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q-Generalizations of a family of harmonic number identities

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ABSTRACT

Paule and Schneider (2003) [3], and Chu (Chu and Donno) (2005) [1] gave a family of wonderful harmonic number identities. Their generalized versions associated with q-harmonic numbers will be established by applying a derivative operator to Watson's q-Whipple transformation.

For classical harmonic numbers defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \in \mathbb{N},$$

there exist eight beautiful identities:

$$\sum_{k=0}^{n} \binom{n}{k}^{-2} \{1 - 2(n-2k)H_k\} = 2\frac{(1+n)^2}{(2+n)} H_{n+1}; \tag{1}$$

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} \{1 - (n-2k)H_k\} = (1+n)H_{n+1}; \tag{2}$$

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$$\sum_{k=0}^{n}\binom{n}{k}\{1+(n-2k)H_k\}=1; \tag{3}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\{1+2(n-2k)H_k\}=0; \tag{4}$$

$$\sum_{k=0}^{n}\binom{n}{k}^3\{1+3(n-2k)H_k\}=(-1)^n; \tag{5}$$

$$\sum_{k=0}^{n}\binom{n}{k}^4\{1+4(n-2k)H_k\}=(-1)^n\binom{2n}{n}; \tag{6}$$

$$\sum_{k=0}^{n}\binom{n}{k}^5\{1+5(n-2k)H_k\}=(-1)^n\sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{n}; \tag{7}$$

$$\sum_{k=0}^{n}\binom{n}{k}^6\{1+6(n-2k)H_k\}=(-1)^n\sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{n}\binom{2n-i}{n}. \tag{8}$$

Thereinto, (3)–(7) first appeared in Paule and Schneider [3]. And Chu [1] created other three ones.

For two complex numbers q and x, define q-harmonic numbers and generalized q-harmonic numbers respectively by

$$\mathcal{H}_0=0 \quad \text{and} \quad \mathcal{H}_n=\sum_{k=1}^{n}\frac{q^k}{1-q^k} \quad \text{for } n\in\mathbb{N};$$

$$\mathcal{H}_0(x)=0 \quad \text{and} \quad \mathcal{H}_n(x)=\sum_{k=1}^{n}\frac{q^k}{1-xq^k} \quad \text{for } n\in\mathbb{N}.$$

When $x=1$, the latter reduce to the former.

Given a differentiable function $f(x)$, define the derivative operator $\mathcal{D}$ by

$$\mathcal{D}f(x)=\frac{d}{dx}f(x)\Big|_{x=1}.$$

Then for $m\in\mathbb{N}$, it is not difficult to show the following two derivatives:

$$\mathcal{D}(qxy;q)_m=-y(qy;q)_m\mathcal{H}_m(y) \quad \text{and} \quad \mathcal{D}(qy/x;q)_m=y(qy;q)_m\mathcal{H}_m(y)$$

where y is a complex number independent of x and the shifted factorial has been defined by

$$(x;q)_m=(1-x)(1-xq)\cdots\left(1-xq^{m-1}\right).$$

Recall Watson's q-Whipple transformation (cf. [2, p. 43]):

$${}_8\phi_7\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}\\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{1+n}\end{matrix}\,\middle|\, q;\frac{q^{2+n}a^2}{bcde}\right] \tag{9a}$$

$$=\frac{(qa;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n}\,{}_4\phi_3\left[\begin{matrix}q^{-n}, & b, & c, & qa/de\\ & qa/d, & qa/e, & q^{-n}bc/a\end{matrix}\,\middle|\, q;q\right]. \tag{9b}$$

By applying the derivative operator $\mathcal{D}$ to this equation, q-generalizations of (1)–(8) can be derived. For simplifying the evaluations, sometimes we shall use directly the following special case of (9a)–(9b) (cf. [2, p. 42]):

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & aq^{1+n} \end{matrix}\,\middle|\, q;\ \frac{q^{1+n}a}{bc}\right] = \frac{(qa;q)_n(qa/bc;q)_n}{(qa/b;q)_n(qa/c;q)_n}. \tag{10}$$

The main theorem can be stated as follows.

Theorem 1. *Define q-binomial coefficient by* $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$ *and denote Kronecker delta by* δ. *Then generalized versions of* (1)–(8) *associated with q-harmonic numbers can be displayed, respectively, as follows*:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{-2} q^{k(n-k-1)} \big\{ \big(1-q^{2k-n}\big)(3k+2\mathcal{H}_k) + (1+2k)q^{2k-n} - 2k \big\} \tag{11a}$$

$$= \frac{(1-q^{-1-n})^2(1-q)}{(q-q^{-1-n})(1-q^{-1})} \{2\mathcal{H}_{n+1} + n + 1\}; \tag{11b}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{-1} q^{-k} \big\{ \big(1-q^{2k-n}\big)(2k+\mathcal{H}_k) + (1+2k)q^{2k-n} - 2k \big\} \tag{12a}$$

$$= \big(q^{-1-n} - 1\big)\mathcal{H}_{n+1}; \tag{12b}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n-1)} \big\{ \big(q^{2k-n} - 1\big)\mathcal{H}_k + (1+k)q^{2k-n} - k \big\} = 1; \tag{13}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{2} q^{k(k-n-1)} \big\{ 2\big(q^{2k-n} - 1\big)\mathcal{H}_k + (1+k)q^{2k-n} - k \big\} = \delta_{0,n}; \tag{14}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{3} q^{k(2k-2n-1)} \big\{ \big(q^{2k-n} - 1\big)(k+3\mathcal{H}_k) + (1+k)q^{2k-n} - k \big\} \tag{15a}$$

$$= (-1)^n q^{-\binom{1+n}{2}}; \tag{15b}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{4} q^{k(2k-2n-1)} \big\{ \big(q^{2k-n} - 1\big)(k+4\mathcal{H}_k) + (1+k)q^{2k-n} - k \big\} \tag{16a}$$

$$= (-1)^n q^{-\binom{1+n}{2}} \begin{bmatrix} 2n \\ n \end{bmatrix}; \tag{16b}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{5} q^{k(3k-3n-1)} \big\{ \big(q^{2k-n} - 1\big)(k+5\mathcal{H}_k) + (1+2k)q^{2k-n} - 2k \big\} \tag{17a}$$

$$= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{2} \begin{bmatrix} n+k \\ n \end{bmatrix} q^{k(k-n)-\binom{1+n}{2}}; \tag{17b}$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}^6 q^{k(3k-3n-1)}\big\{\big(q^{2k-n}-1\big)(k+6\mathcal{H}_k)+(1+2k)q^{2k-n}-2k\big\} \tag{18a}$$

$$=(-1)^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}^2\begin{bmatrix}n+k\\n\end{bmatrix}\begin{bmatrix}2n-k\\n\end{bmatrix}q^{k(k-n)-\binom{1+n}{2}}. \tag{18b}$$

Proof. Performing the substitutions $a \to q^{-n}/x$, $b \to q$, $c \to q$, $d \to q$, $e \to q$ for (9a)–(9b), we can reformulate the resulting identity as

$$\sum_{k=0}^{n}\big\{1-q^{2k-n}/x\big\}\frac{(q^{-n};q)_k}{(q/x;q)_k}\left\{\frac{(q;q)_k}{(q^{-n}/x;q)_k}\right\}^3\left\{\frac{q^{-n-2}}{x^2}\right\}^k$$
$$=\left\{1-\frac{1}{x}\right\}\frac{(x-q^{-1-n})}{(x-q^{-1})}\sum_{i=0}^{n}\frac{(q;q)_i(q^{-n};q)_i(q^{-1-n}/x;q)_i}{(q^2x;q)_i(q^{-n}/x;q)_i(q^{-n}/x;q)_i}q^i.$$

Applying the derivative operator $\mathcal{D}$ to both sides of the last identity, we obtain (11a)–(11b) a generalized version of Eq. (1) in terms of q-harmonic numbers. Other equations that appear in Theorem 1 can also be deduced in the same method. The corresponding replacements are laid out in Table 1. □

Table 1
Detailed substitutions.

Original identity	Corresponding replacements	Resulting identity
(9a)–(9b)	$a \to q^{-n}/x$, $b \to q$, $c \to q$, $d \to q$, $e \to \infty$	(12a)–(12b)
(10)	$a \to q^{-n}/x$, $b \to q$, $c \to \infty$	(13)
(10)	$a \to q^{-n}/x$, $b \to q^{-n}$, $c \to q$	(14)
(10)	$a \to q^{-n}/x$, $b \to q^{-n}$, $c \to \infty$	(15a)–(15b)
(10)	$a \to q^{-n}/x$, $b \to q^{-n}$, $c \to q^{-n}$	(16a)–(16b)
(9a)–(9b)	$a \to q^{-n}/x$, $b \to q^{-n}$, $c \to q^{-n}$, $d \to q^{-n}$, $e \to \infty$	(17a)–(17b)
(9a)–(9b)	$a \to q^{-n}/x$, $b \to q^{-n}$, $c \to q^{-n}$, $d \to q^{-n}$, $e \to q^{-n}$	(18a)–(18b)

We point out that these eight q-harmonic number identities displayed in Theorem 1 reduce to (1)–(8) respectively when $q \to 1$.

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