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Identification problem for damped sine-Gordon equation with point sources

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ABSTRACT

We establish the existence and uniqueness of solutions for sine-Gordon equations in a multidimensional setting. The equations contain a point-like source. Furthermore, the continuity and the Gâteaux differentiability of the solution map is established. An identification problem for parameters governing the equations is set, and is shown to have a solution. The objective function is proved to be Fréchet differentiable with respect to the parameters. An expression for the Fréchet derivative in terms of the solutions of the direct and the adjoint systems is presented. A criterion for optimal parameters is formulated as a bang-bang control principle. An application of these results to the one-dimensional sine-Gordon equation is considered.

1. Introduction

A solid state physics model for the dislocation dynamics of a crystal [3] is the sine-Gordon equation

$$y_{tt} - y_{xx} + \sin y = 0. \tag{1.1}$$

It has attracted considerable attention because of many interesting properties of its solutions. For example, (1.1) is known to propagate solitary waves. While some particular solutions of (1.1) such as a single-soliton, a double-soliton, and some others can be found explicitly, in general there is no closed form solution, see [1].

Various modifications of this equation have also been found to be of interest in several applications. The perturbed sine-Gordon equation describing the influence of variable external force and air damping is

$$y_{tt} - y_{xx} + \sin y = f(t,x) - \alpha y_t. \tag{1.2}$$

This equation also describes the dynamics of Josephson junctions, see [10,13]. In [9] the perturbed sine-Gordon equation

$$y_{tt} - y_{xx} + \sin y = f(t,x) - \alpha y_t - \beta y_{txx} \tag{1.3}$$

is proposed for taking into account the dissipation due to the current along a dielectric barrier in Josephson junctions. In [12] vacuum chamber gaps are studied using the extended sine-Gordon equation

$$y_{tt} - y_{xx} + \sin y = f(t,x) - \alpha y_t - \beta y_{txx} + \sigma\delta(x - x_0)\sin y, \tag{1.4}$$

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where the δ function at $x = x_0$ represents the presence of a gap in the vacuum chamber. We refer to [13] for various results on mathematical properties of sine-Gordon equations.

The subject of our study is identification problems for damped sine-Gordon equations of the type

$$\begin{cases} y_{tt} + \alpha y_t - \beta y_{txx} - y_{xx} + \sin y = f(t,x) + \sigma\delta(x - x_0)\sin y \quad \text{in } Q, \\ y(t,0) = y(t,1) = 0 \quad \text{in } (0,T), \\ y(0,x) = y_0(x), \qquad y_t(0,x) = y_1(x), \quad x \in (0,1), \end{cases} \tag{1.5}$$

where α, β, σ are constants, and $Q = (0,T) \times (0,1)$. In fact, we study more general problems

$$y'' + \alpha y' + \beta A y' + Ay = f + \Phi(y,q),$$
$$y(0) = y_0 \in V = H_0^1(\Omega), \qquad y'(0) = y_1 \in H = L^2(\Omega),$$

see (2.7), in a multidimensional setting. They are fully described in Section 2.

Our main results in Sections 3, 4 are the uniqueness and the existence of solutions y of the nonlinear problem (2.7). The continuous dependence of y on the parameters $q = (\alpha, \beta, \sigma)$ is established in Section 5. The Gâteaux differentiability of the solution map is proved in Section 6.

A study of identification problems for the nonlinear equation (2.7) is conducted in Sections 7 and 8. Let $z_d \in W(0,T)$ defined in (2.2). We want to characterize a parameter set $q^* = (\alpha^*, \beta^*, \sigma^*)$ that minimizes objective function

$$J(q) = \left\| y(q) - z_d \right\|^2_{L^2(0,T;H)} \tag{1.6}$$

over an admissible set.

In Section 7 the identification problem is set up, and the objective function J is shown to be Fréchet differentiable with respect to the parameter q. An expression for the Fréchet derivative DJ in terms of the solutions of the direct and the adjoint systems is presented. A criterion for optimal parameters q^* is formulated in Section 8 as a bang-bang control principle. An application of these results to the sine-Gordon equation (1.5) is considered in Section 9.

Our results for sine-Gordon equations with the external point source are new. Traditionally, Galerkin type methods [11,13] use energy estimates for approximate solutions to conclude that one can extract a subsequence of such solutions weakly convergent to the weak solution of (1.5). However, in our case such an approach is not possible, so we have developed a new method based on a fixed point argument, see Section 4.

For other types of semi-linear second order evolution equations see [6]. The Gâteaux and Fréchet differentiability for similar problems were established by us in [7,8,5,4]. Unlike these results, we exploit a higher regularity of the solutions $y(q)$ to established the Gâteaux differentiability of the solution map without appealing to the Lions Transposition Principle, see Section 6.

Optimization problems considered in this paper are developed for parameters $q = (\alpha, \beta, \sigma) \in \mathbb{R}^3$. The paper serves as a foundation for problems with non-constant parameters. See [4] for such sine-Gordon equations without point sources.

2. Problem setup

Let Ω be an open bounded set in $\mathbb{R}^d$ with a sufficiently regular boundary Γ. Introduce the Hilbert spaces $H = L^2(\Omega)$ with the norm $|u|$ and the inner product $(u,v) = \int_\Omega uv\,dx$, and $V = H_0^1(\Omega)$ with the norm $\|u\|$ and the inner product $((u,v)) = (\nabla u, \nabla v)$. The dual H' is identified with H leading to $V \subset H \subset V'$ with compact, continuous and dense embedding. Hence, there exist constants $K_1 = K_1(\Omega)$ and $K_2 = K_2(\Omega)$ such that

$$|w| \leqslant K_1\|w\|, \quad \text{for any } w \in V, \quad \text{and} \quad \|h\|_{V'} \leqslant K_2|h|, \quad \text{for any } h \in H. \tag{2.1}$$

Let $\langle u, v\rangle$ denote the duality pairing between $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$. From now on the dependency on x is suppressed, and $'$ and $''$ stand for the time derivatives. Let

$$W(0,T) = \left\{u\colon\ u \in L^2(0,T;V),\ u' \in L^2(0,T;V),\ u'' \in L^2\left(0,T;V'\right)\right\}, \tag{2.2}$$

where the derivatives are understood in the distributional sense, see [11]. The standard L^2 norm $\|\cdot\|_2$, and $\|\cdot\|_W$ norm in $W(0,T)$ are introduced and discussed in Lemma 2.2.

To define a weak solution of the 1D damped sine-Gordon equation (1.5) one multiplies it by $v \in V$ and performs integration by parts for the terms containing the second partial derivatives in x.

Assuming that the solution $y \in W \subset L^2(0,T;V)$, the term with the delta function $\sigma\int_0^1 \delta(x - x_0)\sin y(t,x)v(x)\,dx = \sigma \sin y(t,x_0)v(x_0)$ makes sense, since $y(t)$, $v \in V \subset C[0,1]$. Accordingly, for $v, w \in V$, let

$$\left\langle F_0(w), v\right\rangle = \sin w(x_0)v(x_0), \quad x_0 \in [0,1].$$

Then

$$\left|\langle F_0(w), v\rangle\right| \leqslant \max_{[0,1]}|w| \max_{[0,1]}|v| \leqslant c\|w\|\|v\|,$$

since V is continuously embedded into $C[0,1]$. Thus $F_0 : V \to V'$ is well defined on V.

This leads to the following variational formulation for the sine-Gordon equation in the 1D case. Let $\Omega = (0,1)$. Function $y \in W$ is called a weak solution of the problem (1.5), if it satisfies

$$\begin{aligned} &\langle y'', v\rangle + \alpha(y', v) + \beta(\nabla y', \nabla v) + (\nabla y, \nabla v) + (\sin y, v) = \langle f, v\rangle + \sigma\langle F_0(y), v\rangle, \\ &y(0) = y_0 \in V, \qquad y'(0) = y_1 \in H \end{aligned} \tag{2.3}$$

in the distributional sense on $(0,T)$ for any $v \in V$. Here $f \in L^2(0,T;V')$, and $\nabla v = v_x$.

Define $A : V \to V'$ by $\langle Au, v\rangle = (\nabla u, \nabla v) = ((u,v))$, $u, v \in V$. Then (2.3) can be rewritten as an equation for y in V'. This form of the sine-Gordon equation also makes sense in the multidimensional case $\Omega \subset \mathbb{R}^d$, and for functionals more general than F_0.

In all our considerations parameters $q = (\alpha, \beta, \sigma)$ are assumed to be within the admissible set

$$P = \left\{q = (\alpha, \beta, \sigma) \in [\alpha_{\min}, \alpha_{\max}] \times [\beta_{\min}, \beta_{\max}] \times [\sigma_{\min}, \sigma_{\max}] \subset \mathbb{R}^3\right\}, \tag{2.4}$$

where $\beta_{\min} > 0$.

Now we can state the object of our investigation.

Damped sine-Gordon problem. Let $f \in L^2(0,T;V')$ and $q = (\alpha, \beta, \sigma) \in P$. Let $F : V \to V'$ be a Lipschitz continuous function. We study solutions $y \in W(0,T)$ of the damped sine-Gordon problem

$$\begin{aligned} &y'' + \alpha y' + \beta A y' + A y + \sin y = f + \sigma F(y), \\ &y(0) = y_0 \in V, \qquad y'(0) = y_1 \in H, \end{aligned} \tag{2.5}$$

where the equation is satisfied in the sense of distributions on $(0,T)$ with the values in V', see [11,13]. Assumptions on F are stated in Section 6.

It is convenient to consider (2.5) as a special case of a more general equation.

Second order nonlinear damped problem. Let $f \in L^2(0,T;V')$ and $q = (\alpha, \beta, \sigma) \in P$. Let function $\Phi : V \times P \to V'$ be Lipschitz continuous with

$$\left\|\Phi(w_1, q_1) - \Phi(w_2, q_2)\right\|_{V'} \leqslant L\left(\|w_1 - w_2\| + |q_1 - q_2|_{\mathbb{R}^3}\right) \tag{2.6}$$

for some $L > 0$ and any $w_1, w_2 \in V$, $q_1, q_2 \in P$. Consider

$$\begin{aligned} &y'' + \alpha y' + \beta A y' + A y = f + \Phi(y, q), \\ &y(0) = y_0 \in V, \qquad y'(0) = y_1 \in H. \end{aligned} \tag{2.7}$$

The sine-Gordon problem (2.5) is (2.7) with $\Phi(y,q) = \sigma F(y) - \sin y$.

Second order linear damped problem. Let $f \in L^2(0,T;V')$ and $q = (\alpha, \beta, \sigma) \in P$. Let $K > 0$ and time dependent linear operators $B(t) : V \to V'$, $t \in [0,T]$ satisfy

$$\left\|B(t)\right\|_{L(V,V')} \leqslant K, \quad \text{and} \quad t \to B(t)w \text{ is continuous on } [0,T] \tag{2.8}$$

for any $w \in V$.

Consider

$$\begin{aligned} &y'' + \alpha y' + \beta A y' + A y + B y = f, \\ &y(0) = y_0 \in V, \qquad y'(0) = y_1 \in H. \end{aligned} \tag{2.9}$$

The following lemma is of a critical importance throughout the paper. It is used in all uniqueness and existence results.

Lemma 2.1. *Let $w \in W(0,T)$. Then, after a modification on the set of measure zero, $w \in C([0,T];V)$, $w' \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$ one has*

$$\frac{d}{dt}\|w\|^2 = 2\left((w', w)\right) = 2\langle Aw, w'\rangle, \quad \textit{and} \quad \frac{d}{dt}\left|w'\right|^2 = 2\langle w'', w'\rangle. \tag{2.10}$$

Proof. According to [13, Lemma 2.3.2], if $u \in L^2(0,T;V)$ and its derivative $u' \in L^2(0,T;V')$, then u is continuous from $[0,T]$ into H after a modification on a set of measure zero, and it satisfies $d/dt|u|^2 = 2\langle u', u\rangle$. Letting $u = w'$ we get $w' \in C([0,T];H)$ and the second equality in (2.10). For the first equality in (2.10) we can use [13, Lemma 2.3.2] with $V = H = V'$, or just notice that the mapping $x \to \|x\|^2$ is Fréchet differentiable. □

Lemma 2.2. *Let $w \in W(0,T)$. Define*

$$\|w\|_2^2 = \|w\|^2_{L^2(0,T;V)} + \|w'\|^2_{L^2(0,T;V)} + \|w''\|^2_{L^2(0,T;V')} \tag{2.11}$$

and

$$\|w\|^2_{W(0,T)} = \max_{t\in[0,T]}\left[\|w(t)\|^2 + |w'(t)|^2\right] + \|w'\|^2_{L^2(0,T;V)} + \|w''\|^2_{L^2(0,T;V')}. \tag{2.12}$$

Then the norms $\|\cdot\|_2$ and $\|\cdot\|_W = \|\cdot\|_{W(0,T)}$ are equivalent on $W(0,T)$. Moreover $(W(0,T), \|\cdot\|_W)$ is a Banach space.

Proof. By Lemma 2.1 we can assume that the functions w and w' are continuous on $[0,T]$ into V and H correspondingly, so $\|w\|_W$ is well defined.

Note that $\|w\|_2^2 \leqslant T\|w\|_W^2 + \|w\|_W^2$. On the other hand, let $t_0 \in [0,T]$ be such that $\|w(t_0)\| = \min_{0\leqslant t\leqslant T}\|w(t)\|$. Then $T\|w(t_0)\|^2 \leqslant \|w\|_2^2$. Integrate the first equality in (2.10) on $[t_0,t]$ to get

$$\begin{aligned}\|w(t)\|^2 &\leqslant \|w(t_0)\|^2 + 2\int_{t_0}^{t}\|w'(s)\|\|w(s)\|\,ds \leqslant \|w(t_0)\|^2 + 2\int_0^T\|w'(s)\|\|w(s)\|\,ds\\ &\leqslant \frac{1}{T}\|w\|_2^2 + \|w'\|^2_{L^2(0,T;V)} + \|w\|^2_{L^2(0,T;V)} \leqslant \left(\frac{1}{T}+1\right)\|w\|_2^2\end{aligned}$$

for any $t \in [0,T]$.

Similarly, let $t_1 \in [0,T]$ be such that $|w'(t_1)| = \min_{0\leqslant t\leqslant T}|w'(t)|$. Then using (2.10) and (2.1)

$$\begin{aligned}|w'(t)|^2 &\leqslant |w'(t_1)|^2 + 2\int_0^T\|w''(s)\|_{V'}\|w'(s)\|\,ds\\ &\leqslant \frac{1}{T}\|w'\|^2_{L^2(0,T;H)} + \|w''\|^2_{L^2(0,T;V')} + \|w'\|^2_{L^2(0,T;V)} \leqslant \left(\frac{K_1^2}{T}+1\right)\|w\|_2^2\end{aligned}$$

for any $t \in [0,T]$, and the equivalence of the norms follows. Since $(W(0,T), \|\cdot\|_2)$ is a Banach space, the norm equivalence implies the same for $(W(0,T), \|\cdot\|_W)$. □

We also need the following modification of the Gronwall's Lemma, see [2, Section B.2].

Lemma 2.3. *Let $C > 0$ and functions $a(t), b(t) \in L^1[0,T]$ be nonnegative a.e. on $[0,T]$. Suppose that $\eta(t) \in W^{1,1}[0,T]$ is nonnegative on $[0,T]$ and*

$$\eta'(t) + a(t) \leqslant C\big(b(t) + \eta(t)\big) \tag{2.13}$$

is satisfied a.e. on $[0,T]$. Then

$$\eta(t) + \int_0^t a(s)\,ds \leqslant e^{Ct}\left(\eta(0) + C\int_0^t b(s)\,ds\right) \tag{2.14}$$

for any $t \in [0,T]$.

Proof. Multiply (2.13) by e^{-Ct} and rewrite it as

$$\frac{d}{dt}\big(e^{-Ct}\eta(t)\big) + a(t)e^{-Ct} \leqslant Cb(t)e^{-Ct}.$$

The integration over $[0,t]$ gives

$$\eta(t) + \int_0^t a(s)e^{C(t-s)}\,ds \leqslant e^{Ct}\left(\eta(0) + C\int_0^t b(s)e^{-Cs}\,ds\right) \leqslant e^{Ct}\left(\eta(0) + C\int_0^t b(s)\,ds\right)$$

and the result follows. □

3. A priori estimates and uniqueness

In this section we derive various estimates and the uniqueness of the solutions and their approximations for problems (2.7) and (2.9). The function Φ is assumed to satisfy the Lipschitz continuity condition (2.6). The existence of the solutions is established in Section 4.

Theorem 3.1. *Let $T > 0$. Then there exists a constant $C > 0$ independent of $q \in P$ such that*

(i) *Any solution $y \in W(0,T)$ of the nonlinear problem (2.7) satisfies*

$$\|y\|^2_{W(0,t)} \leqslant C^2\left[\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,t;V')} + \left\|\Phi(0,q)\right\|^2_{V'}\right] \tag{3.1}$$

for any $t \in [0,T]$.

(ii) *Let $u_1, u_2 \in W(0,T)$ be two solutions of (2.7) with $f_1, f_2 \in L^2(0,T;V')$ correspondingly. Then their difference $w = u_2 - u_1$ satisfies*

$$\|w\|^2_{W(0,t)} \leqslant C^2\|f_2 - f_1\|^2_{L^2(0,t;V')} \tag{3.2}$$

for any $t \in [0,T]$.

(iii) *The solution y of Eq. (2.7) is unique.*

Proof. (i) If y is a solution of the nonlinear problem (2.7), then

$$\langle y'', y'\rangle + \beta\langle Ay', y'\rangle + \langle Ay, y'\rangle = \langle f, y'\rangle + \langle \Phi(y,q), y'\rangle - \alpha(y', y').$$

By Lemma 2.1

$$\frac{1}{2}\frac{d}{dt}\left(\left|y'\right|^2 + \|y\|^2\right) + \beta\left\|y'\right\|^2 = \langle f, y'\rangle + \langle \Phi(y,q), y'\rangle - \alpha\left|y'\right|^2.$$

We have

$$\begin{aligned}\left|\langle f + \Phi(y,q), y'\rangle\right| &\leqslant \left\|f + \left(\Phi(y,q) - \Phi(0,q)\right) + \Phi(0,q)\right\|_{V'}\left\|y'\right\| \leqslant \left(\|f\|_{V'} + \left\|\Phi(0,q)\right\|_{V'} + L\|y\|\right)\left\|y'\right\| \\ &\leqslant \frac{\beta}{2}\left\|y'\right\|^2 + \frac{3}{2\beta}\left(\|f\|^2_{V'} + \left\|\Phi(0,q)\right\|^2_{V'} + L^2\|y\|^2\right).\end{aligned}$$

Therefore

$$\frac{d}{dt}\left(\left|y'\right|^2 + \|y\|^2\right) + \left\|y'\right\|^2 \leqslant \gamma\left(\|f\|^2_{V'} + \left\|\Phi(0,q)\right\|^2_{V'} + \left|y'\right|^2 + \|y\|^2\right), \tag{3.3}$$

where $\gamma > 0$ depends only on the bounds of the admissible set P.

By Lemma 2.3

$$\begin{aligned}\left|y'(t)\right|^2 + \left\|y(t)\right\|^2 + \int_0^t \left\|y'(s)\right\|^2 ds &\leqslant e^{\gamma t}\left(|y_1|^2 + \|y_0\|^2 + \gamma\int_0^t \left(\left\|f(s)\right\|^2_{V'} + \left\|\Phi(0,q)\right\|^2_{V'}\right) ds\right) \\ &\leqslant e^{\gamma T}\left(|y_1|^2 + \|y_0\|^2 + \gamma\int_0^t \left\|f(s)\right\|^2_{V'} ds + T\gamma\left\|\Phi(0,q)\right\|^2_{V'}\right) \\ &\leqslant C_1\left(|y_1|^2 + \|y_0\|^2 + \int_0^t \left\|f(s)\right\|^2_{V'} ds + \left\|\Phi(0,q)\right\|^2_{V'}\right)\end{aligned} \tag{3.4}$$

for any $t \in [0,T]$.

Let $v \in V$ with $\|v\| \leqslant 1$. Then

$$\langle y'', v\rangle = -\alpha(y', v) - \beta\langle Ay', v\rangle - \langle Ay, v\rangle + \langle f, v\rangle + \langle \Phi(y,q), v\rangle.$$

Using $|v| \leqslant K_1\|v\| = K_1$ we get

$$\left|\langle y'', v\rangle\right| \leqslant K_1\alpha_{\max}\left|y'\right| + \beta_{\max}\left\|y'\right\| + \|y\| + \|f\|_{V'} + L\|y\| + \left\|\Phi(0,q)\right\|_{V'}.$$

Therefore

$$\|y''\|_{V'}^2 \leqslant C_2\big(\|f\|_{V'}^2 + \|\Phi(0,q)\|_{V'}^2 + |y'|^2 + \|y\|^2 + \|y'\|^2\big)$$

and, by (3.4)

$$\int_0^t \|y''(s)\|_{V'}^2\,ds \leqslant C_3\left(|y_1|^2 + \|y_0\|^2 + \int_0^t \|f(s)\|_{V'}^2\,ds + \|\Phi(0,q)\|_{V'}^2\right),$$

where $C_1, C_2, C_3 > 0$ are constants independent of $t \in [0,T]$ or $q \in P$. Adding this inequality to (3.4) proves (3.1).

(ii) The difference $w = u_2 - u_1$ satisfies

$$w'' + \alpha w' + \beta A w' + A w = (f_2 - f_1) + \big(\Phi(u_2,q) - \Phi(u_1,q)\big)$$

with zero initial conditions. An argument similar to the one used in part (i) of the theorem leads to estimate (3.2).

(iii) The uniqueness follows from (3.2). □

For the linear problem (2.9) we have

Theorem 3.2. *Let $T > 0$. Then there exists a constant $C > 0$ independent of $q \in P$ such that*

(i) *Any solution $y \in W(0,T)$ of the linear problem* (2.9) *satisfies*

$$\|y\|_{W(0,t)}^2 \leqslant C^2\big[\|y_0\|^2 + |y_1|^2 + \|f\|_{L^2(0,t;V')}^2\big] \tag{3.5}$$

for any $t \in [0,T]$.

(ii) *Let $u_1, u_2 \in W(0,T)$ be two solutions of* (2.9) *with $f_1, f_2 \in L^2(0,T;V')$ correspondingly. Then their difference $w = u_2 - u_1$ satisfies*

$$\|w\|_{W(0,t)}^2 \leqslant C^2 \|f_2 - f_1\|_{L^2(0,t;V')}^2 \tag{3.6}$$

for any $t \in [0,T]$.

(iii) *The solution y of Eq.* (2.9) *is unique.*

Proof. Let y be a solution of (2.9), then

$$\langle y'', y'\rangle + \beta\langle Ay', y'\rangle + \langle Ay, y'\rangle = \langle f, y'\rangle - \langle By, y'\rangle - \alpha(y', y').$$

By Lemma 2.1

$$\frac{1}{2}\frac{d}{dt}\big(|y'|^2 + \|y\|^2\big) + \beta\|y'\|^2 = \langle f, y'\rangle - \langle By, y'\rangle - \alpha|y'|^2.$$

Since $\|B(t)w\|_{V'} \leqslant K\|w\|$, we have

$$\big|\langle f + By, y'\rangle\big| \leqslant \|f\|_{V'}\|y'\| + K\|y\|\|y'\|$$

and the rest of the proof follows the argument of Theorem 3.1. □

Existence proofs of the next section are based on properties of finite-dimensional approximations y_m of the linear problem (2.9).

Let $\{\lambda_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ be the eigenvalues and the eigenfunctions of the negative Laplacian $-\Delta$ in V, such that $\{w_k\}_{k=1}^\infty$ form an orthonormal basis in H. Then $\{w_k/\sqrt{\lambda_k}\}_{k=1}^\infty$ form an orthonormal basis in V, see [2, Chapter 6].

Fix $m \in \mathbb{N}$ and let $V_m = \operatorname{span}\{w_1, \ldots, w_m\}$. Define

$$P_m h = \sum_{k=1}^m (h, w_k) w_k, \quad h \in H. \tag{3.7}$$

Then $P_m : H \to V_m$ is an orthogonal projection in H and $|P_m h| \leqslant |h|$ for any $h \in H$. Also $P_m : V \to V_m$ is an orthogonal projection in V and $\|P_m v\| \leqslant \|v\|$ for any $v \in V$. Now define the adjoint projector $P_m^* : V' \to V'$ by

$$\langle P_m^* g, v\rangle = \langle g, P_m v\rangle, \quad g \in V',\ v \in V. \tag{3.8}$$

The approximate solution of the linear problem (2.9) is defined to be a function $y_m \in W(0,T)$ that satisfies

$$y_m'' + \alpha y_m' + \beta A y_m' + A y_m + P_m^* B y_m = P_m^* f,$$
$$y_m(0) = P_m y_0, \qquad y_m'(0) = P_m y_1 \tag{3.9}$$

in V'.

Lemma 3.3.

(i) *There exists a unique solution $y_m \in W(0,T)$ of Eq.* (3.9)*.*
(ii) *The solution $y_m(t) \in V_m$ for any $t \in [0,T]$, and $y_m, y_m' \in C([0,T];V)$.*
(iii) *There exists a constant $C > 0$ independent of $m \in \mathbb{N}$ and $q \in P$ such that*

$$\|y_m\|^2_{W(0,t)} \leqslant C^2\big[\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,t;V')}\big] \tag{3.10}$$

is satisfied for every approximate solution y_m.

Proof. First, we notice that the solution of this linear equation is unique because of Theorem 3.2. Now suppose that $z_m(t) = \sum_{k=1}^m g_{km}(t) w_k$ satisfies

$$\big(z_m'', w_k\big) + \alpha\big(z_m', w_k\big) + \beta\big\langle A z_m', w_k\big\rangle + \langle A z_m, w_k\rangle + \langle B z_m, w_k\rangle = \langle f, w_k\rangle,$$
$$z_m(0) = P_m y_0, \qquad z_m'(0) = P_m y_1 \tag{3.11}$$

for $k = 1, 2, \ldots, m$.

To see that the solution $z_m(t)$ also satisfies (3.9) it is enough to establish

$$\big\langle z_m'' + \alpha z_m' + \beta A z_m' + A z_m + P_m^* B z_m, w_k\big\rangle = \big\langle P_m^* f, w_k\big\rangle \tag{3.12}$$

for any $k \in \mathbb{N}$. But for $k \leqslant m$, Eq. (3.12) is the same as (3.9), and for $k > m$ Eq. (3.12) is reduced to $0 = 0$. By the uniqueness we conclude that $y_m(t) = z_m(t) \in V_m$.

Since y_m satisfies (2.9) with $\hat{f} = P_m^* f$, and $\hat{B} = P_m^* B$, estimate established similarly to (3.5) gives

$$\|y_m\|^2_{W(0,t)} \leqslant C^2\big[\|P_m y_0\|^2 + |P_m y_1|^2 + \big\|P_m^* f\big\|^2_{L^2(0,t;V')}\big]. \tag{3.13}$$

Let $v \in V$, $\|v\| = 1$. Then

$$\big|\big\langle P_m^* f, v\big\rangle\big| = \big|\langle f, P_m v\rangle\big| \leqslant \|f\|_{V'}\|P_m v\| \leqslant \|f\|_{V'},$$

since $\|P_m v\| \leqslant \|v\| = 1$. Recalling that $\|P_m y_0\| \leqslant \|y_0\|$ and $|P_m y_1| \leqslant |y_1|$, inequality (3.13) implies (3.10).

Therefore

$$\|z_m\|^2_{W(0,T)} \leqslant C^2\big[\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;V')}\big]. \tag{3.14}$$

For each $m \in \mathbb{N}$, system (3.11) is a Cauchy problem for the system of ordinary differential equations that has a unique solution $z_m(t)$. Because of the energy estimate (3.14) this solution exists on the entire interval $[0,T]$. Furthermore, the component functions g_{km} satisfy $g_{km}'' \in L^2(0,T)$. Thus $g_{km}, g_{km}' \in C[0,T]$, and we conclude that $z_m, z_m' \in C([0,T];V)$, and $z_m'' \in L^2([0,T];V)$. □

4. Existence of solutions

Let function Φ satisfy the Lipschitz continuity condition (2.6). First, we prove the existence for the linear problem (2.9) using approximate solutions. Then the existence for the nonlinear problem (2.5) is established using a fixed point argument.

Theorem 4.1. *There exists a unique solution $y \in W(0,T)$ of the linear problem* (2.9)*. This solution satisfies estimates* (3.5) *and* (3.6)*.*

Proof. The uniqueness and estimates (3.5) and (3.6) have already been established in Theorem 3.2.

By Lemma 3.3 the approximate solutions y_m satisfy

$$\|y_m\|^2_{W(0,T)} \leqslant C^2\big[\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;V')}\big] \tag{4.1}$$

where the constant C is independent of $m \in \mathbb{N}$ and $q \in P$.

This estimate shows that for any $m \in \mathbb{N}$ the approximate solutions y_m belong to the same bounded convex ball $\|w\|_W \leqslant C$ of $W(0,T)$. Since $W(0,T)$ is a reflexive space, there exists a subsequence y_{m_k} of y_m that converges weakly to a function $z \in W(0,T)$. Because this is a distributional convergence, we have

$$\begin{aligned} &y_{m_k} \rightharpoonup z \quad \text{in } L^2(0,T;V), \\ &y'_{m_k} \rightharpoonup z' \quad \text{in } L^2(0,T;V), \\ &y''_{m_k} \rightharpoonup z'' \quad \text{in } L^2\big(0,T;V'\big), \end{aligned} \tag{4.2}$$

where $\rightharpoonup$ indicates the weak convergence. We can also assume that $y_{m_k} \rightharpoonup z$ weak-star in $L^\infty(0,T;V)$ and $y'_{m_k} \rightharpoonup z'$ weak-star in $L^\infty(0,T;H)$.

Because of the assumption (2.8), time-dependent operator B can be considered as a continuous linear operator from $L^2(0,T;V)$ into $L^2(0,T;V')$. Therefore it is weakly continuous. Thus we can pass to the limit in (3.9) as $m \to \infty$ and obtain

$$z'' + \alpha z' + \beta A z' + Az + Bz = f.$$

The satisfaction of the initial conditions $z(0) = y_0$ and $z'(0) = y_1$ follows as in [11,2,6].

Thus z is a solution of (2.9). By Theorem 3.1 the solution z is unique, therefore the convergence in (4.2) as well as for $y_m \to z$, $m \to \infty$ in $L^2(0,T;H)$ is for the entire sequence y_m and not just for its subsequence y_{m_k}. □

A convergence estimate for the sequence y_m is given in Theorem 5.1.

The main result of this section is the following theorem on the existence of solutions for the nonlinear problem (2.7).

Theorem 4.2. *There exists a unique solution $y \in W(0,T)$ of the nonlinear problem* (2.7)*. This solution satisfies estimates* (3.1) *and* (3.2)*.*

Proof. The uniqueness and estimates (3.1) and (3.2) have already been established in Theorem 3.1.

In this proof we reference the linear problem (2.9) assuming $B(t) \equiv 0$. According to Theorem 3.1 there exists a positive constant $C > 0$ such that for any solution $u \in W(0,T)$ of the linear problem

$$\begin{aligned} &u'' + \alpha u' + \beta A u' + Au = f, \\ &u(0) = y_0, \qquad u'(0) = y_1 \end{aligned} \tag{4.3}$$

we have

$$\|u\|^2_{W(0,t)} \leqslant C^2\big[\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,t;V')}\big],$$

and solutions u_1, u_2 of (4.3) corresponding to $f_1, f_2 \in L^2(0,T;V')$ satisfy

$$\|u_2 - u_1\|^2_{W(0,t)} \leqslant C^2 \|f_2 - f_1\|^2_{L^2(0,t;V')}$$

for any $t \in [0,T]$. Note that the same constant C can also be used for estimates (3.1) and (3.2) for the solutions of the nonlinear problem (2.7).

Furthermore, inequality (3.4) shows that there exists a constant $\gamma > 0$ such that solutions u of both linear (2.9) and nonlinear (2.7) problems can be estimated for any $t \in [0,T]$ by

$$\big|u'(t)\big|^2 + \big\|u(t)\big\|^2 + \int_0^t \big\|u'(s)\big\|^2 \leqslant e^{\gamma t}\left(|y_1|^2 + \|y_0\|^2 + \gamma \int_0^t \big\|f(s)\big\|^2_{V'}\,ds\right) \tag{4.4}$$

and

$$\big|u'(t)\big|^2 + \big\|u(t)\big\|^2 + \int_0^t \big\|u'(s)\big\|^2 \leqslant e^{\gamma t}\left(|y_1|^2 + \|y_0\|^2 + \gamma \int_0^t \big(\big\|f(s)\big\|^2_{V'} + \big\|\Phi(0,q)\big\|^2_{V'}\big)\,ds\right) \tag{4.5}$$

correspondingly.

Choose $0 < \delta \leqslant T$ to be such that

$$CL\sqrt{\delta} \leqslant \frac{1}{2} \tag{4.6}$$

and

$$CL\sqrt{\delta}e^{\frac{\gamma T}{2}}\left[\|y_0\|^2+|y_1|^2+\gamma\|f\|^2_{L^2(0,T;V')}+T\gamma\left\|\Phi(0,q)\right\|^2_{V'}\right]^{\frac{1}{2}}\leqslant\frac{1}{2}. \tag{4.7}$$

Let $u_0\in W(0,T)$ be the unique solution of the linear problem

$$\begin{aligned}&u_0''+\alpha u_0'+\beta Au_0'+Au_0=f+\Phi(0,q),\\&u_0(0)=y_0,\qquad u_0'(0)=y_1.\end{aligned} \tag{4.8}$$

Such a solution $u_0\in W(0,T)$ exists by Theorem 4.1.

Let $B(0,\delta)\subset W(0,\delta)$ be the following closed convex subset of the ball centered at u_0

$$B(0,\delta)=\left\{u\in W(0,\delta)\colon\ \|u-u_0\|_{W(0,\delta)}\leqslant 1,\ u(0)=y_0,\ u'(0)=y_1\right\}.$$

For any $u\in B(0,\delta)$ define the operator $G(0,\delta):B(0,\delta)\to W(0,\delta)$ by $G(0,\delta)u=w$ where $w\in W(0,\delta)$ is the unique solution of the linear problem

$$\begin{aligned}&w''+\alpha w'+\beta Aw'+Aw=f+\Phi(u,q),\\&w(0)=y_0,\qquad w'(0)=y_1\end{aligned} \tag{4.9}$$

on $[0,\delta]$.

Then for any $u_1,u_2\in B(0,\delta)$ we have

$$\begin{aligned}\left\|G(0,\delta)(u_2)-G(0,\delta)(u_1)\right\|_{W(0,\delta)}&\leqslant C\left\|\Phi(u_2,q)-\Phi(u_1,q)\right\|_{L^2(0,\delta;V')}\leqslant CL\|u_2-u_1\|_{L^2(0,\delta;V)}\\&\leqslant CL\sqrt{\delta}\|u_2-u_1\|_{L^\infty(0,\delta;V)}\leqslant\frac{1}{2}\|u_2-u_1\|_{W(0,\delta)}.\end{aligned} \tag{4.10}$$

Now we estimate

$$\begin{aligned}\left\|G(0,\delta)(u_0)-u_0\right\|_{W(0,\delta)}&\leqslant C\left\|\Phi(u_0,q)-\Phi(0,q)\right\|_{L^2(0,\delta;V')}\\&\leqslant CL\|u_0\|_{L^2(0,\delta;V)}\leqslant CL\sqrt{\delta}\|u_0\|_{L^\infty(0,\delta;V)}.\end{aligned} \tag{4.11}$$

By (4.4) with $\hat{f}=f+\Phi(0,q)$

$$\left|u_0'(t)\right|^2+\left\|u_0(t)\right\|^2+\int_0^t\left\|u_0'(s)\right\|^2ds\leqslant e^{\gamma\delta}\left(\|y_0\|^2+|y_1|^2+\gamma\|f\|^2_{L^2(0,\delta;V')}+\delta\gamma\left\|\Phi(0,q)\right\|^2_{V'}\right). \tag{4.12}$$

Therefore

$$\|u_0\|_{L^\infty(0,\delta;V)}\leqslant e^{\frac{\gamma T}{2}}\left(\|y_0\|^2+|y_1|^2+\gamma\|f\|^2_{L^2(0,T;V')}+T\gamma\left\|\Phi(0,q)\right\|^2_{V'}\right)^{\frac{1}{2}}.$$

Thus

$$\left\|G(0,\delta)(u_0)-u_0\right\|_{W(0,\delta)}\leqslant\frac{1}{2}.$$

Let $u\in B(0,\delta)$. Then

$$\begin{aligned}\left\|G(0,\delta)(u)-u_0\right\|&\leqslant\left\|G(0,\delta)(u)-G(0,\delta)(u_0)\right\|+\left\|G(0,\delta)(u_0)-u_0\right\|\\&\leqslant\frac{1}{2}\|u-u_0\|_{W(0,\delta)}+\frac{1}{2}\leqslant 1.\end{aligned} \tag{4.13}$$

Thus $G(0,\delta)$ is a contraction mapping on $B(0,\delta)$. By Banach Contraction Mapping Theorem [2, Theorem 9.2.1] there exists a unique fixed point $y\in B(0,\delta)$ of $G(0,\delta)$. By construction $y\in W(0,\delta)$ is the solution of the nonlinear problem (2.7) on the interval $[0,\delta]$.

Suppose that y is the solution of (2.7) on an interval $[0,t_0]$. We are going to show that it can be extended to the interval $[t_0,t_0+\delta]$ with the same δ that was chosen above, following the same method as was used to show the existence of y for $t\in[0,\delta]$.

Let $u_0\in W(t_0,t_0+\delta)$ be defined now as the unique solution of the linear problem

$$\begin{aligned}&u_0''+\alpha u_0'+\beta Au_0'+Au_0=f+\Phi(0,q),\\&u_0(t_0)=y(t_0),\qquad u_0'(t_0)=y'(t_0).\end{aligned} \tag{4.14}$$

Let

$$B(t_0, t_0+\delta) = \{u \in W(t_0, t_0+\delta)\colon \|u-u_0\|_{W(t_0,t_0+\delta)} \leqslant 1,\ u(t_0) = y(t_0),\ u'(t_0) = y'(t_0)\}.$$

For any $u \in B(t_0, t_0+\delta)$ define the operator $G(t_0, t_0+\delta) : B(t_0, t_0+\delta) \to W(t_0, t_0+\delta)$ by $G(t_0, t_0+\delta)u = w$ where $w \in W(t_0, t_0+\delta)$ is the unique solution of the linear problem

$$\begin{aligned} &w'' + \alpha w' + \beta A w' + A w = f + \Phi(u, q), \\ &w(t_0) = y(t_0), \qquad w'(t_0) = y'(t_0). \end{aligned} \tag{4.15}$$

Arguing as in (4.10) and (4.11) we get

$$\big\|G(t_0, t_0+\delta)(u_2) - G(t_0, t_0+\delta)(u_1)\big\|_{W(t_0,t_0+\delta)} \leqslant \frac{1}{2}\|u_2-u_1\|_{W(t_0,t_0+\delta)}$$

and

$$\big\|G(t_0, t_0+\delta)(u_0) - u_0\big\|_{W(t_0,t_0+\delta)} \leqslant CL\sqrt{\delta}\|u_0\|_{L^\infty(t_0,t_0+\delta;V)}.$$

Applying estimate (4.12) on the interval $[t_0, t_0+\delta]$ we get

$$\|u_0\|^2_{L^\infty(t_0,t_0+\delta;V)} \leqslant e^{\gamma\delta}\big(\big\|y(t_0)\big\|^2 + \big|y'(t_0)\big|^2 + \gamma\|f\|^2_{L^2(t_0,t_0+\delta;V')} + \delta\gamma\big\|\Phi(0,q)\big\|^2_{V'}\big).$$

By (4.5)

$$\big\|y(t_0)\big\|^2 + \big|y'(t_0)\big|^2 \leqslant e^{\gamma t_0}\big(\|y_0\|^2 + |y_1|^2 + \gamma\|f\|^2_{L^2(0,t_0;V')} + t_0\gamma\big\|\Phi(0,q)\big\|^2_{V'}\big).$$

Therefore

$$\begin{aligned} \|u_0\|^2_{L^\infty(t_0,t_0+\delta;V)} &\leqslant e^{\gamma(t_0+\delta)}\big(\|y_0\|^2 + |y_1|^2\big) + e^{\gamma(t_0+\delta)}\gamma\|f\|^2_{L^2(0,t_0;V')} + e^{\gamma\delta}\gamma\|f\|^2_{L^2(t_0,t_0+\delta;V')} \\ &\quad + e^{\gamma(t_0+\delta)}t_0\gamma\big\|\Phi(0,q)\big\|^2_{V'} + e^{\gamma\delta}\delta\gamma\big\|\Phi(0,q)\big\|^2_{V'} \\ &\leqslant e^{\gamma T}\big(\|y_0\|^2 + |y_1|^2 + \gamma\|f\|^2_{L^2(0,t_0+\delta;V')} + T\gamma\big\|\Phi(0,q)\big\|^2_{V'}\big) \end{aligned}$$

and

$$\big\|G(t_0, t_0+\delta)(u_0) - u_0\big\|_{W(t_0,t_0+\delta)} \leqslant \frac{1}{2}.$$

Thus $G(t_0, t_0+\delta)$ is a contraction on $B(t_0, t_0+\delta)$ and there exists a fixed point y of $G(t_0, t_0+\delta)$ which is the solution of the nonlinear problem

$$\begin{aligned} &u'' + \alpha u' + \beta A u' + A u = f + \Phi(u, q), \\ &u(t_0) = y(t_0), \qquad u'(t_0) = y'(t_0) \end{aligned} \tag{4.16}$$

on the interval $[t_0, t_0+\delta]$. Therefore the solution $y(t)$ was extended to the interval $[0, t_0+\delta]$. Consequently, the solution y exists on the entire interval $[0, T]$. □

5. Convergence estimates and continuity of the solution map

Now that the existence of the solution of the nonlinear problem is established, we show that the solution can be approximated using a Galerkin method. We also prove that the solution map $q \to y(q)$ is continuous from P into $W(0, T)$.

Theorem 5.1. *Define an approximate solution $y_m \in W(0, T)$, $m \in \mathbb{N}$ for the nonlinear problem* (2.7) *as the solution of*

$$\begin{aligned} &y_m'' + \alpha y_m' + \beta A y_m' + A y_m = P_m^* f + P_m^* \Phi(y_m, q), \\ &y_m(0) = P_m y_0, \qquad y_m'(0) = P_m y_1 \end{aligned} \tag{5.1}$$

that is satisfied in the sense of distributions on $(0, T)$ with the values in V'.

Then

(i) *The solution y of* (2.7) *and its approximations y_m satisfy the following convergence estimate*

$$\begin{aligned} &\max_{t\in[0,T]}\big[\big\|y(t) - y_m(t)\big\|^2 + \big|y'(t) - y_m'(t)\big|^2\big] + \big\|y' - y_m'\big\|^2_{L^2(0,T;V)} \\ &\quad \leqslant C(f, \Phi)\big(\|y_0 - P_m y_0\|^2 + |y_1 - P_m y_1|^2 + \big\|y' - P_m y'\big\|_{L^2(0,T;V)}\big). \end{aligned} \tag{5.2}$$

(ii) *We have* $y_m \rightharpoonup y$ *weakly in* $W(0,T)$ *as* $m \to \infty$.
(iii) *The solution satisfies* $y \in C([0,T];V)$ *and* $y' \in C([0,T];H)$.

Proof. The difference $y - y_m$ satisfies

$$(y-y_m)'' + \beta A(y-y_m)' + A(y-y_m) = f - P_m^* f + \Phi(y,q) - P_m^*\Phi(y_m,q) - \alpha(y-y_m)'.$$

By Lemma 2.1

$$\frac{1}{2}\frac{d}{dt}\left\{\left|y'-y_m'\right|^2 + \|y-y_m\|^2\right\} + \beta\left\|y'-y_m'\right\|^2 = \left\langle f - P_m^* f, y'-y_m'\right\rangle + \left\langle \Phi(y,q) - P_m^*\Phi(y_m,q), y'-y_m'\right\rangle - \alpha\left|y'-y_m'\right|^2. \tag{5.3}$$

Since $y_m' \in V_m$ we have $P_m y_m' = y_m'$. Therefore

$$\left\langle f - P_m^* f, y'-y_m'\right\rangle = \left\langle f, (I-P_m)\left(y'-y_m'\right)\right\rangle = \left\langle f, y' - P_m y'\right\rangle$$

and

$$\begin{aligned}\left\langle \Phi(y,q) - P_m^*\Phi(y_m,q), y'-y_m'\right\rangle &= \left\langle \left(I-P_m^*\right)\Phi(y,q), y'-y_m'\right\rangle + \left\langle P_m^*\left(\Phi(y,q)-\Phi(y_m,q)\right), y'-y_m'\right\rangle \\ &= \left\langle \Phi(y,q), y'-P_m y'\right\rangle + \left\langle \Phi(y,q)-\Phi(y_m,q), P_m\left(y'-y_m'\right)\right\rangle.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left\{\left|y'-y_m'\right|^2 + \|y-y_m\|^2\right\} + \beta\left\|y'-y_m'\right\|^2 \leqslant c\left(\|f\|_{V'}\left\|y'-P_m y'\right\| + \left\|\Phi(y,q)\right\|_{V'}\left\|y'-P_m y'\right\| + L\|y-y_m\|\left\|y'-y_m'\right\| + \left|y'-y_m'\right|^2\right). \tag{5.4}$$

By Theorem 3.1 $\|y\| \leqslant c$. Therefore the integration of (5.3) on $[0,t]$ and the Gronwall's Lemma give

$$\left|y'(t)-y_m'(t)\right|^2 + \left\|y(t)-y_m(t)\right\|^2 \leqslant c(f,\Phi)\left(|y_1 - P_m y_1|^2 + \|y_0 - P_m y_0\|^2 + \left\|y'-P_m y'\right\|_{L^2(0,T;V)}\right). \tag{5.5}$$

This inequality and (5.4) imply (i) of the theorem.

Recall that $y' \in L^2(0,T;V)$. Therefore $\|y' - P_m y'\|_{L^2(0,T;V)} \to 0$ as $m \to \infty$ by the Monotone Convergence Theorem. Since $y_m, y_m' \in C([0,T];V)$ the conclusion (iii) follows from (i).

By (i) we have $y_m \to y$ and $y_m' \to y'$ strongly in $L^2(0,T;V)$ as $m \to \infty$. Thus, to establish the weak convergence in $W(0,T)$ it remains to show that $y_m'' \rightharpoonup y''$ weakly in $L^2(0,T;V')$ as $m \to \infty$.

Let $v \in V$. Then

$$\begin{aligned}\left\langle y''-y_m'', v\right\rangle &= -\alpha\left(y'-y_m', v\right) - \beta\left\langle A\left(y'-y_m'\right), v\right\rangle - \left\langle A(y-y_m), v\right\rangle + \left\langle f - P_m^* f, v\right\rangle \\ &\quad + \left\langle \Phi(y,q) - P_m^*\Phi(y,q), v\right\rangle + \left\langle P_m^*\left(\Phi(y,q)-\Phi(y_m,q)\right), v\right\rangle.\end{aligned} \tag{5.6}$$

Since $\langle f - P_m^* f, v\rangle = \langle f, (I-P_m)v\rangle$, $\langle \Phi(y,q) - P_m^*\Phi(y,q), v\rangle = \langle \Phi(y,q), (I-P_m)v\rangle$ and $|\langle P_m^*(\Phi(y,q)-\Phi(y_m,q)), v\rangle| = |\langle \Phi(y,q)-\Phi(y_m,q), P_m v\rangle| \leqslant L\|y-y_m\|\|v\|$ we conclude that every term in (5.6) approaches zero as $m \to \infty$. Thus (iii) is proved. □

Remark 5.2. Using $\Phi \equiv 0$ and slightly modifying the argument of Theorem 5.1 we obtain that the solution y of the linear problem (2.9) and its approximations y_m defined in (3.9) satisfy the convergence estimates and other conclusions of Theorem 5.1.

Theorem 5.3. *Suppose that function* Φ *satisfies the Lipschitz continuity condition* (2.6)*. Let* $q = (\alpha,\beta,\sigma) \in P$ *and* $y(q) \in W(0,T)$ *be the solution of the nonlinear problem* (2.7)*. Then the solution map* $q \to y(q)$ *from* P *into* $W(0,T)$ *is continuous.*

Moreover, there exists a constant $C > 0$ *independent of* $q \in P$ *such that*

$$\left\|y(q_2) - y(q_1)\right\|_{W(0,T)} \leqslant C|q_2 - q_1|_{\mathbb{R}^3}, \tag{5.7}$$

for any $q_1, q_2 \in P$.

Proof. Let $w = y(q_2) - y(q_1)$. Then

$$w'' + Aw + \beta_2 Aw' = -(\beta_2-\beta_1)Ay'(q_1) - \alpha_2 w' - (\alpha_2-\alpha_1)y'(q_1) + \Phi\left(y(q_2), q_2\right) - \Phi\left(y(q_1), q_1\right) \tag{5.8}$$

with $z(0) = 0$, $z'(0) = 0$. Therefore

$$\frac{1}{2}\frac{d}{dt}\{|w'|^2+\|w\|^2\}+\beta_2\|w'\|^2=-(\beta_2-\beta_1)\langle Ay'(q_1),w'\rangle-\alpha_2|w'|^2-(\alpha_2-\alpha_1)(y'(q_1),w') + \langle\Phi(y(q_2),q_2)-\Phi(y(q_1),q_1),w'\rangle. \quad (5.9)$$

Since

$$|\langle\Phi(y(q_2),q_2)-\Phi(y(q_1),q_1),w'\rangle| \leqslant L(\|w\|+|q_2-q_1|_{\mathbb{R}^3})\|w'\|,$$

the integration of (5.9) on $[0,t]$, (3.1) and the Gronwall's inequality give

$$|w'|^2+\|w\|^2 \leqslant c|q_2-q_1|^2_{\mathbb{R}^3},$$

and it follows that

$$\max_{[0,T]}[|w'|^2+\|w\|^2]+\|w'\|^2_{L^2(0,T;V)} \leqslant c|q_2-q_1|^2_{\mathbb{R}^3}. \quad (5.10)$$

We estimate $\|w''\|_{L^2(0,T;V')}$ by the same method that was used in previous theorems. Let $v\in V$ be such that $\|v\|=1$. Then, from (5.8) we obtain

$$|\langle w'',v\rangle| \leqslant c(\|w\|+\|w'\|+|\beta_2-\beta_1|+|\alpha_2-\alpha_1|+|q_2-q_1|_{\mathbb{R}^3}). \quad (5.11)$$

Now estimates (5.10) and (5.11) give the desired result. □

6. Gâteaux differentiability of the solution map

By Theorem 5.3 the solution map $q\to y(q)$ is continuous from P into $W(0,T)$. Our next result shows that the solution map is also Gâteaux differentiable on P assuming certain differentiability conditions on the nonlinear terms.

Definition 6.1. Function $F:V\to V'$ is called Fréchet differentiable if for every $u\in V$ there exists a bounded linear operator $DF(u):V\to V'$ (the Fréchet derivative of F at u) such that

$$\lim_{\|w-u\|\to 0}\frac{\|F(w)-F(u)-DF(u)(w-u)\|_{V'}}{\|w-u\|}=0. \quad (6.1)$$

Definition 6.2. Function $F:V\to V'$ is called uniformly Fréchet differentiable on V if

$$\|F(w)-F(u)-DF(u)(w-u)\|_{V'} \leqslant M\|w-u\|^2 \quad (6.2)$$

for some $M>0$ and any $w,u\in V$.

Definition 6.3. Let $q^*,q\in P$. The solution map $q\to y(q)$ of P into $W(0,T)$ is said to be Gâteaux differentiable at q^* in the direction $q-q^*$ if there exists a function $Dy(q^*;q-q^*)\in W(0,T)$ such that

$$\lim_{\lambda\to 0}\left\|\frac{y(q^*+\lambda(q-q^*))-y(q^*)}{\lambda}-Dy(q^*;q-q^*)\right\|_{W(0,T)}=0. \quad (6.3)$$

If $q^*\in bnd\,P$ (the boundary of the set P), then it is assumed that $\lambda>0$ in (6.3).

Theorem 6.4. *Suppose that function $F:V\to V'$ satisfies*

(i) *F is bounded on V.*
(ii) *F is uniformly Fréchet differentiable on V.*
(iii) *$\|DF(u)\|_{L(V,V')}\leqslant L$ for some $L>0$ and any $u\in V$.*
(iv) *DF is Lipschitz continuous: there exists $C>0$ such that*

$$\|DF(u)-DF(w)\|_{L(V,V')} \leqslant C\|u-w\|. \quad (6.4)$$

Let $q,q^*\in P$ and $y(q)$ be the solution of the sine-Gordon problem (2.5). Then the solution map $y(q):P\to W(0,T)$ is Gâteaux differentiable on P. Its Gâteaux derivative $z=Dy(q^*;q-q^*)\in W(0,T)$ at q^* in the direction $q-q^*$ is the unique solution of the linear problem

$$z''+\alpha^*z'+\beta^*Az'+Az=\sigma^*DF(y(q^*))z-(\cos y(q^*))z+(\alpha^*-\alpha)y'(q^*) + (\beta^*-\beta)Ay'(q^*)-(\sigma^*-\sigma)F(y(q^*)) \quad (6.5)$$

with $z(0)=0$ and $z'(0)=0$.

Proof. Since $\|DF(u)\|_{L(V,V')} \leqslant L$, it follows from the Mean Value Theorem that function F is Lipschitz continuous on V. For $u \in V$ and $q \in P$ define $\Phi : V \times P \to V'$ by

$$\Phi(u,q) = \sigma F(u) - \sin u. \tag{6.6}$$

With this choice of Φ nonlinear problem (2.7) becomes the sine-Gordon problem (2.5).

Since F is bounded on V, and P is bounded in $\mathbb{R}^3$, function Φ is Lipschitz continuous on $V \times P$. Therefore all the results of the previous sections are applicable to the sine-Gordon problem (2.5). In particular, this problem has a unique solution $y(q) \in W(0,T)$. Furthermore, by Theorem 5.3 we have

$$\big\|y(q_2) - y(q_1)\big\|_{W(0,T)} \leqslant C|q_2 - q_1|_{\mathbb{R}^3}. \tag{6.7}$$

Concerning problem (6.5) note that $y(q^*) = y(t;q^*)$. Therefore (6.5) is a linear problem (2.9) with $B(t)z = \sigma^* DF(y(q^*))z - (\cos y(q^*))z$. It satisfies $\|B(t)\|_{L(V,V')} \leqslant K$ for some $K > 0$. The continuity of $t \to B(t)w \in V'$ follows from the continuity of $t \to y(t;q^*) \in V$ and (6.4). By Theorem 4.1 problem (6.5) has a unique solution $z \in W(0,T)$.

Let $\lambda \neq 0$ and $q_\lambda = q^* + \lambda(q - q^*)$. Then

$$y''\big(q^*\big) + \alpha^* y'\big(q^*\big) + \beta^* Ay'\big(q^*\big) + Ay\big(q^*\big) = f + \sigma^* F\big(y\big(q^*\big)\big) - \sin y\big(q^*\big) \tag{6.8}$$

and

$$y''(q) + \alpha y'(q) + \beta Ay'(q) + Ay(q) = f + \sigma F\big(y(q)\big) - \sin y(q). \tag{6.9}$$

Also

$$y''(q_\lambda) + \alpha_\lambda y'(q_\lambda) + \beta_\lambda Ay'(q_\lambda) + Ay(q_\lambda) = f + \sigma_\lambda F\big(y(q_\lambda)\big) - \sin y(q_\lambda) \tag{6.10}$$

or

$$\begin{aligned} y''(q_\lambda) + \alpha^* y'(q_\lambda) + \beta^* Ay'(q_\lambda) + Ay(q_\lambda) = {} & f + \sigma^* F\big(y(q_\lambda)\big) - \sin y(q_\lambda) + \lambda\big(\alpha^* - \alpha\big)y'(q_\lambda) \\ & + \lambda\big(\beta^* - \beta\big)Ay'(q_\lambda) - \lambda\big(\sigma^* - \sigma\big)F\big(y(q_\lambda)\big). \end{aligned} \tag{6.11}$$

Since $q_\lambda - q^* = \lambda(q - q^*)$ the difference $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$ satisfies

$$\begin{aligned} z_\lambda'' + \alpha^* z_\lambda' + \beta^* Az_\lambda' + Az_\lambda = {} & \frac{\sigma^*}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y\big(q^*\big)\big)\big) - \frac{\sin y(q_\lambda) - \sin y(q^*)}{\lambda} + \big(\alpha^* - \alpha\big)y'(q_\lambda) \\ & + \big(\beta^* - \beta\big)Ay'(q_\lambda) - \big(\sigma^* - \sigma\big)F\big(y(q_\lambda)\big). \end{aligned} \tag{6.12}$$

Therefore the difference $w_\lambda = z_\lambda - z$ satisfies

$$\begin{aligned} w_\lambda'' + \alpha^* w_\lambda' + \beta^* Aw_\lambda' + Aw_\lambda = {} & \frac{\sigma^*}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y\big(q^*\big)\big)\big) - \sigma^* DF\big(y\big(q^*\big)\big)z \\ & - \left(\frac{\sin y(q_\lambda) - \sin y(q^*)}{\lambda} - \big(\cos y\big(q^*\big)\big)z\right) + \big(\alpha^* - \alpha\big)\big(y'(q_\lambda) - y'\big(q^*\big)\big) \\ & + \big(\beta^* - \beta\big)A\big(y'(q_\lambda) - y'\big(q^*\big)\big) - \big(\sigma^* - \sigma\big)\big(F\big(y(q_\lambda)\big) - F\big(y\big(q^*\big)\big)\big). \end{aligned} \tag{6.13}$$

Note that

$$\begin{aligned} \frac{1}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y\big(q^*\big)\big)\big) - DF\big(y\big(q^*\big)\big)z = {} & \frac{1}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y\big(q^*\big)\big) - DF\big(y\big(q^*\big)\big)\big(y(q_\lambda) - y\big(q^*\big)\big)\big) \\ & + DF\big(y\big(q^*\big)\big)(z_\lambda - z) \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} \left(\frac{\sin y(q_\lambda) - \sin y(q^*)}{\lambda} - \big(\cos y\big(q^*\big)\big)z\right) = {} & \frac{1}{\lambda}\big(\sin y(q_\lambda) - \sin y\big(q^*\big) - \big(\cos y\big(q^*\big)\big)\big(y(q_\lambda) - y\big(q^*\big)\big)\big) \\ & + \big(\cos y\big(q^*\big)\big)(z_\lambda - z). \end{aligned} \tag{6.15}$$

Thus (6.13) is

$$\begin{aligned} w''_\lambda + \alpha^* w'_\lambda + \beta^* A w'_\lambda + A w_\lambda = {} & \sigma^* DF\big(y(q^*)\big) w_\lambda - \big(\cos y(q^*)\big) w_\lambda \\ & + \frac{\sigma^*}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y(q^*)\big) - DF\big(y(q^*)\big)\big(y(q_\lambda) - y(q^*)\big)\big) \\ & - \frac{1}{\lambda}\big(\sin y(q_\lambda) - \sin y(q^*) - \big(\cos y(q^*)\big)\big(y(q_\lambda) - y(q^*)\big)\big) \\ & + \big(\alpha^* - \alpha\big)\big(y'(q_\lambda) - y'(q^*)\big) + \big(\beta^* - \beta\big) A\big(y'(q_\lambda) - y'(q^*)\big) \\ & - \big(\sigma^* - \sigma\big)\big(F\big(y(q_\lambda)\big) - F\big(y(q^*)\big)\big). \end{aligned} \tag{6.16}$$

This is a linear equation for w_λ. By Theorem 3.1(i) we have the estimate

$$\begin{aligned} \|w_\lambda\|^2_{W(0,T)} \leqslant c\bigg[& \bigg\| \frac{1}{\lambda}\big(F\big(y(q_\lambda)\big) - F\big(y(q^*)\big) - DF\big(y(q^*)\big)\big(y(q_\lambda) - y(q^*)\big)\big)\bigg\|^2_{L^2(0,T;V')} \\ & + \bigg\| \frac{1}{\lambda}\big(\sin y(q_\lambda) - \sin y(q^*) - \big(\cos y(q^*)\big)\big(y(q_\lambda) - y(q^*)\big)\big)\bigg\|^2_{L^2(0,T;V')} \\ & + \big|\alpha^* - \alpha\big|^2 \big\|y'(q_\lambda) - y'(q^*)\big\|^2_{L^2(0,T;H)} + \big|\beta^* - \beta\big|^2 \big\|y'(q_\lambda) - y'(q^*)\big\|^2_{L^2(0,T;V)} \\ & + \big|\sigma^* - \sigma\big|^2 \big\|y(q_\lambda) - y(q^*)\big\|^2_{L^2(0,T;V)}\bigg] = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{6.17}$$

The last three terms I_3, I_4 and I_5 in (6.17) approach zero as $\lambda \to 0$ by Theorem 5.3.

For the first term I_1 we use the uniform Fréchet differentiability of F to obtain

$$\begin{aligned} I_1 &= \frac{c}{\lambda^2}\int_0^T \big\|F\big(y(s;q_\lambda)\big) - F\big(y(s;q^*)\big) - DF\big(y(s;q^*)\big)\big(y(s;q_\lambda) - y(s;q^*)\big)\big\|^2_{V'}\,ds \\ &\leqslant c\frac{M^2}{\lambda^2}\int_0^T \big\|y(s;q_\lambda) - y(s;q^*)\big\|^4\,ds \leqslant c\frac{M^2 T}{\lambda^2}\Big(\max_{0\leqslant t\leqslant T}\big\|y(t;q_\lambda) - y(t;q^*)\big\|\Big)^4. \end{aligned}$$

By (2.12), $\max_{0\leqslant t\leqslant T}\|y(t;q_\lambda) - y(t;q^*)\| \leqslant \|y(q_\lambda) - y(q^*)\|_{W(0,T)}$. By Theorem 5.3, $\|y(q_\lambda) - y(q^*)\|_{W(0,T)} \leqslant C|q_\lambda - q^*|_{\mathbb{R}^3}$. Therefore

$$I_1 \leqslant c\frac{M^2 T}{\lambda^2}\big|q_\lambda - q^*\big|^4_{\mathbb{R}^3} \leqslant c\lambda^2\big|q - q^*\big|^4_{\mathbb{R}^3}. \tag{6.18}$$

Thus $I_1 \to 0$ as $\lambda \to 0$. For the second term I_2 we use inequality

$$\big|\sin b - \sin a - \cos a(b-a)\big| \leqslant |b-a|^2, \quad a, b \in \mathbb{R}$$

and argue as above to obtain

$$I_2 \leqslant \frac{c}{\lambda^2}\int_0^T \big\|y(s;q_\lambda) - y(s;q^*)\big\|^4\,ds \leqslant \frac{c}{\lambda^2}\big|q_\lambda - q^*\big|^4_{\mathbb{R}^3} \leqslant c\lambda^2\big|q - q^*\big|^4_{\mathbb{R}^3}. \tag{6.19}$$

Thus $I_2 \to 0$ as $\lambda \to 0$.

Therefore $\|w_\lambda\|_{W(0,T)} = \|z_\lambda - z\|_{W(0,T)} \to 0$ as $\lambda \to 0$, and the theorem is proved. □

7. Fréchet differentiability of the objective function

Let $z_d \in W(0,T)$. In Section 8 we study the identification problem of finding a parameter q^* that minimizes objective function

$$J(q) = \big\|y(q) - z_d\big\|^2_{L^2(0,T;H)} \tag{7.1}$$

over $q \in P$. The continuity of J follows from Theorem 5.3.

In this section we show that the objective function is Gâteaux differentiable in P, and Fréchet differentiable in *int* P. Since $P \subset \mathbb{R}^3$ this goal can be restated as establishing the existence of directional derivatives, and proving the differentiability of J. The expressions for the derivatives are given in terms of the solutions $y(q)$ and $p(q)$ of the direct and the adjoint systems respectively.

Given $q \in P$ the adjoint state $p(q) \in W(0,T)$ is defined as a solution of the linear terminal value problem

$$p'' - \alpha p' - \beta A p' + Ap - \sigma (DF)^*\big(y(q)\big)p + \big(\cos y(q)\big)p = y(q) - z_d,$$
$$p(T) = 0, \qquad p'(T) = 0, \tag{7.2}$$

where $(DF)^* : V \to V'$ is the operator adjoint to DF.

Assuming that F satisfies conditions of Theorem 6.4, after the change of variable $s = T - t$ system (7.2) becomes a special case of the linear problem (2.9). Therefore the conclusions of Theorem 4.1 are applicable to (7.2), including the existence and the uniqueness of the solution $p(q)$.

From the definition (7.1) of the functional J we derive the expression for its Gâteaux derivative $DJ(q^*; q - q^*)$ at q^* in the direction $q - q^*$

$$DJ\big(q^*; q - q^*\big) = 2\int_0^T \big(y\big(q^*\big) - z_d, Dy\big(q^*; q - q^*\big)\big)\,dt. \tag{7.3}$$

Let $q \in P$. Define linear operators

$$\mathcal{L}(q)w = w'' + \alpha w' + \beta A w' + Aw - \sigma DF\big(y(q)\big)w + \big(\cos y(q)\big)w \tag{7.4}$$

for $w \in W(0,T)$ satisfying $w(0) = w'(0) = 0$, and

$$\mathcal{L}^*(q)p = p'' - \alpha p' - \beta A p' + Ap - \sigma (DF)^*\big(y(q)\big)p + \big(\cos y(q)\big)p \tag{7.5}$$

for $p \in W(0,T)$ satisfying $p(T) = p'(T) = 0$. Then, for such w and p we have

$$\int_0^T \big\langle \mathcal{L}^*(q)p, w\big\rangle\,dt = \int_0^T \big\langle p, \mathcal{L}(q)w\big\rangle\,dt. \tag{7.6}$$

With this notation Eq. (6.5) for the Gâteaux derivative $z = Dy(q^*, q - q^*)$ becomes

$$\mathcal{L}\big(q^*\big)z = \big(\alpha^* - \alpha\big)y'\big(q^*\big) + \big(\beta^* - \beta\big)Ay'\big(q^*\big) - \big(\sigma^* - \sigma\big)F\big(y\big(q^*\big)\big), \tag{7.7}$$

and Eq. (7.2) for the adjoint state $p(q^*)$ becomes

$$\mathcal{L}^*\big(q^*\big)p\big(q^*\big) = y\big(q^*\big) - z_d. \tag{7.8}$$

Theorem 7.1. *Let $q, q^* \in P$. Suppose that $F : V \to V'$ satisfies conditions of Theorem 6.4. Then objective function $J(q) = \|y(q) - z_d\|^2_{L^2(0,T;H)}$ is Gâteaux differentiable on P, and its Gâteaux derivative $DJ(q^*, q - q^*)$ at q^* in the direction $q - q^*$ is given by*

$$DJ\big(q^*, q - q^*\big) = \big(\alpha - \alpha^*\big)a\big(q^*\big) + \big(\beta - \beta^*\big)b\big(q^*\big) + \big(\sigma - \sigma^*\big)c\big(q^*\big), \tag{7.9}$$

where

$$a\big(q^*\big) = -2\int_0^T \big(y'\big(q^*\big), p\big(q^*\big)\big)\,dt, \tag{7.10}$$

$$b\big(q^*\big) = -2\int_0^T \big\langle Ay'\big(q^*\big), p\big(q^*\big)\big\rangle\,dt \tag{7.11}$$

and

$$c\big(q^*\big) = 2\int_0^T \big\langle F\big(y\big(q^*\big)\big), p\big(q^*\big)\big\rangle\,dt. \tag{7.12}$$

Proof. We have

$$\begin{aligned} DJ(q^*; q-q^*) &= 2\int_0^T \big(y(q^*) - z_d, Dy(q^*; q-q^*)\big)\,dt \\ &= 2\int_0^T \big\langle \mathcal{L}^*(q^*)p(q^*), z\big\rangle dt = 2\int_0^T \big\langle p(q^*), \mathcal{L}(q^*)z\big\rangle dt \\ &= 2(\alpha^* - \alpha)\int_0^T \big(y'(q^*), p(q^*)\big)\,dt + 2(\beta^* - \beta)\int_0^T \big\langle Ay'(q^*), p(q^*)\big\rangle dt \\ &\quad - 2(\sigma^* - \sigma)\int_0^T \big\langle F\big(y(q^*)\big), p(q^*)\big\rangle dt \end{aligned} \tag{7.13}$$

which is (7.9)–(7.12). □

Theorem 7.2. *Let $q \in \operatorname{int} P$. Suppose that $F : V \to V'$ satisfies conditions of Theorem* 6.4*. Then objective function $J(q) = \|y(q) - z_d\|^2_{L^2(0,T;H)}$ is Fréchet differentiable on P, and its Fréchet derivative $DJ(q)$ is given by $DJ(q) = \nabla J(q) = \langle a(q), b(q), c(q)\rangle$, where the functions a, b, c are defined in* (7.10)–(7.12)*.*

Proof. Functions $a(q)$, $b(q)$ and $c(q)$ are the partial derivatives of $J(q)$. Therefore, to show the differentiability of $J(q)$ it is enough to establish their continuity. By Theorem 5.3 the solution map $q \to y(q)$ is continuous from P into $W(0,T)$. Thus, according to (7.10)–(7.12) functions $a(q)$, $b(q)$ and $c(q)$ are continuous, provided the solution map $q \to p(q)$ is continuous from P into $L^2(0,T;V)$.

Let $B(t) : V \to V'$ be defined by

$$B(t) = \sigma(DF)^*\big(y(t;q)\big) - \big(\cos y(t;q)\big)$$

for any $t \in [0,T]$. Since F satisfies conditions of Theorem 6.4, we conclude that operators $B(t)$ satisfy conditions (2.8). Therefore the conclusions of Theorem 4.1 are applicable to (7.2), and we can use inequality (3.2) to estimate

$$\big\|p(q_2) - p(q_1)\big\|_{W(0,T)} \leqslant c\big\|y(q_2) - y(q_1)\big\|_{L^2(0,T;V')} \leqslant c|q_2 - q_1|_{\mathbb{R}^3}$$

for $q_1, q_2 \in P$. Thus $q \to p(q)$ is continuous as claimed, and the theorem follows. □

8. Optimal parameters

In this section we assume that $q^* \in P$ is an optimal parameter for the problem

$$J(q^*) = \inf_{q\in P} J(q) = \inf_{q\in P}\big\|y(q) - z_d\big\|^2_{L^2(0,T;H)}. \tag{8.1}$$

By Theorem 5.3 the solution map $q \to y(q)$ is continuous on P, therefore the minimization problem (8.1) has a solution.

Theorem 8.1. *Suppose that function F satisfies conditions of Theorem* 6.4*, and $q^* \in P$ be an optimal parameter of* (8.1)*. Let the functions $a(q)$, $b(q)$ and $c(q)$ be defined by* (7.10)–(7.12)*.*

(i) *If $q^* \in \operatorname{int} P$, then*

$$a(q^*) = b(q^*) = c(q^*) = 0. \tag{8.2}$$

(ii) *If $q^* \in P$, then*

$$(\alpha - \alpha^*)a(q^*) + (\beta - \beta^*)b(q^*) + (\sigma - \sigma^*)c(q^*) \geqslant 0 \tag{8.3}$$

for any $q \in P$.

(iii) *The optimality condition* (8.3) *can be restated as a bang-bang control principle*

$$\alpha^* = \frac{1}{2}\big\{\operatorname{sign}\big(a(q^*)\big) + 1\big\}\alpha_{\max} - \frac{1}{2}\big\{\operatorname{sign}\big(a(q^*)\big) - 1\big\}\alpha_{\min}, \tag{8.4}$$

if $a(q^*) \neq 0$.

$$\beta^* = \frac{1}{2}\{\operatorname{sign}(b(q^*)) + 1\}\beta_{\max} - \frac{1}{2}\{\operatorname{sign}(b(q^*)) - 1\}\beta_{\min}, \tag{8.5}$$

if $b(q^*) \neq 0$.

$$\sigma^* = \frac{1}{2}\{\operatorname{sign}(c(q^*)) + 1\}\sigma_{\max} - \frac{1}{2}\{\operatorname{sign}(c(q^*)) - 1\}\sigma_{\min}, \tag{8.6}$$

if $c(q^*) \neq 0$.

Proof. The necessary optimality condition for q^* is $DJ(q^*; q - q^*) \geqslant 0$ for any $q \in P$. According to Theorem 7.1 it takes the form

$$(\alpha - \alpha^*)a(q^*) + (\beta - \beta^*)b(q^*) + (\sigma - \sigma^*)c(q^*) \geqslant 0 \tag{8.7}$$

for any $q = (\alpha, \beta, \delta) \in P$. If $q^* \in int\, P$, then (8.7) can be satisfied only if $a(q^*) = b(q^*) = c(q^*) = 0$.

Recall that the admissible set P was defined in (2.4). Choose $q = (\alpha, \beta^*, \delta^*) \in P$. Then (8.7) becomes $(\alpha^* - \alpha)a(q^*) \geqslant 0$ for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. If $\alpha^* \in (\alpha_{\min}, \alpha_{\max})$ then we must have $a(q^*) = 0$. If $a(q^*) > 0$ then $\alpha^* = \alpha_{\max}$. If $a(q^*) < 0$ then $\alpha^* = \alpha_{\min}$. Thus the case $a(q^*) \neq 0$ can be compactly written as (8.4). A similar argument is used for $b(q^*) \neq 0$ and $c(q^*) \neq 0$. □

9. Application to damped sine-Gordon equation

Following the setup for the one-dimensional sine-Gordon problem in Section 2, let $u, w, v \in V = H_0^1(0, 1)$. Fix $x_0 \in [0, 1]$ and define nonlinear functional $F_0 : V \to V'$ by

$$\langle F_0(w), v\rangle = \sin w(x_0)v(x_0).$$

Then $|\langle F_0(w), v\rangle| = |\sin w(x_0)||v(x_0)| \leqslant \|v\|$. Therefore $\|F_0(w)\|_{V'} \leqslant 1$ for any $w \in V$. Thus F_0 is bounded on V. Also

$$\left|\langle F_0(u) - F_0(w), v\rangle\right| \leqslant \left|u(x_0) - w(x_0)\right|\left|v(x_0)\right| \leqslant \|u - w\|\|v\|.$$

Therefore $\|F_0(u) - F_0(w)\|_{V'} \leqslant \|u - w\|$ and functional F_0 is Lipschitz continuous on V.

Let $q \in P$. To simplify the formulas assume that $\sigma \in [\sigma_{\min}, \sigma_{\max}] \subset [-1, 1]$. Define

$$\langle \Phi(w, q), v\rangle = \sigma\langle F_0(w), v\rangle - (\sin w, v).$$

Then

$$\left|\langle \Phi(w, q), v\rangle\right| \leqslant \left\|F_0(w)\right\|_{V'} + |\sin w| \leqslant 2.$$

Therefore $\|\Phi(w, q)\|_{V'} \leqslant 2$. Since

$$\begin{aligned}\left|\langle \Phi(w, q) - \Phi(u, q^*), v\rangle\right| &\leqslant \left(\left\|F_0(w) - F_0(u)\right\|_{V'} + \left|q - q^*\right| + |w - u|\right)\left|v(x_0)\right| \\ &\leqslant c\left(\|u - w\| + \left|q - q^*\right|\right)\|v\|,\end{aligned} \tag{9.1}$$

function Φ is Lipschitz continuous on $V \times P$.

Theorem 9.1. *One-dimensional damped sine-Gordon problem* (1.5) *has a unique weak solution* $y(q)$ *defined in* (2.3).

Proof. Apply Theorem 4.2. □

We claim that F_0 is Fréchet differentiable on V, and that its Fréchet derivative is given by

$$\langle DF_0(u)w, v\rangle = \cos u(x_0)w(x_0)v(x_0).$$

Indeed, fix $u \in V$, and let linear operator $T_u : V \to V'$ be defined by

$$\langle T_u w, v\rangle = \cos u(x_0)w(x_0)v(x_0), \quad v, w \in V.$$

Then

$$\left|\langle T_u w, v\rangle\right| \leqslant \|w\|\|v\|.$$

Therefore $\|T_u w\|_{V'} \leqslant \|w\|$, and $\|T_u\|_{L(V,V')} \leqslant 1$.

From inequality

$$\left|\sin b - \sin a - \cos a(b-a)\right| \leqslant |b-a|^2, \quad a, b \in \mathbb{R}$$

we get

$$\begin{aligned}\left|\left\langle F_0(w) - F_0(u) - T_u(w-u), v\right\rangle\right| &\leqslant \left|\sin w(x_0) - \sin u(x_0) - \cos u(x_0)\big(w(x_0) - u(x_0)\big)\right|\left|v(x_0)\right| \\ &\leqslant \left|w(x_0) - u(x_0)\right|^2\left|v(x_0)\right| \leqslant \|w-u\|^2\|v\|. \end{aligned} \tag{9.2}$$

Thus

$$\left\|F_0(w) - F_0(u) - T_u(w-u)\right\|_{V'} \leqslant \|w-u\|^2. \tag{9.3}$$

This means that F_0 is uniformly Fréchet differentiable on V, and $DF_0(u) = T_u$. Furthermore, inequality (9.3) shows that (6.2) is satisfied with $M = 1$.

Similarly

$$\left|\left\langle\big(DF_0(u_2) - DF_0(u_1)\big)w, v\right\rangle\right| \leqslant \left|\cos u_2(x_0) - \cos u_1(x_0)\right|\|w\|\|v\| \leqslant \|u_2 - u_1\|\|w\|\|v\|.$$

Therefore $\|(DF_0(u_2) - DF_0(u_1))w\|_{V'} \leqslant \|u_2 - u_1\|\|w\|$, and

$$\left\|DF_0(u_2) - DF_0(u_1)\right\|_{L(V,V')} \leqslant \|u_2 - u_1\|.$$

Thus inequality (6.4) is satisfied with $C = 1$, and DF_0 is Lipschitz continuous.

Theorem 9.2. *Minimization problem*

$$J\big(q^*\big) = \min_{q\in P} J(q) = \min_{q\in P} \int_Q \left|y(t,x;q) - z_d(t,x)\right|^2 dx\,dt, \quad Q = [0,1]\times[0,T],$$

for the one-dimensional sine-Gordon problem (1.5) *has a solution* $q^* \in P$.

Furthermore, if the optimal coefficient $q^* \in \operatorname{int} P$, *then it can be characterized by*

$$\nabla J\big(q^*\big) = \big\langle a\big(q^*\big), b\big(q^*\big), c\big(q^*\big)\big\rangle = 0,$$

where

$$a\big(q^*\big) = -2\int_Q y_t\big(t,x;q^*\big)p\big(t,x;q^*\big)\,dx\,dt,$$

$$b\big(q^*\big) = -2\int_0^T y_{xt}\big(t,x;q^*\big), p\big(t,x;q^*\big)\,dx\,dt,$$

and

$$c\big(q^*\big) = 2\int_0^T \sin y\big(t,x_0;q^*\big)p\big(t,x_0;q^*\big)\,dt.$$

If $q^* \in \operatorname{bnd} P$ *(the boundary of the set* P*), then the solution map* $q \to y(q)$ *is Gâteaux differentiable, and the optimal coefficient* q^* *is characterized by* (8.3).

Proof. Use Theorem 8.1. □

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