A generalization of a theorem of Bochner

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Abstract

We describe all polynomial solutions to the general second order operator equation in the Askey–Wilson operators, thus extending a theorem of Bochner from differential operators to Askey–Wilson divided difference operators.

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1. Introduction

A problem that attracted the attention of several mathematicians is to study polynomial solutions to equations of the type [4,10,1, Section 7, 18]

\[ f(x)Ty(x) + g(x)Sy(x) + h(x)y(x) = \lambda_n y(x), \]  

(1.1)

where \( S \) and \( T \) are linear operators which map a polynomial of precise degree \( n \) to a polynomial of exact degree \( n - 1 \), and \( n - 2 \), respectively. Moreover, \( f, g, h \) are polynomials and \( \{\lambda_n\} \) is a sequence of constants. We require \( f, g, h \) to be independent of \( n \) and demand for every \( n \) Eq. (1.1) has a polynomial solution of exact degree \( n \). It is tacitly assumed that \( S \) reduces the degree of a polynomial by 1 while \( T \) reduces the degree of a polynomial 2.

Lemma 1.1. Let \( S, T \) be as above. If (1.1) has a polynomial solution of exact degree \( n \) for all \( n \) then \( f \) and \( g \) have degrees at most 2 and 1, respectively, and we may take \( \lambda_0 = 0 \) and \( h \equiv 0 \).

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Proof. By adding \(-\lambda_0 y(x)\) to both sides of (1.1) we may assume that \(\lambda_0 = 0\). Let \(y_n(x) = x^n + \text{lower order terms}\), be a solution of (1.1). The result follows from substituting \(y_0(x) = 1\), \(y_1(x) = x + a\), \(y_2(x) = x^2 + bx + c\) in (1.1). \(\square\)

Lemma 1 appeared in the literature several times for special \(T\) and \(S\), see [1,4].

We shall denote the exact degree of a polynomial \(p\) by \(\text{deg}(p)\). The theorem of Bochner alluded to in the title is Theorem 1.2 below. The notation \(\{P_n^{(x,\beta)}(x)\}, \{L_n^{(x)}(x)\}, \{y_n(x; a)\}\) for Jacobi, Laguerre polynomials, and Bessel polynomials is the same as in [2,19,20,7], see also [14].

Theorem 1.2 (Bochner [4]). Let \(S = \frac{d}{dx}, T = S^2\). Then \(\lambda_n\) and a solution \(y_n\) are given by:

1. \(f(x) = 1 - x^2\), \(g(x) = \beta - \alpha - x(\alpha + \beta + 2)\), \(\lambda_n = -n(n + \alpha + \beta + 1)\), \(y_n = P_n^{(\alpha,\beta)}(x)\),
2. \(f(x) = x^2\), \(g(x) = ax + 1\), \(\lambda_n = n(n + a - 1)\), \(y_n(x) = y_n(x; a, 1)\),
3. \(f(x) = x^2\), \(g(x) = x\), \(\lambda_n = n^2\), \(y_n(x) = x^n\),
4. \(f(x) = x\), \(g(x) = 1 + \alpha - x\), \(\lambda_n = -n\), \(y_n(x) = L_n^{(\alpha)}(x)\),
5. \(f(x) = 1\), \(g(x) = -2x\), \(\lambda_n = -2n\), \(y_n(x) = H_n(x)\).

The following motivation explains Theorem 1.2. We seek a polynomial solution of degree \(n\) to the differential equation

\[
f(x)y''(x) + g(x)y'(x) = \lambda y(x).
\]

We know that one of the coefficients in \(f\) or \(g\) is not zero, hence there is no loss of generality in choosing it equal to 1. Thus \(f\) and \(g\) contain four free parameters. The scaling \(x \rightarrow ax + b\) of the independent variable absorbs two of the four parameters. The eigenvalue parameter \(\lambda\) is then uniquely determined by equating coefficients of \(x^n\) in (1.2) since \(y\) has degree \(n\). This reduces (1.2), in general, to a Jacobi differential equation whose polynomial solution, in general, is a Jacobi polynomial. The other cases are special or limiting cases of Jacobi polynomials.

It must be emphasized that Bochner’s theorem classifies second order differential equations of Sturm–Liouville type with polynomial solutions. Several authors assumed that \(f\) and \(g\) have degrees 2 and 1, respectively, then proceeded to find orthogonal polynomial solutions of (1.1) for special \(S\) and \(T\). The interested reader may consult [1,15–18].

In Section 2 we define the Askey–Wilson operators and state some of their properties. Our main result extends Bochner’s theorem to the Askey–Wilson operators and will be stated and proved in Section 3. Since no scaling in \(x\) is allowed in the presence of an Askey–Wilson operator, we expect the eigenfunctions to depend on four parameters.
Recently Grunbaum and Haine [8,9] have studied the bispectral problem of finding simultaneous solutions to the eigenvalue problem \( Lp_n(x) = \lambda_n p_n(x) \) and \( Mp_n(x) = x p_n(x) \), where \( L \) is a second order Askey–Wilson operator and \( M \) is a second order difference equation in \( n \). This is different from what we do here because we just classify solutions to (2.6) without requiring that they also solve a second order difference equation in \( n \) which is also of Sturm–Liouville type.

2. The Askey–Wilson operators

We follow the notation in [11]. Given a polynomial \( f \) we set \( \tilde{f}(e^{i\theta}) := f(x) \), \( x = \cos \theta \), that is

\[
\tilde{f}(z) = f((z + 1/z)/2), \quad z = e^{i\theta}.
\]  

(2.1)

This means that we think of \( f(\cos \theta) \) as a function of \( e^{i\theta} \). In this notation the Askey–Wilson divided difference operator \( D_q \) is

\[
(D_q f)(x) := \frac{\tilde{f}(q^{1/2} e^{i\theta}) - \tilde{f}(q^{-1/2} e^{i\theta})}{\tilde{e}(q^{1/2} e^{i\theta}) - \tilde{e}(q^{-1/2} e^{i\theta})}, \quad x = \cos \theta,
\]  

(2.2)

with

\[
e(x) = x.
\]  

(2.3)

A calculation reduces (2.2) to

\[
(D_q f)(x) = \frac{\tilde{f}(q^{1/2} e^{i\theta}) - \tilde{f}(q^{-1/2} e^{i\theta})}{(q^{1/2} - q^{-1/2}) i \sin \theta}, \quad x = \cos \theta.
\]  

(2.4)

An averaging operator associated with the Askey–Wilson operator is

\[
(\mathcal{A}_q f)(x) = \frac{1}{2} [\tilde{f}(q^{1/2} e^{i\theta}) + \tilde{f}(q^{-1/2} e^{i\theta})].
\]  

(2.5)

As \( q \to 1 \), \( D_q \) and \( \mathcal{A}_q \) tend to \( d/dx \) and the identity operator, respectively.

It is important to note that although we use \( x = \cos \theta \), \( \theta \) is not necessarily real. In fact \( e^{i\theta} \) is defined as

\[
e^{i\theta} = x + \sqrt{x^2 - 1},
\]

and the branch of the square root is taken such that \( \sqrt{x^2 - 1} \approx x \) as \( x \to \infty \). The basis

\[
\phi_n(x; a) := (ae^{i\theta}, ae^{-i\theta}, q)_n, \quad n = 0, 1, \ldots,
\]  

(2.6)

plays the role played by \( \{x - a^n\} \) in calculus in the sense

\[
D_q \phi_n(x; a) = \frac{-2a(1 - q^n)}{(1 - q)} \phi_{n-1}(x; a q^{1/2}).
\]  

(2.7)

Moreover

\[
\mathcal{A}_q \phi_n(x; q) = [1 - axq^{-1/2}(1 + q^n) + a^2 q^{n-1}] \phi_{n-1}(x; a q^{1/2}).
\]  

(2.8)

The Askey–Wilson polynomials satisfy

\[
p_n(\cos \theta; a, b, c, d|q) := 4\phi_3 \left( \begin{array}{c} q^{-n}, abc dq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} \right) q, q \right) .
\]  

(2.9)
Remark 2.1. It is important to note that both $\mathcal{D}_q$ and $\mathcal{A}_q$ are invariant under $q \to q^{-1}$. On the other hand

$$(\lambda; 1/q)_n = (-\lambda)^n q^{n(1-n)/2}(1/\lambda; q)_n,$$

and (2.9) implies

$$p_n(x; a, b, c, d | 1/q) = p_n(x; 1/a, 1/b, 1/c, 1/d | q).$$

(2.10)

3. Main result

We consider the operator equation (1.1) with

$$T = \mathcal{D}_q^2, \quad S = \mathcal{A}_q \mathcal{D}_q,$$

that is

$$f(x) \mathcal{D}_q^2 y_n(x) + g(x) \mathcal{A}_q \mathcal{D}_q y_n(x) + h(x) y_n(x) = \lambda_n y_n(x).$$

(3.1)

Such equations are satisfied by the Askey–Wilson polynomials [3,6], and arise in solving the Bethe ansatz equations for a generalization of the XXZ model, see [12].

Recall that the Askey–Wilson polynomials satisfy

$$\pi_2(x) \mathcal{D}_q^2 y(x) + \pi_1(x) \mathcal{A}_q \mathcal{D}_q y(x) = \lambda_n y(x),$$

(3.2)

with

$$\pi_2(x) = -q^{-1/2} [2(1 + \sigma_4)x^2 - (\sigma_1 + \sigma_3)x - 1 + \sigma_2 - \sigma_4],$$

$$\pi_1(x) = \frac{2}{1-q} [2(\sigma_4 - 1)x + \sigma_1 - \sigma_3],$$

$$\lambda_n = \frac{4q(1 - q^{-n})(1 - \sigma_4q^{n-1})}{(1-q)^2}. $$

(3.3)

Here $\sigma_j$, is the $j$th elementary symmetric function of the Askey–Wilson parameters $a, b, c, d$ [3].

We now come to our generalization of Theorem 1.2. Lemma 1.1 implies that $h \equiv 0$, $\deg(f) \leq 2$ and $\deg(g) \leq 1$, and $\lambda_0 = 0$. To match (3.2) with (3.1), let

$$f(x) = f_0 x^2 + f_1 x + f_2, \quad g(x) = g_0 x + g_1.$$  

(3.4)

If $2q^{1/2} f_0 + (1-q) g_0 \neq 0$ then through a suitable multiplier we can assume that $2q^{1/2} f_0 + (1-q) g_0 = -8$ and then determine the $\sigma$’s uniquely, hence we determine the parameters $a, b, c, d$ up to permutations. As in [5] we then prove that $\lambda_n$ is given by (3.3) and (3.2) has only one polynomial solution, an Askey–Wilson polynomial of degree $n$. If $2q^{1/2} f_0 + (1-q) g_0 = 0$ but $|f_0| + |g_0| \neq 0$ then we let $q = 1/p$, and apply (2.10) and Remark 2.1 to see that (3.2) is transformed to a similar equation where the $\sigma$’s are elementary symmetric functions of $1/a, 1/b, 1/c, 1/d$ and $q$ is replaced by $1/q$. Finally if $f_0 = g_0 = 0$, then $\lambda_n = 0$ for all $n$ and with $u = \mathcal{D}_q y$ we see that

$$(f_1 x + f_2) \mathcal{D}_q u(x) + g_1 \mathcal{A}_q u(x) = 0.$$ 

(3.5)
Substituting $u(x) = \sum_{k=0}^{n} u_k \phi_k(x; a)$ in (3.5) and equating coefficients of $\phi_k$ for all $k$ we see that it is impossible to find polynomial solutions to (3.5) of all degrees. This establishes the following theorem.

**Theorem 3.1.** Given an equation of the form (3.1), it has a polynomial solution $y_n(x)$ of degree $n$ for every $n$, $n = 0, 1, \ldots$, if and only if $y_n(x)$ is a multiple of $p_n(x; a, b, c, d)$ for some parameters $a, b, c, d$, including limiting cases as one or more of the parameters tends to $\infty$. In all these cases (3.1) can always be reduced to (3.2), or a special or limiting case of it.

**Remark 3.2.** It is important to note that solutions to (3.2) may not satisfy the orthogonality relation for the Askey–Wilson polynomials [3, 13]. For example $r_n(x) = \lim_{d \to \infty} p_n(x; a, b, c, d)$ satisfies

$$2x r_n(x) = A_n r_{n+1}(x) + C_n r_{n-1}(x) + [a + a^{-1} - A_n - C_n] r_n(x),$$

with

$$A_n = \frac{(1 - ab q^n)(1 - ac q^n)}{abc q^{2n}}, \quad C_n = \frac{(1 - q^n)(1 - bc q^{n-1})}{abc q^{2n-1}}.$$  

(3.7)

Therefore the corresponding moment problem is indeterminate for $q \in (0, 1)$ and $\max \{ab, ac, ad\} < 1$. On the other hand if $q > 1$, the moment problem is determinate for $\min \{ab, ac, ad\} > 1$. In fact the latter polynomials are special Askey–Wilson polynomials.

**References**


