The Optimal Age-Dependent Checkpoint Strategy for a Stochastic System Subject to General Failure Mode

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Received March 14, 2000

In this paper, we consider an age-dependent stochastic system with checkpointing and rollback recovery, subject to general failure mode. In the case where the queueing effect for transaction processing is not remarkable, we investigate the stochastic behavior of the system and derive analytically the optimal checkpointing strategy which maximizes the system availability. When the queueing effect due to the idle period in the system is remarkable, the optimal checkpoint strategy is characterized approximately.
1. INTRODUCTION

Checkpointing and rollback recovery are commonly used techniques for improving the reliability/maintainability of fault-tolerant computing systems. Especially, when the file system to write and/or read data is designed in terms of maintainability, checkpoint generations play an important role to limit the amount of data processing for the recovery actions after system failure occurrence. If the generations of checkpoints are frequently executed, a larger overhead by them will be incurred. Conversely, if only a few checkpoints are generated, a larger overhead after system failures will be required in the rollback recovery actions. Hence, it is very significant, from the practical point of view, to determine theoretically the optimal checkpoint interval taking account of the trade-off between two kinds of overhead factors above. In fact, several studies of determining the optimal checkpoint sequence were developed in the literature of computer science.

First, Young [14] obtained the optimal checkpoint interval approximately for the computation restart after system failures. Chandy et al. [2, 3], Gelenbe and Derochette [4], Gelenbe [5], Grassi et al. [9], Kulkarni et al. [10], and Nicola and Van Spanje [11] proposed performance evaluation models for database recovery and calculated the optimal checkpoint intervals which maximize and minimize the system availability and the mean overhead during the normal operation, respectively. Toueg and Babaoglu [13] derived a dynamic programming algorithm which minimizes expected execution times of tasks placing checkpoints between two consecutive tasks under very general assumptions. Baccelli [1], Gelenbe and Hernandez [6], and Sumita et al. [12] extended the standard periodic checkpoint model by Gelenbe [5], where the underlying system failures occur in accordance with a non-homogeneous Poisson process, a homogeneous Markov process, etc. Recently, Ziv and Bruck [15] pointed out several problems in earlier checkpoint models and suggested an on-line algorithm to generate the optimal checkpoint sequence.

An interesting attempt was made by Goes and Sumita [7] and Goes [8]. They proposed new checkpoint models dependent on the system age, instead of periodic checkpointing methods [4–6, 9, 10, 13, 14]. The main advantage for the periodic checkpointing is that it is easier to administer in general, since the checkpoint is placed periodically even if a number of system failures occur during the checkpoint interval. However, it should be noted that in periodic checkpoint models, the information recovered by the rollback procedure after a system failure is not always backed-up in the secondary medium. This fact will affect the data maintainability strongly and will reduce the checkpointing effect, especially, in an unreliable system circumstance. Unfortunately, the age-dependent checkpoint model by Goes and Sumita [7] was incomplete to overcome the problem mentioned above.
More specifically, in the literature [7], it was as if a checkpoint took place after recovery had taken place. In addition, Goes and Sumita [7] developed a performance evaluation model under the simple assumption that the system failure occurs according to the homogeneous Poisson process. This assumption is rather questionable in actual file management (see [1, 6, 12, 13]).

In this paper, we develop a new age-dependent checkpoint model subject to general failure mode. Especially, it should be noted that the present model is improved in terms of possibility of back-up just after the rollback procedure. In the case where the queueing effect for transaction processing is not remarkable, we investigate the stochastic behavior of the system and derive analytically the optimal checkpoint strategy which maximizes the system availability. When the queueing effect due to the idle period in the system is remarkable, we provide the approximation form of the optimal checkpoint interval.

2. MODEL DESCRIPTION

Let us consider a stochastic file system to write and/or read data with failure interruptions. Suppose that the file system is operated in continuous time. System failures occur at the i.i.d. random variables, where the inter-failure time distribution $G(t)$ is absolutely continuous and monotonically increasing with $G(0) = 0$ and $\lim_{t \to \infty} G(t) = 1$, having probability density function $g(t) = dG(t)/dt$, failure rate $r(t) = g(t)/G(t)$, and finite mean $1/\lambda (>0)$, where in general $\psi(\cdot) = 1 - \phi(\cdot)$. Upon a failure, a rollback recovery takes place where both the information of transactions saved at the last checkpoint creation and recorded in the log file are used for restoring the system to a usable state. A checkpoint is placed just after the rollback recovery, and all information is backed-up from the primary memory to the secondary medium. This modelling approach can present a real file recovery scheme and satisfies our intuition well.

Define $\{X(t), t \geq 0\}$ to be the cumulative operation time for the file system at time $t$ since the last checkpoint. That is, we denote the elapsed time since the last checkpoint by $X(t)$. The length of the rollback recovery is assumed to depend on the number of transactions executed, i.e., on the value of $X(t)$ at the time of failure. We extensively employ a generic random variable $V_x$ denoting the length of the rollback recovery given that a failure occurs at time $t$ with $X(t) = x$. The probability distribution of $V_x$ is denoted by $B_x(y) = \Pr\{V_x \leq y\}$ with probability density function $b_x(y)$ and hazard rate $\xi_x(y) = b_x(y)/B_x(y)$.

Intervals between two consecutive checkpoints are determined by the total operation time excluding rollback periods. The $i$th checkpoint is
generated as soon as the total operation time since the \((i - 1)\)st checkpoint reaches the length \(S_i (i = 1, 2, \ldots)\) or the system failure occurs before the total operation time reaches the length \(S_i\). Assume that \(S_i (i = 1, 2, \ldots)\) constitutes a sequence of random variables with common distribution \(A(x) = \Pr\{S_i \leq x\}\) with probability density function \(a(x)\) and hazard rate \(\eta(x) = a(x)/A(x)\). Times (overheads) required for creating checkpoints also form a sequence of i.i.d. random variables \(C_i (i = 1, 2, \ldots)\) with \(W(z) = \Pr\{C_i \leq z\}\) and probability density function \(w(z)\). In the latter discussion, we drop the subscripts of \(S_i\) and \(C_i\).

For convenience, define the stochastic process \(\{I(t), t \geq 0\}\) as

\[
I(t) = \begin{cases} 
0 & \text{if a checkpoint is being created at time } t \\
1 & \text{if the file system is operating normally at time } t \\
2 & \text{if the file system is recovering from a system failure at time } t.
\end{cases}
\]  

To simplify the discussion, we assume \(X(t) = 0\) at the end of state \(I(t) = 0\) and \(I(0) = 1\). Further, we define the stochastic process \(\{Y(t), t \geq 0\}\) as the elapsed time since the last failure interruption, given \(I(t) = 2\). In other words, \(Y(t)\) is defined to be zero if \(I(t) \neq 2\). Now we are in the position to characterize the stochastic system by the trivariate process \(\{X(t), Y(t), I(t), t \geq 0\}\). Because of the renewal nature of both checkpointing and recovery, the stochastic behavior of the trivariate process has a cyclic structure as depicted in Fig. 1.

3. AVAILABILITY ANALYSIS

Of interest in this section is the effective operating time \(S_{\text{eff}}\) describing the cumulative time between the checkpoints. Define

\[
F_0(t) = \Pr\{I(t) = 0\} = \Pr\{S_{\text{eff}} \leq t\},
\]  

FIG. 1. Configuration of a stochastic system.
Define the Laplace transform of the function

\[ F_1(x, t) = \Pr\{X(t) \leq x, I(t) = 1\}, \quad (3) \]

\[ F_2(x, y, t) = \Pr\{X(t) \leq x, Y(t) \leq y, I(t) = 2\}, \quad (4) \]

and

\[ f_0(t) = \frac{\partial}{\partial t} F_0(t), \quad (5) \]

\[ f_1(x, t) = \frac{\partial}{\partial x} F_1(x, t), \quad (6) \]

\[ f_2(x, y, t) = \frac{\partial^2}{\partial x \partial y} F_2(x, y, t). \quad (7) \]

Defining the Laplace transform of the function \( \psi(t) \) by \( \hat{\psi}(s) = \int_0^\infty \exp\{-st\} \times \psi(t) \, dt \). With the well-known state space method, we obtain the following results.

**Lemma 3.1.** The Laplace transform of the effective operation time \( S_{\text{eff}} \) is

\[
E[\exp(-sS_{\text{eff}})] = \int_0^\infty \exp(-sx) \overline{G}(x) dA(x) \\
+ \int_0^\infty \exp(-sx) \overline{A}(x) \hat{\beta}_1(s) dG(x). \quad (8)
\]

**Proof.** From \( f_2(x, y, t) = f_2(x, 0, t - y) \overline{B}_1(y) \) and \( f_2(x, 0, t) = r(x)f_1(x, t) \), we have \( f_2(x, y, t) = r(x)f_1(x, t - y) \overline{B}_1(y) \). Similarly, we can see that

\[ f_0(t) = \int_0^\infty f_1(x, t) \eta(x) dx + \int_0^\infty \int_0^\infty f_2(x, y, t) \xi(y) dy dx. \quad (9) \]

From these relations, we have

\[ f_0(t) = \int_0^\infty f_1(x, t) \eta(x) dx + \int_0^\infty \int_0^\infty r(x)f_1(x, t - y) \beta_1(y) dy dx. \quad (10) \]

Taking the Laplace transform of both sides in Eq. (10), one sees that

\[ \hat{f}_0(s) = \int_0^\infty \hat{f}_1(x, s) \eta(x) dx + \int_0^\infty \int_0^\infty r(x)\hat{f}_1(x, s) \beta_1(s) dx. \quad (11) \]

In order to obtain \( \hat{f}_1(x, s) \), we investigate the infinitesimal behavior in the interval \([t, t + \Delta]\). Then, it is immediate to get

\[ f_1(x + \Delta, t + \Delta) = \{1 - r(x)\Delta\} \{1 - \eta(x)\Delta\} f_1(x, t) + o(\Delta). \quad (12) \]

Dividing both sides of Eq. (12) by \( \Delta \) and letting \( \Delta \to 0 \), we have the partial differential equation

\[
\frac{\partial}{\partial x} \hat{f}_1(x, s) + s \hat{f}_1(x, s) - f_1(x, 0) = -\{r(x) + \eta(x)\} \hat{f}_1(x, s). \quad (13)
\]
Solving this equation with boundary conditions \( f_1(x, 0) = 0 \) for \( x > 0 \) and \( f_1(0, t) = \delta(t) \) for \( t \geq 0 \) or \( \hat{f}_1(0, s) = 1 \), where \( \delta(\cdot) \) is the usual delta function, then we find

\[
\hat{f}_1(x, s) = \exp\{-sx\} \bar{G}(x) \bar{A}(x).
\] (14)

Substituting Eq.(14) into Eq.(11), the proof is completed.

**Corollary 3.2.** The mean effective operation time and the variance are

\[
E[S_{eff}] = \int_0^\infty \{G(x) + E[V_1]g(x)\} \bar{A}(x)dx
\] (15)

and

\[
\text{Var}[S_{eff}] = E[S_{eff}^2] - \{E[S_{eff}]\}^2,
\] (16)

respectively, where

\[
E[S_{eff}^2] = \int_0^\infty \{2x(G(x) + E[V_1]g(x)) + E[V_1^2]g(x)\} \bar{A}(x)dx.
\] (17)

The proof is omitted for brevity.

From the results above, we formulate the ergodic probabilities in

\[
\Pi_0 = \lim_{t \to \infty} \Pr\{I(t) = 0\} = \frac{E[C]}{E[S_{eff}] + E[C]},
\] (18)

\[
\Pi_1 = \lim_{t \to \infty} \Pr\{I(t) = 1\} = \frac{\int_0^\infty G(x) \bar{A}(x)dx}{E[S_{eff}] + E[C]},
\] (19)

\[
\Pi_2 = \lim_{t \to \infty} \Pr\{I(t) = 2\} = \frac{\int_0^\infty g(x)E[V_1] \bar{A}(x)dx}{E[S_{eff}] + E[C]}.
\] (20)

The system availability is defined by \( \Pi_1 \), which denotes the probability that the system is operating in the steady-state. Then, the optimal prespecified checkpoint interval which maximizes \( \Pi_1 \) can be characterized as follows.

**Lemma 3.3.** Define the functions \( \kappa_1(\tau) = \int_0^\tau G(x)dx \) and \( \kappa_2(\tau) = \int_0^\tau \{G(x) + E[V_1]g(x)\}dx + E[C] \). There exists \( \tau \) which maximizes \( \kappa_1(\tau)/\kappa_2(\tau) \) and let it be \( T (>0) \). Then \( A(x) \) which maximizes the system availability \( \Pi_1 \) is

\[
A(x) = U(x - T) = \begin{cases} 1 & \text{if } x \geq T \\ 0 & \text{otherwise} \end{cases}
\] (21)

where \( U(\cdot) \) is the unit function.
Proof. From an intuitive argument with an additional parameter \( \tau (>0) \), the system availability in Eq. (19) can be rewritten by

\[
\Pi_1 = \frac{\int_0^\infty \kappa_1(\tau)dA(\tau)}{\int_0^\infty \kappa_2(\tau)dA(\tau)}.
\]

(22)

Since the function \( \kappa_1(\tau)/\kappa_2(\tau) \) is continuous in \( \tau \), there exists \( \tau = T \) which maximizes \( \kappa_1(\tau)/\kappa_2(\tau) \). Thus, it is found that \( \kappa_1(\tau)/\kappa_2(\tau) \leq \kappa_1(T)/\kappa_2(T) \) for arbitrary \( \tau (>0) \). This leads to

\[
\Pi_1 = \Pi_1(A(x)) = \frac{\int_0^\infty \kappa_1(\tau)dA(\tau)}{\int_0^\infty \kappa_2(\tau)dA(\tau)}
\]

\[
\leq \frac{\kappa_1(T)}{\kappa_2(T)} = \Pi_1(U(x - T))
\]

(23)

and the result is obtained.

From Lemma 3.3, the randomized policy \( A(x) \) is translated to the constant policy \( T \), and we can replace \( \Pi_1 \) by \( \Pi_1(T) \), where

\[
\Pi_1 = \Pi_1(T) = \frac{\int_0^T \overline{G}(x)dx}{\int_0^T \left[ \overline{G}(x) + \text{E}[V_x]g(x) \right]dx + \text{E}[C]}.
\]

(24)

Then, the problem can be reduced to the simple algebraic one, i.e.,

\[
\max_{0<T<\infty} \Pi_1(T).
\]

Define the functions \( h(x) = \text{E}[V_x]r(x) \) and

\[
q(T) = \int_0^T \text{E}[V_x]g(x)dx + \text{E}[C] - h(T) \int_0^T \overline{G}(x)dx.
\]

(25)

The following results give the optimal checkpoint strategy which maximizes the system availability.

**Theorem 3.4.** There exists at least one optimal checkpoint interval which maximizes the system availability. If \( q(\infty) < 0 \), then it should be finite.

Proof. Taking the logarithm of \( \Pi_1(T) \) and differentiating it with respect to \( T \), we have

\[
\frac{d \log \Pi_1(T)}{dT} = \frac{\overline{G}(T)q(T)}{\Pi_n(T)\Pi_d(T)},
\]

(26)

where

\[
\Pi_n(T) = \int_0^T \overline{G}(x)dx (> 0),
\]

(27)

\[
\Pi_d(T) = \int_0^T \left\{ \overline{G}(x) + \text{E}[V_x]g(x) \right\}dx + \text{E}[C] (> 0).
\]

(28)
For a sufficiently small $T$, it is seen that

$$\frac{d \log \Pi_1(T)}{dT} \simeq \frac{\tilde{G}(T)q(0)}{\Pi_a(T)\Pi_d(T)} > 0 \quad (29)$$

and that there exists at least one optimal checkpoint interval $T^*$ $(0 < T^* \leq \infty)$. On the other hand, for a sufficiently large $T$, we find

$$\frac{d \log \Pi_1(T)}{dT} \simeq \frac{\tilde{G}(T)q(\infty)}{\Pi_a(T)\Pi_d(T)} \quad (30)$$

Hence, if $q(\infty) < 0$, then the optimal checkpoint interval $T^*$ is finite, i.e., $0 < T^* < \infty$.

**Theorem 3.5.** (i) Suppose that the function $h(x)$ is strictly increasing. If $q(\infty) < 0$, then there exists a finite and unique optimal checkpoint interval $T^*$ $(0 < T^* < \infty)$ satisfying the non-linear equation $q(T^*) = 0$, and the corresponding system availability is

$$\Pi_1(T^*) = \frac{1}{1 + h(T^*)}. \quad (31)$$

(ii) If $q(\infty) \geq 0$ with strictly increasing $h(x)$ or if the function $h(x)$ is decreasing, then the optimal checkpoint interval is $T^* \to \infty$, i.e., no checkpoint should be generated and the corresponding system availability is

$$\Pi_1(\infty) = \frac{1/\lambda}{1/\lambda + E[C] + \int_0^\infty E[V_x]g(x)dx}. \quad (32)$$

*Proof.* Differentiating $\Pi_1(T)$ with respect to $T$ and setting this equal to zero implies $q(T) = 0$. If $dh(x)/dx > 0$, then $dq(T)/dT < 0$ and the function $\Pi_1(T)$ is strictly concave in $T$. Since it necessarily holds that $q(0) > 0$, if $q(\infty) < 0$, then there exists a finite and unique optimal checkpoint interval $T^*(0 < T^* < \infty)$ which satisfies $q(T^*) = 0$. The maximum system availability can be obtained from $q(T^*) = 0$. On the other hand, if $dh(x)/dx > 0$ and $q(\infty) \geq 0$ or if $dh(x)/dx \leq 0$, then $d\Pi_1(T)/dT \geq 0$ or the function $\Pi_1(T)$ is strictly increasing in $T$, and the result is proved.

In this section, we have considered the sample model where the file system is operating in continuous time. However, it should be noted that the actual file management system should be treated as an intermittently used system. That is, the file system is alternatively operative and inoperative and strictly speaking, should be formulated as a queueing model. In the following section, we re-formulate the underlying problem taking account of the queueing effect and derive the optimal checkpoint strategy approximately.
4. QUEUEING MODEL

Suppose that transactions arrive at the system according to a homogeneous Poisson process with intensity $\nu$ ($>0$) and that the processing times are i.i.d. random variables having the common exponential distribution with mean $1/\mu$ ($>0$), where $\rho = \nu/\mu$ is called the utilization factor or the traffic intensity. In other words, the files system under consideration is modelled by the $M/M/1/\infty$ queueing system. Denote the number of transactions existing in the system at time $t$ by $\{N(t), t \geq 0\}$, where $N(0) = 0$. Define the joint conditional probability

$$F(n, x, t) = \Pr\{N(t) = n, X(t) \leq x | I(t) = 1\}, \quad (33)$$

where

$$f(n, x, t) = \frac{\partial}{\partial x} F(n, x, t). \quad (34)$$

Also, we define

$$f(n, x) = \lim_{t \to \infty} f(n, x, t), \quad (35)$$

$$\hat{f}(n, s) = \int_0^\infty \exp\{-sx\} f(n, x) dx, \quad (36)$$

$$\pi(u, x) = \sum_{n=0}^\infty u^n f(n, x), \quad (37)$$

$$\hat{\pi}(u, s) = \sum_{n=0}^\infty u^n \hat{f}(n, s). \quad (38)$$

In the time interval $[t, t + \Delta)$, let us consider the case of $n > 0$ and $x > 0$. Then, a simple probabilistic argument yields

$$f(n, x + \Delta, t + \Delta) = (1 - \nu\Delta)(1 - \mu\Delta)(1 - r(x)\Delta)(1 - \eta(x)\Delta)f(n, x, t)$$

$$+ \nu\Delta f(n - 1, x, t) + \mu\Delta f(n + 1, x, t) + o(\Delta). \quad (39)$$

If $n = 0$ and $x > 0$, one sees that

$$f(0, x + \Delta, t + \Delta) = (1 - \nu\Delta)(1 - r(x)\Delta)(1 - \eta(x)\Delta)f(0, x, t)$$

$$+ \mu\Delta f(1, x, t) + o(\Delta). \quad (40)$$

If $n \geq 0$ and $x > 0$, it holds that $f(n, x, 0) = 0$. On the other hand, for $n \geq 0$ and $x = 0$, one has

$$f(n, 0, t) = \int_0^\infty \int_0^\infty \sum_{j=0}^{\infty} \frac{\nu^j}{j!} \exp\{-\nu z\} f(n - j, x, t - z) \eta(x) w(z) dx dz$$

$$+ \int_0^\infty \int_0^\infty \sum_{j=0}^{\infty} \frac{\nu^j}{j!} \exp\{-\nu z\} f(n - j, x, t - z)$$

$$\times r(x)b_j(z) * w(z) dz dx, \quad (41)$$
where $b_x(z) \ast w(z)$ denotes the convolution of the functions $b_x(z)$ and $w(z)$.

In the case of $n > 0$ and $x > 0$, letting $\Delta \to 0$, we have the partial differential equation

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) f(n, x, t) = -\{v + \mu + r(x) + \eta(x)\} f(n, x, t)
+ vf(n - 1, x, t) + \mu f(n + 1, x, t). \tag{42}
$$

Similarly, in the case of $n = 0$ and $x > 0$, one finds

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) f(0, x, t) = -\{v + r(x) + \eta(x)\} f(0, x, t) + \mu f(1, x, t). \tag{43}
$$

We assume throughout the remaining part of this paper that the queueing system is ergodic.

**Lemma 4.1.** If there exist the stationary solutions in Eqs. (41), (42), and (43), then

$$
\hat{\pi}(u, s) = \left\{ \mu \left( 1 - \frac{1}{u} \right) \hat{f}(0, s) + \int_0^\infty \left\{ \hat{w}(\nu(1-u)) - \exp\{-sx\} \right\} \eta(x) \pi(u, x) dx 
+ \int_0^\infty \left\{ \hat{w}(\nu(1-u)) \hat{b}_x(\nu(1-u)) - \exp\{-sx\} \right\} r(x) \pi(u, x) dx \right\}
\bigg/ \left\{ s + v(1-u) + \mu \left( 1 - \frac{1}{u} \right) \right\}, \tag{44}
$$

where

$$
\hat{\pi}(u, 0) = \left\{ \mu \left( 1 - \frac{1}{u} \right) \hat{f}(0, 0) + \int_0^\infty \left[ \hat{w}(\nu(1-u)) - 1 \right] \eta(x) \pi(u, x) dx 
+ \int_0^\infty \left[ \hat{w}(\nu(1-u)) \hat{b}_x(\nu(1-u)) - 1 \right] r(x) \pi(u, x) dx \right\}
\bigg/ \left\{ v(1-u) + \mu \left( 1 - \frac{1}{u} \right) \right\}, \tag{45}
$$

$$
\hat{w}(\nu(1-u)) = \int_0^\infty \exp\{-\nu(1-u)z\} w(z) dz, \tag{46}
$$

$$
\hat{b}(\nu(1-u)) = \int_0^\infty \exp\{-\nu(1-u)z\} b_x(z) dz. \tag{47}
$$
Proof. In the case of \( n \geq 0 \) and \( x = 0 \), one gets
\[
\begin{align*}
f(n, 0, s) &= \int_{0}^{\infty} \int_{0}^{\infty} \sum_{j=0}^{n} \frac{\nu z}{j!} \exp\{-\nu + s\}z \\
&\quad \times \int_{0}^{\infty} \int_{0}^{\infty} \nu z \exp\{-\nu + s\}z \hat{f}(n - j, x, s) \\
&\quad \times r(x)b_{n}(x) \ast w(z)dxdz.
\end{align*}
\]
From the equation above, it is seen that
\[
\begin{align*}
f(n, 0) &= \lim_{t \to \infty} s\hat{f}(n, 0, s) = \lim_{t \to \infty} f(n, 0, t) \\
&= \int_{0}^{\infty} \int_{0}^{\infty} \sum_{j=0}^{n} \frac{\nu z}{j!} \exp\{-\nu z\}f(n - j, x)\eta(x)w(z)dxdz \\
&\quad \times r(x)b_{n}(x) \ast w(z)dxdz.
\end{align*}
\]
Consequently, from the definition, we calculate
\[
\begin{align*}
\pi(u, 0) &= \sum_{n=0}^{\infty} u^n f(n, 0) \\
&= \hat{w}(\nu(1 - u)) \int_{0}^{\infty} \{\eta(x) + r(x)b_{n}(r(1 - u))\} \pi(u, x)dx.
\end{align*}
\]
When \( n > 0 \) and \( x > 0 \), from Eq. (42), one finds
\[
\begin{align*}
s\hat{f}(n, x, s) - f(n, x, 0) + \frac{\partial}{\partial x} \hat{f}(n, x, s) \\
&= -\{\nu + \mu + r(x) + \eta(x)\} \hat{f}(n, x, s) \\
&\quad + v\hat{f}(n - 1, x, s) + \mu \hat{f}(n + 1, x, s).
\end{align*}
\]
Taking \( t \to \infty \), Eq. (51) is reduced to
\[
\begin{align*}
\frac{\partial}{\partial x} f(n, x) &= -\{\nu + \mu + r(x) + \eta(x)\}f(n, x) \\
&\quad + vf(n - 1, x) + \mu f(n + 1, x).
\end{align*}
\]
That is,
\[
\begin{align*}
s\hat{f}(n, s) - f(n, 0) &= -(\nu + \mu)\hat{f}(n, s) + v\hat{f}(n - 1, s) + \mu \hat{f}(n + 1, s) \\
&\quad - \int_{0}^{\infty} \exp\{-sx\}(r(x) + \eta(x))f(n, x)dx.
\end{align*}
\]
Also, in the case of \( n = 0 \) and \( x > 0 \), one obtains, from the similar procedure,

\[
s\dot{f}(0, s) - f(0, 0) = -v\dot{f}(0, s) + \mu \dot{f}(1, s) - \int_{0}^{\infty} \exp\{-sx\}(r(x) + \eta(x)) f(0, x) dx. \tag{54}
\]

Using Eqs. (50), (54), and (55), we can derive the results above.

**Lemma 4.2.** If there exist the stationary solutions in Eqs. (41), (42), and (43), then

\[
\pi(1, x) = \sum_{n=0}^{\infty} f(n, x) = \frac{G(x)}{G(\infty)} A(x). \tag{55}
\]

**Proof.** From the familiar renewal argument, the result is immediate.

**Theorem 4.3.** There exist the stationary solutions in Eqs. (41), (42), and (43), iff

\[
\frac{\nu}{\mu} < \frac{\Pi_1}{T}. \tag{56}
\]

**Proof.** From Lemmas 4.1 and 4.2 and

\[
\lim_{u \to 1} \lim_{s \to 0} \hat{\pi}(u, s) = 1, \tag{57}
\]

we obtain

\[
\dot{f}(0, 0) = 1 - \left( \frac{\nu}{\mu} \right) \frac{1}{\Pi_1}. \tag{58}
\]

Replacing \( \Pi_1 \) by \( \Pi_1(T) \), we obtain the result, if the probability that the operative system is idle in the steady-state is strictly positive, i.e., \( \dot{f}(0, 0) = \int_{0}^{\infty} f(0, x) dx > 0 \), and vice versa.

Next, we specify the random variable \( V_x \) denoting the length of the rollback recovery. Following the literature [1–8, 11, 12, 14], define the affine form

\[
E[V_x] = \alpha x + \beta, \quad \alpha > 0, \beta \geq 0, \tag{59}
\]

where the first term denotes the mean time necessary to re-execute transactions which were processed in time interval \([0, x]\), and the second term is a fixed time associated with re-loading the information stored at the checkpoint back into primary memory.

Let \( k (>0) \) be the proportion of transactions which have to be re-processed after a system failure. The recovery time required in the rollback procedure should be measured by the cumulative operating time during
the busy period. That is, the recovery parameter $\alpha$ has to be a function of arrival and processing parameters $\nu$ and $\mu$. When the system can be regarded to be continuously operating (heavy loaded), roughly speaking, the parameter $\alpha$ can be approximated as constant, i.e., $\alpha = k$, and the optimal checkpoint interval is given by Theorem 3.5. However, if the system is influenced by the arrival/process of transactions and is light loaded, the parameter $\alpha$ depends on the length of busy period. Chandy et al. [2, 3] and Gelenbe and Derochette [4] assumed that the recovery parameter is proportional to the utilization factor $\rho$ which denotes the rate that the system is busy in the steady-state, i.e., $\alpha \approx k\rho$. Further, Gelenbe [5] introduced the ergodic conditional probability that the system is idle given that the system is operating and developed a sophisticated approximation scheme. In fact, since it is difficult to catch completely the transient behavior of the busy/idle period even for the $M/M/1/\infty$ queueing system, these approximation schemes are useful to represent the system availability under an intermittently used environment.

For studying the optimal checkpoint interval, it is necessary to derive the ergodic conditional probability $p^*$ that the file system is idle given that the state of the process $\{I(t), t \geq 0\}$ is in 1. Define the ergodic conditional probability by

$$p^* = \int_0^\infty f(0, x)dx = \hat{f}(0, 0).$$

Following Gelenbe [5], one then has

$$\alpha \approx k(1 - p^*).$$

The following are the main results of this paper.

**Theorem 4.4.**

$$p^* = 1 - \frac{\rho}{\Pi_1(T)}.$$ 

**Theorem 4.5.** Under the approximation scheme $\alpha \approx k(1 - p^*)$, define the function

$$q_p(T) = \left\{1 - k\rho Tr(T)\right\}\left\{\int_0^T \rho \bar{G}(x)dx + \beta G(T) + E[C]\right\}$$

$$- \left\{1 + \beta r(T)\right\}\left\{(1 - k\rho) \int_0^T \rho \bar{G}(x)dx + k\rho T \bar{G}(T)\right\}.$$ 

(i) Suppose that the function $h(x) = (\alpha x + \beta) r(x)$ is strictly increasing. If $q_p(\infty) < 0$, then there exists a finite and unique optimal checkpoint interval.
$T^* \ (0 < T^* < \infty)$ satisfying the non-linear equation $q_p(T^*) = 0$, and the corresponding system availability is

$$\Pi_1(T^*) = \frac{1 - k\rho T^* r(T^*)}{1 + \beta r(T^*)}. \quad (64)$$

(ii) If $q_p(\infty) \geq 0$ with strictly increasing $h(x)$ or if the function $h(x)$ is decreasing, then the optimal checkpoint interval is $T^* \rightarrow \infty$, i.e., no checkpoint should be generated and the corresponding system availability is

$$\Pi_1(\infty) = \frac{(1 - k\rho)/\lambda}{1/\lambda + \beta + E[C]} \quad (65)$$

The above results will be useful to generate the age-dependent optimal checkpoint approximately for an intermittently operated file system subject to both general failure mode and arrival of transactions.

ACKNOWLEDGMENTS

The authors thank two editors of this journal, William F. Ames and George Leitmann, who kindly gave us a chance to present this paper. This work is based upon the support by Grant-in-Aids for Scientific Research from the Ministry of Education, Sports, Science and Culture of Japan under Grants 11780331, 10558059, and by the Research Program 1999 under the Institute for Advanced Studies of the Hiroshima Shudo University, Hiroshima, Japan. Also, the first author is supported in part by Telecommunication Advancement Foundation, Tokyo, Japan.

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