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Perturbing non-real eigenvalues of non-negative real matrices

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Abstract

Let $\sigma = (\rho, b + \mathrm{i}c, b - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$ be the spectrum of an entry non-negative matrix and $t \geqslant 0$. Laffey [T.J. Laffey, Perturbing non-real eigenvalues of nonnegative real matrices, Electron. J. Linear Algebra 12 (2005) 73–76] has shown that $\sigma = (\rho + 2t, b - t + \mathrm{i}c, b - t - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$ is also the spectrum of some non-negative matrix. Laffey (2005) has used a rank one perturbation for small t and then used a compactness argument to extend the result to all non-negative t. In this paper, a rank two perturbation is used to deduce an explicit and constructive proof for all $t \geqslant 0$.

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1. Introduction

A matrix $A = (a_{ij})_{n\times n} \geqslant 0$ if $a_{ij} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$. A list σ of n complex numbers is said to be *realizable* if σ is the spectrum of a non-negative real matrix. Denote by $\mathbb{N}_n$ the collection of all *n-tuples* list, of complex numbers, which are realizable. Denote e_i the ith unit vector and I_n the $n \times n$ identity matrix.

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Guo [2] (Refs. [1,4]) has given the following result:

Theorem 1.1. *Let $\sigma = (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$ be the spectrum of a non-negative matrix A. If λ_2 is real, then $(\rho + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n)$ are realizable for all $t \geqslant 0$.*

Laffey [3] has extended the result to

Theorem 1.2. *Let A be an $n \times n$ non-negative real matrix with spectrum σ and Perron root ρ. Let $b \pm \mathrm{i}c$, where b and c are real and $\mathrm{i} = \sqrt{-1}$, be a pair of non-real eigenvalues of A. Then, for all $t \geqslant 0$, replacing $\rho, b \pm \mathrm{i}c$ in σ by $\rho + 2t, b - t \pm \mathrm{i}c$, respectively, while keeping the other entries of σ unchanged, again yields the spectrum of an $n \times n$ non-negative matrix.*

In Laffey's paper [3], a rank one perturbation has been applied to A to first prove Theorem 1.2 for sufficiently small $t > 0$, and then a compactness argument is used to extend the result to all $t > 0$. In this paper, we apply a rank two perturbation to A and directly prove Theorem 1.2 for all non-negative t. Our proof is constructive; thus one can easily find a non-negative matrix to realize the perturbed spectrum list.

2. Proof of theorem – a rank two perturbation

Let $t > 0$. Let A be an $n \times n$ non-negative matrix with the spectrum $\sigma = (\rho, b + \mathrm{i}c, b - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$, where ρ is the Perron root, b and c are real, and $\mathrm{i} = \sqrt{-1}$. We assume that $c > 0$. By [1] Lemma 2.2, we can assume that the Perron eigenvector of A is $e = (1, 1, \ldots, 1)^{\mathrm{T}}$, i.e., $Ae = \rho e$. Let the Jordan canonical form of A be

$$\Lambda = \begin{pmatrix} \rho & & & & & \\ & b & c & & & \\ & -c & b & & * & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$$

and let $P = (e, u, v, w_4, \ldots, w_n)$ be $n \times n$ non-singular real matrix such that $P \Lambda P^{-1} = A$, where

$$u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}, \quad v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$$

be real vectors such that $u \pm \mathrm{i}v$ are eigenvectors of A corresponding to the eigenvalues $b \pm \mathrm{i}c$.

Let

$$\det(i, j, k) = \begin{vmatrix} 1 & u_i & v_i \\ 1 & u_j & v_j \\ 1 & u_k & v_k \end{vmatrix} = (u_i - u_k)(v_j - v_k) - (v_i - v_k)(u_j - u_k)$$

for any $1 \leqslant i, j, k \leqslant n$. Without loss of generality, we assume

$$\Delta = \det(1, 2, 3) = \max_{1, \leqslant i, j, k \leqslant n} \det(i, j, k). \tag{2.1}$$

Since P is non-singular it easy to see that $\Delta = \det(1, 2, 3) > 0$.

Let

$$X^{\mathrm{T}} = (x_1, x_2, x_3, 0, \ldots, 0)P = (0, t, 0, *, \ldots, *),$$
$$Y^{\mathrm{T}} = (y_1, y_2, y_3, 0, \ldots, 0)P = (0, 0, t, *, \ldots, *),$$

where

$$x_1 = \frac{t}{\Delta}(v_2 - v_3), \quad x_2 = \frac{t}{\Delta}(v_3 - v_1), \quad x_3 = \frac{t}{\Delta}(v_1 - v_2),$$
$$y_1 = -\frac{t}{\Delta}(u_2 - u_3), \quad y_2 = -\frac{t}{\Delta}(u_3 - u_1), \quad y_3 = -\frac{t}{\Delta}(u_1 - u_2).$$

Then we let

$$w_1 = u_1x_1 + v_1y_1,$$
$$w_2 = u_2x_2 + v_2y_2,$$
$$w_3 = u_3x_3 + v_3y_3$$

such that

$$W^{\mathrm{T}} = (w_1, w_2, w_3, 0, \ldots, 0)P = (2t, *, *, \ldots, *).$$

Now we have[1]

$$\begin{aligned}
&P(e_1W^{\mathrm{T}} - e_2X^{\mathrm{T}} - e_3Y^{\mathrm{T}})P^{-1} \\
&= P\begin{pmatrix} 2t & * & * & * & \cdots & * \\ 0 & -t & 0 & * & \cdots & * \\ 0 & 0 & -t & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^{-1} \\
&= \begin{pmatrix} w_1 - u_1x_1 - v_1y_1 & w_2 - u_1x_2 - v_1y_2 & w_3 - u_1x_3 - v_1y_3 & 0 & \cdots & 0 \\ w_1 - u_2x_1 - v_2y_1 & w_2 - u_2x_2 - v_2y_2 & w_3 - u_2x_3 - v_2y_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ w_1 - u_nx_1 - v_ny_1 & w_2 - u_nx_2 - v_ny_2 & w_3 - u_nx_3 - v_ny_3 & 0 & \cdots & 0 \end{pmatrix}.
\end{aligned}$$

From (2.1), we have $\Delta = \det(1, 2, 3) \geqslant \det(i, 2, 3)$ and it implies

$$u_1(v_2 - v_3) - v_1(u_2 - u_3) \geqslant u_i(v_2 - v_3) - v_i(u_2 - u_3), \quad 1 \leqslant i \leqslant n. \tag{2.2}$$

Thus for $1 \leqslant i \leqslant n$

$$w_1 - u_ix_1 - v_iy_1 = \frac{t}{\Delta}((u_1(v_2 - v_3) - v_1(u_2 - u_3)) - (u_i(v_2 - v_3) - v_i(u_2 - u_3))) \geqslant 0.$$

From (2.1), we have $\Delta = \det(1, 2, 3) \geqslant \det(1, i, 3)$ and it implies

$$u_2(v_3 - v_1) - v_2(u_3 - u_1) \geqslant u_i(v_3 - v_1) - v_i(u_3 - u_1), \quad 1 \leqslant i \leqslant n. \tag{2.3}$$

Thus for $1 \leqslant i \leqslant n$

$$w_2 - u_ix_2 - v_iy_2 = \frac{t}{\Delta}((u_2(v_3 - v_1) - v_2(u_3 - u_1)) - (u_i(v_3 - v_1) - v_i(u_3 - u_1))) \geqslant 0.$$

From (2.1), we have $\Delta = \det(1, 2, 3) \geqslant \det(1, 2, i)$ and it implies

$$u_3(v_1 - v_2) - v_3(u_1 - u_2) \geqslant u_i(v_1 - v_2) - v_i(u_1 - u_2), \quad 1 \leqslant i \leqslant n. \tag{2.4}$$

[1] In the paper, e_i refer to standard basis elements.

Thus for $1 \leqslant i \leqslant n$

$$w_3 - u_i x_3 - v_i y_3 = \frac{t}{\Delta}((u_3(v_1 - v_2) - v_3(u_1 - u_2)) - (u_i(v_1 - v_2) - v_i(u_1 - u_2))) \geqslant 0.$$

So we have

$$P(e_1 W^{\mathrm{T}} - e_2 X^{\mathrm{T}} - e_3 Y^{\mathrm{T}}) P^{-1} \geqslant 0.$$

Therefore,

$$A(t) = P(\Lambda + e_1 W^{\mathrm{T}} - e_2 X^{\mathrm{T}} - e_3 Y^{\mathrm{T}}) P^{-1} = A + P(e_1 W^{\mathrm{T}} - e_2 X^{\mathrm{T}} - e_3 Y^{\mathrm{T}}) P^{-1} \geqslant 0.$$

It is easy to see that $A(t) = P(\Lambda + e_1 W^{\mathrm{T}} - e_2 X^t\mathrm{T} - e_3 Y^{\mathrm{T}}) P^{-1}$ has the spectrum $(\rho + 2t, b - t + \mathrm{i}c, b - t - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$. So we complete the proof of Theorem 1.2.

3. Remarks

To completely extend Theorem 1.1, the following result is interesting and needs to be improved.

Proposition 3.1. *Let A be an $n \times n$ non-negative real matrix with Perron root ρ and the spectrum $(\rho, b + \mathrm{i}c, b - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$, where b is real and $c > 0$. Then there is a constant $C > 0$ such that the list $(\rho + Ct, b + t + \mathrm{i}c, b + t - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$ is realizable for all $t \geqslant 0$.*

Proof. We use the same notation as in Section 2. Taking

$$z_1 = u_2 x_2 + v_2 y_2 + u_3 x_3 + v_3 y_3,$$
$$z_2 = u_3 x_3 + v_3 y_3 + u_1 x_1 + v_1 y_1,$$
$$z_3 = u_1 x_1 + v_1 y_1 + u_2 x_2 + v_2 y_2.$$

We have

$$Z^{\mathrm{T}} = (z_1, z_2, z_3, 0, \ldots, 0) P = (4t, *, *, \ldots, *).$$

From (2.3) and (2.4), we have

$$u_i(v_2 - v_3) - v_i(u_2 - u_3) \geqslant -(u_2(v_3 - v_1) - v_2(u_3 - u_1)) - (u_3(v_1 - v_2) - v_3(u_1 - u_2))$$

and this implies

$$z_1 + u_i x_1 + v_i y_1 \geqslant 0, \quad 1 \leqslant i \leqslant n.$$

Similarly, we have

$$z_2 + u_i x_2 + v_i y_2 \geqslant 0, \quad 1 \leqslant i \leqslant n,$$

and

$$z_3 + u_i x_3 + v_i y_3 \geqslant 0, \quad 1 \leqslant i \leqslant n.$$

Thus,

$$P(e_1 Z^{\mathrm{T}} + e_2 X^{\mathrm{T}} + e_3 Y^{\mathrm{T}}) P^{-1} \geqslant 0.$$

Therefore,

$$A(t) = P(\Lambda + e_1 Z^{\mathrm{T}} + e_2 X^{\mathrm{T}} + e_3 Y^{\mathrm{T}}) P^{-1} = A + P(e_1 Z^{\mathrm{T}} + e_2 X^{\mathrm{T}} + e_3 Y^{\mathrm{T}}) P^{-1} \geqslant 0.$$

The matrix $P(\Lambda + e_1 Z^{\mathrm{T}} + e_2 X^{\mathrm{T}} + e_3 Y^{\mathrm{T}})P^{-1}$ has the spectrum $(\rho + 4t, b + t + \mathrm{i}c, b + t - \mathrm{i}c, \lambda_4, \ldots, \lambda_n)$. So the proof is complete. □

It would be interesting to know if the constant C in Proposition 3.1 can be improved to be 1 or 2. Also further research is necessary to consider the perturbations of imaginary parts.

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