}$ are mean square derivations of $u^{(1)}$, $u^{(2)}$, $k^{(1)}$, $k^{(2)}$, $V^{(2)}$ respectively; $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$ are the correlation coefficients of $k^{(1)}$ and $u^{(1)}$, $k^{(2)}$ and $u^{(2)}$ respectively.

In the following, RFM is explicitly applied to the analytical bounds and estimates considered. The method of computing mean values is essentially equivalent to a deterministic evaluation of the bounds and estimates. Hence, attention will be focused to the evaluation of the mean square deviation. For conciseness, only SC and DF estimates will be explicitly analyzed. The application of the overall approach to the remaining estimates and bounds follow in a straightforward fashion from the methods employed in these two applications.

2.3. Self-Consistent model

The mean values μ_{k^*} and μ_{u^*} can be directly obtained from the formulas in Table 1. Note that SC model is a direct but implicit result where k^* and u^* are coupled each other, and one cannot directly solve the mean square deviations from Eqs. (1) and (2) in Appendix. Firstly, one can add k^* or u^* to both sides of the corresponding equation in Table 1 to obtain

$$k^* = k^* + \sum_{i=1}^{2} V^{(i)} \frac{k^{(i)} - k^*}{k^{(i)} + g(u^*)} = G\left(k^*, u^*, k^{(1)}, k^{(2)}, V^{(2)}\right), \quad (3)$$

$$u^* = u^* + \sum_{i=1}^{2} V^{(i)} \frac{u^{(i)} - u^*}{u^{(i)} + f(k^*, u^*)} = F\left(k^*, u^*, u^{(1)}, u^{(2)}, V^{(2)}\right). \quad (4)$$

From Eqs. (3) and (4), one can then obtained the partial differential Eqs. (3)–(6) in the Appendix, where $g = g(u^*)$, $f = f(k^*, u^*)$. The partial differential equations $\frac{\partial k^*}{\partial k^{(2)}}, \frac{\partial u^*}{\partial k^{(2)}}, \frac{\partial k^*}{\partial u^{(1)}}, \frac{\partial u^*}{\partial u^{(1)}}, \frac{\partial k^*}{\partial u^{(2)}}, \frac{\partial u^*}{\partial u^{(2)}}$ can also be obtained in a similar fashion. Note that only $k^{(1)}$ is the variable in Eqs. (3), (4) of the Appendix and $k^{(2)}$, $u^{(1)}$, $u^{(2)}$, $V^{(2)}$, k^*, u^* are taken as their mean values, and only $V^{(2)}$ is the variable in Eqs. (5) and (6) of Appendix. Let common constants in these partial differential equations be $a = \frac{\partial f}{\partial k^*}\Big|_{\mu_{k^{(1)}}, \mu_{u^{(1)}}, \mu_{k^{(2)}}, \mu_{u^{(2)}}, \mu_{V^{(2)}}}$, $b = \frac{\partial f}{\partial u^*}|_{\mu_{k^{(1)}}, \mu_{u^{(1)}}, \mu_{k^{(2)}}, \mu_{u^{(2)}}, \mu_{V^{(2)}}}$, $e_1 = \frac{V^{(1)}}{\mu_{u^{(1)}} + f(\mu_{k^*}, \mu_{u^*})}$, $h_1 = \frac{V^{(1)}\left(\mu_{u^{(1)}} - \mu_{u^*}\right)}{\left(\mu_{u^{(1)}} + f(\mu_{k^{(1)}}, \mu_{u^{(1)}})\right)^2}$, $i_1 = \frac{V^{(2)}_{\#}}{\mu_{u^{(2)}} + f(\mu_{k^*}, \mu_{u^*})}$, $j_1 = \frac{\mu_{V^{(2)}}\left(\mu_{u^{(2)}} - \mu_{u^*}\right)}{\left(\mu_{u^{(2)}} + f\left(\mu_{k^*}, \mu_{u^*}\right)\right)^2}$, $l_1 = \frac{\mu_{u^{(2)}} - \mu_{u^*}}{\mu_{u^{(2)}} + f\left(\mu_{k^*}, \mu_{u^*}\right)}$, $n_1 = \frac{\mu_{u^{(1)}} - \mu_{u^*}}{\mu_{u^{(1)}} + f(\mu_{k^*}, \mu_{u^*})}$, $e_2 = \frac{V^{(1)}}{\mu_{k^{(1)}} + g(\mu_{u^*})}$, $h_2 = \frac{V^{(1)}(\mu_{k^{(1)}} - \mu_{k^*})}{(\mu_{k^{(1)}} + g(\mu_{u^*}))^2}$, $i_2 = \frac{\mu_{V^{(2)}}}{\mu_{k^{(2)}} + g(\mu_{u^*})}$, $j_2 = \frac{\mu_{V^{(2)}}\left(\mu_{k^{(2)}} - \mu_{k^*}\right)}{\left(\mu_{k^{(2)}} + g(\mu_{u^*})\right)^2}$, $l_2 = \frac{\mu_{k^{(2)}} - \mu_{k^*}}{\mu_{k^{(2)}} + g(\mu_{u^*})}$, $n_2 = \frac{\mu_{k^{(1)}} - \mu_{k^*}}{\mu_{k^{(1)}} + g(\mu_{u^*})}$, where $V^{(1)} = 1 - \mu_{V^{(2)}}$.

After the partial differential equations are rewritten as the Eqs. (7)–(16) in Appendix where $c_0 = 2(d-1)/d$, one can obtain $\{x_i\} = \{c_{i0}\} + [c_{i,j}]_{10\times 10} \cdot \{x_i\}$ $(i = 1,2,\ldots,10; j = 1,2,\ldots,10)$, where $\{c_{i0}\} = \{e_2 - h_2, i_2 - j_2, 0, 0, l_2 - n_2, 0, 0, e_1 - h_1, i_1 - j_1, l_1 - n_1\}^T$ is a column vector and elements of the matrix $[c_{i,j}]_{10\times 10}$ are $c_{1,1} = c_{2,2} = c_{3,3} = c_{4,4} = c_{5,5} = 1 - e_1 - b \cdot h_1 - i_1 - b \cdot j_1$, $c_{1,6} = c_{2,7} = c_{3,8} = c_{4,9} = c_{5,10} = -a \cdot (h_1 + j_1)$, $c_{6,1} = c_{7,2} = c_{8,3} = c_{9,4} = c_{10,5} = -c_0 \cdot (h_2 + j_2)$, $c_{6,6} = c_{7,7} = c_{8,8} = c_{9,9} = c_{10,10} = 1 - e_2 - i_2$, and other elements are zero. Let $x_1 = \frac{\partial k^*}{\partial k^{(1)}}$, $x_2 = \frac{\partial u^*}{\partial k^{(1)}}$, $x_3 = \frac{\partial k^*}{\partial k^{(2)}}$, $x_4 = \frac{\partial u^*}{\partial k^{(2)}}$, $x_5 = \frac{\partial k^*}{\partial u^{(1)}}$, $x_6 = \frac{\partial u^*}{\partial u^{(1)}}$, $x_7 = \frac{\partial k^*}{\partial u^{(2)}}$, $x_8 = \frac{\partial u^*}{\partial u^{(2)}}$, $x_9 = \frac{\partial k^*}{\partial V^{(2)}}$, $x_{10} = \frac{\partial u^*}{\partial V^{(2)}}$, and one can obtain $\{x_i\}$ from $\{x_i\} = K^{-1}\{c_{i0}\}$ by solving the system of linear equations, where the matrix $K = I - [c_{i,j}]_{10\times 10}$ and I is a 10×10 identity matrix.

One can finally get mean square deviations of k^* and u^* for SC model by substituting $\frac{\partial k^*}{\partial k^{(1)}}, \frac{\partial u^*}{\partial k^{(1)}}, \frac{\partial k^*}{\partial V^{(2)}}, \frac{\partial u^*}{\partial V^{(2)}}, \frac{\partial k^*}{\partial k^{(2)}}, \frac{\partial u^*}{\partial k^{(2)}}, \frac{\partial k^*}{\partial u^{(1)}}, \frac{\partial u^*}{\partial u^{(1)}}, \frac{\partial k^*}{\partial u^{(2)}}, \frac{\partial u^*}{\partial u^{(2)}}$ into Eqs. (1) and (2) of Appendix.

2.4. Differential model

For DF model in Table 1, the expressions for the mean square deviations are implicit and coupled as well. Firstly, the mean values μ_{k^*} and μ_{u^*} of k^* and u^* can be obtained by the integral of $V^{(2)}$ from 0 to 1.0. In order to evaluate σ_{k^*}, note that one can obtain the Eqs. (17), (18) in Appendix.

In the same way, the additionally required terms $\frac{\partial}{\partial k^{(2)}}\left(\frac{\partial k^*}{\partial V^{(2)}}\right)$, $\frac{\partial}{\partial u^{(1)}}\left(\frac{\partial k^*}{\partial V^{(2)}}\right)$, $\frac{\partial}{\partial u^{(2)}}\left(\frac{\partial k^*}{\partial V^{(2)}}\right)$, $\frac{\partial}{\partial k^{(2)}}\left(\frac{\partial u^*}{\partial V^{(2)}}\right)$, $\frac{\partial}{\partial u^{(1)}}\left(\frac{\partial u^*}{\partial V^{(2)}}\right)$, $\frac{\partial}{\partial u^{(2)}}\left(\frac{\partial u^*}{\partial V^{(2)}}\right)$ can be obtained and can also be rewritten as $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial u^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial u^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial k^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial u^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial u^{(2)}}\right)$. Note that these are differential equations that need to be solved in a similar fashion to the evaluation of the estimate, namely by numerical integration. For instance, for $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(1)}}\right)$, one needs to integrate $\frac{\partial k^*}{\partial k^{(1)}}$ with respect to $V^{(2)}$. For this purpose, let $c_1 = \frac{k^{(2)} - k^*}{k^{(2)} + g}$, $c_2 = \frac{u^{(2)} - u^*}{u^{(2)} + f}$, $c_3 = k^* + g$, $c_4 = u^* + f$, $c_5 = \frac{k^{(2)} - k^*}{(k^{(2)} + g)^2}$, $c_6 = \frac{u^{(2)} - u^*}{(u^{(2)} + f)^2}$, $c_7 = u^{(2)} + f$, $c_8 = k^{(2)} + g$, $c_9 = \frac{\partial f}{\partial k^*}$, $c_{10} = \frac{\partial f}{\partial u^*}$, where k^*, u^*, $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ in $c_1 \sim c_{10}$ are taken as their mean values, so $c_1 \sim c_{10}$ are ten constants in differential equations $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial u^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial u^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial k^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial k^{(2)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial u^{(1)}}\right)$, $\frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial u^{(2)}}\right)$. In order to obtain $\frac{\partial k^*}{\partial k^{(1)}}$, $\frac{\partial k^*}{\partial k^{(2)}}$, $\frac{\partial k^*}{\partial u^{(1)}}$, $\frac{\partial k^*}{\partial u^{(2)}}$, $\frac{\partial u^*}{\partial k^{(1)}}$, $\frac{\partial u^*}{\partial k^{(2)}}$, $\frac{\partial u^*}{\partial u^{(1)}}$, $\frac{\partial u^*}{\partial u^{(2)}}$, differential equations are then integrated from $V^{(2)} = 0$ to $V^{(2)} = 1$. Due to $k^* = k^{(1)}$, $u^* = u^{(1)}$ at $V^{(2)} = 0$, integral initial conditions to be employed are $\frac{\partial k^*}{\partial k^{(1)}} = 1$, $\frac{\partial u^*}{\partial k^{(1)}} = 0$, $\frac{\partial k^*}{\partial k^{(2)}} = 0$, $\frac{\partial u^*}{\partial k^{(2)}} = 0$, $\frac{\partial k^*}{\partial u^{(1)}} = 0$, $\frac{\partial u^*}{\partial u^{(1)}} = 1$, $\frac{\partial k^*}{\partial u^{(2)}} = 0$, $\frac{\partial u^*}{\partial u^{(2)}} = 0$ respectively.

In the present work, $\frac{\partial k^*}{\partial k^{(1)}}$, $\frac{\partial k^*}{\partial k^{(2)}}$, $\frac{\partial k^*}{\partial u^{(1)}}$, $\frac{\partial k^*}{\partial u^{(2)}}$, $\frac{\partial u^*}{\partial k^{(1)}}$, $\frac{\partial u^*}{\partial k^{(2)}}$, $\frac{\partial u^*}{\partial u^{(1)}}$, $\frac{\partial u^*}{\partial u^{(2)}}$ will be computed using the fourth order Runge–Kutta numerical integration method. The mean square deviations of k^* and u^* are obtained from Eqs. (1) and (2) in Appendix. Note that $\frac{\partial k^*}{\partial V^{(2)}}$ and $\frac{\partial u^*}{\partial V^{(2)}}$ can be obtained by directly substituting mean values of $k^{(2)}$, $u^{(1)}$, $u^{(2)}$, $V^{(2)}$, k^*, u^* into $\frac{\partial k^*}{\partial V^{(2)}}$ and $\frac{\partial u^*}{\partial V^{(2)}}$ in Table 1.

3. Numerical examples

In this section, the RFM results are demonstrated numerically. Here the input parameters of the two constituents in the heterogeneous material are taken to be $\{d, k^{(1)}, u^{(1)}, k^{(2)}, u^{(2)}\} = \{3, 4, 1, 120, 20\}$, where the physical units of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$ and $u^{(2)}$ are GPa. When taking variation coefficients of random variables

Table 3
Computational results of effective bulk and shear moduli k^* and u^* by different methods.

	RV−	RV+	HS−	HS+	MW	SC	DF
Results of Random Factor Method							
μ_{k^*}	7.7419	62	8.8842	24.602	8.8842	12.9001	10.0448
σ_{k^*}	0.7502	9.0024	0.7724	2.3044	0.7724	0.9166	0.8112
μ_{u^*}	1.9048	10.5000	2.8009	8.0161	2.8009	4.8851	3.4726
σ_{u^*}	0.1819	1.5009	0.2809	1.1725	0.2245	0.4406	0.2791
Results of Monte-Carlo Method							
μ_{k^*}	7.7359	62.0569	8.8688	23.9588	8.8722	12.8662	9.8220
σ_{k^*}	0.7478	9.0502	0.7939	2.5078	0.7962	0.9853	0.8295
μ_{u^*}	1.9024	10.5081	2.7900	8.0118	2.7901	4.8679	3.3745
σ_{u^*}	0.1817	1.5005	0.2441	1.0677	0.2454	0.4560	0.2766

Table 4
Computational results of different random models by Random Factor Method.

Model		k^*	u^*
1	$\alpha_{k^{(1)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.7340$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.1017$
	$\alpha_{u^{(1)}} = \alpha_{k^{(2)}} = \alpha_{u^{(2)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 0.0547$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0$
2	$\alpha_{k^{(2)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.0425$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0$
	$\alpha_{u^{(1)}} = \alpha_{k^{(1)}} = \alpha_{u^{(2)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 0.7177$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.214$
3	$\alpha_{u^{(1)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.1118$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.2320$
	$\alpha_{k^{(1)}} = \alpha_{k^{(2)}} = \alpha_{u^{(2)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 0$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.0796$
4	$\alpha_{u^{(2)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.0359$
	$\alpha_{u^{(1)}} = \alpha_{k^{(2)}} = \alpha_{k^{(1)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 1.1410$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.7742$
5	$\alpha_{V^{(2)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.9357$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.3260$
	$\alpha_{u^{(1)}} = \alpha_{k^{(2)}} = \alpha_{u^{(2)}} = \alpha_{v^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 3.3182$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.8851$
6	$\alpha_{k^{(1)}} = \alpha_{u^{(1)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.7697$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.2756$
	$\alpha_{k^{(2)}} = \alpha_{u^{(2)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 0.5472$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.0796$
7	$\alpha_{k^{(2)}} = \alpha_{u^{(2)}} = 0.1$	$\mu_{k^*_{HS-}} = 8.8842, \sigma_{k^*_{HS-}} = 0.0425$	$\mu_{u^*_{HS-}} = 2.8009, \sigma_{u^*_{HS-}} = 0.0359$
	$\alpha_{k^{(1)}} = \alpha_{u^{(1)}} = \alpha_{V^{(2)}} = 0$	$\mu_{k^*_{HS+}} = 24.0602, \sigma_{k^*_{HS+}} = 1.4922$	$\mu_{u^*_{HS+}} = 8.0161, \sigma_{u^*_{HS+}} = 0.7798$

Table 5
mean square deviations of k^* and u^* for SC model.

Results (MCM)	$V^{(2)}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
σ_{k^*}	0.4544	0.5279	0.6061	0.7328	0.9853	1.6258	3.0448	5.3586	9.7083
$\sigma^{\#}_{k^*}$	0.4312	0.4767	0.5151	0.5661	0.6797	0.9167	1.3615	2.1972	3.5922
$100 \times \left(\sigma_{k^*} - \sigma^{\#}_{k^*}\right)/\sigma^{\#}_{k^*}$	5.38	10.74	17.67	29.45	44.96	77.35	123.64	143.88	170.26
σ_{u^*}	0.1213	0.1481	0.1852	0.2590	0.4560	0.8376	1.3542	1.8693	2.4728
$\sigma^{\#}_{u^*}$	0.1120	0.1314	0.1577	0.2064	0.3003	0.4677	0.6896	0.9605	1.2259
$100 \times \left(\sigma_{u^*} - \sigma^{\#}_{u^*}\right)/\sigma^{\#}_{u^*}$	8.3	12.71	17.44	25.48	51.85	79.09	96.37	94.62	101.71

Here $\sigma^{\#}_{k^*}$ and $\sigma^{\#}_{u^*}$ are mean square deviations when correlation coefficients $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0$.

Table 6
computational results of mean square deviation of k^* and u^*.

σ_{k^*}	ρ										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
σ_{k^*}	0.680	0.893	0.917	0.922	0.954	0.972	0.995	1.033	1.045	1.053	1.075
$100 \times \left(\sigma_{k^*} - \sigma^{\wedge}_{k^*}\right)/\sigma^{\wedge}_{k^*}$	0	31.32	34.85	35.59	40.29	42.94	46.32	51.91	53.68	54.85	58.09
σ_{u^*}	0.3	0.436	0.442	0.439	0.451	0.448	0.445	0.461	0.461	0.458	0.463
$100 \times (\sigma_{u^*} - \sigma^{\wedge}_{u^*})/\sigma^{\wedge}_{u^*}$	0	45.33	47.33	46.33	50.33	49.33	48.33	53.67	53.67	52.67	54.33

Here ρ means $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}}$; $\sigma^{\wedge}_{k^*}$ and $\sigma^{\wedge}_{u^*}$ are mean square deviations when $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0$.

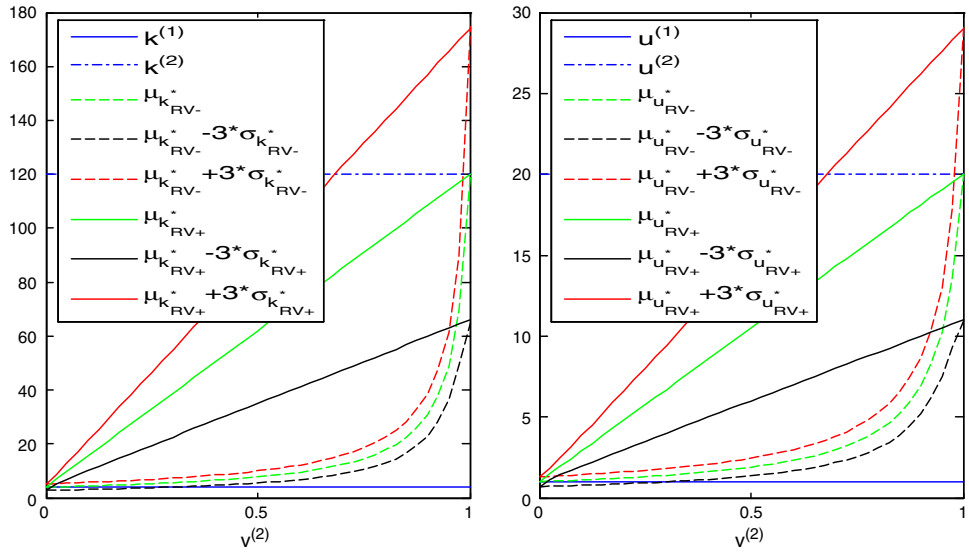
as $\alpha_{k^{(1)}} = \sigma_{k^{(1)}}/\mu_{k^{(1)}} = 0.1$, $\alpha_{u^{(1)}} = \sigma_{u^{(1)}}/\mu_{u^{(1)}} = 0.1$, $\alpha_{k^{(2)}} = \sigma_{k^{(2)}}/\mu_{k^{(2)}} = 0.15$, $\alpha_{u^{(2)}} = \sigma_{u^{(2)}}/\mu_{u^{(2)}} = 0.15$, $\alpha_{V^{(2)}} = \sigma_{V^{(2)}}/\mu_{V^{(2)}} = 0.05$ simultaneously, the random homogenization results are presented in Tables 3–7 and Figs. 1–7. Results obtained from two different methods are generally given in the form of mean value μ_{y^*}, upper bound $\mu_{y^*} + 3\sigma_{y^*}$ and lower bound $\mu_{y^*} - 3\sigma_{y^*}$ according to $\pm 3\sigma$ rule.

3.1. Computational results of analytical bounds and estimates by two methods

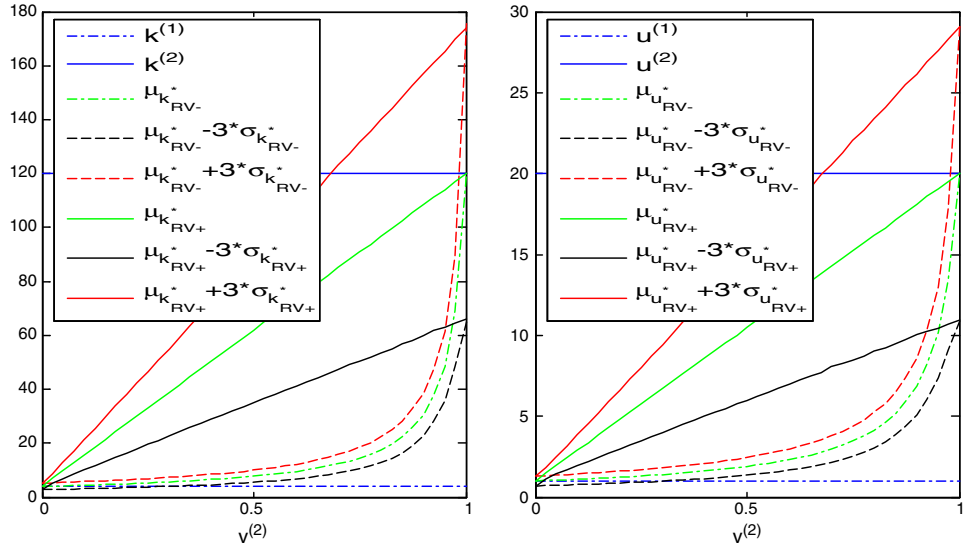
In Table 3, random effective bounds and estimates for a single microstructure are given when $V^{(2)} = 0.5$ and correlation coefficients $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0.5$.

Table 7
correlation coefficients of k^* and u^* when $V^{(2)} = 0.5$ by Monte-Carlo Method.

	$\rho_{k^{(1)}u^{(1)}} = 0.1$ $\rho_{k^{(2)}u^{(2)}} = 0.1$	$\rho_{k^{(1)}u^{(1)}} = 0.5$ $\rho_{k^{(2)}u^{(2)}} = 0.1$	$\rho_{k^{(1)}u^{(1)}} = 0.1$ $\rho_{k^{(2)}u^{(2)}} = 0.5$	$\rho_{k^{(1)}u^{(1)}} = 0.5$ $\rho_{k^{(2)}u^{(2)}} = 0.5$	$\rho_{k^{(1)}u^{(1)}} = 0.9$ $\rho_{k^{(2)}u^{(2)}} = 0.1$	$\rho_{k^{(1)}u^{(1)}} = 0.1$ $\rho_{k^{(2)}u^{(2)}} = 0.9$	$\rho_{k^{(1)}u^{(1)}} = 1$ $\rho_{k^{(2)}u^{(2)}} = 1$
$\rho_{k^*_{DF}u^*_{DF}}$	0.3812	0.6542	0.4019	0.6734	0.8717	0.4329	0.9612
$\rho_{k^*_{SC}u^*_{SC}}$	0.6039	0.6809	0.6547	0.7318	0.7669	0.7140	0.8895
$\rho_{k^*_{MW}u^*_{MW}}$	0.2925	0.6124	0.2885	0.6290	0.9046	0.3068	0.9883
$\rho_{k^*_{HS+}u^*_{HS+}}$	0.8390	0.8468	0.9079	0.9118	0.8545	0.9669	0.9929
$\rho_{k^*_{HS-}u^*_{HS-}}$	0.2862	0.6206	0.2971	0.6167	0.9035	0.3048	0.9879
$\rho_{k^*_{HS+}u^*_{HS-}}$	0.2283	0.3252	0.2389	0.3219	0.4023	0.2581	0.4076
$\rho_{k^*_{HS-}u^*_{HS+}}$	0.0383	0.0586	0.0677	0.0884	0.0756	0.0982	0.1495
$\rho_{k^*_{RV+}u^*_{RV+}}$	0.0970	0.0991	0.4997	0.5017	0.0976	0.8996	0.9999
$\rho_{k^*_{RV-}u^*_{RV-}}$	0.0990	0.4969	0.1012	0.4960	0.8960	0.1028	0.9996
$\rho_{k^*_{RV+}u^*_{RV-}}$	0.0093	0.0146	0.0457	0.0535	0.0260	0.0730	0.1036
$\rho_{k^*_{RV-}u^*_{RV+}}$	0.0016	0.0236	0.0249	0.0444	0.0436	0.0529	0.0881

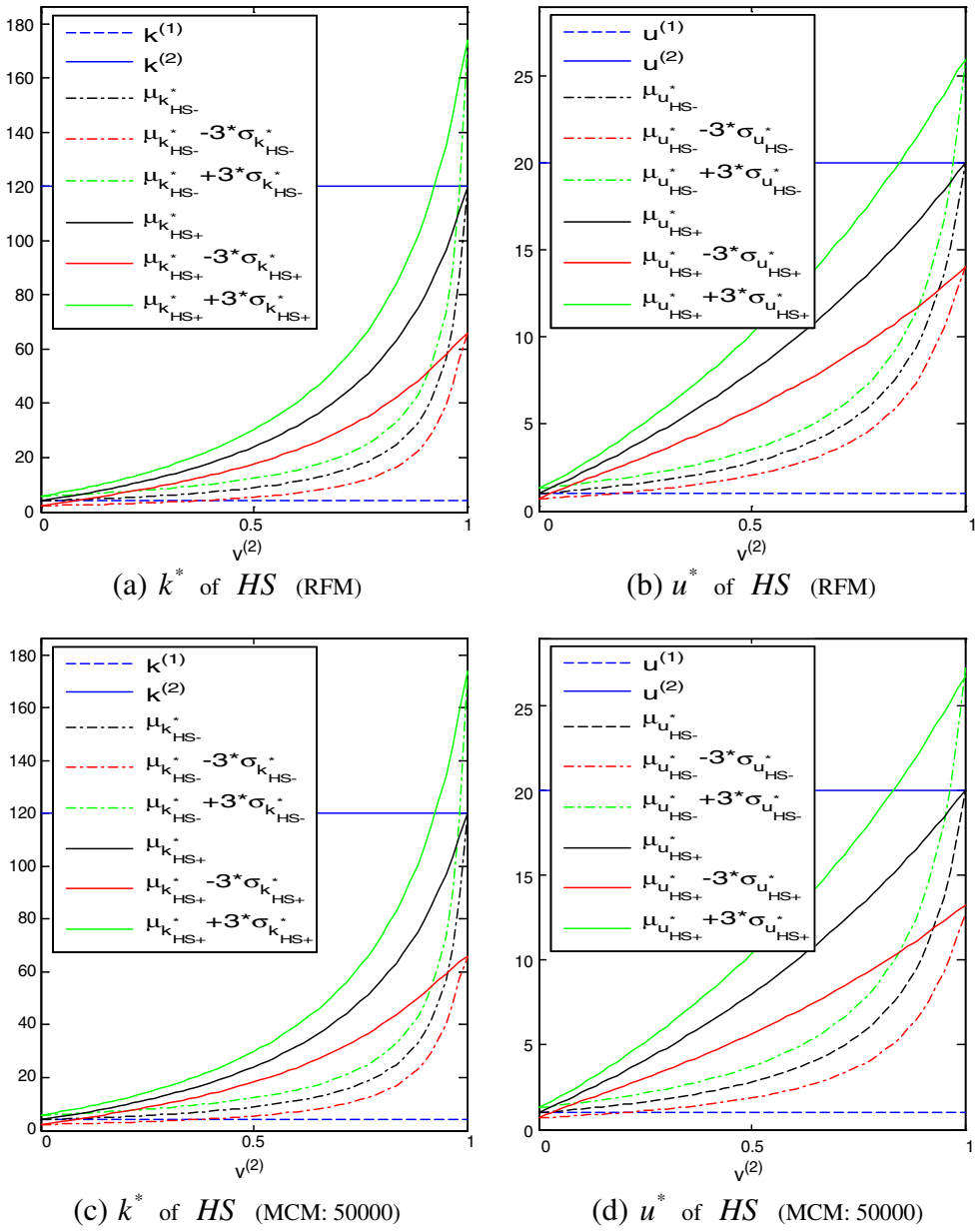


(a) k^* of RV (RFM) (b) u^* of RV (RFM)



(c) k^* of RV (MCM: 50000) (d) u^* of RV (MCM: 50000)

Fig. 1. Curves of RV model.



(a) k^* of HS (RFM) (b) u^* of HS (RFM)

(c) k^* of HS (MCM: 50000) (d) u^* of HS (MCM: 50000)

Fig. 2. Curves of HS model.

In Figs. 1–5, curves of k^* and u^* of RV, HS, MW, SC and DF models obtained by the two methods are given when correlation coefficients $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0.5$.

From Table 3 and Figs. 1–5, one observes that.

(1) As analytical bounds, HS is tighter than RV. From Table 3, one can get $\sigma_{k^*_{RV+}} - \sigma_{k^*_{RV-}} > \sigma_{k^*_{HS+}} - \sigma_{k^*_{HS-}}$, which is showed in Figs. 1,2, e.g. the distances between curves $\mu_{k^*_{RV-}} - 3\sigma_{k^*_{RV-}}$ and $\mu_{k^*_{RV+}} - 3\sigma_{k^*_{RV+}}$ are greater than those between curves $\mu_{k^*_{HS-}} - 3\sigma_{k^*_{HS-}}$ and $\mu_{k^*_{HS+}} - 3\sigma_{k^*_{HS+}}$. That is, compared with RV, with the tightening of mean value HS, its upper bounds and lower bounds tighten too. But one cannot get the similar conclusion about estimates MW, SC and DF models.

(2) Generally, computational results and curves obtained by RFM are in very good agreement with those obtained by MCM.

3.2. Effects of randomness on final results

In order to inspect the impact of random variables on random effective properties k^* and u^*, different random models are taken for HS bounds by RFM in Table 4 when $V^{(2)} = 0.5$ and $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0.5$. From the results of random models 1–5, when random dispersion of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ is equal, both $k^{(1)}$ and $u^{(1)}$ has a greater impact on $k^*_{HS^-}$ and $u^*_{HS^-}$ respectively, while $k^{(2)}$ and $u^{(2)}$ has the greater impact on $k^*_{HS^+}$ and $u^*_{HS^+}$ respectively, but one can draw the conclusion from σ_{k^*} and σ_{u^*} that the randomness of $V^{(2)}$ has the greatest effect on k^* and u^*, that is, for constituent with greater bulk and shear moduli, its random volume fraction should be paid more attention to. From the results of random models 6–7, when the random dispersion of $k^{(1)}$ and $u^{(1)}$ equals that of $k^{(2)}$ and $u^{(2)}$, the former has the greater effect on $k^*_{HS^-}$ and $u^*_{HS^-}$, while the latter has the greater effect on $k^*_{HS^+}$ and $u^*_{HS^+}$. In a word, the randomness of different constituents and their volume fractions has the obvious impacts on different effective properties,

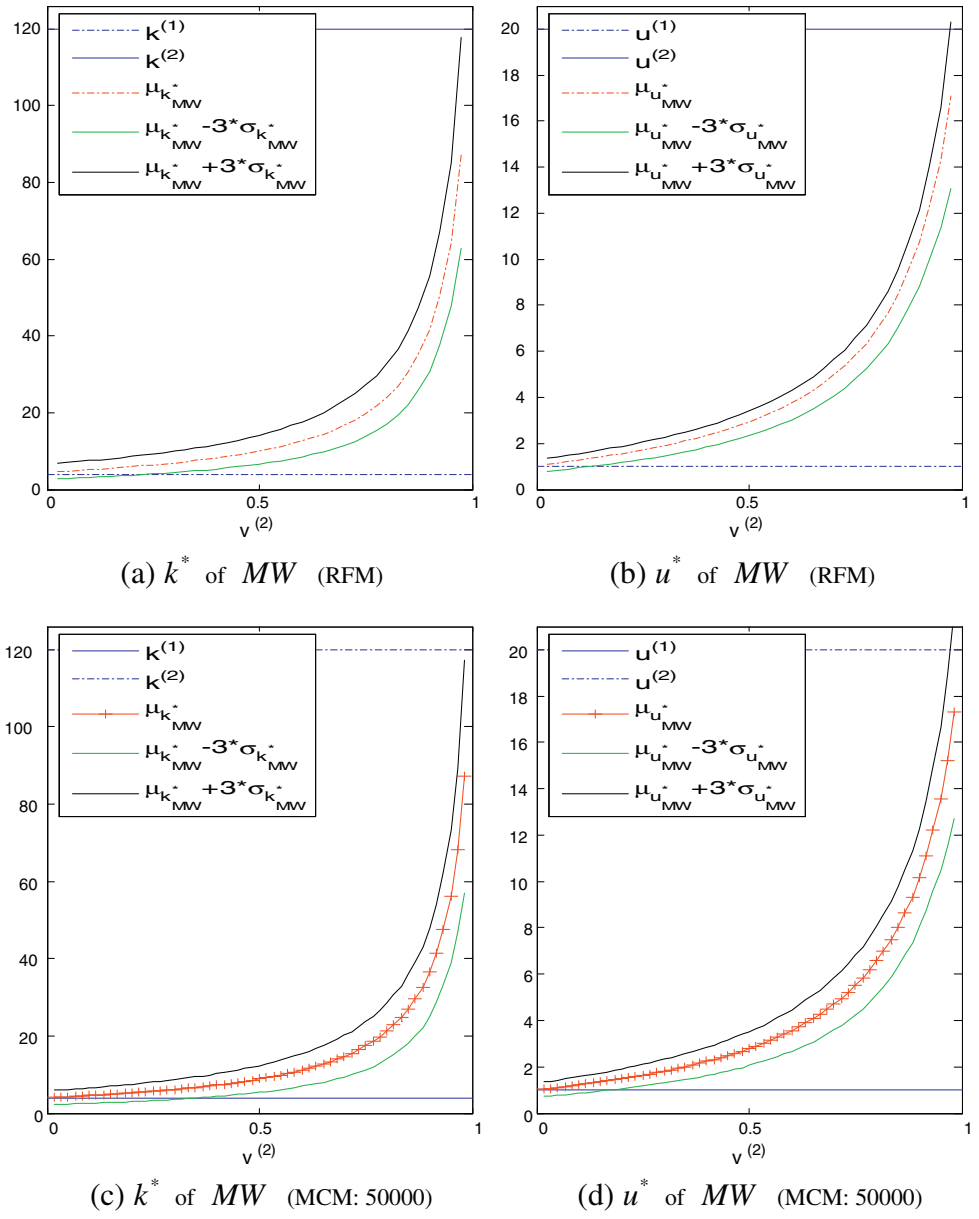


Fig. 3. Curves of MW model.

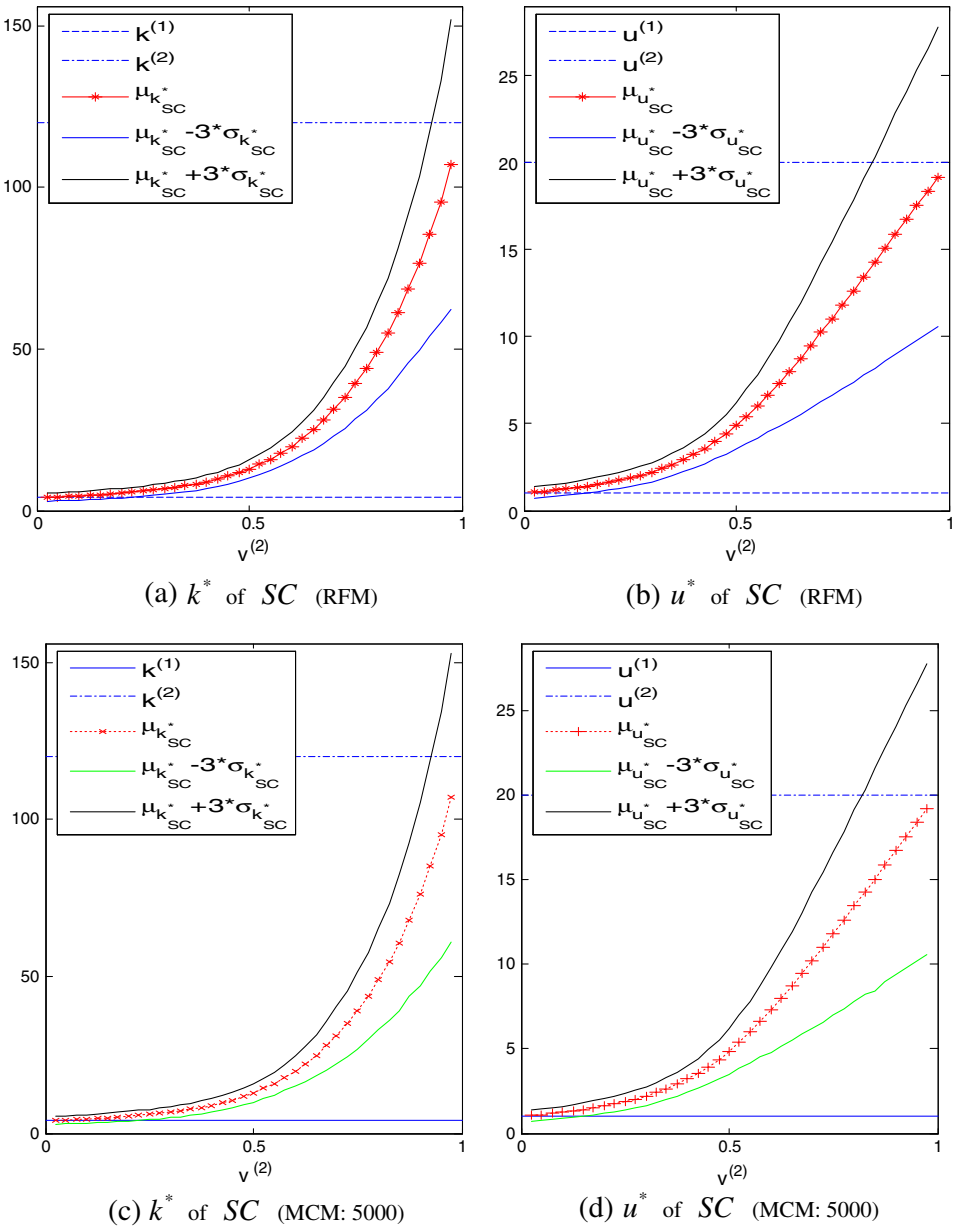
which should be carefully considered in the designs of composite materials and structures and even in the structural probability analysis. Note the impact can also be inspected by MCM.

3.3. Correlation of random effective properties

Correlation coefficient is typically referred to as linear correlation coefficient which is a measurement of the linear correlation degree between random variables. In order to inspect the influence of correlation among random variables on effective properties, mean square deviations of SC model are computed by MCM. The number of realizations in MCM is 5000. σ_{k^*} and σ_{u^*} in Table 5 are obtained when $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0.5$ and taking $V^{(2)}$ as different values, and results in Table 6 are given when $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$ are taken as different values and $V^{(2)} = 0.5$.

From Table 5, effects of correlation coefficients on σ_{k^*}, σ_{u^*}, $\sigma^{\#}_{k^*}$ and $\sigma^{\#}_{u^*}$ obviously increase with the increase of $V^{(2)}$, and the effects are greatest when $V^{(2)} = 0.9$ and σ_{k^*} and σ_{u^*} are much greater than $\sigma^{\#}_{k^*}$ and $\sigma^{\#}_{u^*}$ at this time. Moreover, it demonstrates once again the conclusion from Table 4 that the randomness of $V^{(2)}$ has a greater influence on randomness of k^* and u^*. From Table 6, with the change from non-correlation to correlation among random variables, effects of ρ on σ_{k^*} and σ_{u^*} increase and this effect is maximized when $\rho = 1$, which demonstrates that the correlation between random variables should not be omitted.

Since the correlation among random variables is considered, the random effective properties definitely have correlation too. In Table 7, the correlation coefficients of k^* and u^* of DF, SC, MW, HS, RV models are given when taking different correlation coefficients $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$ at $V^{(2)} = 0.5$. Simulation times of MCM for every model are the same as those in Figs. 1–5. The curves of correlation coefficients of RV, MW, SC and DF are given in Figs. 6,7 by MCM. It should be noted that if one can get the joint probability density function of k^* and u^* when assuming that probability distribution density of every random variable is known, the correlation coefficient of k^* and u^* can then be obtained by probability theory. But this approach cannot be pursued in the context of RFM. So the work in Table 7 and Figs. 6,7 can only be done by MCM.



(a) k^* of SC (RFM) (b) u^* of SC (RFM)

(c) k^* of SC (MCM: 5000) (d) u^* of SC (MCM: 5000)

Fig. 4. Curves of SC model.

From Table 7 and Figs. 6,7, the following conclusions can be obtained.

(1) Generally, the strong correlation exist between $k^*_{HS^+}$ and $u^*_{HS^+}$, k^*_{SC} and u^*_{SC} even if $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$ are very small, ($\rho_{k^*_{SC}u^*_{SC}} = 0.6039$ and $\rho_{k^*_{HS^+}u^*_{HS^+}} = 0.8390$ when $\rho_{k^{(1)}u^{(1)}} = \rho_{k^{(2)}u^{(2)}} = 0.1$); the correlations of k^*_{DF} and u^*_{DF}, $k^*_{HS^-}$ and $u^*_{HS^-}$, k^*_{MW} and u^*_{MW} take the second place; so these correlations should be fully considered in the design of materials and structures.

(2) For RV model, the correlation coefficients $\rho_{k^*_{RV^+}u^*_{RV^+}}$ and $\rho_{k^*_{RV^-}u^*_{RV^-}}$ clearly illustrate the increasing trend with the increase of the $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$; the correlation of $k^*_{RV^+}$ and $u^*_{RV^-}$, $k^*_{RV^-}$ and $u^*_{RV^+}$ can be omitted; $\rho_{k^{(1)}u^{(1)}}$ has a greater effect on $\rho_{k^*_{RV^-}u^*_{RV^-}}$ while $\rho_{k^{(2)}u^{(2)}}$ has a greater effect on $\rho_{k^*_{RV^+}u^*_{RV^+}}$.

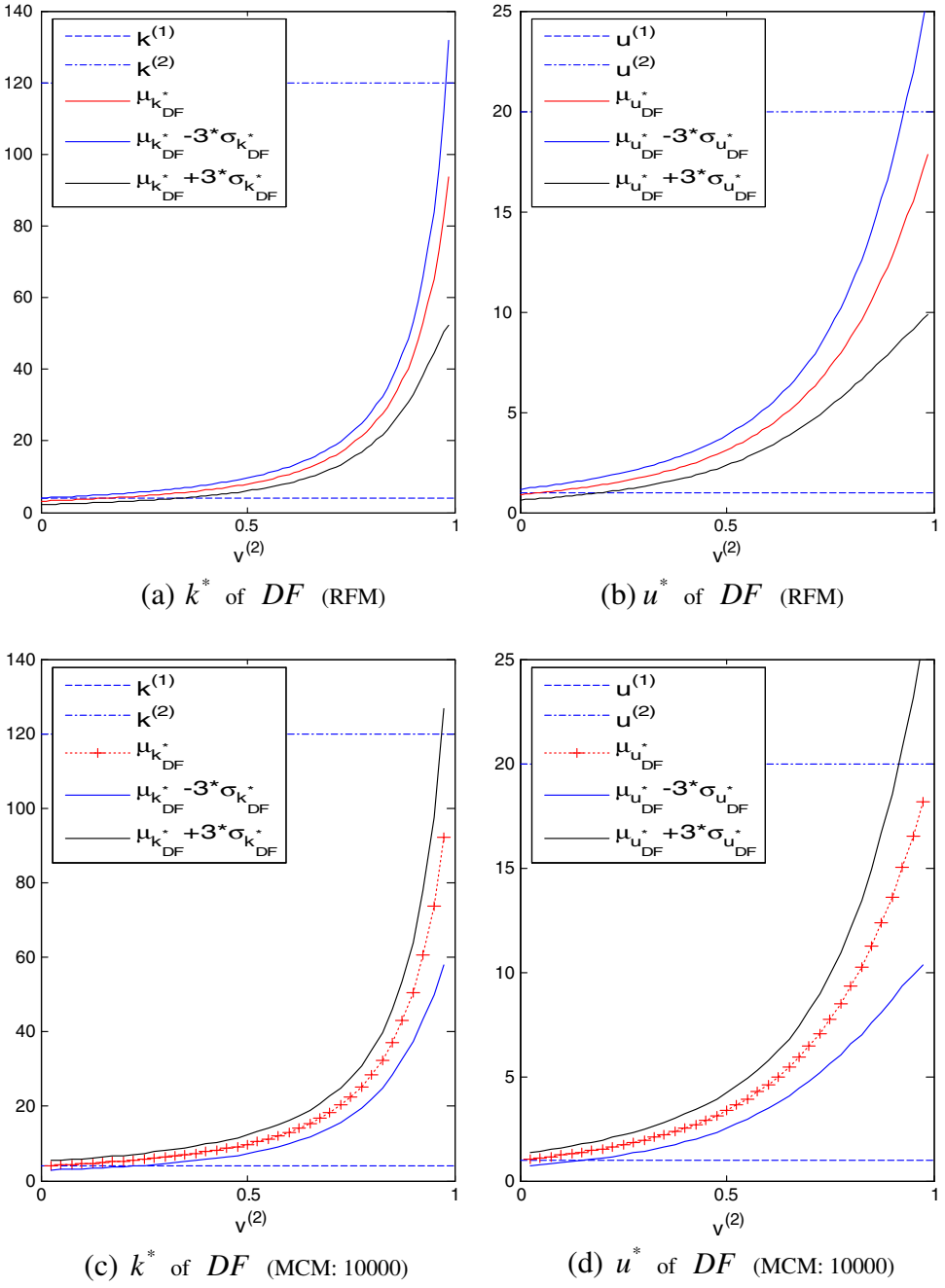
(3) For HS model, the correlation between $k^*_{HS^+}$ and $u^*_{HS^+}$ is strongest and its change with $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$ is not very obvious, and the correlation between $k^*_{HS^-}$ and $u^*_{HS^-}$ comes second; $\rho_{k^{(1)}u^{(1)}}$ has a stronger effect on $\rho_{k^*_{HS^-}u^*_{HS^-}}$ than $\rho_{k^{(2)}u^{(2)}}$ does; the correlations of $k^*_{HS^-}$ and $u^*_{HS^+}$, especially the correlation between $k^*_{HS^+}$ and $u^*_{HS^-}$, can be omitted.

(4) For MW, SC and DF models, correlations of k^*_{MW} and u^*_{MW}, k^*_{DF} and u^*_{DF} are affected strongly by $\rho_{k^{(1)}u^{(1)}}$; all correlations increase with the increase of $\rho_{k^{(1)}u^{(1)}}$ and $\rho_{k^{(2)}u^{(2)}}$.

Because the accuracy of results obtained by MCM definitely depends on the simulation number, there are local fluctuations in the curves of Figs. 6 and 7. With the increase of simulation number, curves in Figs. 8 and 9 are much smoother than those in Figs. 6 and 7, but the cost of the former is much more expensive than that of the latter.

4. Conclusions

The subject of homogenization is devoted to the determination of the properties of a homogeneous material that approximates the behavior of the original heterogeneous problem. These properties



(a) k^* of DF (RFM) (b) u^* of DF (RFM)

(c) k^* of DF (MCM: 10000) (d) u^* of DF (MCM: 10000)

Fig. 5. Curves of DF model.

are termed macroscopic, effective, apparent properties in various sources. Concatenated in a single sentence, these are the properties that "appear" to approximate the "effects" of the microscale features on the "macroscale". In linear elasticity problem, these properties can be illuminated by analytical bounds (Reuss–Voigt bounds and Hashin–Shtrikman bounds) and estimates (Maxwell model, Self-Consistent model and Differential model) after the homogenization. In this work, the homogenization of heterogeneous materials with randomness was pursued in the context of linear elasticity based on analytical estimates and bounds. In order to compute the mean value and mean square deviation associated with the effective properties when one or more micromechanical variables display randomness, two methods were employed and compared. The Random Factor Method (RFM) was observed to deliver rapid numerical results of comparable accuracy with the computationally intensive Monte-Carlo Method (MCM). For example, for the Self-Consistent estimate, the time consumed by RFM is only 0.83% of that cost by MCM based on 5000 realizations to get the curves and computational results when considering all randomness and correlation among materials' variables at the same time. On the other hand, although the correlation of constituents' properties can be fully considered by RFM during the solution of the mean square deviation of the effective properties, the correlation coefficients among the effective properties cannot be computed because one cannot obtain the probability distribution density of the effective properties by RFM, which is the limitation to the application of RFM. MCM is a more suitable numerical simulation method for this purpose.

Generally, the effective properties are not normal random variables even if constituents' parameters are normal variables, except

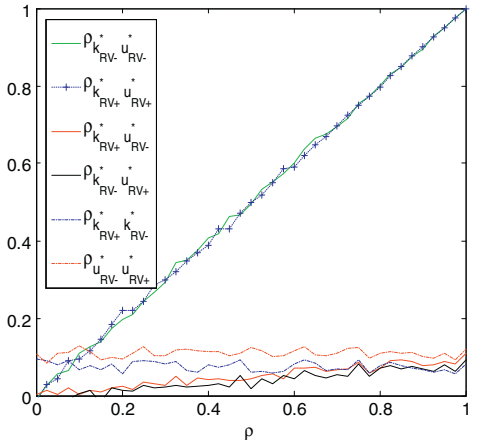


Fig. 6. Correlation coefficient of RV (MCM: 100,000).

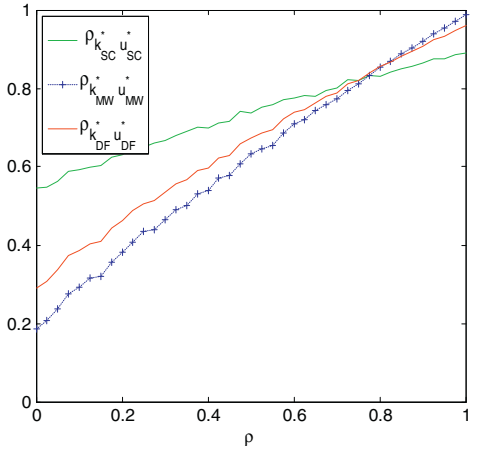


Fig. 7. Correlation coefficient of MW, SC, DF (MCM: 10,000).

that the system is linear and has only one Gaussian random variable in the input. But the effective properties can still be similarly assumed as Gaussian variables especially when samples large enough are taken for RV, HS, MW, SC and DF models by MCM according to the central limit theorem. Based on this assumption, the mean value and mean square deviation can be obtained by several mature methods and $\pm 3\sigma$ rule is used to illustrate the random dispersion degree of the effective properties. Another possible consideration to replace the $\pm 3\sigma$ rule may be the use of the third central probabilistic moment in the future work.

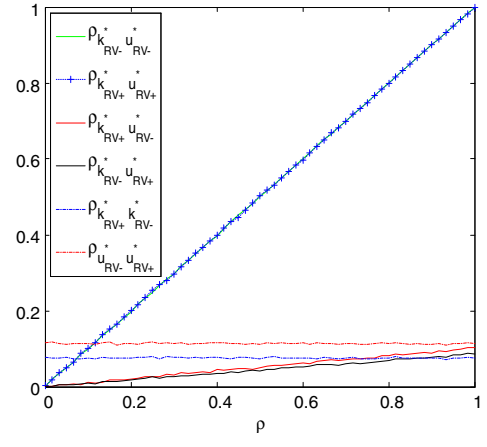


Fig. 8. Correlation coefficient of RV (MCM: 300,000).

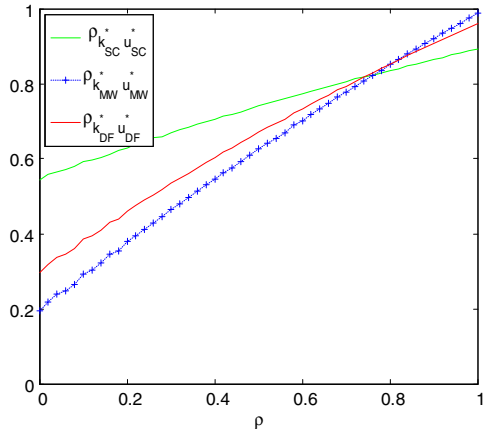


Fig. 9. Correlation coefficient of MW, SC, DF (MCM: 100,000).

From the tables and figures, the randomness of different constituents' properties and the volume fractions has the obvious impact on different effective properties, e.g. volume fraction $V^{(2)}$ has the greatest effect on k^* and u^*, moreover, the correlations existing among $k^*_{HS^+}$ and $u^*_{HS^+}$, k^*_{SC} and u^*_{SC}, k^*_{DF} and u^*_{DF}, $k^*_{HS^-}$ and $u^*_{HS^-}$, k^*_{MW} and u^*_{MW} cannot be omitted, etc. All of these conclusions should be fully considered in the engineering application of materials and structures.

Although the results presented in this work were limited to linear elasticity analysis based on analytical estimates and bounds, the RFM is extensible to the homogenization of a heterogeneous medium with arbitrary microstructure in the generally inelastic finite deformation regime. Computational approaches that are suitable to this purpose are currently being pursued by the authors. In a word, the Random Factor Method, when applicable, is an efficient tool of assessing uncertainty in the response of composite materials.

Appendix A

$$\begin{aligned}\sigma^2_{k^*} = &\left(\frac{\partial k^*}{\partial k^{(1)}}\right)^2 \cdot \sigma^2_{k^{(1)}} + \left(\frac{\partial k^*}{\partial k^{(2)}}\right)^2 \cdot \sigma^2_{k^{(2)}} + \left(\frac{\partial k^*}{\partial u^{(1)}}\right)^2 \cdot \sigma^2_{u^{(1)}} \\ &+ \left(\frac{\partial k^*}{\partial u^{(2)}}\right)^2 \cdot \sigma^2_{u^{(2)}} + \left(\frac{\partial k^*}{\partial V^{(2)}}\right)^2 \cdot \sigma^2_{V^{(2)}} \\ &+ \left(\frac{\partial k^*}{\partial k^{(1)}}\right) \cdot \left(\frac{\partial k^*}{\partial u^{(1)}}\right) \cdot \sigma_{k^{(1)}} \cdot \sigma_{u^{(1)}} \cdot \rho_{k^{(1)}u^{(1)}} + \left(\frac{\partial k^*}{\partial k^{(2)}}\right) \left(\frac{\partial k^*}{\partial u^{(2)}}\right)^2 \\ &\cdot \sigma_{k^{(2)}} \cdot \sigma_{u^{(2)}} \cdot \rho_{k^{(2)}u^{(2)}}, \end{aligned} \tag{1}$$

$$\begin{aligned}\sigma^2_{u^*} = &\left(\frac{\partial u^*}{\partial k^{(1)}}\right)^2 \cdot \sigma^2_{k^{(1)}} + \left(\frac{\partial u^*}{\partial k^{(2)}}\right)^2 \cdot \sigma^2_{k^{(2)}} + \left(\frac{\partial u^*}{\partial u^{(1)}}\right)^2 \cdot \sigma^2_{u^{(1)}} \\ &+ \left(\frac{\partial u^*}{\partial u^{(2)}}\right)^2 \cdot \sigma^2_{u^{(2)}} + \left(\frac{\partial u^*}{\partial V^{(2)}}\right)^2 \cdot \sigma^2_{V^{(2)}} \\ &+ \left(\frac{\partial u^*}{\partial k^{(1)}}\right) \cdot \left(\frac{\partial u^*}{\partial u^{(1)}}\right) \cdot \sigma_{k^{(1)}} \cdot \sigma_{u^{(1)}} \cdot \rho_{k^{(1)}u^{(1)}} + \left(\frac{\partial u^*}{\partial k^{(2)}}\right) \left(\frac{\partial u^*}{\partial u^{(2)}}\right) \\ &\cdot \sigma_{k^{(2)}} \cdot \sigma_{u^{(2)}} \cdot \rho_{k^{(2)}u^{(2)}}, \end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial k^*}{\partial k^{(1)}} &= \frac{\partial G}{\partial k^*}\frac{\partial k^*}{\partial k^{(1)}} + \frac{\partial g}{\partial u^*}\frac{\partial u^*}{\partial k^{(1)}} + \frac{\partial G}{\partial k^{(1)}} \\ &= \frac{V^{(1)}}{k^{(1)}+g} - \frac{V^{(1)}(k^{(1)}-k^*)}{(k^{(1)}+g)^2} \\ &\quad + \frac{\partial k^*}{\partial k^{(1)}} \cdot \left(1 - \frac{V^{(1)}}{k^{(1)}+g} - \frac{V^{(2)}}{k^{(2)}+g}\right) \\ &\quad - \frac{\partial u^*}{\partial k^{(1)}}\left(\frac{V^{(1)}(k^{(1)}-k^*)}{(k^{(1)}+g)^2}\frac{\partial g}{\partial u^*} + \frac{V^{(2)}(k^{(2)}-k^*)}{(k^{(2)}+g)^2}\frac{\partial g}{\partial u^*}\right), \end{aligned} \tag{3}$$

$$\frac{\partial u^*}{\partial k^{(1)}} = \frac{\partial F}{\partial u^*}\frac{\partial u^*}{\partial k^{(1)}} + \frac{\partial f}{\partial k^*}\frac{\partial u^*}{\partial k^{(1)}} + \frac{\partial f}{\partial u^*}\frac{\partial k^*}{\partial k^{(1)}} + \frac{\partial F}{\partial k^{(1)}}$$
$$= \frac{\partial u^*}{\partial k^{(1)}}\cdot\left(1 - \frac{V^{(1)}}{u^{(1)}+f} - \frac{V^{(1)}(u^{(1)}-u^*)}{(u^{(1)}+f)^2}\frac{\partial f}{\partial u^*} - \frac{V^{(2)}}{u^{(2)}+f} - \frac{V^{(2)}(u^{(2)}-u^*)}{(u^{(2)}+f)^2}\frac{\partial f}{\partial u^*}\right) - \frac{\partial k^*}{\partial k^{(1)}}\left(\frac{V^{(1)}(u^{(1)}-u^*)}{(u^{(1)}+f)^2}\frac{\partial f}{\partial k^*} + \frac{V^{(2)}(u^{(2)}-u^*)}{(u^{(2)}+f)^2}\frac{\partial f}{\partial k^*}\right), \tag{4}$$

$$\frac{\partial k^*}{\partial V^{(2)}} = \frac{\partial G}{\partial k^*}\frac{\partial k^*}{\partial V^{(2)}} + \frac{\partial g}{\partial u^*}\frac{\partial u^*}{\partial V^{(2)}} + \frac{\partial G}{\partial V^{(2)}}$$
$$= \frac{k^{(2)}-k^*}{k^{(2)}+g} - \frac{k^{(1)}-k^*}{k^{(1)}+g} + \frac{\partial k^*}{\partial V^{(2)}}\cdot\left(1 - \frac{V^{(1)}}{k^{(1)}+g} - \frac{V^{(2)}}{k^{(2)}+g}\right) - \frac{\partial u^*}{\partial V^{(2)}}\left(\frac{V^{(1)}(k^{(1)}-k^*)}{(k^{(1)}+g)^2}\frac{\partial g}{\partial u^*} + \frac{V^{(2)}(k^{(2)}-k^*)}{(k^{(2)}+g)^2}\frac{\partial g}{\partial u^*}\right), \tag{5}$$

$$\frac{\partial u^*}{\partial V^{(2)}} = \frac{\partial F}{\partial u^*}\frac{\partial u^*}{\partial V^{(2)}} + \frac{\partial f}{\partial u^*}\frac{\partial u^*}{\partial V^{(2)}} + \frac{\partial f}{\partial k^*}\frac{\partial k^*}{\partial V^{(2)}} + \frac{\partial F}{\partial V^{(2)}}$$
$$= \frac{u^{(2)}-u^*}{u^{(2)}+f} - \frac{u^{(1)}-u^*}{u^{(1)}+f} + \frac{\partial u^*}{\partial V^{(2)}}\cdot\left(1 - \frac{V^{(1)}}{u^{(1)}+f} - \frac{V^{(1)}(u^{(1)}-u^*)}{(u^{(1)}+f)^2}\frac{\partial f}{\partial u^*} - \frac{V^{(2)}}{u^{(2)}+f} - \frac{V^{(2)}(u^{(2)}-u^*)}{(u^{(2)}+f)^2}\frac{\partial f}{\partial u^*}\right) - \frac{\partial k^*}{\partial V^{(2)}}\cdot\left(\frac{V^{(1)}(u^{(1)}-u^*)}{(u^{(1)}+f)^2}\frac{\partial f}{\partial k^*} + \frac{V^{(2)}(u^{(2)}-u^*)}{(u^{(2)}+f)^2}\frac{\partial f}{\partial k^*}\right), \tag{6}$$

$$\frac{\partial k^*}{\partial k^{(1)}} = (e_2 - h_2) + \frac{\partial k^*}{\partial k^{(1)}}(1 - e_2 - i_2) - \frac{\partial u^*}{\partial k^{(1)}}(h_2 + j_2)\cdot c_0, \tag{7}$$

$$\frac{\partial k^*}{\partial k^{(2)}} = (i_2 - j_2) + \frac{\partial k^*}{\partial k^{(2)}}(1 - e_2 - i_2) - \frac{\partial u^*}{\partial k^{(2)}}(h_2 + j_2)\cdot c_0, \tag{8}$$

$$\frac{\partial k^*}{\partial u^{(1)}} = 0 + \frac{\partial k^*}{\partial u^{(1)}}(1 - e_2 - i_2) - \frac{\partial u^*}{\partial u^{(1)}}(h_2 + j_2)\cdot c_0, \tag{9}$$

$$\frac{\partial k^*}{\partial u^{(2)}} = 0 + \frac{\partial k^*}{\partial u^{(2)}}(1 - e_2 - i_2) - \frac{\partial u^*}{\partial u^{(2)}}(h_2 + j_2)\cdot c_0, \tag{10}$$

$$\frac{\partial k^*}{\partial V^{(2)}} = (l_2 - n_2) + \frac{\partial k^*}{\partial v^{(2)}}(1 - e_2 - i_2) - \frac{\partial u^*}{\partial v^{(2)}}(h_2 + j_2)\cdot c_0, \tag{11}$$

$$\frac{\partial u^*}{\partial k^{(1)}} = 0 + \frac{\partial u^*}{\partial k^{(1)}}(1 - e_1 - h_1\cdot b - i_1 - j_1\cdot b) - \frac{\partial k^*}{\partial k^{(1)}}(h_1\cdot a + j_1\cdot a), \tag{12}$$

$$\frac{\partial u^*}{\partial k^{(2)}} = 0 + \frac{\partial u^*}{\partial k^{(2)}}(1 - e_1 - h_1\cdot b - i_1 - j_1\cdot b) - \frac{\partial k^*}{\partial k^{(2)}}(h_1\cdot a + j_1\cdot a), \tag{13}$$

$$\frac{\partial u^*}{\partial u^{(1)}} = (e_1 - h_1) + \frac{\partial u^*}{\partial u^{(1)}}(1 - e_1 - h_1\cdot b - i_1 - j_1\cdot b) - \frac{\partial k^*}{\partial u^{(1)}}(h_1\cdot a + j_1\cdot a), \tag{14}$$

$$\frac{\partial u^*}{\partial u^{(2)}} = (i_1 - j_1) + \frac{\partial u^*}{\partial u^{(2)}}(1 - e_1 - h_1\cdot b - i_1 - j_1\cdot b) - \frac{\partial k^*}{\partial u^{(2)}}(h_1\cdot a + j_1\cdot a), \tag{15}$$

$$\frac{\partial u^*}{\partial V^{(2)}} = (l_1 - n_1) + \frac{\partial u^*}{\partial v^{(2)}}(1 - e_1 - h_1\cdot b - i_1 - j_1\cdot b) - \frac{\partial k^*}{\partial v^{(2)}}(h_1\cdot a + j_1\cdot a), \tag{16}$$

$$\frac{\partial}{\partial k^{(1)}}\left(\frac{\partial k^*}{\partial V^{(2)}}\right) = \frac{\partial}{\partial V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(1)}}\right) = \frac{1}{1-V^{(2)}}\left(\frac{\partial k^*}{\partial k^{(1)}} + \frac{\partial g}{\partial u^*}\cdot\frac{\partial u^*}{\partial k^{(1)}}\right)\cdot\frac{k^{(2)}-k^*}{k^{(2)}+g} + \frac{k^*+g}{1-V^{(2)}}\cdot\frac{1}{k^{(2)}+g}\left(-\frac{\partial k^*}{\partial k^{(1)}}\right) - \frac{k^*+g}{1-V^{(2)}}\cdot\frac{k^{(2)}-k^*}{(k^{(2)}+g)^2}\cdot\frac{\partial g}{\partial u^*}\cdot\frac{\partial u^*}{\partial k^{(1)}}, \tag{17}$$

$$\frac{\partial}{\partial k^{(1)}}\left(\frac{\partial u^*}{\partial V^{(2)}}\right) = \frac{\partial}{\partial V^{(2)}}\left(\frac{\partial u^*}{\partial k^{(1)}}\right) = \frac{1}{1-V^{(2)}}\left(\frac{\partial u^*}{\partial k^{(1)}} + \frac{\partial f}{\partial k^*}\cdot\frac{\partial k^*}{\partial k^{(1)}} + \frac{\partial f}{\partial u^*}\cdot\frac{\partial u^*}{\partial k^{(1)}}\right)\cdot\frac{u^{(2)}-u^*}{u^{(2)}+f} + \frac{u^*+f}{1-V^{(2)}}\cdot\frac{1}{u^{(2)}+f}\left(-\frac{\partial u^*}{\partial k^{(1)}}\right) - \frac{u^*+f}{1-V^{(2)}}\cdot\frac{u^{(2)}-u^*}{(u^{(2)}+f)^2}\cdot\left(\frac{\partial f}{\partial k^*}\cdot\frac{\partial k^*}{\partial k^{(1)}} + \frac{\partial f}{\partial u^*}\cdot\frac{\partial u^*}{\partial k^{(1)}}\right). \tag{18}$$

References

Aboudi, J., 1991. Mechanics of Composites Materials: A Unified Micromechanical Approach. Elesevier, Amsterdam.

Ali Touran, Edward P., 1992. Monte Carlo technique with correlated random variables. J. Constr. Eng. Manage. 118, 258–272.

Eshelby, J.D., 1957. The elastic field of an ellipsoidal inclusion, and related problems. Proc. Royal Soc. A241, 376–396.

Gao, W., Chen, J.J., Ma, J., Liang, Z.T., 2004. Dynamic response analysis of stochastic frame structures under nonstationary random excitation. AIAA J. 42 (9), 1818–1822.

Hashin, Z., Shtrikman, S., 1962. On some variational principles in anisotropic and nonhomogeneous elasticity. J. Mech. Phys. Solids 10, 335–342.

Idiart, M.I. et al., 2006. Macroscopic behavior and field fluctuations in viscoplastic composites: second order estimates versus full-field simulations. J. Mech. Phys. Solids. 54, 1029–1063.

Kaminski, M., 2001. Stochastic finite element method homogenization of heat conduction problem in fiber composites. Struct. Eng. Mech. 11, 373–392.

Kaminski, M., 2005. Multiscale homogenization of *n*-component composites with semi-elliptical stochastic interface defects. Int. J. Solids Struct. 42 (11–12), 3590–3751.

Kaminski, M., 2007. Generalized perturbation-based stochastic finite element method in elastostatics. Comput. Struct. 85, 586–594.

Kaminski, M., Kleiber, M., 2000. Perturbation based stochastic finite element method for homogenization of two-phase elastic composites. Comput. Struct. 78, 811–826.

Kanit, T., 2006. Apparent and effective physical properties of heterogeneous materials: representativity of samples of two materials from food industry. Comput. Methods Appl. Mech. Eng. 195, 3960–3982.

Lux, J. et al., 2006. Macroscopic thermal properties of real fibrous materials: volume averaging method and 3D image analysis. Int. J. Heat Mass Transfer 49, 1958–1973.

Ma, J., Chen, J.J., Gao, W., 2006. Dynamic response analysis of fuzzy stochastic truss structures under fuzzy stochastic excitation. Comput. Mech. 38, 283–292.

Markovic, D., Ibrahimbegovic, A., 2006. Complementary energy based FE modeling of coupled elasto-plastic and damage behavior for continuum microstructure computations. Comput. Methods Appl. Mech. Eng. 195, 5077–5093.

Mori, T., Tanaka, K., 1973. Average stress in matrix and average energy of materials with misfitting inclusions. Acta Metall. Sinica (English letters) 21, 571–574.

Mura, T., 1987. Micromechanics of Defects in Solids. Martinus Nijhoff, Dordrecht.

Nemat-Nasser, S., Hori, M., 1999. Micromechanics: Overall Properties of Heterogeneous Solids. Elsevier, Amsterdam.

Ostoja-Starzewski, M., 2002. Microstructural randomness versus representative volume element in thermomechanics. J. Appl. Mech. 69, 25–35.

Reuss, A., 1929. Berechnung der Fliessgrenz von Mischkristallen auf Grund der Plastizitatsbedingung fur Einkristalle. Z. Angew. Math. Mech. 9, 49–58.

Sakata, S., Ashida, F., Kojima, T., Zako, M., 2008. Influence of uncertainty in microscopic material property on homogenized elastic property of unidirectional fiber reinforced composites. Theor. Appl. Mech. 56, 67–76.
Sakata, S., Ashida, F., Kojima, T., Zako, M., 2008. Three-dimensional stochastic analysis using a perturbation-based homogenization method for homogenized elastic property of inhomogeneous material considering microscopic uncertainty. Int. J. Solid Struct. 45, 894–907.
Sakata, S., Ashida, F., Zako, M., 2008. Stochastic response analysis of FRP using the second-order perturbation-based homogenization method. J. Solid Mech. Mater. Eng. 2, 70–81.
Samaey, G., Kevrikidis, I.G., Roose, D., 2006. Patch dynamics with buffers for homogenization problems. J. Comput. Phys. 213, 264–287.
Steeves, C.A., Fleck, N.A., 2006. In-plane properties of composite laminates with through-thickness pin reinforcement. Int. J. Solids Struct. 43, 3197–3212.
Torquato, S., 2002. Random Heterogeneous Materials: Microstructure and Macroscopic Properties. Springer.
Voigt, W., 1889. Uber die Beziehung zwischen den beiden Elastizitatskonstanten isotroper Korper. Wied. Ann. 38, 573–587.
Xu, X.F., Brady, L.G., 2005. Computational stochastic homogenization of random media elliptic problems using Fourier Galerkin method. Finite Elem. Anal. Des. 42, 613–622.
Zohdi, I.I., Wriggers, P., 2005. Introduction to Computational Micromechanics. Springer, Berlin, Heidelberg, New York.