Extendable local partial clones

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Abstract

We investigate interpolation and extrapolation properties of composition-closed sets of partial operations defined on an infinite set $E$. Considering local completeness (interpolation property) the structure of a maximal local partial clone is described via its intersections with the full partial clones on every finite $k$-element ($k \geq 2$) subset of $E$. The criteria are established which characterize any finite domain partial operation that can be extended to an everywhere defined operation from the same local partial clone as well as the criteria describing a local partial clone (called extendable) in which every finite domain partial operation is extendable (extrapolation properties). Next the full list of partial orders on the countable set such that the partial clones of their partial $n$-endomorphisms are extendable is obtained. Finally, based on these criteria the description of all extendable maximal local partial clones defined on the countable set is provided.

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1. Introduction and basic definitions

Let $E$ be an infinite set. For an integer $n \geq 1$ an $n$-ary partial operation $f$ on $E$ (an $n$-ary partial function of infinite-valued logic) is a map from a subset Dom($f$) of $E^n$ (called the domain of $f$) into $E$, $f : \text{Dom}(f) \to E$. Denote $P^n(E)$ the set of all $n$-ary partial operations on $E$ including the empty operation $p_n$ having an empty domain. Set $P(E) = \bigcup_{n \geq 1} P^n(E)$. Furthermore, denote $O^n(E) = \{f \in P^n(E) : \text{Dom}(f) = E^n\} (n = 1, 2, \ldots )$. Then $O(E) = \bigcup_{n \geq 1} O^n(E)$ is the set of all (everywhere defined) operations on $E$.

A set $F, F \subseteq P(E)$, of partial operations (everywhere defined operations) closed under compositions of its elements and containing all projections is called a partial clone (respectively, a clone) on $E$. Then $P(E)(O(E))$ is called the full partial clone (respectively, the full clone). Denote $[F]$ a partial clone generated by the set $F$. Next a partial clone (clone) $A$ is called maximal if there is no partial clone (respectively, clone) $B$ such that $A \subset B \subset P(E)$ (respectively, $A \subset B \subset O(E)$).

The interpolation property of composition of operations defined on an infinite set is described by a topological (which is non-algebraic, see [3,7]) closure system called local closure. For any $f \in P^n(E) (n \geq 1)$ and a finite non-single subset $A, A \subset E$, denote by $f|_A$ the restriction of $f$ to the set $A : \text{Dom}(f|_A) = \text{Dom}(f) \cap A^n$ and $f|_A = f$ on $\text{Dom}(f|_A)$. Then local closure of a set $F, F \subseteq P(E)$, denote it $\text{Loc}(F)$, is defined as follows: $g \in \text{Loc}(F)$ if for every finite non-single subset $A, A \subset E$, and $g|_A$ is non-void there exists $f \in [F]$ such that $g|_A = f$ on $\text{Dom}(g|_A)$.

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Clearly \( \text{Loc}(F) = \text{Loc}([F]) \), so it is the property of partial clones. It is easy to verify that this definition is equivalent to the conventional one (for partial clones, e.g., see [12] as well as for clones, see [7,15,16]), where instead of the restriction of \( g \) to \( A^m \), \( 2 \leq |A| < \infty \), the restriction of \( g \) to any finite subset \( B, B \subseteq E^m \), can be presented as a composition of elements from \( F \).

Partial clone (clone) \( A \) is called local if \( \text{Loc}(A) = A \). Next a local partial clone (local clone) \( A \) is called maximal if there is no such local partial clone (respectively, local clone) \( B \) that \( A \subset B \subseteq P(E) \) (respectively, \( A \subset B \subseteq O(E) \)), i.e., \( P(E) \) covers \( A \) in the lattice of all local partial clones (respectively, \( O(E) \) covers \( A \) in the lattice of all local clones). A set of partial operations \( F \) is locally complete if \( \text{Loc}(F) = P(E) \). Unlike the finite case (\( 2 \leq |E| < \infty \), see [11,5]) the general local completeness criterion cannot be based merely on the full description of maximal local partial clones since there are local partial clones which are contained in no maximal local partial clone [8,12].

So a set \( F, F \subseteq P(E) \), is locally complete if and only if: (a) \( F \) is contained in no maximal local partial clone; (b) \( F \) is not contained in a set of directed upward by inclusion (filter) local partial clones (in particular, increasing chains) with the locally complete union and, in addition, the members of this set are contained in no maximal local partial clone.

Notice that the last condition from (b) is essential since from the results of [8] we get that for any infinite set \( E \) there exists a filter of local partial clones with the locally complete union having all its elements included in some maximal local partial clones.

The relational description of all maximal local partial clones (maximal incomplete locally closed sets of partial operations) was presented in [12]. In this paper we identify the structure of a maximal local partial clone \( B \) via partial clones \( B|A \) consisting of all such restrictions of its elements to every non-single \( A, A \subset E, 2 \leq |A| < \infty \), which have also \( A \) as their range. It is shown that a local partial clone is maximal if and only if for any \( A \subset E, 2 \leq |A| < \infty \), partial clone \( B|A \) either is the full partial clone (equals to \( P(A) \)) or is a maximal partial clone on the finite \( A \) and the second condition holds at least for some \( A \). Next we consider the following problems general solutions to which are given in terms of relations (see Propositions 3.2 and 3.4).

**Extension problem.** Let \( A \) be a local partial clone, \( f \in A(|\text{Dom}(f)| < \infty) \). Find the necessary and sufficient conditions under which \( f \) has an everywhere defined extension \( g \in A \cap O(E) \).

**Extendibility of local partial clones problem.** Determine the criterion for a local partial clone to be extendable, i.e., every \( f \in A(|\text{Dom}(f)| < \infty) \) can be extended to some \( g \in A \cap O(E) \).

The most significant results in solving these problems were obtained in the case of the countable set \( E \) (see Theorem 3.12), where similar to the finite case [10] the criteria were established based on elimination of the existential quantifier in \( (\exists, \& , = ) \)-formulas of restricted first order predicate calculus over the relations which are invariant to the partial clone \( A \). Based on these criteria the full description of extendable maximal local partial clones defined on the countable set was obtained (see Theorem 4.6).

Note that the above problems are related to the problem with important applications in the field of operations research and artificial intelligence: to recognize whether a partial Boolean operation has an extension in the given (not necessarily composition-closed) class of everywhere defined Boolean operations, which has been studied in [2,4] (see also references in the latter paper) with respect to computational complexity.

## 2. Maximal local partial clones

Now we introduce the relational description of local partial clones.

**Definition 2.1.** An \( n \)-ary partial operation \( f \) preserves an \( m \)-ary relation (predicate) \( R(n, m \geq 1) \) defined on the set \( E \) if

\[
\begin{align*}
f(x_{11}, \ldots , x_{ni}) &= y_1 \& \ldots \& f(x_{1m}, \ldots , x_{nm}) = y_m \& R(x_{11}, \ldots , x_{1m}) \& \ldots , \\
& \& R(x_{n1}, \ldots , x_{nm}) \rightarrow R(y_1, \ldots , y_m)
\end{align*}
\]  

(1)

holds for all \( x_{ij}, y_j \in E \ (1 \leq i \leq n, 1 \leq j \leq m) \).

Note that the empty operations preserve each relation. This definition is equivalent to the one (e.g., see [5,8]) involving all matrices \( [x_{ij}] \) \( 1 \leq i \leq n, 1 \leq j \leq m \) over \( E \) whose rows are \( m \)-tuples from \( R \) and the result of the application of \( f \) to all columns, whenever exists, must also be an \( m \)-tuple from \( R \). If \( f \in O^n(k) \), then (1) becomes the notion (see [1,14]) of “an algebraic operation preserving a relation”.

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Denote \(\text{Pol}(R) = \{ f \in P(E) : f \text{ preserves } R \} \) and \(\text{Pol}'(R) = \{ f \in O(E) : f \text{ preserves } R \} \). Clearly \(\text{Pol}'(R) = \text{Pol}(R) \cap O(k)\). Note that each set of the form \(A = \text{Pol}(R)\) is a partial clone and of the form \(B = \text{Pol}'(R)\) is a clone. In this case we say that a relation \(R\) determines a partial clone \(A\) and also a clone \(B\). Moreover, from Definition 2.1 we obtain that \(\text{Pol}(R)\) is a local partial clone, as well as \(\text{Pol}'(R)\) is a local clone. Next for every non-void set \(\mathcal{R}\) of relations on \(E\) set \(\text{Pol}(\mathcal{R}) = \bigcap_{R \in \mathcal{R}} \text{Pol}(R)\) and also \(\text{Pol}'(\mathcal{R}) = \bigcap_{R \in \mathcal{R}} \text{Pol}'(R)\). Furthermore, from (1) we obtain that each partial clone of the form \(\text{Pol}(\mathcal{R})\) is actually restriction-closed, i.e., if \(f \in \text{Pol}(\mathcal{R})\), then for each restriction \(g \in \text{Dom}(f)\) \((g = f \text{ on } \text{Dom}(g))\) \(g \in \text{Pol}(\mathcal{R})\). For \(F \subseteq P(E)\) denote \(\text{Inv}(F) = \{ R : \text{each } f \in F \text{ preserves } R \}\). The functors \(\text{Pol}\) and \(\text{Inv}\) (\(\text{Pol}'\) and \(\text{Inv}\)) establish Galois connection between the lattices of local partial clones and Galois-closed sets of relations (respectively, between the lattices of local clones and Galois-closed sets of relations).

Example 2.2. An \(h\)-ary relation \(D(h \geq 1)\) is called a diagonal if it is complete or empty or there exists an equivalence relation \(\varepsilon\) on the numbers of its coordinates \(\{1, 2, \ldots, h\}\) \((h \geq 2)\) such that \((a_1, \ldots, a_h) \in D\) if and only if \(a_i = a_j\) whenever \((i, j) \in \varepsilon\) \((1 \leq i, j \leq h)\). The main property of these relations is: \(\text{Pol}(D) = P(E)\) and also \(\text{Pol}'(D) = O(E)\) if and only if \(D\) is a diagonal relation. In addition, it is easy to check that every non-void diagonal can be obtained by some \&-formula from the equality relation \(x = y\).

Straight from Definition 2.1 we obtain such property:

\[ f \in P(E) \text{ preserves } R \text{ if and only if } f|_A \text{ preserves } R \text{ for all } A \subseteq E, \ 2 \leq |A| < \infty. \quad (2) \]

Then from this statement applying properties of \(\text{Pol}\) and \(\text{Inv}\) we get

\[ f \in \text{Loc}(F) \iff f \in \text{Pol Inv}(F). \quad (3) \]

Using the stipulation (3) we obtain the proposition [8] (the case of local clones see in [7]).

Proposition 2.3. A partial clone (clone) on an infinite set \(E\) is local if and only if it has the form \(\text{Pol}(\mathcal{R})\) (respectively, \(\text{Pol}'(\mathcal{R})\)).

Next we introduce the following operations on relations from an arbitrary set \(\mathcal{R}\), under which the Galois-closed set \(\text{Inv Pol}(\mathcal{R})\) is closed:

(a) formation of \((\&,-)\)-formulas of restricted quantifier-free first order calculus, i.e., each relation which can be presented as a \((\&,-)\)-formula (including arbitrary identifications and permutations of coordinates) over \(\mathcal{R}\);

(b) infinite intersection \(\bigcap_{i \in I} R_i\) of the set \(\{ R_i : i \in I \}\) of the cardinality \(\text{card}(I) \leq \text{card}(E)\) consisting of relations having the same arity;

(c) direct limit (union) \(\uparrow\) of the directed upward system (filter) \(\{ R_i : i \in I \}\) of relations of the same arity (i.e., for every \(i, j \in I\) there exists \(k \in I\) such that \(R_i \cup R_j \subseteq R_k\)); we set \(\uparrow \{ R_i : i \in I \} = \bigcup_{i \in I} R_i\).

Proposition 2.4 (Romov [8]). Every \(R \in \text{Inv Pol}(\mathcal{R})\) is presented as a direct limit of infinite intersections of \((\&,-)\)-formulas over the set \(\mathcal{R}\).

Proposition 2.5 (Romov [8,12]). Each maximal local partial clone is determined by a relation on \(E\) which is minimal via expressibility by \((\&,-)\)-formulas (called, in brief, minimal relations).

Next we introduce the classes of relations determining all maximal local partial clones [12]:

\(K_1\): any binary areflexive, i.e., \(R(x, x) \equiv \emptyset\), and symmetric relation without cycles of an odd length.

\(K_2\): any binary areflexive and asymmetric, i.e., \(R(x, y) \& R(y, x) \equiv \emptyset\), relation without paths of length two.

\(Q\): any binary relation, distinct from the equality, which is reflexive, antisymmetric, i.e., \(R(x, y) \& R(y, x) \equiv x = y\), and has no cycles of a finite length.

\(H\): all non-full relations of arity \(h, 2 \leq h\), which are totally symmetric (stable under all permutations of coordinates) and totally reflexive (contain all tuples with at least two equal coordinates); if \(h = 1\), then \(H\) consists of all non-empty proper subsets of \(E\).
Set $T_h(x_1, \ldots, x_h) \equiv \bigvee_{1 \leq i \leq j < h} (x_i = x_j)(h \geq 2)$. An $h$-ary $(3 \leq h)$ relation $R$ is areflexive, if $T_h(x_1, \ldots, x_h)\& R(x_1, \ldots, x_h) \equiv \emptyset$, and normal, if for every permutation $s$ of its variables we have either (a) $R(x_{s1}, \ldots, x_{sh}) \equiv R(x_1, \ldots, x_h)$ or (b) $R(x_{s1}, \ldots, x_{sh}) \& R(x_1, \ldots, x_h) \equiv \emptyset$. In addition, let $G(R)$ be the group of symmetry of $R$ consisting of all permutations $s$ with the property $(a)$. Set an $h$-ary relation $(h$-orbit of the group $G(R))$ on the finite set $E(h) = \{1, 2, \ldots, h\}$: there exists $s \in G(R)$ such that $(a_1, \ldots, a_h) = (s_1, s_2, \ldots, s_h)$. Denote $R|_A$ a restriction of $R$ to a finite set $A$: $(a_1, \ldots, a_h) \in R|_A \iff (a_1, \ldots, a_h) \in R \cap A^h$.

$R_1$: $R$ of arity $h \geq 3$ is areflexive and normal and for any finite $A, C \subseteq E, |A| > h$, whenever $R|_A \neq \emptyset$, there exists an epimorphism $\Psi: A \rightarrow E(h)$ such that

$$\Psi(R|_A) \equiv \{(\Psi a_1, \ldots, \Psi a_h) : (a_1, \ldots, a_h) \in R|_A\} = \text{Orb}(G(R)).$$

$R_2$: a relation $Q \in R_2$ of arity $h \geq 3$ has the form $Q = R \cup D$, where $R$ is areflexive of arity $h$ and $D$ is a non-full diagonal. Next let $G(D)$ be the symmetry group of $D$ and $D(h)$ be the diagonal on $E(h)$ of the same type as $D$ (having the same equal coordinates). Finally, $Q \in R_2$ if and only if:

1. $R$ is normal and $G(R) \subseteq G(D)$;
2. for every finite non-empty $A, C \subseteq E, |A| > h$, whenever $R|_A \neq \emptyset$, and every non-void $M \subseteq R|_A$ there exists an epimorphism $\Psi: A \rightarrow E(h)$ such that

$$\Psi M \subseteq \text{Orb}(G(R)) \cup D(h) \quad \text{and} \quad \Psi M \cap \text{Orb}(G(R)) \neq \emptyset.$$

$R_3$: a relation $Q \in R_3$ of arity four has the form: either $R \cup H_1 \equiv R(x, y, u, z) \land x = y \land u = z \land x = u \land y = z$, or $R \cup H_2 \equiv R(x, y, u, z) \lor x = y \land u = z \land x = u \land y = z$, where $R$ is areflexive of arity four. Then $Q \equiv R \cup H_i \in R_3$ $(i = 1, 2)$ if and only if:

1. $R$ is normal and $G(R)$ coincides with the group of symmetry of $H_i$ $(i = 1, 2)$;
2. for every non-empty $A, C \subseteq E, |A| > h$, whenever $R|_A \neq \emptyset$, and every non-void $M \subseteq R|_A$ there exists an epimorphism $\Psi: A \rightarrow E(h)$ such that $\Psi M \subseteq \text{Orb}(G(R)) \cup H_i(4)$ and $\Psi M \cap \text{Orb}(G(R)) \neq \emptyset$, where $H_i(4)$ is $H_i$ defined as the corresponding conjunction of diagonals on the four-element set $\{1, 2, 3, 4\}$ $(i = 1, 2)$.

**Theorem 2.6** (Romov [12]). A local partial clone defined on an infinite set $E$ is maximal if and only if it is presented in the form $\text{Pol}(R)$, where $R \in K_1 \cup K_2 \cup Q \cup H \cup R_1 \cup R_2 \cup R_3$.

Note that for relations defined on a fixed finite set $A$, $3 \leq |A| < \infty$, we obtain the definition of seven classes of relations $(K_1, K_2, Q, H, R_1, R_2,$ and $R_3)$ which determine all, but one, maximal partial clones of the full partial clone $P(A)$ on the set $A$ (see [111]).

In what follows, we will investigate the structure of maximal local partial clones via their intersections with the full partial clones $P(A)$ for every finite subset $A, C \subseteq E, 2 \leq |A| < \infty$. For a partial clone $B$ we define $B|_A = B \cap P(A)$, a partial clone on the finite set $A, C \subseteq E, 2 \leq |A| < \infty$, which consists of all restrictions of elements $B$ on the set $A$ that also have the range in $A$. Moreover, if $B$ is local, then $B|_A$ is restriction-closed.

**Definition 2.7.** A partial clone $B$ is hereditary if for every finite subset $A, C \subseteq E, 2 \leq |A| < \infty, B|_A$ is either the full partial clone or a maximal partial clone on $A$ and, moreover, the second condition holds for at least one finite subset $B$ of $E$.

Straight from this definition one can obtain such corollary.

**Corollary 2.8.** Let $B$ be hereditary and for some finite subset $B \subset E, 2 \leq |B| < \infty$, $B|_B$ be a maximal partial clone on $B$. Then $B|_C$ is also maximal partial clone on every finite set $C, B \subset C \subseteq E$.

**Proposition 2.9.** Any hereditary local partial clone is a maximal local partial clone.

**Proof.** We will prove that for a hereditary local partial clone $B$ $\text{Loc}(B \cup \{f\}) = P(E)$ holds for every $f \notin B$, which is equivalent to the statement of this proposition. First from the definition of a local partial clone we get that if
Proof of the Fact. 

Corollary 2.10. If \( f \notin B \) for some finite set \( A \), then we choose the finite set \( B \) such that the range of \( f|_A \) is in \( B \) and also \( A \subseteq B \), i.e., \( g \equiv (f|_A) \in P(B) \). Hence \( g \notin B \). Next since \( B \) is hereditary \( B|_B \) is a maximal partial clone on \( B \) and \( B|_B \cup \{g\} = P(B) \). Hence from \( B|_B \cup \{g\} \subseteq P(B) \) we obtain that \( P(B) = P(B) \) and so \( P(B) \) is a maximal partial clone.

Next we need the following fact.

Fact. \( P(D) \subseteq Loc(B \cup \{f\}) \) for every finite subset \( B \subset D \subset E \).

Proof of the Fact. First we get \( g \notin D \). Otherwise \( g \in B \), since \( B \) is restriction-closed. Then from Corollary 2.8 we obtain \( B|_D \cup \{g\} = P(D) \). Hence \( B|_D \cup \{g\} \subseteq \{B \cup \{g\} \}_{B} = P(D) \) and \( P(D) \subseteq Loc(B \cup \{f\}) \).

Clearly \( Loc(B \cup \{f\}) = P(E) \) is equivalent to: for each \( h \in P(E) \) and any finite subset \( A \subseteq E \), \( 2 \leq |A| < \infty \), a finite domain operation \( h|_A \in Loc(B \cup \{f\}) \). Then we choose the finite set \( D \) such that: (a) \( A \cup B \subseteq D \); (b) the range of \( h|_A \) is in \( D \). Next applying the Fact we obtain \( h|_A \in P(D) \subseteq Loc(B \cup \{f\}) \) which completes the proof of the proposition.

Let \( \Phi(E) = O(E) \cup \{p_n : n = 1, 2, \ldots\} \) be a partial clone consisting of \( O(E) \) and all empty (void) operations. It is known [13] that \( \Phi(E) \) is a maximal partial clone. Moreover, since \( \Phi(E)|_A = \Phi(A) = O(A) \cup \{p_n : n = 1, 2, \ldots\} \) is a maximal partial clone for every finite non-single \( A \) (e.g., see [5,11]) it is also hereditary. In addition, it is easy to check that \( \Phi(E) \) is the only one hereditary partial clone with the property \( \Phi(E)|_A = \Phi(A) \). Indeed, if for some hereditary \( B \), other than \( \Phi(E) \), there exists finite \( A \) and \( B|_A = \Phi(A) \), then for any finite non-single \( B \) we have \( B|_B = \Phi(B) \). Otherwise, if \( B|_B = P(B) \), then \( B|_{A\cup B} = P(A \cup B) \), which contradicts Corollary 2.8. Thus, we obtain the corollary.

Corollary 2.10. \( \Phi(E) \) is the only one hereditary partial clone such that \( Loc(\Phi(E)) = P(E) \).

For a hereditary non-local partial clone \( B \), other than \( \Phi(E) \), it is easy to verify that:

\[
Loc(B) = Loc(\cup\{B|_A : A \subseteq E, 2 \leq |A| < \infty\}).
\]

Next the same way as in the proof of Proposition 2.9 we get for hereditary \( B \), other than \( \Phi(E) \), that \( Loc(\cup\{B|_A : A \subseteq E, 2 \leq |A| < \infty\}) \) is a maximal local partial clone.

Corollary 2.11. The local closure of any hereditary partial clone, except for \( \Phi(E) \), is a maximal local partial clone.

Now we prove the converse statement to Proposition 2.9.

Proposition 2.12. Every maximal local partial clone is hereditary.

Proof. For every local partial clone \( B = Pol(R) \) and \( A \subseteq E \), \( 2 \leq |A| < \infty \), straight from the Definition 2.1 we obtain

\[
B|_A = Pol(R|_A) \cap P(A).
\]

Now it suffices to consider relations from the seven classes (see Theorem 2.6) that determine all maximal local partial clones.

Lemma 2.13. If \( R \in K_1 \cup K_2 \cup Q \cup H \cup R_1 \cup R_2 \cup R_3 \), then for any finite \( A \subseteq E \), \( 2 \leq |A| < \infty \), \( R|_A \) either is a minimal relation on \( A \) (from the same class of relations defined on \( A \)) or it is a diagonal.

Proof. Let \( R \in Q \). Clearly each non-void \( R|_A \) is also reflexive, antisymmetric and does not contain cycles. If \( |A| = 2 \), then \( R|_A \) is either the equality relation or the minimal relation \( \{(a, a), (a, b), (b, b)\} \) on \( A = \{a, b\} \) (e.g., see [8]). In the general case \( |A| \geq 3 \), if \( R|_A \) is not the equality, then it is a minimal relation from the class \( Q \) on the finite set \( A \).

The same idea is applicable to the relations from classes \( K_1, K_2 \) and \( H \) taking into account that the following properties of a relation of arity \( h \geq 2 \), namely, areflexive, asymmetric, totally reflexive and totally symmetric, are stable.
under non-void restrictions of this relation to any subset $A$ ($|A| \geq h$). In fact, these properties are defined via first order calculus sentences with the prefix consisting of only universal quantifiers.

If $R \in R_1$ and $|A| = k < h$, then $R|_A$ is empty. Next if $k \geq h$, then $R|_A$ is either empty or a minimal relation on $A$ (see the definition of the class $R_1$ on a finite set $A$, $|A| \geq h$, [11]).

If $R \in R_2 \cup R_3$, then, in case $|A| = k < h$, $R|_A$ is a diagonal for $R \in R_2$ and either $H_1$ or $H_2$ for $R \in R_3$, the last two are minimal relations on $A$ [11]. If $k \geq h \geq 3$, then $R|_A$ is either a diagonal or a minimal relation on $A$ (see the description of classes $R_2$ and $R_3$ on a finite set $A$, $k \geq 3$, [11]). □

Next we apply this Lemma and Definition 2.7. □

Finally, from Propositions 2.9 and 2.12 we obtain the theorem.

**Theorem 2.14.** A local partial clone is maximal if and only if it is hereditary.

For the countable set $E = \{0, 1, \ldots, k, \ldots\}$ we choose $B = \{i_1, \ldots, i_m\}$ from Definition 2.7 and put $p = \max(i_1, \ldots, i_m)$. Then from Theorem 2.14 and Corollary 2.8 we get the corollary.

**Corollary 2.15.** A local partial clone $B$ defined on the countable set $E = \{0, 1, \ldots, k, \ldots\}$ is maximal if and only if there exists a positive integer $p \geq 1$ such that for every $k \geq p$ the partial clone $B|_{E(k)}$ defined on $E(k) = \{0, 1, \ldots, k-1\}$ is maximal.

From the stipulation (4) and Theorem 2.14 we obtain the corollary.

**Corollary 2.16.** A relation $R$ on $E$ is minimal if and only if for every $A, A \subset E, 2 \leq |A| < \infty$, $R|_A$ either is a minimal relation on $A$ or it is a diagonal and, moreover, the first condition holds for at least one finite set $A$.

Let $B$ be a hereditary maximal partial clone, distinct from $\Phi(E)$. Then from Corollary 2.11 we obtain that $B$ is actually a local partial clone. Then from Propositions 2.3 and 2.12 we get the corollary.

**Corollary 2.17.** A maximal partial clone is hereditary if and only if it either coincides with $\Phi(E)$, or is determined by a finite arity relation.

Notice that the description of all maximal partial clones determined by finite arity relations was obtained in [13].

**Remark.** It was mentioned previously that every local partial clone is restriction-closed. Moreover, for a finite set $E$ these two notions coincide. For an infinite set $E$, we state without proof the following result: every maximal partial clone, distinct from $\Phi(E)$ and those determined by finite arity relations (the description of all such clones is in [13]), is restriction-closed, but not local.

3. Extension of finite domain partial operations

Let $f \in \text{Pol}(\mathfrak{R})$ be a finite domain partial operation, i.e., $|\text{Dom}(f)| < \infty$.

**Definition 3.1.** $f \in \text{Pol}(\mathfrak{R})$, $|\text{Dom}(f)| < \infty$, is extendable in the local partial clone $B = \text{Pol}(\mathfrak{R})$ if there exists an everywhere defined operation $g \in B \cap O(E)$ such that $f = g$ on $\text{Dom}(f)$.

**Proposition 3.2.** An $n$-ary ($n \geq 1$) partial operation $f$, $|\text{Dom}(f)| < \infty$, is extendable in $B = \text{Pol}(\mathfrak{R})$ if and only if $f \in \text{PolInv}(B \cap O(E))$, i.e., $f$ preserves any relation from $\text{Inv}(B \cap O(E))$.

**Proof.** ($\Rightarrow$) Let $g$ be an everywhere defined extension of $f$ in $B$, i.e., $f = g$ on $\text{Dom}(f)$ and $g \in B \cap O(E)$. Clearly $g$ preserves all relations from $\text{Inv}(B \cap O(E))$. Next $f$, as a restriction of $g$, also preserves any relation from $\text{Inv}(B \cap O(E))$.

($\Rightarrow$) We arrange the finite set $\text{Dom}(f) \subset E^n$, $|\text{Dom}(f)| = m \geq 1$, as an $n \times m$ matrix $[\text{Dom}(f)]$ having $m$ $n$-tuples as its columns. Then for $n$-ary $g \in O(E)$ applying $g$ columnwise we get $g([\text{Dom}(f)]) = r \in E^m$. Next we define an
Proposition 3.2 and obtain that actually

\[ G_{n,m} = \{ r : \exists g \in B \cap O^n(E)g([\text{Dom}(f)]) = r \}. \tag{5} \]

Since all operations from \( B \cap O(E) \) preserve \( G_{n,m} \) we have \( G_{n,m} \in \text{Inv}(B \cap O(E)) \). Hence \( f \) also preserves \( G_{n,m} \) and \( f([\text{Dom}(f)]) = r \in G_{n,m} \). Next from the definition of \( G_{n,m} \) we obtain that there exists \( g \in B \cap O(E) \) and \( f = g \) on \( \text{Dom}(f) \). \( \square \)

**Definition 3.3.** A local partial clone is extendable if any finite domain partial operation from it is extendable in this partial clone.

**Proposition 3.4.** A local partial clone \( B \) is extendable if and only if \( \text{Inv}(B) = \text{Inv}(B \cap O(E)) \).

**Proof.** (\( \Rightarrow \)) Assume that \( \text{Inv}(B) \subseteq \text{Inv}(B \cap O(E)) \). Then there exists \( R \in \text{Inv}(B \cap O(E)) \) which is not preserved by some partial operation from \( B \). However, from Proposition 3.2 we get that every finite domain \( f \in B \) preserves \( R \). Hence we have \( h \in B \cap O(E), \text{Dom}(h) \) is an infinite set, that \( h \) does not preserve \( R \). Then we apply stipulation (2) and Proposition 3.2 and obtain that actually \( h \) preserves \( R \). Contradiction.

(\( \Leftarrow \)) For any \( f \in B \), \( |\text{Dom}(f)| < \infty \), we consider \((n, m)\)-graph \( G_{n,m} \) defined in (5). From \( \text{Inv}(B) = \text{Inv}(B \cap O(E)) \) we obtain that \( G_{n,m} \in \text{Inv}(B) \). Hence \( f \) preserves \( G_{n,m} \). Next we use the final part of the proof of Proposition 3.2. \( \square \)

Let \( E = \{0, 1, \ldots, k, \ldots\} \) be the countable set and \( \mathcal{R} \) be a set of relations defined on \( E \). We introduce the characteristics of the set \( \text{Inv Pol}^f(\mathcal{R}) \) via operations on relations which do not change the inclusion property for the set \( \text{Inv Pol}^f(\mathcal{R}) \):

(a) formation of \((\exists, \&, =)\)-formulas of the first order predicate calculus (including arbitrary permutations and identifications of coordinates in relations);
(b) infinite intersections of relations of the same arity (for \( E \) countable, it suffices to consider only countable intersections (see [8]));
(c) direct limits (unions) of directed upward systems of relations of the same arity (for \( E \) countable, it suffices to consider increasing by inclusion countable chains \( \{R_i : R_i \subseteq R_{i+1}, i = 1, 2, \ldots\} \) of relations and, consequently, limits \( \uparrow R_i = \bigcup_{i \geq 1} R_i \)).

Next we describe \( \text{Inv Pol}^f(\mathcal{R}) \) using operations (a), (b) and (c) (see also [6]).

**Theorem 3.5.** Let \( \mathcal{R} \) be a set of relations defined on the countable set \( E \). Then every \( Q \in \text{Inv Pol}^f(\mathcal{R}) \) is presented as a direct limit of relations constructed by \((\exists, \&, =)\)-formulas and infinite intersections of relations from \( \mathcal{R} \).

**Proof.** For each \( n \geq 1 \) we arrange all \( n \)-tuples from \( E^n \) in lexicographic order \( E^n = \{r_1^n, \ldots, r_m^n, \ldots\} \), where \( r_1^n = (0, \ldots, 0), r_2^n = (0, \ldots, 0, 1), r_3^n = (0, \ldots, 1, 0) \) and so on. Consider an \( m \)-ary relation \( G_{n,m} \), an \((n, m)\)-graph of \( \mathcal{A} = \text{Pol}^f(\mathcal{R}) \) defined on \( m (m \geq 1) \) first \( n \)-tuples according to that order. These \( n \)-tuples form an \( n \times m \) matrix \( [r_1^n, \ldots, r_m^n] \), to which \( n \)-ary operations from \( A \) are applied columnwise:

\[ G_{n,m} = G_{n,m}(A \cap O(E)) = \{ r : \exists g \in A \cap O^n(E)g([r_1^n, \ldots, r_m^n]) = r \} \quad (n, m \geq 1). \]

**Lemma 3.6.** Each \( G_{n,m} \) \((n, m \geq 1)\) can be presented as an infinite intersection of relations constructed by \((\exists, \&, =)\)-formulas and infinite intersections of relations from \( \mathcal{R} \).

**Proof of the Lemma.** Consider the set \( \{T_{n,m}^\hat{\lambda} : \hat{\lambda} \in \lambda \} \) of all relations of arity \( m \geq 1 \) constructed by \((\exists, \&, =)\)-formulas and infinite intersections over \( \mathcal{R} \) such that for \( n \)-tuples \( q_i \) \((i = 1, \ldots, n)\), which being arranged by rows form an \( n \times m \) matrix \( [r_1^n, \ldots, r_m^n] \), \( q_i = (r_1^n(i), \ldots, r_m^n(i)) \) \((i = 1, \ldots, n)\), we have \( q_1, \ldots, q_n \in T_{n,m}^\hat{\lambda}(\hat{\lambda} \in \lambda) \). Next we define the relation:

\[ R_{n,m} \equiv \bigcap \{T_{n,m}^\hat{\lambda} : \hat{\lambda} \in \lambda \}. \tag{6} \]
From the properties of operations (a) and (b) over relations we have $R_{n,m} \in \text{Inv Pol}^f(\mathcal{R})$. Hence $G_{n,m} \subseteq R_{n,m}$ $(n, m \geq 1)$. Then we need the following fact.

**Fact 1.** $\exists x_m R_{n,m}(x_1, \ldots, x_m) \equiv R_{n,m-1}(x_1, x_m, \ldots, x_{m-1})$ $(n \geq 1, m \geq 2)$.

**Proof of Fact 1.** Clearly $R_{n,m-1} \subseteq (\exists x_m)T_{n,m}^\lambda(n, \lambda \in A)$. Then $[r^n_1, \ldots, r^n_{m-1}] \subseteq \exists x_m R_{n,m}$ the relation $\exists x_m R_{n,m}$ is in (6) for $R_{n,m-1}$ and we obtain $R_{n,m-1} \subseteq \exists x_m R_{n,m}$ $(n \geq 1, m \geq 2)$. If there exists an $(m - 1)$-tuple $r \in (\exists x_m R_{n,m}) \setminus R_{n,m-1}$, then $r \not\in T_{n,m}^\lambda$ for some $(\exists, \&)$-formula $T_{n,m-1}^\lambda(\mu \in A) \subseteq (6)$ involved in the definition (6) for the relation $R_{n,m-1}$. Then from (6) we have $[r^n_1, \ldots, r^n_{m-1}] \subseteq s T_{n,m}^\lambda$. Hence we get $[r^n_1, \ldots, r^n_{m-1}] \subseteq s T_{n,m-1}^\lambda(x_1, \ldots, x_{m-1})$ and $(x_m = x_m)$ and so the relation $T_{n,m-1}^\lambda \& x_m = x_m$ is also included in the definition (6) for $R_{n,m}$. Hence $\exists x_m R_{n,m} = (\exists x_m) \cap T_{n,m}^\lambda \subseteq (\exists x_m)T_{n,m}^\lambda \subseteq (\exists x_m)T_{n,m-1}^\lambda \& (x_m = x_m) = T_{n,m-1}^\lambda$. Contradiction. □

Next we will prove that $G_{n,m} \equiv R_{n,m}$ for all $n, m \geq 1$. Assume the converse: for some $n$ and $m$ there exists $r_m = (r(1), \ldots, r(m)) \in R_{n,m} \setminus G_{n,m}$. Then we define a partial $n$-ary relation $f$ as follows: $f([r^n_1, \ldots, r^n_{m-1}]) = r_m$, with the domain $\text{Dom}(f) = [r^n_1, \ldots, r^n_{m-1}]$. Next from the Fact 1 there exists $r_{m+1} = (r(1), \ldots, r(m), r(m + 1)) \in R_{n,m+1}$. Hence we can extend $f$ on the $n$-tuple $r_{m+1}$ such that $f([r^n_1, \ldots, r^n_{m+1}]) = r_{m+1} \in R_{n,m+1} \setminus G_{n,m+1}$. Now $f$ is defined on $m + 1$ first $n$-tuples from $E^n$. Repeating this procedure we obtain an everywhere defined $n$-ary relation $f$ such that $f([r^n_1, \ldots, r^n_k]) \in R_{n,s}$ for every $s \geq 1$.

**Fact 2.** The above defined $f$ preserves each $R \in \mathcal{R}$, i.e., $f \in \text{Pol}^f(\mathcal{R})$.

**Proof of Fact 2.** We will apply the statement which is equivalent to Definition 2.1: $f \in O^n(E)(n \geq 1)$ preserves $R$ of arity $m \geq 1$ if and only if for every, not necessarily distinct, $n$ $m$-tuples $q_i = (q_i(1), \ldots, q_i(m)) \in R$ $(i = 1, \ldots, n)$ we have $f(q_1, \ldots, q_n) = (f(q_1(1), \ldots, q_1(m)), \ldots, f(q_n(1), \ldots, q_n(m))) \in R$.

First we consider the $n \times m$ matrix $[q_i(j)]$ $((1 \leq i \leq n, 1 \leq j \leq m)$ having all $m$ columns $(q_i(1), \ldots, q_i(m))$ $(i = 1, \ldots, m)$ distinct. Let $\{i_1, \ldots, i_m\}$ be the numbers of these $m$-tuples in the ordering of $E^n$, $s = \max(i_1, \ldots, i_m), s \geq m$, and $x: \{1, \ldots, m\} \rightarrow \{i_1, \ldots, i_m\}$ be the one-to-one correspondence. Next we construct the relation:

$$Q(x_1, \ldots, x_m) \equiv \exists x_{j_1} \ldots x_{j_m} R_{n,s}(x_1, \ldots, x_m),$$

where $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, s\}$.

Then we define the following relation:

$$S_n(x_1, \ldots, x_m) \equiv Q(x_{s-1(i_1)}, \ldots, x_{s-1(i_m)}).$$

Clearly $S_n$ contains all $m$-tuples $q_1, \ldots, q_n$. In addition, similar to the proof of the Fact 1 one can show that $S_n$ is built by the analogue of the formula (6) from relations containing $[q_1, \ldots, q_n]$ and constructed by $(\exists, \&)$-formulas and infinite intersections over $\mathcal{R}$. Moreover, since the relation $T(x_1, \ldots, x_i) \equiv R(x_1, \ldots, x_{2m})$ is involved in the formula (6) for $R_{n,s}$ straight from (7) and (8) we obtain that $S_n \subseteq R$.

Next since $f([r^n_1, \ldots, r^n_m]) \in R_{n,s}$ we get $f(q_1, \ldots, q_n) = f([r^n_1, \ldots, r^n_m]) \in S_n$. Finally $f(q_1, \ldots, q_n) \in R$. In case, when some $n$-tuples (columns) in $[q_i(j)]$ $(1 \leq i \leq n, 1 \leq j \leq m)$ are equal we use the “duplication” of coordinates in $S_n$, i.e., conjunction $S_n \& (x_i = x_j)$ if $(q_i(1), \ldots, q_i(m)) = (q_j(1), \ldots, q_j(m))$ $(i, j = 1, \ldots, m)$. Then again $S_n \subseteq R$ and $f(q_1, \ldots, q_n) \in R$. This completes the proof of the Fact 2. □

From the Fact 2 we obtain that $f([r^n_1, \ldots, r^n_m]) = r \in G_{n,m}$ and hence from this contradiction we get $G_{n,m} \equiv R_{n,m}$ for all $n, m \geq 1$. This proves the Lemma. □

From the Lemma we obtain the following fact.

**Fact 3.** For each $m$-tuple $r \in S_n$ there exists $g \in A \cap O^n(E)$ such that $g(q_1, \ldots, q_n) = r(n, m \geq 1)$.

Next we choose $R = \{q_1, \ldots, q_n, \ldots\} \in \text{Inv Pol}^f(R)$ of arity $m \geq 1$. For any first $n$ $m$-tuples $q_1, \ldots, q_n$ using (7) and (8) we construct the relation $S_n$ of arity $m$ such that $q_1, \ldots, q_n \in S_n$. Then from the Fact 3 we obtain $S_n \subseteq R$ and also taking into account properties of composition of operations $S_n \subseteq S_{n+1}$ $(n = 1, 2, \ldots)$. Finally, $R = \uparrow S_n = \bigcup_{n \geq 1} S_n$. This completes the proof of the Theorem 3.5. □
Remark 1. Notice that for the countable set $E$ the characteristics of Galois-closed sets of the form $\text{Inv Pol}'(\mathfrak{R})$ have some similarities with the finite case (see [1]). Indeed, it is still sufficient to use $(\exists, \&,, =)$-formulas of the first order predicate calculus, although combining them with infinite operations on relations (direct limits and intersections). For an uncountable set $E$ the things become more complicated, namely, for the description of $\text{Inv Pol}'(\mathfrak{R})$ one has to utilize formulas with an infinite existential quantifier prefix (e.g., see [17]), which is obviously beyond the scope of the first order predicate calculus.

Applying Theorem 3.5 we obtain the criterion of extendibility for a local partial clone defined on the countable set in the form similar to the finite case [10].

Theorem 3.7. $B = \text{Pol}(\mathfrak{R})$, defined on the countable set, is extendable if and only if the closure of $\text{Inv Pol}(\mathfrak{R})$ via application of all $(\exists, \&,, =)$-formulas coincides with $\text{Inv Pol}(\mathfrak{R})$.

Proof. $(\Rightarrow)$ From Proposition 3.4 we have $\text{Inv}(B) = \text{Inv}(B \cap O(E))$. Since $\text{Inv}(B \cap O(E))$ is closed under application of $(\exists, \&,, =)$-formulas the same holds for $\text{Inv Pol}(\mathfrak{R}) = \text{Inv}(B)$.

$(\Leftarrow)$ Each relation constructed by a $(\exists, \&,, =)$-formula over $\mathfrak{R}$ belongs to $\text{Inv Pol}(\mathfrak{R})$. Then this is also true for infinite intersections of these relations (see Proposition 2.4), as well as $(\exists, \&,, =)$-formulas applied to those intersections. And, finally, directed limits of such relations also are in $\text{Inv Pol}(\mathfrak{R})$. Hence from Theorem 3.5 any relation from $\text{Inv Pol}'(\mathfrak{R})$ belongs to $\text{Inv Pol}(\mathfrak{R})$. Next we apply Proposition 3.4. □

Corollary 3.8. $B = \text{Pol}(\mathfrak{R})$, defined on the countable set, is extendable if and only if for every $m$-ary ($m \geq 2$) relation $Q \in \text{Inv Pol}(\mathfrak{R})$ the relation $(\exists x)Q(x, y, \ldots)$ also belongs to $\text{Inv Pol}(\mathfrak{R})$.

Corollary 3.9. If $f \in B = \text{Pol}(\mathfrak{R})$, $|\text{Dom}(f)| < \infty$, does not preserve some relation obtained by $(\exists, \&,, =)$-formula over $\mathfrak{R}$, then $f$ is not extendable in $B$.

Corollary 3.10. If $B = \text{Pol}(\mathfrak{R})$ is extendable, then each $f \in B$, $|\text{Dom}(f)| < \infty$, preserves all relations obtained by application of all $(\exists, \&,, =)$-formulas to $\text{Inv Pol}(\mathfrak{R})$.

Now we consider partial clones determined by a single relation. We will use the following proposition.

Proposition 3.11 (Romov [8]). If $B = \text{Pol}(R)$, defined on an arbitrary infinite set, then $\text{Inv Pol}(R)$ coincides with the set of all relations constructed by $(\&,, =)$-formulas from the relation $R$.

Theorem 3.12. A partial clone $B = \text{Pol}(R)$, defined on the countable set $E$, is extendable if and only if each relation constructed by a $(\exists, \&,, =)$-formula from $R$ is equivalent to a relation constructed by some $(\&,, =)$-formula from $R$.

Proof. We will prove the equivalent statement (see Proposition 3.4): $\text{Inv Pol}(R) = \text{Inv Pol}'(R) \iff$ each relation constructed by a $(\exists, \&,, =)$-formula from $R$ is equivalent to a relation constructed by some $(\&,, =)$-formula from $R$.

$(\Rightarrow)$ Clearly the set $\text{Inv Pol}'(R)$ contains all relations constructed by $(\exists, \&,, =)$-formulas from $R$ (this is even true for any infinite set $E$). Based on Proposition 3.11 each $Q \in \text{Inv Pol}(R)$ can be presented as $(\&,, =)$-formula from $R$. Since $\text{Inv Pol}(R) = \text{Inv Pol}'(R)$ this covers also all relations constructed by $(\&,, =)$-formulas from $R$.

$(\Leftarrow)$ Applying Theorem 3.5 (characteristics of the set $\text{Inv Pol}'(R)$) and Propositions 2.4 and 3.11 (characteristics of the set $\text{Inv Pol}(R)$) we obtain $\text{Inv Pol}(R) = \text{Inv Pol}'(R)$. Indeed, in this case there is only a finite number of fixed arity relations constructed by $(\exists, \&,, =)$-formulas from $R$, since each of them is equivalent to a relation constructed by some $(\&,, =)$-formula. Hence we cannot apply infinite intersections and limits. □

Notice that the part $(\Rightarrow)$ of the proof of Theorem 3.12 is valid for any infinite set $E$.

Corollary 3.13. If there exists a non-diagonal relation $Q$ constructed by a $(\exists, \&,, =)$-formula from $R$ such that $Q$ cannot be represented by any $(\&,, =)$-formula from $R$, then $B = \text{Pol}(R)$ defined on an arbitrary infinite set $E$ is not extendable.
Remark 2. Note that the main results on extendibility, in particular Theorems 3.7 and 3.12, are also valid for any finite set $E$ while merely replacing the notion of local partial clone by a restriction-closed partial clone defined on a finite set $E$, $|E| = k \geq 2$ (see [10]). However, the difference is: in the finite case Theorem 3.12 holds for any restriction-closed partial clone $B = Pol(\mathbb{R})$, while for the countable case it is true only for $B = Pol(R)$, determined by a single relation.

3.1. Extendable partial orders

We apply the results of this section to a partial order relation $x \leq y$ (reflexive, antisymmetric and transitive binary relation) on an arbitrary infinite set $E$. Let $x^{n,y}$ be the $n$th power of $x \leq y$ defined on $n$-tuples of $E^n$ ($n \geq 1$). A partial (everywhere defined) mapping $g: (E^n; x^{n,y}) \rightarrow (E; x \leq y)$ is called a partial $n$-endomorphism (respectively, n-endomorphism) of the partial order if for every $a, b \in \text{Dom}(g) \subseteq E^n a^{n,b}$ implies $g(a) \leq g(b)$, i.e., $g$ is a partial homomorphism (homomorphism) of these algebraic systems. Next considering $g$ as a partial $n$-ary operation on $E$ it is easy to check that $g \in \text{Pol}(x \leq y)$ and vice versa. A partial order $x \leq y$ is called directed upward (downward) [3] if for every $a, b \in E$ there exists $c \in E$ such that $a \leq c$ and $b \leq c$ (respectively, $c \leq a$ and $c \leq b$). If the two conditions hold simultaneously we call $x \leq y$ directed. Next a directed downward (directed upward) partial order is called totally directed downward (totally directed upward) if for any pair $(a, b)$ and $(b \leq a)$ (i.e., $a, b$: incomparable) there exists no $c \in E$ ($c \neq a, b$) such that $a \leq c \land b \leq c$ (respectively, $c \leq a \land c \leq b$).

Consider the case, when $x \leq y$ does not belong to the classes of directed, totally directed downward or totally directed upward partial orders. Then there are $a, b \in E$ $(a, b$: incomparable) such that there is no element $c \in E$ $(c \neq a, b)$ that either holds $(a) c \leq a \land c \leq b$ or $(b) a \leq c \land b \leq c$. Suppose that the condition $(a)$ is not true. Then in order to avoid totally directed upward and downward partial orders we have to add either one of these two conditions: (1) there exists a pair $(d, e)$ ($d, e$: incomparable) such that $c \leq d \land c \leq e$ for some $c \in E$; (2) there exists a pair $(d, e)$ ($d, e$: incomparable) such that $d \leq c \land e \leq c$ for some $c \in E$ and, in addition, the condition $(b)$ also is not true. Note that the outcome when we first consider $(b)$ to be not true is similar to cases (1) and (2).

First consider case (1). We construct the following relation by $(\exists, \&,-)$-formula:

$$R(y, z) \equiv (\exists x) x \leq y \land x \leq z. \quad (9)$$

Clearly $y = z \subset R(y, z)$. Since $c \leq d$ $(c \neq d)$ implies $(c, d) \in R$ we have $y \leq z \subseteq R(y, z)$. Then from (9) we obtain $(a, b), (b, a) \notin R$ and so $R$ is non-full relation. Hence $R$ is not a diagonal. Moreover, from (1) we obtain $(d, e), (e, d) \in R$ (where $d$ and $e$ are incomparable). Hence $y \leq z \subseteq R(y, z)$. Next it is obvious that any binary non-diagonal relation obtained from $y \leq z$ by some $(\&,-)$-formula is $y \leq z$ itself or its inverse. Finally, we apply Corollary 3.13.

The case (2) is handling the same way by constructing the following relation:

$$Q(y, z) \equiv (\exists x) y \leq x \land z \leq x. \quad (10)$$

Then it is proven that $y \leq z \subset Q(y, z) \subset y = y \land z = z$. Hence applying Corollary 3.13 we get the proposition.

Proposition 3.14. For any partial order on an infinite set which is neither directed nor totally directed downward nor totally directed upward there exists a finite domain partial n-endomorphism $(n \geq 1)$ that is not extendable to an everywhere defined n-endomorphism.

Now we consider the countable set $E$.

Theorem 3.15. A partial clone $\text{Pol}(x \leq y)$ defined on the countable set $E$ is extendable if and only if $x \leq y$ belongs to the three classes of partial orders: (a) directed; (b) totally directed upward; (c) totally directed downward.


$(\Leftarrow)$ In order to use Theorem 3.12 we will show that any $(\exists, \&,-)$-formula from $x \leq y$ that belongs to the classes (a), (b) or (c) is equivalent to some $(\&,-)$-formula constructed from the same relation. In addition, it is obvious that for relations which are repetition-free (without equal coordinates) we may consider $(\exists,\&)$-formulas, since the equality relation adds only equal coordinates. Next it suffices to check $(\exists,\&)$-formulas having only one quantifier in the prefix form: $(\exists x_s) \&_{(i,j) \in G} x_i \leq y_j$, where $G$ is a binary relation (graph) of indices.
We have such identities: (1) for \( R \) defined in (9): \( R(x, y) \equiv x = x \& y = y \), if \( x \leq y \) belongs to the classes (a) or (b), and \( R(x, y) \equiv x = y \), if \( x \leq y \) belongs to (c); (2) for \( Q \) defined in (10): \( Q(x, y) \equiv x = x \& y = y \), if \( x \leq y \) belongs to (a) or (c), and \( Q(x, y) \equiv x = y \), if \( x \leq y \) belongs to (b); (3) \( S(u, v, y, z) \equiv (\exists x) y \leq x \& z \leq x \& x \leq u \& x \leq v \equiv y \leq u \& y \leq v \& z \leq u \& z \leq v \) for \( x \leq y \) from (a); \( S(u, v, y, z) \equiv y \leq u \& y \leq v \& z \leq u \& z \leq v \& u = v \) for \( x \leq y \) from (b); \( S(u, v, y, z) \equiv y \leq u \& y \leq v \& z \leq u \& z \leq v \& y = z \) for \( x \leq y \) from (c).

These three cases cover all possible combinations in which a bounded variable \( x \) is occurring in any \&-formula from \( z \leq y \). Hence for any partial order from (a), (b) or (c) we have

\[
(\exists x_i) & (i, j) \in G, x_i \leq y_j \equiv & (i, j) \in T, x_i \leq y_j \& D, \tag{11}
\]

where \( D \) is a diagonal, \( T \) is the relation on the same set of indices, but \( s \), and if \( s \neq i, j \), then \( (i, j) \in G \Leftrightarrow (i, j) \in T \), in addition, if \( (i, s) \in G \& (s, j) \in G \), then \( (i, j) \in T \).

Next we apply Theorem 3.12. \( \square \)

Note that there are such cases when we are able to prove extendibility directly for any infinite set.

**Corollary 3.16.** For any linear order \( \leq \) defined on an arbitrary infinite set, \( \text{Pol}(x \leq y) \) is extendable.

**Proof.** Consider \( f \in \text{Pol}(x \leq y), |\text{Dom}(f)| < \infty \), of arity \( n (n \geq 1) \) which is taking on exactly \( k \) different values \( (k \geq 1) \) as its range. Without loss of generality we denote them \( 1, 2, \ldots, k \) and arrange in such order \( 1 \leq 2 \leq \cdots \leq k \). Then we apply a \( k \)-step procedure to extend \( f \) on all \( n \)-tuples from \( E^n \): (1) put \( f(b) = k \) for every \( b \in E^n \) such that there exists \( a \in \text{Dom}(f) \) and \( a \preceq b \& f(a) = k \), denote \( \text{Dom}_1(f) \) the new domain of \( f \); (2) put \( f(b) = k - 1 \) for every \( b \in E^n \setminus \text{Dom}_1(f) \) such that \( a \preceq b \) and \( f(a) = k - 1 \); (3) put \( f(b) = 2 \) for every \( b \in E^n \setminus \text{Dom}_{k-2}(f) \) such that \( a \preceq b \) and \( f(a) = 2 \); (4) put \( f(b) = 1 \) for the rest \( n \)-tuples. Assume that for the extended \( f \) holds \( f \notin \text{Pol}(x \leq y) \), i.e., for some \( a \preceq b \) we have \( f(a) = i > f(b) = j \). Since the step \( k - i + 1 \) preceded the step \( k - j + 1 \), \( f(a) = i \) implies \( f(b) = j \). Contradiction. \( \square \)

4. **Extendable maximal local partial clones**

Now we will investigate extendable maximal local partial clones.

**Lemma 4.1.** If \( A = \text{Pol}(R) \) defined on an arbitrary infinite set \( E \) is an extendable maximal local partial clone, then \( A \cap O(E) \) is a maximal local clone on \( E \).

**Proof.** Assume that \( A \cap O(E) \) is not a maximal local clone. Then there exists a local clone \( B \) such that \( A \cap O(E) \subset B \subset O(E) \), and \( B = \text{Pol}^l(Q) \), where \( Q \) is a non-diagonal relation. Hence \( Q \in \text{Inv Pol}^l(R) = \text{Inv}(A \cap O(E)) \). From Proposition 3.4 we have \( Q \in \text{Inv}(A) = \text{Inv}(A \cap O(E)) \), or \( Q \in \text{Inv}(R) \). Next since \( A \) is a maximal local partial clone \( Q \) is obtained by some \((\&, =)\)-formula from \( R \) and vice versa (see Proposition 1.4). Hence \( \text{Pol}(Q) = \text{Pol}(R) \). Finally, \( A \cap O(E) = \text{Pol}(R) \cap O(E) = \text{Pol}(Q) \cap O(E) = B \). Contradiction. \( \square \)

Although the full description of the set of maximal local clones has not been obtained yet, the set of relations on an arbitrary infinite set \( E \), called **generic**, was produced by Rosenberg and Szabo [16]. This set contains some classes of relations, which determine maximal local clones, as well as relations determining classes of local clones in which all other maximal local clones are contained along with the local clones that are not included in any maximal local clone.

Next we consider all relations from the generic set and apply several criteria for selecting or rejecting relations in order to find the ones determining extendable maximal local partial clones. Although there is an infinite set of relations that determine the same maximal local clone and they in turn may determine different local partial clones, we will show that it is sufficient to check for extendibility only relations which determine maximal local partial clone. Namely, we have the following Lemma.

**Lemma 4.2.** If \( \text{Pol}^l(R) \) is a maximal local clone, but \( \text{Pol}(R) \) is not a maximal local partial clone, then any non-full partial clone \( \text{Pol}(Q) \), where \( \text{Pol}(R) \subset \text{Pol}(Q) \) and \( \text{Pol}^l(R) = \text{Pol}^l(Q) \), is not extendable.
Proof. Since \( \text{Pol}(R) \subset \text{Pol}(Q) \) a non-diagonal relation \( Q \) is obtained by some \((\&\), \(=\))-formula from \( R \), but the converse is not true (see Proposition 1.4). Hence \( R \notin \text{Inv Pol}(Q) \). At the same time \( R \in \text{Inv Pol}^{1}(Q) \), since \( \text{Pol}^{1}(R) = \text{Pol}^{1}(Q) \). Thus, we have \( \text{Inv Pol}(Q) \subset \text{Inv Pol}^{1}(Q) \). Next we apply Proposition 3.4.

In addition, we will use such result for checking extendibility of maximal partial clones.

**Lemma 4.3.** Let \( \text{Pol}(R) \) be a maximal local partial clone on an arbitrary infinite set. If there exists a non-diagonal \( Q \) constructed from \( R \) by a \((3, \&\), \(=\))-formula such that \( R \) itself cannot be obtained from \( Q \) by some \((\&\), \(=\))-formula, then \( \text{Pol}(R) \) is not extendable.

Proof. If \( \text{Pol}(R) \) is a local partial clone that is extendable, then any \( Q \) constructed by a \((\exists, \&\), \(=\))-formula from \( R \) can be obtained by some \((\&\), \(=\))-formula (see Corollary 3.13).

Since \( \text{Pol}(R) \) is a maximal local partial clone \( R \), in turn, is obtained from a non-diagonal \( Q \) by a \((\&\), \(=\))-formula.

Thus, we have such criteria for rejecting \( R \) taken from the generic set:

1. If \( R \) is not a minimal relation with respect to expressibility by \((\&\), \(=\))-formulas, then \( R \) is rejected (see Proposition 2.5 and Lemma 4.2; valid for all infinite sets).
2. If there exists \( Q \) constructed by a \((\exists, \&\), \(=\))-formula from \( R \) such that it cannot be obtained from \( R \) by any \((\&\), \(=\))-formula, then \( R \) is rejected (see Corollary 3.13; valid for any infinite set).
3. Let \( R \) be a minimal relation and a non-diagonal \( Q \) is constructed by a \((\exists, \&\), \(=\))-formula from \( R \). If \( R \) cannot be obtained from \( Q \) by some \((\&\), \(=\))-formula, then \( R \) is rejected (see Lemma 4.3; valid for any infinite set).

Criteria for selecting \( R \):

4. If \( R \) is minimal and it is shown that any \( f \in \text{Pol}(R) \), \( |\text{Dom}(f)| < \infty \), can be extended to some \( g \in \text{Pol}(R) \cap O(E) \), then \( R \) is accepted (see Definition 3.1; valid for any infinite set).
5. If any \((\exists, \&\), \(=\))-formula from a minimal relation \( R \) is equivalent to some \((\&\), \(=\))-formula from \( R \), then \( R \) is accepted (see Theorem 3.7; valid only for the countable set).

Note that if a relation \( Q \) without equal coordinates (repetition free) is obtained by a \((\exists, \&\), \(=\))-formula (by a \((\&\), \(=\))-formula) from \( R \), then the same relation can be obtained by some \((\exists, \&\), \(=\))-formula (respectively \&-formula) from \( R \) (e.g., see [11]). This implies that in the above criteria it suffices to consider \((\exists, \&\)-formulas instead of \((\exists, \&\), \(=\))-formulas.

Next we use some definitions from [16]. A binary relation \( R \) is called **locally bounded** if for every finite \( A \subset E \) there exist \( a, b \in E \) such that \( A \times \{a\} \subseteq R \) and \( \{b\} \times A \subseteq R \). A totally reflexive and totally symmetric \( R \in \mathcal{H} \) of arity \( h \geq 2 \) (see Section 2) is **locally central** if for every finite \( A \subset E \) there exist \( a \in E \) such that \( A^{h-1} \times \{a\} \subseteq R \). Denote \( Q = Q \times Q = \exists yQ(x, y)\&\&Q(y, z) \), then \( Q = Q \times \cdots \times Q \) \( n \) times \((n \geq 2) \). A reflexive symmetric binary relation \( Q \) has **diameter** \( n \) if \( Q \subseteq Q^2 \subseteq \cdots \subseteq Q^n \equiv E^2 \) (i.e., \( Q^n \) is the full binary relation) and **infinite diameter** if \( Q^n \subset E^2 \) for all \( n = 1, 2, \ldots \) but \( \bigcup_{n \geq 1} Q^n = E^2 \). Now we introduce the generic system [16]:

1. (1) proper unary relations;
2. (a) locally bounded partial orders;
   (b) proper equivalence relations;
   (c) graphs of a fixed point free permutations on \( E \) whose cycles are all of the same prime length;
3. (3) quaternary relations \( Q(x, y, u, v) \equiv x + y = u + v \), where \( \langle E, + \rangle \) is an abelian group which is either an elementary \( p \)-group or is torsion-free and divisible;
4. (4) locally central \( h \)-ary relations \((h \geq 2) \);
5. (5) ternary relations with the properties: (i) \( x = y \subset Q(x, y, z) \); (ii) \( Q(x, y, z)\&\&Q(y, x, z) \equiv Q(x, y, z) \); (iii) \( \exists Q(x, t, z)\&\&Q(y, t, z) \subseteq Q(x, y, z) \) and, (iv) for every finite \( A \subset E \) there exist \( a \in A \) such that \( A^2 \times \{a\} \subseteq Q \);
6. (a) binary reflexive symmetric relations of infinite diameter;
   (b) locally bounded reflexive antisymmetric binary relations of diameter 2;
   (c) locally bounded areflexive antisymmetric binary relations;
   (d) locally bounded areflexive symmetric binary relations;
(7) graphs of a fixed point free permutations on $E$ with cycles that are all infinite;
(8) $h$-ary relations from $H$ that are not locally central ($h = 3, 4, \ldots$).

Now using criteria (1)–(5) we will investigate classes (1)–(8).

(1) It is obvious that any finite domain partial operation $f$ preserving a proper unary relation $M, M \subseteq E$, is extended to everywhere defined operation preserving $M$, e.g., set $f(a_1, \ldots, a_n) = a \in M$ for any $(a_1, \ldots, a_n) \in E^n \setminus \text{Dom}(f)$ ($n \geq 1$). In addition, these relations, denote the set of them by $H_1$, belong to the class $H, H_1 \subseteq H$. Criterion (4) for selecting.

(2a) See the previous section (Section 3.1). Clearly the notion of a directed partial order is equivalent to a locally bounded partial order. Then from Theorem 3.15 $\text{Pol}(R)$, where $R$ is locally bounded (directed) partial order, is extendable for the countable set $E$. Denote $O_{\text{b}}$ the class of such relations. We have $O_{\text{b}} \subset Q$. Criterion (5) for the countable set.

(2b) Clearly any finite domain $n$-ary $f$ ($n \geq 1$) preserving a proper equivalence relation can be extended to an everywhere defined operation also preserving this relation. To prove this it suffices to consider the presentation $E^n = \bigcup_{(i_1, \ldots, i_n) \in A^n} B_{i_1} \times \cdots \times B_{i_n}$ as a disjoint union of products of $n$ blocks of the equivalence relation, where $E = \bigcup_{i \in A} B_i$ is the direct sum of its blocks. So $f$ can be extended uniformly on every product $B_{i_1} \times \cdots \times B_{i_n}(i_1, \ldots, i_n \in A)$. Denote $E$ the class of proper equivalence relations. We have $E \subseteq H$. Criterion (4) valid for all infinite sets.

(2c) Let $R(x, y)$ be the graph of a fixed point free permutation of degree $p \geq 3$. Then consider areflexive ternary relation $T(x, y, z) = R(x, y) \& R(y, z)$. Clearly each identification of $T$ to a binary relation is void and $R$ cannot be obtained from $T$ by any ($\&$, $=$)-formula. Hence $R$ is not a minimal relation (see Proposition 2.5). Criterion (1) for rejecting.

For any fixed point free permutation $\varphi$ of the degree $p = 2$ it is easy to verify that every finite domain $n$-ary operation $f$ ($n \geq 1$) preserving its graph $R(x, y) \Leftrightarrow y = \varphi x$ can be extended to some $g \in \text{Pol}(R) \cap O(E)$. Indeed, the set $E$ can be presented as a disjoint union of pairs $\{(a_1, \ldots, a_n), (\varphi a_1, \ldots, \varphi a_n)\}$ and so if $g(a_1, \ldots, a_n) = a$, then we set $g(\varphi a_1, \ldots, \varphi a_n) = \varphi a (a \in E)$. Denote $P_2$ the class of graphs $R(x, y) \Leftrightarrow y = \varphi x$. We have $P_2 \subset K_1$ (see Theorem 2.5). Criterion (4) for selecting $R$; it is valid for all infinite sets.

(3) For every quaternary $Q(x, y, z) \equiv x + y = z + u$, where $(E; +)$ is an abelian group which is either an elementary $p$-group ($p \geq 3$) or a torsion-free and divisible, consider the relation:

$$L(x, y, z) \equiv Q(x, y, z) \equiv x + y = 2z.$$ 

Clearly $L$ is not a diagonal, since there are 3-tuples on $E$ not belonging to $L$, as well as 3-tuples with distinct elements actually belonging to $L$. Next we will prove that $Q$ cannot be obtained by any ($\&$, $=$)-formula from $L$. The latter implies criterion (1) for rejecting.

First consider an elementary $p$-group, $p \geq 5$. We choose a 4-tuple with distinct coordinates $r = (0, a, (p - 1)a, 2a) \in Q$, where $a \neq 0$. If $Q(x, y, z, u)$ is constructed by a $\&$-formula from $L$, then clearly this formula has at least two conjunctive terms presenting different relations, e.g., $T(x, y, z, u) \equiv y + z = 2x & x + z = 2u$ (here each conjunctive term is considered as a 4-ary relation by adding dummy coordinates to it). Then it is easy to verify that for every $p > 5$ there is only one connection between any three coordinates of the 4-tuple $r$ with respect to the relation $L$, namely, $y + z = 2x \Leftrightarrow a + (p - 1)a = 2 \times 0$. Hence any other conjunctive term does not include $r$ and, thus, $Q$ cannot be presented as $\&$-formula from $L$.

Next for $p = 5$ we choose $r = (0, a, 4a, 2a) \in Q \cap T$ and obtain that there are no other possible conjunctive terms containing $r$. Hence only $T$ could be equal to $Q$ among all $\&$-formulas from $L$. At the same time, the 4-tuple $q = (4a, 2a, 0, a) \in Q$ does not belong to $T$.

The same stipulation as in the case $p > 5$ can be applied to a torsion free divisible group while considering the 4-tuple $r = (0, a, -a, 2a) \in Q$.

If $p = 3$, then consider a 4-tuple $(a, b, b, a) \in Q$, where $a \neq b$ and $a \neq 2b$. It is easy to check that any three coordinates of this 4-tuple cannot be a 3-tuple from $L$, e.g., if $(a, b, b) \in L$, then $a + b = 2b$ which yields $a = b$. Hence $Q$ cannot be obtained by a $\&$-formula from $L$.

Now consider an elementary abelian 2-group determined by the relation:

$$Q(x, y, u, z) \equiv x + y = u + z \equiv x + y + u + z = 0.$$
We will construct a 6-ary relation \( S \) by some \((\exists, \&)-formula\) from \( Q \) and prove that \( S \) cannot be obtained from \( Q \) by any \&-formula. See criterion (2) for rejecting \( Q \). We have:

\[
S \equiv \exists x (x_1 + x_2 + x_3 + x = 0 \& x_4 + x_5 + x_6 + x = 0) \equiv x_1 + x_2 + x_3 = x_4 + x_5 + x_6. \tag{12}
\]

To prove the above statement it suffices to show that for every conjunctive term \( Q(x_1, \ldots, x_6) (i_1, \ldots, i_4 \in \{1, 2, \ldots, 6\}) \) in any \&-formula of arity 6 constructed from \( Q \) there exists a 6-tuple from \( S \) that does not belong to this term. First clearly any identification of coordinates in \( Q \) yields a diagonal. Then we check this condition for the term \( Q(x_1, x_2, x_4, x_5) \) (two pairs of variables from the both sides of equality sign in (12)). Now for the 6-tuple \( (a, 0, a, a, a, a) \in S \) we choose \( (a, 0, a, a) \notin Q(x_1, x_2, x_4, x_5) \). Next for the term \( Q(x_1, x_2, x_3, x_5) \) (3 variables from one side of equality in (12)) we choose the 6-tuple \( (0, a, a, a, 0) \in S \) and obtain \( (0, a, a, a) \notin Q(x_1, x_2, x_3, x_5) \).

(4) Let \( R \) be a locally central \( h \)-ary \((h \geq 2)\) relation. Denote the class of such relations by \( C_{lb} \). Clearly \( C_{lb} \subseteq H \). Next we will show that an application of \( \exists x \) to any \&-formula from \( R \) yields either the full relation, if all conjunctive terms of it contain the bounded variable \( x \), or a \&-formula consisting of all terms which do not contain the variable \( x \). Since \( R \) is totally symmetric and totally reflexive (i.e., \( R(x, x, x_3, \ldots, x_{h-2}) \) is the full \((h - 1)\)-ary relation) in order to prove that it suffices to check that the following relation:

\[
Q \equiv (\exists x) R(x, x_1, \ldots, x_{h-1}) \& R(x, x_h, \ldots, x_{2h-2}) \& \cdots \& R(x, x_{(h-1)(m-1)+1}, \ldots, x_{m(h-1)}) \tag{13}
\]

is the full relation for any \( m \geq 1 \) and any arrangement of \( m(h - 1) \) free variables in it.

Let \( s \) be the arity of \( Q \) (in case, when all free variables are distinct \( s = m(h - 1) \)). Then for each \( s \)-tuple \((a_1, \ldots, a_s) \in E^s \) we choose the finite set \( B = \{a_1, \ldots, a_t \} \) consisting of all different elements from this \( s \)-tuple. Next since \( R \) is locally central there exists \( c \in E \) such that \( c \times B^{h-1} \subseteq R \). Hence from (13) we get \((a_1, \ldots, a_s) \in Q \) and \( Q \) is the full relation.

Thus, \( Q \) is accepted due to the criterion (5) which is valid for the countable set.

Notice that in case \( Q \in C_{lb} \) from the Lemma 4.1 we obtain another proof that the local clones \( \text{Pol}'(Q) \) are maximal on the countable set (see [15] for any infinite set).

**Corollary 4.4.** \( \text{Pol}'(Q), Q \in C_{lb}, \) is a maximal local clone on the countable set.

(5) From the properties (i), (ii) and (iii) we obtain that \( Q \) has the following form:

\[
Q(x, y, z) \equiv \bigcup \{ R_i(x, y) \& z = a_i : \text{for all } a_i \in E \}, \tag{14}
\]

where \( R_i \) are binary equivalence relations on \( E \) corresponding to each element of \( E \).

Indeed, from (i) we get that each \( R_i \) is reflexive, from (ii) we obtain that \( R_i \) is symmetric, and from (iii) it follows that \( R_i \) is transitive. In addition, from (i) we get that the set \( \{a_i \} \) covers all elements of \( E \).

Next we consider \( T(x, y) \equiv Q(x, y, y) \). From (i) we obtain that \( x = y \subseteq T(x, y) \subseteq x = x \& y = y \) (complete binary relation). If \( T \) is complete, then from the presentation (14) it is easy to get that all \( R_i \) are also complete and hence \( Q \) is the complete ternary relation. In case, when \( T \) is non-diagonal, i.e., \( x = y \subseteq T(x, y) \subseteq x = x \& y = y \), \( \text{Pol}'(T) \) is included in some local clone determined by a binary relation from classes (2a)–(2c), (6a)–(6d) or (7) (see [16], Section 3). Hence in what follows we only consider \( Q \) with the properties: \( x = y \equiv Q(x, y, y) \) and \( Q(x, y, x) \equiv x = y \). This implies the fact.

**Fact 1.** \( Q(x, y, z) \equiv x = y \lor R(x, y, z) \), where \( R \) is an areflexive ternary relation,

From the presentation (14) one can easily get the following fact.

**Fact 2.** Each equivalence relation \( R_i \) in (14) has \( a_i \) as its single block.

From this fact we get another fact.

**Fact 3.** \( Q(x, y, z) \& x = z \equiv x = y = z \) and also \( Q(x, y, z) \& y = z \equiv x = y = z \).

Now we construct a quaternary relation \( S \) by a \((\exists, \&)-formula\) from \( Q \):

\[
S(x, y, u, v) \equiv \exists z Q(x, y, z) \& Q(z, u, v).
\]
We will show that $S$ cannot be obtained by some $\&$-formula from $Q$, which yields the criterion (2) for rejecting this type of relations. Clearly we have $S(x, y, u, u) \equiv Q(x, y, u)$. Hence $S$ is not a diagonal. Then applying Fact 1 and properties of the first order calculus operations we obtain:

$$S(x, y, u, v) \equiv \exists z (x = y \lor R(x, y, z)) \& \exists z (z = u \lor R(z, u, v)) \equiv \exists z (x = y \& z = u \lor R(x, y, z) \& z = u \lor x = y \& R(z, u, v) \lor R(x, y, z) \& R(z, u, v)) \equiv x = y \lor R(x, y, u) \lor \exists z R(x, y, z) \& R(z, u, v). \quad (15)$$

Hence $Q(x, y, u) \subseteq S(x, y, u, v)$ (on the left, $Q$ has a dummy variable $v$, which is equivalent to the conjunction with $v = v$). At the same time there exists a 4-tuple $(b, c, a, d) \in S(b \neq c)$ which from Fact 3 does not belong to $Q(x, y, u) \& v = v$. Indeed, let $(b, c, d) \in R$, then from (iv) we choose $a \in E$ such that for $B = \{c, d\}$ $(c \neq d)$ $B^2 \times \{a\} \subset Q$ and so $(d, c, a) \in R$. Then from (15) we get $(b, c, d) \in R \& (d, c, a) \in R \rightarrow (b, c, a) \in S$. Hence $Q(x, y, u) \subseteq S(x, y, u, u)$ and so $Q(x, y, u)$ cannot be a conjunctive term in any $\&$-formula for $S$ constructed from $Q$. Then each conjunctive term in such $\&$-formula must contain $Q(x, y, u)$ which leads to contradiction with Facts 1 and 3, e.g., if $Q(x, y, u) \subseteq Q(y, z, x)$, then by identification of $x$ and $y$ we get $Q(x, x, u) \equiv x = x \& u = u \supset Q(x, z, x) \equiv x = z$.

(6a) We have $R(x, y) \subseteq R^2(x, y) \subseteq \ldots$, where $R^2(x, y) \equiv \exists z R(x, z) \& R(z, y)$ is defined by $(\exists, \&)$-formula from $R$. Clearly $R^2$ cannot be obtained from $R$ by some $\&$-formula (actually, the symmetric $R$ itself and diagonals are the only binary relations constructed from $R$ by any $\&$-formula). Criterion (2) for rejecting.

(6b) From the results of [12] (Lemma 2) we obtain that any reflexive and antisymmetric $R$ is not minimal unless $R \in Q$ (the class of reflexive, antisymmetric binary relations without cycles of finite length).

**Lemma 4.5.** Any $R \in Q$ with the diameter 2 is a transitive binary relation.

**Proof.** First we will prove that for $R \in Q$ with the diameter 2 and any pair $(a, b) \in E^2, a \neq b$, holds either $(a, b) \in R$ or $(b, a) \in R$. Indeed, if $(a, b) \in E^2 \setminus R$, then since $R$ has diameter 2 there exists $c \neq a, b$ such that $(a, c) \in R$ and $(c, b) \in R$. If we assume that $(a, b) \in E^2 \setminus R$, then applying the same property (namely, $R$ has diameter 2) we obtain that there exists $d, d \neq c$ (antisymmetric) such that $(b, d) \in R$ and $(d, a) \in R$. Hence $R$ contains a finite cycle $(a, c, b, d, a)$. Contradiction.

Next if for three different elements $a, b, c \in E$ we have $(a, b) \in R$ and $(b, c) \in R$, then $(c, a) \in R$. Otherwise, from the above we get $(c, a) \in R$ and, consequently, there is a cycle $(a, b, c, a)$ of the length 3 in $R$. Hence $R$ is a transitive relation. □

From Lemma 4.5 it follows that each $R \in Q$ with the diameter 2 belongs to the class $O_{lb}$ (locally bounded partial order relations, see case (2a)).

(6c) Among all relations from this class only those belonging to the class $K_2$ (areflexive, asymmetric without paths of length 2, see Theorem 2.6) are actually minimal. Then we will show that each $R \in K_2$ is not bounded. Indeed, if $R$ is bounded, then for each $(a, b) \in R$ there exists $c, c \neq a, b$ (areflexive) such that $(c, a) \in R$ and $(c, b) \in R$. Hence we obtain the path $(c, a, b) \in R$ of the length 2. Criterion (1) for rejecting.

(6d) Here only relations from the class $K_1$ (areflexive, symmetric without cycles of an odd length, see Theorem 2.6) are actually minimal. Then, since $R$ is locally bounded, for any $(a, b) \in R$ there exists $d \neq a, b$ (areflexive) such that $(a, d) \in R$ and $(b, d) \in R$ and, consequently $(d, b) \in R$ (R is symmetric). Hence we obtained the cycle $(a, d, b, a)$ in $R$ of the length 3. Contradiction with $R \in K_1$. Criterion (1) for the rejection.

(7) For the graph $R(x, y)$ of the fixed point free permutation with infinite cycles we put $Q(x, y, z) \equiv R(x, y) \& R(y, z)$. Clearly $Pol(R) \subset Pol(Q)$ (one cannot obtain $R$ from $Q$ by identification of coordinates). Hence $R$ is not a minimal relation. Criterion (1) for rejecting.

(8) First we consider totally reflexive and totally symmetric relation $T_h(x_1, \ldots, x_h) \equiv \lor_{1 \leq i \leq j \leq h} (x_i = x_j) (h \geq 3)$ that does not contain $h$-tuples with distinct coordinates. It is known (see [15] and [9]) that $[Pol^h(T_h) : h = 3, 4, \ldots]$ form an increasing chain of local clones which are not included in any maximal local clone. Then we apply Lemma 4.1 for rejecting those relations.

Next if $R \in H_{\leq 3}(T_h)$ of arity $h \geq 3$ is not locally central, then there exists a finite set $B = \{a_1, \ldots, a_n\} (n \geq h)$ such that for any $a \in E$ we have $[a] \times B^{h-1} \notin R$. Then we construct an $n$-ary $(n \geq h)$ relation by $(\exists, \&)$-formula from $R$:

$$S_n(x_1, \ldots, x_n) \equiv \exists x \& \lor_{1 \leq i_1 \leq \cdots \leq i_{h-1} \leq n} R(x, x_{i_1}, \ldots, x_{i_{h-1}}). \quad (16)$$
From (16) one can get that \((a_1,\ldots,a_n) \in S_n\) is equivalent to the existence of \(a \in E\) such that \([a] \times B^{h-1} \subset R\), where \(B = \{a_1,\ldots,a_n\}\) is the above defined set. Hence \((a_1,\ldots,a_n) \notin S_n\) and a totally symmetric \(S_n\) \((n \geq h \geq 3)\) is an incomplete, non-diagonal relation.

Now we will check whether \(R\) of arity \(h \geq 3\) can be obtained from \(S_n\) \((n > h)\) by some \&-formula that yields to identification of coordinates in \(S_n\) to arity \(h\). Next it is easy to verify that \(S_h(x_1,\ldots,x_h) = \exists x \&_1 \leq k \prec \leq h \leq h \exists x, x_{1i},\ldots,x_{hi-1} R (x, x_1,\ldots,x_{hi-1})\) is the unique relation obtained by any identification of \(S_n\) \((n > h)\) to arity \(h\), since each term \(R(x, x_1, x_{i},\ldots)\) is the complete relation. Clearly \(R \subseteq S_h\). For \(n = h\), since \(R\) and \(S_h\) are both totally symmetric \(R\) can be obtained from \(S_h\) by a \&-formula if and only if \(R \equiv S_h\). Next we have two cases:

1. \(R \subset S_h\). Then a minimal relation \(R\) cannot be obtained from \(S_h\) by any \&-formula. Criterion (3) for the rejection.
2. \(R \equiv S_h\). This is equivalent to such statement: for any \(B = \{a_1,\ldots,a_n\}\) such that there exists \(a \in E\) with the property \(a \times B^{h-1} \subset R\) we have \((a_1,\ldots,a_n) \in R\). Since for the identification of coordinates we have \(\text{id}(x_h = x_{h+1}) S_{h+1}(x_1,\ldots,x_{h+1}) \equiv S_h(x_1,\ldots,x_h)\) the relation \(S_{h+1}\) is a non-diagonal (any identification of coordinates in a diagonal is also a diagonal).

Next consider an \((h + 1)\)-ary relation \(S(x_1,\ldots,x_{h-3},u,v,y,z)\) which is built by \((\exists,\&)-formula\) from \(R\) in the same way as \(S_{h+1}\) in (16) with the exception of two conjunctive terms \(R(x, x_1,\ldots,x_{h-3},u,v)\) and \(R(x, x_1,\ldots,x_{h-3},y,z)\) that are not included in the \((\exists,\&)-formula\) for \(S\). Clearly \(S_{h+1} \subseteq S\). We will use the following Fact whose proof relies on the totally symmetric and reflexive properties of \(R\).

**Fact 4.** \(\text{id}(u = v) S \equiv \text{id}(y = z) S\) is the complete \(h\)-ary relation and any other identification of \(2\) coordinates in \(S(x_1,\ldots,x_{h-3},u,v,y,z)\) is equal to \(S_h\).

Since any identification of \(2\) variables of \(S_{h+1}\) is \(S_h\) from the Fact 4 we obtain \(S_{h+1} \subset S\). Next consider the possibility of constructing \(S\) by some \&-formula from \(R\). We have the following Fact.

**Fact 5.** If \(S(y,z,\ldots)\) of arity \(n(n > h)\) is presented as a \&-formula from totally symmetric and reflexive relation \(R\) of arity \(h \geq 3\) and \(\text{id}(y = z) S\) is the complete \((n - 1)\)-ary relation, then each incomplete conjunctive term in the above \&-formula contains explicitly variables \(y\) and \(z\).

Indeed, if it is not true for some conjunctive term \(R\), then the application of \(\text{id}(y = z)\) to this \&-formula does not yield the complete relation rather the result is included in \(R\).

Next if \(S(x_1,\ldots,x_{h-3},u,v,y,z)\) is represented by some \&-formula from \(R\), then from the Facts 4 and 5 variables \(u,v,y,z\) are included in every conjunctive term of this \&-formula. For \(h = 3\) this means that there is no such formula, and for \(h = 4\) we get \(S(u,v,y,z) \equiv R(u,v,y,z) \equiv S_4\) which contradicts \(S_4 \subset S\). Finally, for \(h \geq 5\) we have \(S(x_1,\ldots,x_{h-3},u,v,y,z) \equiv \&R(u,v,y,z,\ldots)\). At the same time \(\text{id}(u = y) S \equiv S_h\) (Fact 4) on the left and the complete relation \((R\) is totally reflexive\) on the right side of this identity. Contradiction. Criterion (2) for rejecting.

Now we summarize the results of this section.

**Theorem 4.6.** A maximal local partial clone defined on the countable set is extendable if and only if a relation from the following classes determines it:

1. \(H_1\): all unary proper relations;
2. \(E\): all proper equivalence relations;
3. \(P_2\): all graphs of fixed point free permutations of degree 2;
4. \(O_{lh}\): all partial orders that are directed (locally bounded);
5. \(C_{lb}\): all locally bounded central relations of arity \(h \geq 2\).

**Proposition 4.7.** Every maximal local partial clone, other than \(\text{Pol}(R)\), \(R \in H_1 \cup E \cup P_2 \cup O_{lb} \cup C_{lb}\), which is defined on an arbitrary infinite set, contains a finite domain partial operation that cannot be extended to everywhere defined operation from the same partial clone.

**Corollary 4.8.** For any uncountable set \(E\), \(\text{Pol}(R), R \in H_1 \cup E \cup P_2\), is an extendable maximal local partial clone.
Notice that in order to complete the description of all extendable maximal local partial clones defined on any infinite set we only need to investigate the extendibility of maximal local partial clones of the form $\text{Pol}(R)$, $R \in \mathcal{O}_{lb} \cup \mathcal{C}_{lb}$, defined on an uncountable infinite set.

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References