



Mathematical modal logic: A view of its evolution

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Abstract

This is a survey of the origins of mathematical interpretations of modal logics, and their development over the last century or so. It focuses on the interconnections between *algebraic* semantics using Boolean algebras with operators and *relational* semantics using structures often called *Kripke models*. It reviews the ideas of a number of people who independently contributed to the emergence of relational semantics, and compares them with the work of Kripke. It concludes with an account of several applications of modal model theory to mathematics and theoretical computer science.

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1. Introduction

Modal logic was originally conceived as the logic of *necessary* and *possible* truths. It is now viewed more broadly as the study of many linguistic constructions that qualify the truth conditions of statements, including statements concerning knowledge, belief, temporal discourse, and ethics. Most recently, modal symbolism and model theory have been put to use in computer science, to formalise reasoning about the way programs behave and to express dynamical properties of transitions between states.

Over a period of three decades or so from the early 1930s there evolved two kinds of mathematical semantics for modal logic. *Algebraic* semantics interprets modal connectives as operators on Boolean algebras. *Relational* semantics uses relational structures, often called *Kripke models*, whose elements are thought of variously as being possible worlds, moments of time, evidential situations, or states of a computer. The two approaches are intimately related: the subsets of a relational structure form a modal algebra (Boolean algebra with operators), while conversely any modal algebra can be embedded into an

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algebra of subsets of a relational structure via extensions of Stone’s Boolean representation theory. Techniques from both kinds of semantics have been used to explore the nature of modal logic and to clarify its relationship to other formalisms, particularly first and second order monadic predicate logic.

The aim of this article is to review these developments in a way that provides some insight into how the present came to be as it is. The pervading theme is the *mathematics* underlying modal logic, and this has at least three dimensions. To begin with there are the new mathematical ideas: when and why they were introduced, and how they interacted and evolved. Then there is the use of methods and results from other areas of mathematical logic, algebra and topology in the analysis of modal systems. Finally, there is the application of modal syntax and semantics to study notions of mathematical and computational interest.

There has been some mild controversy about priorities in the origin of relational model theory, and space is devoted to this issue in Section 4. An attempt is made to record in one place a sufficiently full account of what was said and done by early contributors to allow readers to make their own assessment (although the author does give his).

Despite its length, the article does not purport to give an encyclopaedic coverage of the field. For instance, there is much about temporal logic (see *Gabbay et al.* [77]) and logics of knowledge (see *Fagin et al.* [64]) that is not reported here, while the surface of modal *predicate* logic is barely scratched, and proof theory is not discussed at all. I have not attempted to survey the work of the present younger generation of modal logicians (see *Chagrov and Zakharyashev* [34], *Kracht* [141], and *Marx and Venema* [171], for example). There has been little by way of historical review of work on intensional semantics over the last century, and no doubt there remains room for more.

2. Beginnings

2.1. What is a modality?

Modal logic began with Aristotle’s analysis of statements containing the words “necessary” and “possible”.¹ These are but two of a wide range of *modal connectives*, or *modalities* that are abundant in natural and technical languages. Briefly, a modality is any word or phrase that can be applied to a given statement S to create a new statement that makes an assertion about the *mode of truth* of S : about when, where or how S is true, or about the circumstances under which S may be true. Here are some examples, grouped according to the subject they are naturally associated with

tense logic: henceforth, eventually, hitherto, previously, now,
tomorrow, yesterday, since, until, inevitably, finally,
ultimately, endlessly, it will have been, it is being ...

¹ For the early history of modal logic, including the work of Greek and medieval scholars, see *Bochenski 1961* [20] and *Kneale and Kneale 1962* [134]. The Historical Introduction to Lemmon [156] gives a brief but informative sketch.

deontic logic:	it is obligatory/forbidden/permitted/unlawful that
epistemic logic:	it is known to X that, it is common knowledge that
doxastic logic:	it is believed that
dynamic logic:	after the program/computation/action finishes, the program enables, throughout the computation
geometric logic:	it is locally the case that
metalogic:	it is valid/satisfiable/provable/consistent that

The key to understanding the *relational* modal semantics is that many modalities come in dual pairs, with one of the pair having an interpretation as a universal quantifier (“in all ...”) and the other as an existential quantifier (“in some ...”). This is illustrated by the following interpretations, the first being famously attributed to Leibniz (see Section 4).

necessarily	in all possible worlds
possibly	in some possible world
henceforth	at all future times
eventually	at some future time
it is valid that	in all models
it is satisfiable that	in some model
after the program finishes	after all terminating executions
the program enables	there is a terminating execution such that

It is now common to use the symbol \Box for a modality of universal character, and \Diamond for its existential dual. In systems based on classical truth-functional logic, \Box is equivalent to $\neg\Diamond\neg$, and \Diamond to $\neg\Box\neg$, where \neg is the negation connective. Thus “necessarily” means “not possibly not”, “eventually” means “not henceforth not”, a statement is valid when its negation is not satisfiable, etc.

Notation

Rather than trying to accommodate all the different notations used for truth-functional connectives by different authors over the years, we will fix on the symbols \wedge , \vee , \neg , \rightarrow and \leftrightarrow for conjunction, disjunction, negation, (material) implication, and (material) equivalence. The symbol \top is used for a constant true formula, equivalent to any tautology, while \perp is a constant false formula, equivalent to $\neg\top$. We also use \top and \perp as symbols for truth values.

The standard syntax for propositional modal logic is based on a countably infinite list p_0, p_1, \dots of *propositional variables*, for which we typically use the letters p, q, r . Formulas are generated from these variables by means of the above connectives and the symbols \Box and \Diamond . There are of course a number of options about which of these to take as primitive symbols, and which to define in terms of primitives. When describing the work of different authors we will sometimes use their original symbols for modalities, such as M for *possibly*, L or N for *necessarily*, and other conventions for deontic and tense logics.

The symbol \Box^n stands for a sequence $\Box\Box\cdots\Box$ of n copies of \Box , and likewise \Diamond^n for $\Diamond\Diamond\cdots\Diamond$ (n times).

A systematic notation will also be employed for Boolean algebras: the symbols $+$, \cdot , $-$ denote the operations of sum (join), product (meet), and complement in a Boolean algebra, and 0 and 1 are the greatest and least elements under the ordering \leq given by $x \leq y$ iff $x \cdot y = x$. The supremum (sum) and infimum (product) of a set X of elements will be denoted $\sum X$ and $\prod X$ (when they exist).

2.2. MacColl's iterated modalities

The first substantial algebraic analysis of modalised statements was carried out by Hugh MacColl, in a series of papers that appeared in *Mind* between 1880 and 1906 under the title *Symbolical Reasoning*,² as well as in other papers and his book [163] of 1906. MacColl symbolised the conjunction of two statements a and b by their concatenation ab , used $a + b$ for their disjunction, and wrote $a : b$ for the statement “ a implies b ”, which he said could be read “if a is true, then b must be true”, or “whenever a is true, b is also true”. The equation $a = b$ was used for the assertion that a and b are equivalent, meaning that each implies the other. Thus $a = b$ is itself equivalent to the “compound implication” $(a : b)(b : a)$, an observation that was rendered symbolically by the equation $(a = b) = (a : b)(b : a)$.

MacColl wrote a' for the “denial” or “negative” of statement a , and stated that $(a' + b)'$ is equivalent to ab' . However, while $a' + b$ is a “necessary consequence” of $a : b$ (written $(a : b) : a' + b$), he argued that the two formulas are not equivalent because their denials are not equivalent, claiming that the denial of $a : b$ “only asserts the *possibility* of the combination ab' ”, while the denial of $a' + b$ “asserts the *certainty* of the same combination”.³

Boole had written $a = 1$ and $a = 0$ for “ a is true” and “ a is false”, giving a temporal reading of these as *always true* and *always false* respectively (see *Boole 1854* [21], Chapter XI). MacColl invoked the letters ε and η to stand for certainty and impossibility, initially describing them as replacements for 1 and 0, and then introduced a third letter θ to denote a statement that was neither certain nor impossible, and hence was “a *variable* (neither always true nor always false)”. He wrote the equations $(a = \varepsilon)$, $(b = \eta)$ and $(c = \theta)$ to express that a is a certainty, b is an impossibility, and c is a variable. Then he changed these to the symbols a^ε , b^η , c^θ , and went on to write a^τ for “ a is true” and a^l for “ a is false”, noting that a true statement is “not necessarily a certainty” and a false one is “not necessarily impossible”. In these terms he stated that $a : b$ is equivalent both to $(a.b')^\eta$ (“it is impossible that a and not b ”) and to $(a' + b)^\varepsilon$ (“it is certain that either not a or b ”).

Once the step to this superscript notation had been taken, it was evident that it could be repeated, giving an easy notation for iterations of modalities. MacColl gave the example of $A^{\eta l \varepsilon \varepsilon}$ as “it is certain that it is certain that it is false that it is impossible that A ”, abbreviated this to “it is certain that a is certainly possible”, and observed that

² A listing of these papers is given in the Bibliography of *Lewis 1918* [158] and on p. 132 of Church's bibliography in volume 1 of *The Journal of Symbolic Logic*. A comprehensive bibliography of MacColl's works is given in *Astroh and Klüwer 1998* [3].

³ This appears to conflict with his earlier claim that the denial of $a' + b$ is equivalent to ab' . “Actuality” may be a better word than “certainty” to express what he meant here (see *Mind*, Vol. 5, 1880, p. 54).

Probably no reader—at least no English reader, born and brought up in England—can go through the full unabbreviated translation of this symbolic statement $A^{nl\epsilon\epsilon}$ into ordinary speech without being forcibly reminded of a certain nursery composition, whose ever-increasing accumulation of *thats* affords such pleasure to the infantile mind; I allude, of course, to “The House that Jack Built”. But trivial matters in appearance often supply excellent illustrations of important general principles.⁴

There has been a recent revival of interest in MacColl, with a special issue of the Nordic Journal of Philosophical Logic⁵ devoted to studies of his work. In particular the article *Read 1998* [213] analyses the principles of modal algebra proposed by MacColl and argues that together they correspond to the modal logic T, later developed by Feys and von Wright, that is described at the end of Section 2.4 below.

2.3. The Lewis systems

MacColl’s papers are similar in style to earlier nineteenth century logicians. They give a descriptive account of the meanings and properties of logical operations but, in contrast to contemporary expectations, provide neither a formal definition of the class of formulas dealt with nor an axiomatisation of operations in the sense of a rigorous deduction of theorems from a given set of principles (axioms) by means of explicitly stated rules of inference. The first truly modern formal axiom systems for modal logic are due to C.I. Lewis, who defined five different ones, S1–S5, in Appendix II of the book *Symbolic Logic* [159] of 1932 that he wrote with C.H. Langford. Lewis had begun in a paper of 1912 [157, p. 522] with a concern that

the expositors of the algebra of logic have not always taken pains to indicate that there is a difference between the algebraic and ordinary meanings of implication.

He observed that the algebraic meaning, as used in the *Principia Mathematica* of Russell and Whitehead, leads to the “startling theorems” that a false proposition implies any proposition, and a true proposition is implied by any proposition. These so-called *paradoxes of material implication* take the symbolic forms

$$\begin{aligned} \neg\alpha &\rightarrow (\alpha \rightarrow \beta), \\ \alpha &\rightarrow (\beta \rightarrow \alpha). \end{aligned}$$

For Lewis the ordinary meaning of “ α implies β ” is that β can be *validly inferred*⁶ from α , or is *deducible*⁷ from α , an interpretation that he considered was not subject to these paradoxes. Taking “ α implies β ” as synonymous with “either not- α or β ”, he distinguished *extensional* and *intensional* meanings of disjunction, providing two meanings for “implies”. Extensional disjunction is the usual truth-functional “or”, which gives the *material*

⁴ *Mind* (New Series), Vol. 9, 1900, p. 75.

⁵ Vol 3 no. 1, Dec 1998, available at <http://www.hf.uio.no/filosofi/njpl/vol3no1/index.html>.

⁶ [157, p. 527].

⁷ [159, p. 122].

(algebraic) implication synonymous with “it is false that α is true and β is false”. Intensional disjunction

is such that at least one of the disjoined propositions is “necessarily” true.⁸

That reading produces Lewis’ “ordinary” implication, which he also dubbed “strict”, meaning that “it is *impossible* (or *logically inconceivable*) that α is true and β is false”.

The system of Lewis’s 1918 book *A Survey of Symbolic Logic* [158] used a primitive *impossibility* operator to define strict implication. This later became the system S3 of [159], which introduced instead the symbol \diamond for possibility, but Lewis decided that he wished S2 to be regarded as the correct system for strict implication. The systems were defined with negation, conjunction, and possibility as their primitive connectives, but he made no use of a symbol for the dual combination $\neg\diamond\neg$.¹⁰ For strict implication the symbol $\text{—}\mathfrak{3}$ was used, with $\alpha \text{—}\mathfrak{3} \beta$ being a definitional abbreviation for $\neg\diamond(\alpha \wedge \neg\beta)$. Strict equivalence ($\alpha = \beta$) was defined as $(\alpha \text{—}\mathfrak{3} \beta) \wedge (\beta \text{—}\mathfrak{3} \alpha)$.

Here now are definitions of S1–S5 in Lewis’s style, presented both to facilitate discussion of later developments and to convey some of the character of his approach. System S1 has the axioms¹¹

$$\begin{aligned} &(p \wedge q) \text{—}\mathfrak{3} (q \wedge p), \\ &(p \wedge q) \text{—}\mathfrak{3} p, \\ &p \text{—}\mathfrak{3} (p \wedge p), \\ &((p \wedge q) \wedge r) \text{—}\mathfrak{3} (p \wedge (q \wedge r)), \\ &((p \text{—}\mathfrak{3} q) \wedge (q \text{—}\mathfrak{3} r)) \text{—}\mathfrak{3} (p \text{—}\mathfrak{3} r), \\ &(p \wedge (p \text{—}\mathfrak{3} q)) \text{—}\mathfrak{3} q, \end{aligned}$$

where p, q, r are propositional variables, and the following rules of inference.

- *Uniform substitution* of formulas for propositional variables.
- *Substitution of strict equivalents*: from $(\alpha = \beta)$ and γ infer any formula obtained from γ by substituting β for some occurrence(s) of α .
- *Adjunction*: from α and β infer $\alpha \wedge \beta$.
- *Strict detachment*: from α and $\alpha \text{—}\mathfrak{3} \beta$ infer β .¹²

System S2 is obtained by adding the axiom $\diamond(p \wedge q) \text{—}\mathfrak{3} \diamond p$ to the basis for S1. S3 is S1 plus the axiom $(p \text{—}\mathfrak{3} q) \text{—}\mathfrak{3} (\neg\diamond q \text{—}\mathfrak{3} \neg\diamond p)$. S4 is S1 plus $\diamond\diamond p \text{—}\mathfrak{3} \diamond p$, or equivalently $\Box p \text{—}\mathfrak{3} \Box\Box p$. S5 is S1 plus $\diamond p \text{—}\mathfrak{3} \Box\diamond p$.

⁸ [157, p. 523].

⁹ [159, p. 161].

¹⁰ The dual symbol \Box was later devised by F.B. Fitch and first appeared in print in 1946 in a paper of R. Barcan. See footnote 425 of *Hughes and Cresswell* [121].

¹¹ Originally $p \text{—}\mathfrak{3} \neg\neg p$ was included as an axiom, but this was shown to be redundant by McKinsey (1934).

¹² Lewis used the name “Inference” for the rule of strict detachment. He also used “assert” rather than “infer” in these rules.

The axioms for S4 and S5 were first proposed for consideration as further postulates in a paper of Oskar Becker in 1930 [7]. His motivation was to find axioms that reduced the number of logically non-equivalent combinations that could be formed from the connectives “not” and “impossible”. He also considered the formula $p \rightarrow \neg \diamond \neg \diamond p$, and called it the “Brouwersche axiom”. The connection with Brouwer is remote: if “not” is translated to “impossible” ($\neg \diamond$), and “implies” to its strict version, then the intuitionistically acceptable principle $p \rightarrow \neg \neg p$ becomes the Brouwersche axiom.

2.4. Gödel on provability as a modality

Gödel in [83] reviewed Becker’s article [7]. In reference to Becker’s discussion of connections between modal logic and intuitionistic logic he wrote

It seems doubtful, however, that the steps here taken to deal with this problem on a formal plane will lead to success.

He subsequently took up this problem himself with great success, and at the same time simplified the way that modal logics are presented. The Lewis systems contain all truth-functional tautologies as theorems, but it requires an extensive analysis to demonstrate this.¹³ Such effort would be unnecessary if the systems were defined by directly extending a basis for the standard propositional calculus. That approach was first used in the 1933 note “An interpretation of the intuitionistic propositional calculus” [84], published in the proceedings of Karl Menger’s mathematical colloquium at the University of Vienna for 1931–32. Gödel formalised assertions of *provability* by a propositional connective B (from “beweisbar”), reading $B\alpha$ as “ α is provable”. He defined a system which has, in addition to the axioms and rules of ordinary propositional calculus, the axioms

$$\begin{aligned} Bp &\rightarrow p, \\ Bp &\rightarrow (B(p \rightarrow q) \rightarrow Bq), \\ Bp &\rightarrow BBp, \end{aligned}$$

and the inference rule: *from α infer $B\alpha$* . He stated that this system is equivalent to Lewis’ S4 when $B\alpha$ is translated as $\Box\alpha$.¹⁴ Then he gave the following two translations of propositional formulas

$p \mid p$	$p \mid p$
$\neg\alpha \mid \neg B\alpha$	$\neg\alpha \mid B\neg B\alpha$
$\alpha \rightarrow \beta \mid B\alpha \rightarrow B\beta$	$\alpha \rightarrow \beta \mid B\alpha \rightarrow B\beta$
$\alpha \vee \beta \mid B\alpha \vee B\beta$	$\alpha \vee \beta \mid B\alpha \vee B\beta$
$\alpha \wedge \beta \mid \alpha \wedge \beta$	$\alpha \wedge \beta \mid B\alpha \wedge B\beta$

and asserted that in each case the translation of any theorem of Heyting’s intuitionistic propositional calculus¹⁵ is derivable in his system, adding that “presumably” the converse

¹³ See *Hughes and Cresswell* [121, pp. 218–223].

¹⁴ More precisely, he stated that it is equivalent to Lewis’s System of Strict Implication supplemented by Becker’s axiom $\Box p \rightarrow \neg \Box \neg p$. It is unlikely that he was aware of the name “S4” at that time.

¹⁵ Heyting published this calculus in 1930.

is true as well. He also asserted that the translation of $p \vee \neg p$ is not derivable, and that a formula of the form $B\alpha \vee B\beta$ is derivable only when one of $B\alpha$ and $B\beta$ is derivable. Proofs of these claims first appeared in *McKinsey and Tarski* [178] (see Section 3.2).

Those familiar with later developments will recognise the pregnancy of this brief note of scarcely more than a page. Its translations provided an important connection between intuitionistic and modal logic that contributed to the development both of topological interpretations and of Kripke semantics for intuitionistic logic. Its ideas also formed the precursor to the substantial branch of modal logic concerned with the modality “it is provable in Peano arithmetic that”. We will return to these matters below (see Section 3.2, 7.5, 7.6).

It is now standard practice to present modal logics in the axiomatic style of Gödel. The notion of a *logic* refers to any set A of formulas that includes all truth-functional tautologies and is closed under the rules of uniform substitution for variables and detachment for material implication. The formulas belonging to A are the A -*theorems*, and are also said to be A -*provable*. A logic is called *normal* if it includes Gödel’s second axiom, which is usually presented (with \Box in place of B) as

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),$$

and has the rule of *Necessitation*: from α infer $\Box\alpha$. S5 can be defined as the normal logic obtained by adding the axiom $p \rightarrow \Box\Diamond p$ to Gödel’s axiomatisation of S4. Following Becker [7], $p \rightarrow \Box\Diamond p$ is called the *Brouwerian* axiom. The smallest normal logic is commonly called K, in honour of Kripke. The normal logic obtained by adding the first Gödel axiom $\Box p \rightarrow p$ to K is known as T. That system was first defined by Feys¹⁶ in 1937 by dropping Gödel’s third axiom from S4. T is equivalent to the system M of *von Wright 1951* [271]. The *Brouwerian system* B is the normal logic obtained by adding the Brouwerian axiom to T.

The first formulation of the *non-normal* systems S1–S3 in the Gödel style was made in *Lemmon 1957* [153], which also introduced a series of systems E1–E5 designed to be “epistemic” counterparts to S1–S5. These systems have no theorems of the form $\Box\alpha$, and in place of Necessitation they have the rule *from $\alpha \rightarrow \beta$ infer $\Box\alpha \rightarrow \Box\beta$* . Lemmon suggests that they capture the reading of \Box as “it is scientifically but not logically necessary that”.

3. Modal algebras

Modern propositional logic began as algebra, in the thought of Boole. We have seen that the same was true for modern modal logic, in the thought of MacColl. By the time that the Lewis systems appeared, algebra was well-established as a postulational science, and the study of the very notion of *an abstract algebra* was being pursued see *Birkhoff 1933* [13] and *1935* [14]. Over the next few years, algebraic techniques were applied to the study of modal systems, using *modal algebras*: Boolean algebras with an additional operation

¹⁶ Who called it “t”.

to interpret \diamond . During the same period, representation theories for various lattices with operators were developed, beginning with the Stone representation of Boolean algebras (Stone 1936 [239]), and these were to have a significant impact on semantical studies of modal logic.

3.1. McKinsey and the finite model property

McKinsey in a 1941 paper [174] showed that there is an algorithm for deciding whether any given formula is a theorem of S2, and likewise for S4. His method was to show that if a formula is not a theorem of the logic, then it is falsified by some finite model which satisfies the logic. This property was dubbed the *finite model property* in 1958 by Ronald Harrop [109], who proved the general result that any finitely axiomatisable propositional logic Λ with the finite model property is decidable. The gist of Harrop’s argument was that finite axiomatisability guarantees that Λ is effectively enumerable, while the two properties together guarantee the same for the complement of Λ . By enumerating the finite models and the formulas, and at the same time systematically testing formulas for satisfaction by these models, a list can be effectively generated of those formulas that are falsified by some finite model which satisfies the axioms of Λ . By the finite model property this is just a listing of all the non-theorems of Λ .

McKinsey actually showed something stronger: the size of a falsifying model for a non-theorem α is bounded above by a number that depends computably on the size of α . Thus to decide if α is a theorem it suffices to generate all finite models up to a prescribed bound. However this did not yield a *feasible* algorithm: the proof for S2 gave an upper bound of $2^{2^{n+1}}$, doubly exponential in the number n of subformulas of α .

McKinsey’s construction is worth outlining, since it was an important innovation that has been adapted numerous times to other propositional logics (as he suggested it might be), and has been generalised to other contexts, as we shall see. He used models of the form $(K, D, -, *, \cdot)$, called *matrices*, where $-, *, \cdot$ are operations on a set K for evaluating the connectives \neg, \diamond , and \wedge , while D is a set of *designated* elements of K . A formula α is *satisfied* by such a matrix if every assignment of elements of K to the variables of α results in α being evaluated to a member of the subset D . These structures abstract from the tables of values, with designated elements, used to define propositional logics and prove the independence of axioms. Their use as a general method for constructing logical systems is due to Alfred Tarski.¹⁷

A logic is *characterised* by a matrix if the matrix satisfies the theorems of the logic and no other formulas. Structures of this kind had been developed for S2 in 1937 by Huntington [122], who gave the concrete example of K being the class of “propositions” and D the subclass of those that are “asserted” or “demonstrable”, describing this subclass as “corresponding roughly to the Frege assertion sign”.

¹⁷ The historical origins of the “matrix method” are described in *Lukasiewicz and Tarski 1930* [162]. See footnotes on pages 40 and 43 of the English translation of this article in *Tarski 1956* [250].

A matrix is *normal* if

$$\begin{aligned} x, y \in D & \text{ implies } x \cdot y \in D, \\ x, (x \Rightarrow y) \in D & \text{ implies } y \in D, \\ (x \Leftrightarrow y) \in D & \text{ implies } x = y, \end{aligned}$$

where $(x \Rightarrow y) = \neg^*(x \cdot \neg y)$ and $(x \Leftrightarrow y) = (x \Rightarrow y) \cdot (y \Rightarrow x)$ are the operations interpreting strict implication and strict equivalence in K . These closure conditions on D are intended to correspond to Lewis' deduction rules of adjunction, strict detachment, and substitution of strict equivalents. In a normal S2-matrix, $(K, -, \cdot)$ is a Boolean algebra in which D is a filter. Hence the greatest element 1 is always designated. McKinsey showed that there exists an infinite¹⁸ normal matrix that characterises S2, using what he described as an unpublished method due to Lindenbaum that was explained to him by Tarski and which applies to any propositional calculus that has the rule of uniform substitution for variables. Taking $(K, -, *, \cdot)$ as the algebra of formulas, with $\neg\alpha = \neg\alpha$, $*\alpha = \diamond\alpha$ and $\alpha \cdot \beta = \alpha \wedge \beta$, and with D as the set of S2-theorems, gives a characteristic S2-matrix which satisfies all but the last normality condition on D . Since that condition is needed to make the matrix into a Boolean algebra, it is imposed by identifying formulas α, β whenever $(\alpha \Leftrightarrow \beta) \in D$. The resulting quotient matrix is the one desired, and is what is now widely known as the *Lindenbaum algebra* of the logic. Its designated elements are the equivalence classes of the theorems.

Now if α is a formula that not an S2-theorem, then there is some evaluation in this Lindenbaum algebra that fails to satisfy α . Let x_1, \dots, x_n be the values of all the subformulas of α in this evaluation, and let K_1 be the Boolean subalgebra generated by the $n + 1$ elements $x_1, \dots, x_n, *0$. Then K_1 has at most $2^{2^{n+1}}$ members. Define an element of K_1 to be designated iff it was designated in the ambient Lindenbaum algebra. McKinsey showed how to define an operation $*_1$ on K_1 such that $*_1x = *x$ whenever x and $*x$ are both in K_1 :

$$*_1x = \prod \{ *y \in K_1 : x \leq y \in K_1 \}.$$

The upshot was to turn K_1 into a finite S2-matrix in which the original falsifying evaluation of α can be reproduced.

This same construction shows that S4 has the finite model property, with the minor simplification that the element $*0$ does not have to be worried about, since $*0 = 0$ in any normal S4-matrix (so the computable upper bound becomes 2^{2^n}). The Lindenbaum algebra for S4 has only its greatest element designated, i.e., $D = \{1\}$, because $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ is an S4-theorem whenever α and β are, putting all theorems into the same equivalence class. This is a fact that applies to any logic that has the rule of Necessitation, and it allows algebraic models for normal logics to be confined to those that just designate 1.

3.2. Topology for S4

Topological interpretations of modalities were given in a 1938 paper of Tang Tsao-Chen [244], which proposed that “the algebraic postulates for the Lewis calculus of strict

¹⁸ Dugundi in 1940 [53] had proved that none of S1–S5 has a *finite* characteristic matrix.

implication” be the axioms for a Boolean algebra with an additional operation x^∞ having $x^\infty \cdot x = x^\infty$ and $(x \cdot y)^\infty = x^\infty \cdot y^\infty$. The symbol \diamond was used for the dual operation $\diamond x = -(-x)^\infty$. The notation $\vdash x$ was defined to mean that $1^\infty \leq x$, and it was shown that $\vdash x$ holds whenever x is any evaluation of a theorem of S2. In effect this says that putting $D = \{x: 1^\infty \leq x\}$ turns one of these algebras into an S2-matrix. In fact if $1^\infty = 1$, or equivalently $\diamond 0 = 0$, it also satisfies S4. But S4 was not mentioned in this paper.

A “geometric” meaning was proposed for the new operations by taking x^∞ to be the interior of a subset x of the Euclidean plane, in which case $\diamond x$ is the *topological closure* of x , i.e., the smallest closed superset of x . If the greatest element 1 of the algebra is the whole plane, or any open set, then in that case $1^\infty = 1$, but it is evident that Tang did not intend this, since the paper has a footnote explaining that another geometric meaning of x^∞ can be obtained by letting 1^∞ be some subset of the plane, possibly even a one-element subset, and defining x^∞ to be $x \cdot 1^\infty$. (This construction could be carried out in any Boolean algebra by fixing 1^∞ arbitrarily.) It appears then that the best way to understand Tang’s first geometric meaning is that the ambient Boolean algebra should be the powerset algebra $\mathcal{P}(S)$ of all subsets of some subset S of the Euclidean plane, with “interior” and “closure” being taken in the subspace topology on S .

Now a well-known method, due to Kuratowski, for defining a topology on an arbitrary set S is to give a *closure operation* $X \mapsto \mathbf{C}X$ on subsets X of S , i.e., an operation satisfying $\mathbf{C}\emptyset = \emptyset$, $\mathbf{C}(X \cup Y) = \mathbf{C}X \cup \mathbf{C}Y$ and $X \subseteq \mathbf{C}X = \mathbf{C}X$. Then a set X is *closed* iff $\mathbf{C}X = X$, and *open* iff its complement in S is closed. Any topological space can be presented in this way, with $\mathbf{C}X$ being the topological closure of X .

McKinsey and Tarski’s 1944 paper [176] undertook an abstract algebraic study of closure operations by defining a *closure algebra* to be any Boolean algebra with a unary operation \mathbf{C} satisfying Kuratowski’s axioms. The operation $*$ on an S4-matrix satisfies these axioms, and McKinsey had shown in his work [174] on S4 that any *finite* normal S4-matrix can be represented as the closure algebra of all subsets of some topological space, using the representation of a finite Boolean algebra as the powerset algebra of its set of atoms. McKinsey and Tarski now extended this representation to arbitrary closure algebras. Combining the Stone representation of Boolean algebras with the idea of the $*_1$ -operation from McKinsey’s finite model construction they showed that any closure algebra is isomorphic to a subalgebra of the closure algebra of subsets of some topological space. They gave a deep algebraic analysis of the class of closure algebras, including such results as the following.

- (1) The closure algebra of any zero-dimensional dense-in-itself subspace of a Euclidean space (e.g., Cantor’s discontinuum or the space of points with rational coordinates) includes isomorphic copies of all finite closure algebras as subalgebras.
- (2) Every finite closure algebra is isomorphic embeddable into the closure algebra of subsets of some *open* subset of Euclidean space.
- (3) An equation that is satisfied by the closure algebra of any Euclidean space is satisfied by every closure algebra.
- (4) An equation that is satisfied by all finite closure algebras is satisfied by every closure algebra (this is an analogue of McKinsey’s finite model property for S4).

- (5) If an equation of the form $\mathbf{C}\sigma \cdot \mathbf{C}\tau = 0$ is satisfied by all closure algebras, then so is one of the equations $\sigma = 0$ and $\tau = 0$.

The proof of result (5) involved taking the direct product of two closure algebras that each reject one of the equations $\sigma = 0$ and $\tau = 0$, and then embedding this direct product into another closure algebra that is *well-connected*, meaning that if x and y are non-zero elements, then $\mathbf{C}x \cdot \mathbf{C}y \neq 0$. The result itself is equivalent to the assertion that if the equation $\mathbf{l}\sigma + \mathbf{l}\tau = 1$ is satisfied by all closure algebras, then so is one of the equations $\sigma = 1$ and $\tau = 1$, where $\mathbf{l} = \mathbf{-C-}$ is the abstract *interior* operator dual to \mathbf{C} . This is an algebraic version of one of the facts about S4 stated in *Gödel 1933* [84] (see later in this section).

In a sequel article of 1946 [177], McKinsey and Tarski studied the algebra of *closed* (i.e., $\mathbf{C}x = x$) elements of a closure algebra. These form a sublattice with operations $x - y = \mathbf{C}(x \cdot -y)$ and $\ominus x = 1 - x = \mathbf{C-}x$. An axiomatisation of these algebras was given in the form of an equational definition of certain *Brouwerian* algebras of the type $(K, +, \cdot, -, 1)$, and a proof that every Brouwerian algebra is isomorphic to a subalgebra of the Brouwerian algebra of closed sets of some topological space. Results were proven for Brouwerian algebras that are analogous to results (1)–(5) above for closure algebras, with the analogue of (5) being:

- (6) If the equation $\sigma \cdot \tau = 0$ is satisfied by all Brouwerian algebras, then so is one of the equations $\sigma = 0$ and $\tau = 0$.

Brouwerian algebras are so named because they provide models of the intuitionistic propositional calculus IPC. This works in a way that is *dual* to the method that has been described for evaluating modal formulas, in that 0 is the unique designated element; \wedge is interpreted as the lattice sum/join operation $+$; \vee is interpreted as lattice product/meet \cdot ; \rightarrow is interpreted as the operation \div defined by $x \div y = y - x$; and \neg is interpreted as the unary operation $x \div 1 = \ominus x$.

The algebra of *open* (i.e., $\mathbf{l}x = x$) elements of a closure algebra also form a sublattice that is a model of intuitionistic logic. It relates more naturally to the Boolean semantics in that 1 is designated and \wedge and \vee are interpreted as \cdot and $+$. Implication is interpreted by the operation $x \Rightarrow y = \mathbf{l}(-x + y) = \mathbf{-C}(x \cdot -y)$ and negation by $\neg x = x \Rightarrow 0 = \mathbf{l-}x$. This topological interpretation had been developed in the mid-1930's by Tarski [245] and Marshall Stone [240] who independently observed that the lattice $\mathcal{O}(S)$ of open subsets of a topological space S is a model of IPC under the operations just described. Tarski took this further to identify a large class of spaces, including all Euclidean spaces, for which $\mathcal{O}(S)$ exactly characterises IPC.

The abstract algebras $(K, +, \cdot, \Rightarrow, 0)$ that can be isomorphically embedded into ones of the type $\mathcal{O}(S)$ form an equationally defined class. They are commonly known as *Heyting algebras*, or *pseudo-Boolean algebras*. The relationship between Brouwerian and Heyting algebras as models is further clarified by the description of Kripke's semantics for IPC given in Section 7.6.

McKinsey and Tarski applied their work on the algebra of topology to S4 and intuitionistic logic in their 1948 paper [178], which uses closure algebras with just 1 designated to model S4, and Brouwerian algebras in the manner just explained to model Heyting's

calculus. Using various of the results (1)–(4) above, it follows that S4 is characterised by the class of (finite) closure algebras, as well as the closure algebra of any Euclidean space, or of any zero-dimensional dense-in-itself subspace of Euclidean space. Hence in view of result (5), the claim of Gödel 1933 [84] follows: if $\Box\alpha \vee \Box\beta$ is an S4-theorem, then so is one of α and β , therefore so is one of $\Box\alpha$ and $\Box\beta$ by the rule of Necessitation. Similarly, result (6) gives a proof of the *disjunction property* for IPC: if $\alpha \vee \beta$ is a theorem, then so is one of α and β . The final section of the paper uses the relationships between Brouwerian and closure algebras to verify the correctness of the two translations of IPC into S4 conjectured in Gödel’s paper, and introduced a new one:

$$\begin{array}{l|l} p & \Box p \\ \neg\alpha & \Box\neg\alpha \\ \alpha \rightarrow \beta & \Box(\alpha \rightarrow \beta) \quad (\text{i.e., } \alpha \multimap \beta) \\ \alpha \vee \beta & \alpha \vee \beta \\ \alpha \wedge \beta & \alpha \wedge \beta \end{array}$$

It is this translation that inspired Kripke [147] to derive his semantics for intuitionistic logic from his model theory for S4 (see Section 7.6).

Another significant result of the 1948 paper is that S5 is characterised by the class of all closure algebras in which each closed element is also open. Structures of this kind were later dubbed *monadic algebras* by Halmos in his study of the algebraic properties of quantifiers (see Halmos 1962 [102]). The connection is natural: the modalities \Box and \Diamond have the same formal properties in S5 as do the quantifiers \forall and \exists in classical logic. The *polyadic* algebras of Halmos and the *cylindric* algebras of Tarski and his co-researchers (see Henkin, Monk and Tarski 1971 [112]) have a family of pairwise commuting closure operators for which each closed element is open.

Any Boolean algebra can be made into a monadic algebra by defining $\mathbf{C}0 = 0$ and otherwise $\mathbf{C}x = 1$. These are the *simple*¹⁹ monadic algebras. Let \mathfrak{A}_n be the simple monadic algebra defined on the finite Boolean algebra with n atoms, viewed as a matrix with only 1 designated. Then S5 is characterised by the set of all these \mathfrak{A}_n ’s. This was shown by Schiller Joe Scroggs in [219] (1951), written as a Masters thesis under McKinsey’s direction, whose analysis established that every finite monadic algebra is a direct product of \mathfrak{A}_n ’s. Scroggs used this to prove that each proper extension of S5 is equal to the logic characterised by some \mathfrak{A}_n , and so has a finite characteristic matrix. By “extension” here is meant any logic that includes all S5-theorems and is closed under the rules of uniform substitution for variables and detachment for material implication. Scroggs was able to show from this characterisation that any such extension of S5 is closed under the Necessitation rule as well, and so is a normal logic.

Another notable paper on S5 algebras from this era is Davis 1954 [48], based on a 1950 doctoral thesis supervised by Garrett Birkhoff. This describes the correspondence between equivalence relations on a set and S5 operations on its powerset Boolean algebra; a correspondence between algebras with two S5 operations and the *projective* algebras of Everett and Ulam 1946 [63]; and the use of several S5 operators to provide a Boolean model of features of first-order logic.

¹⁹ In the technical algebraic sense of having no non-trivial congruences.

3.3. BAOs: The theory of Jónsson and Tarski

The notion of a *Boolean algebra with operators* (BAO) was introduced by Jónsson and Tarski in the 1948 abstract [127], with the details of their announced results being presented in 1951 [128]. That work contains representations of algebras that could immediately have been applied to give new characterisations of modal systems. But the paper was overlooked by modal logicians, who were still publishing re-discoveries of some of its results fifteen years later.

A unary function f on a Boolean algebra is an *operator* if it is additive, i.e., $f(x + y) = f(x) + f(y)$. f is *completely additive* if $f(\sum X) = \sum f(X)$ whenever $\sum X$ exists, and is *normal* if $f(0) = 0$. A function of more than one argument is an operator/is completely additive/is normal when it has the corresponding property separately in each argument. A BAO is an algebra $\mathfrak{A} = (\mathfrak{B}, f_i: i \in I)$, where the f_i 's are all operators on the Boolean algebra \mathfrak{B} .

The *Extension Theorem* of Jónsson and Tarski showed that any BAO \mathfrak{A} can be embedded isomorphically into a complete and atomic BAO \mathfrak{A}^σ which they called a *perfect extension* of \mathfrak{A} . The construction built on Stone's embedding of a Boolean algebra \mathfrak{B} into a complete and atomic one \mathfrak{B}^σ , with each operator f_i of \mathfrak{A} being extended to an operator f_i^σ on \mathfrak{B}^σ that is *completely* additive, and is normal if f_i is normal. The notion of perfect extension was defined by three properties that determine \mathfrak{A}^σ uniquely up to a unique isomorphism over \mathfrak{A} and give an algebraic characterisation of the structures that arise from Stone's topological representation theory. These properties can be stated as follows.

- (i) For any distinct atoms x, y of \mathfrak{A}^σ there exists an element a of \mathfrak{A} with $x \leq a$ and $y \leq -a$.
- (ii) If a subset X of \mathfrak{A} has $\sum X = 1$ in \mathfrak{A}^σ , then some finite subset X_0 of X has $\sum X_0 = 1$.
- (iii) $f_i^\sigma(x) = \prod \{f_i(y): x \leq y \in \mathfrak{A}^\sigma\}$ when f_i is n -ary and the terms of the n -tuple x are atoms or 0.

Property (i) corresponds to the Hausdorff separation property of the Stone space of \mathfrak{B} , while (ii) is an algebraic formulation of the compactness of that space. The meaning of (iii) will be explained below.

Jónsson and Tarski showed that any equation satisfied by \mathfrak{A} will also be satisfied by \mathfrak{A}^σ if it does not involve Boolean complementation (i.e., refers only to $+$, \cdot , 0 , 1 and the operators f_i). More generally, perfect extensions were shown to preserve any implication of the form $(t = 0 \rightarrow u = v)$ whose terms t, u, v do not involve complementation. They then established a fundamental representation of normal n -ary operators in terms of $n + 1$ -ary relations. This was based on a bijective correspondence between normal *completely* additive n -ary operators f on a powerset Boolean algebra $\mathcal{P}(S)$ and $n + 1$ -ary relations $R_f \subseteq S^{n+1}$. Here

$$R_f(x_0, \dots, x_{n-1}, y) \quad \text{iff} \quad y \in f(\{x_0\}, \dots, \{x_{n-1}\}).$$

Under this bijection an arbitrary $R \subseteq S^{n+1}$ corresponds to the n -ary operator f_R on $\mathcal{P}(S)$, where

$$y \in f_R(X_0, \dots, X_{n-1}) \quad \text{iff} \quad R(x_0, \dots, x_{n-1}, y) \text{ for some elements } x_i \in X_i.$$

Thus any relational structure $\mathfrak{S} = (S, R_i: i \in I)$ whatsoever gives rise to the complete atomic BAO

$$\text{Cm } \mathfrak{S} = (\mathcal{P}(S), f_{R_i}: i \in I)$$

of all subsets of S with the completely additive normal operators f_{R_i} . Conversely, any complete and atomic BAO whose operators are normal and completely additive was shown to be isomorphic to $\text{Cm } \mathfrak{S}$ for some structure \mathfrak{S} [128, Theorem 3.9]. This representation is relevant to an understanding of the *incompleteness* phenomenon to be discussed later in Section 6.1. When applied to the perfect extension \mathfrak{A}^σ of a BAO \mathfrak{A} , it can be seen as defining a relational structure on the Stone space of \mathfrak{A} . This is now known as the *canonical structure* of \mathfrak{A} , denoted $\text{Cst } \mathfrak{A}$, and its role will be explained further in Section 6.5. The above property (iii) expresses the fact that in $\text{Cst } \mathfrak{A}$, if R is the relation corresponding to some n -ary operator f_i^σ , then for each point y the set

$$\{(x_0, \dots, x_{n-1}): R(x_0, \dots, x_{n-1}, y)\}$$

is closed in the n -fold product of the Stone space topology.

$\text{Cm } \mathfrak{S}$ is the *complex algebra* of \mathfrak{S} , and any subalgebra of $\text{Cm } \mathfrak{S}$ is a *complex algebra*. This terminology derives from an old usage of the word “complex” introduced into group theory by Frobenius in the (pre-set-theoretic) 1880’s to mean a collection of elements in a group. The *binary product*

$$HK = \{hk: h \in H \text{ and } k \in K\}$$

of subsets (complexes) H, K of a group G is precisely the operator f_R on $\mathcal{P}(G)$ corresponding to the *ternary graph*

$$R = \{(h, k, hk): h, k \in G\}$$

of the group operation.

Combining the Extension Theorem with the representation of a complete atomic algebra (like \mathfrak{A}^σ) as one of the form $\text{Cm } \mathfrak{S}$, Jónsson and Tarski established that

every BAO with normal operators is isomorphic to a subalgebra of the complex algebra of a relational structure.

The case $n = 1$ of this analysis of operators is highly germane to modal logic: the algebraic semantics discussed so far has been based on interpreting \diamond as an operator on a Boolean algebra, and a normal one in the case of S4 and S5. Jónsson and Tarski observed that basic properties of a binary relation $R \subseteq S^2$ correspond to simple equational properties of the operator f_R . Thus R is reflexive iff the BAO $(\mathcal{P}(S), f_R)$ satisfies $x \leq f x$, and transitive iff it satisfies $f f x \leq x$. Hence $\text{Cm}(S, R)$ is a closure algebra iff R is reflexive and transitive, i.e., a *quasi-ordering*. Since these conditions $x \leq f x$ and $f f x \leq x$ are preserved by perfect extensions, it followed [128, Theorem 3.14] that

every closure algebra is isomorphic to a subalgebra of the complex algebra of a quasi-ordered set.

This result, along with the Extension Theorem and the representation of a normal BAO as a complex algebra, were all stated in the 1948 abstract [127].

A number of other properties of R were discussed in [128], including *symmetry*. This was shown to be characterised by *self-conjugacy* of f_R , meaning that $\text{Cm}(S, R)$ satisfies the condition $f(x) \cdot y = 0$ iff $x \cdot f(y) = 0$, which can be expressed equationally, for example by $f0 = 0$ and $fx \cdot y \leq f(x \cdot fy)$. The characterisation was used to give a representation of certain two-dimensional cylindric algebras as complex algebras over a pair of equivalence relations. Self-conjugacy of an operator is also equivalent to the equation $x \cdot f - fx = 0$, corresponding to the Brouwerian modal axiom $p \rightarrow \Box \Diamond p$. In closure algebras this is equivalent to every closed element being open: a self-conjugate closure algebra is the same thing as a monadic algebra.

As already mentioned, this study of BAOs was later overlooked. *Dummett and Lemmon 1959* [54] makes extensive use of complex algebras over quasi-orderings in studying extensions of S4, but makes no mention of the Jónsson–Tarski article, taking its lead instead from the McKinsey–Tarski papers and a construction in *Birkhoff 1948* [16] that gives a correspondence between partial orderings (i.e., antisymmetric quasi-orderings) and closure operations of certain topologies on a set. The same omission occurs in [155], which re-proves the representation of a unary operator on a Boolean algebra as a complex algebra over a binary relation, although it does extend the result by allowing the operator to be non-normal (see Section 5.1).

3.4. Could Tarski have invented Kripke semantics?

A question like this can only remain a matter of speculation. But it is not just idle speculation, given that Tarski had worked on modal logic during the same period, and given his pioneering role in the development of model theory, including the formalisation of the notions of truth and satisfaction in relational structures.

The Jónsson–Tarski work on closure algebras applies immediately to the McKinsey–Tarski results on modal logic to show that S4 is characterised by the class of complex algebras of quasi-orderings. It can also be applied to show that S5 is characterised by the class of complex algebras of equivalence relations. Now the complex algebra of an equivalence relation R is a subdirect product of the complex algebras of the equivalence classes of R , each of which is a set on which R is universal. Moreover, the complex algebra of a universal relation is a *simple* monadic algebra. These observations could have been used to give a more accessible approach to the structural analysis of S5-algebras that appears in *Scroggs 1951* [219].

But the Jónsson–Tarski paper makes no mention of modal logic at all. Jónsson [125] has explained that their theory evolved from Tarski’s research on the algebra of binary relations, beginning with the finite axiom system in *Tarski 1941* [246] which was designed to formalise the calculus of binary relations that had been developed in the nineteenth century by De Morgan, Peirce and Schröder. The primitive notions of that paper were those of Boolean algebra together with the binary operation $R_1; R_2$ of relational composition, the unary operation R^\sim of inversion, and the distinguished constant $1'$ for the identity relation. Tarski asked whether any model of his axiom was representable as an algebra of actual binary relations. He later gave an equational definition of a *relation algebra* as an abstract

BAO $(\mathfrak{B}, ;, \sim, 1')$ that forms an involuted monoid under $;$, \sim , $1'$ and satisfies the condition $x \sim; -(x; y) \leq -y$. Concrete examples include the set $\mathcal{P}(S \times S)$ of all binary relations on a set S and, more generally, the set $\mathcal{P}(E)$ of subrelations of an equivalence relation E on S . Any algebra isomorphic to a subalgebra of the normal BAO $(\mathcal{P}(E), ;, \sim, 1')$ is called *representable*, and Tarski's representation question became the problem of whether every abstract relation algebra is representable in this sense.²⁰

Late in 1946 Tarski communicated to Jónsson a proof that every relation algebra is embeddable in a complete and atomic one. That construction became the prototype for the Jónsson–Tarski Extension Theorem for BAOs (see *Jónsson 1993* [125, Section 1.2]). The second part of their joint work [129] is entirely devoted to relation algebras and their representations.

It appears then that in developing his ideas on BAOs Tarski was coming from a different direction: modal logic was not on the agenda. According to *Copeland* [39, p. 13], Tarski told Kripke in 1962 that he was unable to see a connection with what Kripke was then doing.

4. Relational semantics

Leibniz had a good deal to say about possible worlds, including that the actual world is the best of all of them. Apparently he never *literally* described necessary truths as being “true in all possible worlds”, but he did say of them that

Not only will they hold as long as the world exists, but also they would have held if God had created the world according to a different plan.

He defined a truth as being necessary when its opposite implies a contradiction, and also said that there are as many worlds as there are things that can be conceived without contradiction (see *Mates 1986* [172, pp. 72–73, 106–107]).

This way of speaking has provided the motivation and intuitive explanation for a mathematical semantics of modality using relational structures that are now often called *Kripke models*. A formula is assigned a truth-value relative to each point of a model, and these points are thought of as being possible worlds or states of affairs.

An account will now be given of the contribution of Saul Kripke, followed by a survey of some of its “anticipations”.

4.1. Kripke's relatively possible worlds

Kripke's first paper in 1959 [142] on modal logic gave a semantics for a quantificational version of S5 that included propositional variables as the case $n = 0$ of n -ary predicate variables. A *complete assignment* for a formula α in a non-empty set D was defined to be

²⁰ This was answered negatively by Lyndon (1950). Work of Tarski, Monk and Jónsson eventually showed that the representable relation algebras form an equational class that is not finitely axiomatisable, with any equational definition of it requiring infinitely many variables.

any function that assigns an element of D to each free individual variable in α , a subset of D^n to each n -ary predicate variable occurring in α , and a truth-value (\top or \perp) to each propositional variable of α . A *model* of α in D is a pair (G, K) , where K is a set of complete assignments that all agree on their treatment of the free individual variables of α , and G is an element of K . Each member H of K assigns a truth value to each subformula of α , by induction on the rules of formation for formulas. The truth-functional connectives and the quantifiers \forall, \exists behave as in standard predicate logic, and the key clause for modality is that

H assigns \top to $\Box\beta$ iff every member of K assigns \top to β .

A formula α is *true*²¹ in a model (G, K) over D iff it is assigned \top by G ; *valid* over D iff true in all of its models in D ; and *universally valid* iff valid in all non-empty sets D .

An axiomatisation of the class of universally valid formulas was given, with the completeness proof employing the method of *semantic tableaux* introduced in *Beth 1955* [12]. It was then observed that for purely propositional logic this could be turned into a truth table semantics. A complete assignment becomes just an assignment of truth values to the variables in α , i.e., a row of a truth table, and a model (G, K) is just a classical truth table with some (but not all) of the rows omitted and G some designated row. Formula $\Box\beta$ is assigned \top in every row if β is assigned \top in every row of the table; otherwise it is assigned \perp in every row. The resulting notion of “S5-tautology” precisely characterises the theorems of propositional S5, a result that Kripke had in fact obtained first, before, as he explained in footnote 4,

acquaintance with Beth’s paper led me to generalize the truth tables to semantic tableaux and a completeness theorem.

Kripke’s informal motivation for these models was that the assignment G represents the “real” or “actual” world, and the other members of K represent worlds that are “conceivable but not actual”. Thus $\Box\beta$ is “evaluated as true when and only when β holds in all conceivable worlds”. The lack of any further structure on K reflects the assumption that “any combination of possible worlds may be associated with the real world”.

The abstract [143] appearing later in 1959 announced the availability of “appropriate model theory” and completeness theorems for a raft of modal systems, including S2–S5, the Feys/von Wright system T (or M), Lemmon’s E-systems, systems with the Brouwerian axiom, deontic systems, and others. Various extensions to quantificational logic with identity were described, and it was stated that “the methods for S4 yields a semantical apparatus for Heyting’s system which simplifies that of Beth”. The details of this programme appeared in the papers of 1963 [145,146] and 1965 [147,148].

The normal propositional logics S4, S5, T and B are the main focus of [145], which defines a *normal model structure* as a triple (G, K, R) with $G \in K$ and R a *reflexive* binary relation on K . A *model* for a propositional formula α on this structure is a function $\Phi(p, H)$ taking values in $\{\top, \perp\}$, with p ranging over variables in α and H ranging over

²¹ Actually “*valid* in a model” was used here, but changed to “*true*” in the 1963 paper [145].

K . This is extended to assign a truth value $\Phi(\beta, H)$ to each subformula β of α and each $H \in K$, with

$$\Phi(\Box\beta, H) = \top \quad \text{iff} \quad \Phi(\beta, H') = \top \text{ for all } H' \in K \text{ such that } HRH'.$$

α is *true* in the model if $\Phi(\alpha, G) = \top$.

In addition to the introduction of the relation R , the other crucial conceptual advance here is that the set K of “possible worlds” is no longer a collection of value assignments, but is permitted to be an arbitrary set. This allows that there can be different worlds that assign the same truth values to atomic formulas. As to the relation R , Kripke’s intuitive explanation is as follows [145, p. 70]:

we read “ H_1RH_2 ” as H_2 is “possible relative to H_1 ”, “possible in H_1 ” or “related to H_1 ”; that is to say, every proposition true in H_2 is to be possible in H_1 . Thus the “absolute” notion of possible world in [142] (where every world was possible relative to every other) gives way to relative notion, of one world being possible relative to another. It is clear that every world H is possible relative to itself; for this simply says that every proposition *true* in H is *possible* in H . In accordance with this modified view of “possible worlds” we evaluate a formula A as *necessary* in a world H_1 if it is *true* in every world possible relative to $H_1 \dots$ Dually, A is *possible* in H_1 iff there exists H_2 , possible relative to H_1 , in which A is *true*.

Semantic tableaux methods are again used to prove completeness theorems: a formula is true in all models iff it is a theorem of T; true in all transitive models iff it is an S4-theorem, true in all symmetric models iff a B-theorem, and true in all transitive and symmetric models iff an S5-theorem. The arguments also give decision procedures, and show that attention can be restricted to models that are *connected* in the sense that each $H \in K$ has GR^*H , where R^* is the *ancestral* or reflexive-transitive closure of R . Kripke notes that

in a connected model in which R is an equivalence relation, *any* two worlds are related. This accounts for the adequacy, for S5, of the model theory of [142].

An illustration of the tractability of the new model theory is given by a new proof of the deduction rule in S4 that if $\Box\alpha \vee \Box\beta$ is deducible then so is one of α and β . If neither α nor β is derivable then each has a falsifying S4-model. Take the disjoint union of these two models and add a new “real” world that is R -related to everything. The result is an S4-model falsifying $\Box\alpha \vee \Box\beta$. This argument is much easier to follow than the McKinsey–Tarski construction involving well-connected algebras described in Section 3.2, and it adapts readily to other systems.

Other topics discussed include the presentation of models in “tree-like” form, and the association with each model structure of a matrix, essentially the modal algebra of all functions $\rho: K \rightarrow \{\top, \perp\}$, which are called *propositions*, with the ones having $\rho(G) = \top$ being designated. A model can then be viewed as a device for associating a proposition $H \mapsto \Phi(p, H)$ to each propositional variable p . The final section of the paper raises the possibility of defining new systems by imposing various requirements on R , and concludes that

If we were to drop the condition that R be reflexive, this would be equivalent to abandoning the modal axiom $\Box A \rightarrow A$. In this way we could obtain systems of the type required for deontic logic.

Non-normal logics are the subject of [148], which focuses mainly on Lewis's S2 and S3 and the corresponding systems E2 and E3 of *Lemmon 1957* [153]. The E-systems have no theorems of the form $\Box\alpha$, and this suggests to Kripke the idea of allowing worlds in which any formula beginning with \Box is false, and hence any beginning with \Diamond , even $\Diamond(p \wedge \neg p)$, is true. A model structure now becomes a quadruple (G, K, R, N) with N a subset of K , to be thought of as a set of *normal* worlds, and R a binary relation on K as before, but now required to be reflexive on N only. The semantic clause for \Box in a model on such a structure is modified by stipulating that

$$\Phi(\Box\beta, H) = \top \text{ iff } H \text{ is normal, i.e., } H \in N, \text{ and } \Phi(\beta, H') = \top \text{ for all } H' \in K \text{ such that } HRH';$$

and hence

$$\Phi(\Diamond\beta, H) = \top \text{ iff } H \text{ is non-normal or else } \Phi(\beta, H') = \top \text{ for some } H' \in K \text{ such that } HRH'.$$

This has the desired effect of ensuring $\Phi(\Box\beta, H) = \perp$ and $\Phi(\Diamond\beta, H) = \top$ whenever H is non-normal. Thus in a non-normal world, even a contradiction is *possible*.

These models characterise E2, and the ones in which R is transitive characterise E3. Requiring that the “real” world G belongs to N gives models that characterise S2 and S3 in each case.²² A number of other systems are discussed and applications given, including a proof of a long-standing conjecture that the Feys-von Wright system has no finite axiomatisation with detachment as its sole rule of inference.

Kripke's semantics for quantificational modal logic is presented in [146]. A model structure now has the added feature of a function assigning a set $\psi(H)$ to each $H \in K$. Intuitively, $\psi(H)$ is the set of all individuals existing in H , and it provides the range of values for a variable x when a formula beginning with $\forall x$ is evaluated at H . A model now assigns to each n -ary predicate letter and each $H \in K$ an n -ary relation on the set $\bigcup\{\psi(H') : H' \in K\}$ of individuals that exist in any world. Axioms are given for quantificational versions of the basic modal logics and it is stated that the completeness theorems of [145] can be extended to them. An indication of how that would work can be obtained from Kripke's [147], which gives a tableaux completeness proof for his semantics for Heyting's intuitionistic predicate calculus.

²² A semantics for S1 was devised in 1969 by Max Cresswell, modifying Kripke's S2-models to allow some formulas $\Diamond\beta$ to be false in a non-normal world under certain restrictions, defined with the help of a *neighbourhood* relation $R' \subseteq K \times \mathcal{P}(K)$. See [41,44].

4.2. So who invented relational models?

Kripke’s abstract [143] notes that “for systems based on S4, S5, and M, similar work has been done independently and at an earlier date by K. J. J. Hintikka”. This acknowledgement is repeated in footnote 2 of [145], where he draws attention to prior work by a number of researchers, including Bayart, Jónsson and Tarski, and Kanger, explaining that his own work was done independently of all of them. He states that the 1957 modelling of Kanger [132] “though more complex, is similar to that in the present paper”, and also records that he discovered the Jónsson–Tarski paper when his own was almost finished.

Key ideas surrounding relational interpretations of modality had occurred to several people. In the next few sections we survey some of this background, before expressing a view about the relative significance of Kripke’s work.

As mathematics progresses, notions that were obscure and perplexing become clear and straightforward, sometimes even achieving the status of “obvious”. Then hindsight can make us all wise after the event. But we are separated from the past by our knowledge of the present, which may draw us into “seeing” more than was really there at the time. This should be borne in mind in reading what follows.

4.3. Carnap and Bayart on S5

A *state-description* is defined by Rudolf Carnap in a 1946 paper [32] and a 1947 book [33] to be set of sentences which consists of exactly one of α and $\neg\alpha$ for each *atomic* α . State-descriptions are said to “represent Leibniz’s possible worlds or Wittgenstein’s possible states of affairs”. A sentence is called *L-true* if it holds in every state-description, this being “an explicatum for what Leibniz called necessary truth and Kant analytic truth” [33, p. 8].

Of course it needs to be explained what it is to hold in a state-description. An atomic sentence holds in a state description iff it belongs to it, the conditions for the connectives \neg , \wedge , and \vee are as expected, and the criterion for Carnap’s necessity connective N is that

$N\alpha$ holds in every state-description if α holds in every state-description; otherwise, $N\alpha$ holds in no state-description

[32, D9-5i]; [33, 41-1]. His list of L-truths [32, p. 42], [33, p. 186] includes the axioms for S5, and he also notes the similarity between N and \forall , and between \diamond and \exists under this semantics. The 1946 paper observes that there is a procedure for deciding L-truth that is “*theoretically* effective”: if a sentence α has n atomic components then there are 2^n state-descriptions that have to be considered in evaluating it, and therefore 2^{2^n} possibilities for the *range* of α , which is the set of state-descriptions in which α holds. We can examine all possibilities to see if the range includes all state-descriptions. Carnap defines a version of S5 which he calls MPC and proves that it is complete with respect to his semantics, by a reduction of formulas to a normal form²³ which also gives a decision procedure that is

²³ Called *modal conjunctive normal form* in Hughes and Cresswell [121], where a variant of the proof is given on p. 116.

practicable, i.e., sufficiently short for modal sentences of ordinary length.

He attributes the completeness result to a paper of Mordchaj Wajsberg from 1933. Footnote 8 of [32] gives a description of Wajsberg’s system and also contains the information that Carnap constructed MPC independently in 1940 and later found that it was equivalent to Lewis’s S5.

A contribution to possible worlds model theory that has been largely overlooked is the work of the Belgian logician A. Bayart, whose papers of 1958 [5] and 1959 [6] gave a semantics for a version of second order quantificational S5, and a complete axiomatisation of it using a Gentzen-style sequent calculus. The models used allow a restricted range of interpretation of predicate variables. This idea had been introduced in *Henkin 1950* [111] to give a completeness result for non-modal higher order logic, and Bayart commented [6, p. 100] that he had just adapted Henkin’s theorem to S5.²⁴ The other source of motivation he gives [5, p. 28] is Leibniz’s definition of necessity as truth in all possible worlds,²⁵ and his bibliography cites the items [32,33] of Carnap.

In Bayart’s theory a *universe* U is defined to be a disjoint pair A, B of sets, with members of A called *individuals* and members of B called *worlds* (“mondes”). An n -place *intensional predicate* is a function of $n + 1$ arguments, taking the values “true” or “false”, having a world as its first argument, and having individuals as the remaining arguments when $n \neq 0$. A *value system* relative to U is a function S assigning a member of A to each individual variable, and an n -place intensional predicate to each n -place predicate variable. The notion of a formula being true or false for the universe U , the world M and the value system S —or more briefly for UMS —is defined in the expected way for the non-modal connectives and quantifiers, including quantifiers binding predicate variables. For modalized formulas Lp and Mp it is declared that

$$Lp \text{ is true for } UMS \quad \text{iff} \quad \text{for every world } M' \text{ of } U, p \text{ is true for } UM'S;$$

$$Mp \text{ is true for } UMS \quad \text{iff} \quad \text{for some world } M' \text{ of } U, p \text{ is true for } UM'S.$$

A formula is *valid* in the universe U if it is true for UMS for every world M and value system S of U .

Bayart used the notation \ddot{a}, I, \ddot{e} for a Gentzen sequent, with \ddot{a} (the antecedent) and \ddot{e} (the consequent) being finite sequences of formulas, and I a separating symbol. The sequent is true in UMS if some member of \ddot{a} is false or else some member of \ddot{e} is true. He adopted the axiom schema \ddot{p}, I, \ddot{p} and a system of twenty-five deduction rules, showing in [5] that all deducible sequents are valid in all universes. There are four modal rules, allowing the introduction of the modalities L and M into antecedents and consequents:

$$\frac{p, \ddot{a}, I, \ddot{e}}{Lp, \ddot{a}, I, \ddot{e}} \quad \frac{p, \ddot{a}, I, \ddot{e}}{Mp, \ddot{a}, I, \ddot{e}} \quad \frac{\ddot{a}, I, \ddot{e}, p}{\ddot{a}, I, \ddot{e}, Lp} \quad \frac{\ddot{a}, I, \ddot{e}, p}{\ddot{a}, I, \ddot{e}, Mp}$$

The last two rules are subject to the restriction that any formula appearing in \ddot{a} or \ddot{e} must be “couverte”, meaning that it is formed from formulas of the types Lq and Mq using

²⁴ “En réalité notre exposé n’est qu’une adaptation du théorème de Henkin à la logique modale S5.”

²⁵ “... en nous inspirant de la définition Leibnizienne du nécessaire, comme étant ce qui est vrai dans tous les mondes possibles.”

only the non-modal connectives and quantifiers. Such a formula has the same truth value in UMS and $UM'S$ for all worlds M, M' .

The second paper proved the completeness of this sequent system for validity in certain *quasi-universes* obtained by allowing predicate variables to take values in a restricted class of intensional predicates. From this it was shown that the first order fragment of the system is complete for validity in all universes. The method used was subsequently generalised in *Cresswell 1967* [40] to obtain a completeness theorem for the relational semantics of a first order version of the modal logic T (see Section 5.1).

It is worth recording Bayart's explanation of why the set of worlds of a universe $U = A, B$ is essential to this theory. He considered the possibility of dispensing with B , requiring a value system S to interpret an n -place predicate variable as an *extensional* predicate (i.e., a truth-valued function on A^n), and modelling the necessity modality by declaring that

$$Lp \text{ is true of } US \quad \text{iff} \quad p \text{ is true of } US' \text{ for every value system } S'.$$

He noted that this interpretation fails to validate the formula

$$\exists y L(bx \vee \neg by)$$

(where b is a unary predicate variable), a formula that is valid according to the above semantics. His explanation of the flaw in this alternative approach is that it gives Lp the same meaning as the universal closure of p (i.e., $\forall v_1 \dots \forall v_n p$, where v_1, \dots, v_n are the free variables of p), and confuses necessity with validity.

4.4. Meredith, Prior and Geach

Arthur Prior wrote in 1967 [209, p. 42] that

In some notes made in 1956, C.A. Meredith related modal logic to what he called the 'property calculus'.

This material was made available by Prior as a one-page departmental mimeograph [179] which was published much later in the collection [38]. Its basic idea was to express modal formulas in the first-order language of a binary predicate symbol U , beginning with the following definitions, in which L and M are connectives for necessity and possibility (but the other notation is that of this paper rather than the original Polish):

$$\begin{aligned} (\neg p)a &= \neg(pa), \\ (p \rightarrow q)a &= (pa) \rightarrow (qa), \\ (Lp)a &= \forall b(Uab \rightarrow pb), \\ (Mp)a &= (\neg L\neg p)a = \exists b(Uab \wedge pb). \end{aligned}$$

Possible axioms for U are then listed:

1. $Uab \vee Uba$,
2. $Uab \rightarrow (Ubc \rightarrow Uac)$,
3. $Uab \rightarrow (Ucb \rightarrow Uac)$,

4. Uaa ,
5. $Uab \rightarrow Uba$,

and it is noted that “1 gives 4”; “3, 4 give 5”; and “3, 5 give 2”. The notes are written in this telegraphic style with no interpretation of the symbolism, but presumably “ pa ” may be read “ a has property p ”.

It is stated that quantification theory alone allows the derivation of

$$(L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq))a,$$

and then formal deductions are given of $(Lp \rightarrow p)a$ using 4; of $(Lp \rightarrow LLP)a$ using 2; of $(MLp \rightarrow Lp)a$ using 2 and 5; and of $\forall apa$ from $(Lp)a$ using 1 and 5. The conclusion is as follows:

Thus 1, or 4, gives T; 1, 2 or 4, 2 gives S4; 1, 3 or 4, 3 gives S5; and 1, 3 (but not 4, 3) gives the equivalence of the above $(Lp)a$ with the usual S5 $(Lp)a$, i.e., $\forall apa$.

Prior’s 1962 article “Possible Worlds” [207] gives a fuller exposition of this *U-calculus*, saying (p. 37) “This whole symbolism I owe to C.A. Meredith”. He applies an interpretation of the predicate U , suggested to him by P.T. Geach in 1960,²⁶ as a relation of *accessibility*. Here is Prior’s account of that interpretation.

Suppose we define a ‘possible’ state of affairs or world as one which can be reached from the world we are actually in. What is meant by reaching or travelling to one world from another need not here be amplified; we might reach one world from another merely in thought, or we might reach it more concretely in some dimension-jumping vehicle dreamed up by science-fiction (the case originally put by Geach), or we might reach it simply by the passage of time (one important sense of ‘possible state of affairs’ is ‘possible outcome of the present state of affairs’). What I want to amplify here is the idea (the core of Geach’s suggestion) that we may obtain different modal systems, different versions of the logic of necessity and possibility, by making different assumptions about ‘world-jumping’.

Prior was the founder of tense logic (also known as temporal logic). He wanted to analyse the arguments of the Stoic logician Diodorus Chronos, who had defined a proposition to be *possible* if it either is true or will be true. Prior conceived the idea of using a logical system with temporal operators analogous to those of modal logic, and thus introduced the connectives

- F it will be the case that,
- P it has been the case that,
- G it will always be the case that,
- H it has always been the case that.

Here F and P are “diamond” type modalities, with duals G and H respectively. In the 1958 paper “The Syntax of Time-Distinctions” [206] a propositional logic called the *PF*-

²⁶ This date is given in [208, p. 140], where the acknowledgement of Meredith is repeated once more.

calculus is defined.²⁷ It is a normal logic with respect to G and H , has the axioms $Gp \rightarrow Fp$, $FFp \rightarrow Fp$ and $Fp \rightarrow FFp$, as well as an “interaction” axiom $p \rightarrow GPP$ and a *Rule of Analogy* allowing that from any theorem another may be deduced by replacing F by P and vice versa.

This system is then interpreted into what Prior calls the *l-calculus*, a first-order language whose variables x, y, z range over *dates*, and which has a binary symbol l taking dates as arguments, with the expression lxy being read “ x is later than y ”.²⁸ Variables p, q, r stand for propositions considered as functions of dates, with the expression px being read “ p at x ”. The following interpretations are given of propositional formulas, using an arbitrarily chosen date variable z to represent “the date at which the proposition under consideration is uttered”.

$$\begin{aligned} Fp & \exists x(lxz \wedge px), \\ Pp & \exists x(lzx \wedge px), \\ Gp & \forall x(lxz \rightarrow px), \\ Hp & \forall x(lzx \rightarrow px). \end{aligned}$$

Prior observes that the interpretations of some theorems of the *PF-calculus* are provable in the *l-calculus* just from the usual axioms and rules for quantificational logic. This applies to any *PF*-theorem derivable from the basis for normal logics together with the interaction axiom $p \rightarrow GPP$ and the rule of Analogy. He then states that the interpretation of $Gp \rightarrow Fp$ requires for its proof the axiom $\exists x lxz$ (“infinite extent of the future”), and that $FFp \rightarrow Fp$ depends similarly on *transitivity*: $lxy \rightarrow (lyz \rightarrow lxz)$, while $Fp \rightarrow FFp$ depends on the density condition $lxz \rightarrow \exists y(lxy \wedge lyz)$.

The modality M of possibility is given a temporal reading by defining Mp to be an abbreviation for $p \vee Fp \vee Pp$, i.e., “ p is true at some time, past present or future”. This makes the dual Lp equivalent to $p \wedge Gp \wedge Fp$, “at all times, p ”. Prior notes that to derive the S5-principle $M\neg Mp \rightarrow \neg Mp$, which is “clearly a law” under this interpretation of M , requires *trichotomy*: $x = y \vee lxy \vee lyx$. His explorations here are quite tentative. For instance he defines *asymmetry*: $lxy \rightarrow \neg lyx$, but makes no use of it, and he fails to note that the S4-principle $MMp \rightarrow Mp$ also depends on trichotomy and not just transitivity.

Why did Prior give such unequivocal credit to Meredith for the 1956 *U-calculus*? The puzzle about this is that his paper on the *l-calculus*, although published in 1958, was presented much earlier, on 27 August 1954, as his Presidential Address to the New Zealand Philosophy Congress at the Victoria University of Wellington. Perhaps he was crediting Meredith with the extension of the symbolism to *modal* logic as he understood it, i.e., the logic of necessity and possibility, as distinct from tense logic. The *l-calculus* was intended to describe a very specific situation: an ordered system of dates or moments in time that forms an “infinite and continuous linear series” [206, p. 115]. In the absence of any corresponding interpretation of the *U*-predicate, the purely formal application of the symbolism by Meredith may have been seen by Prior as a significant advance.

Prior made much use of *l* and *U* calculi in his papers and books on tense logic. He did not however pursue their implicit relational model theory, and would not have thought it

²⁷ The contents of this paper are reviewed in *Prior 1967* [209, pp. 34–41].

²⁸ Prior notes that the structure of the calculus would be unchanged if l were read “is earlier than”.

philosophically worthwhile to do so. Although he described the l -calculus as “a device of considerable metalogical utility” [206, p. 115], he went on to deny that the interpretation of the PF -calculus within the l -calculus has any metaphysical significance as an

explanation of what we mean by “is”, “has been” and “will be”.

On the contrary he proposed that what was needed was an interpretation in the reverse direction (p. 116):

the l -calculus should be exhibited as a logical construction out of the PF -calculus.

This proposal became a major programme for Prior. He used formulas like $p \wedge \neg Pp \wedge \neg Fp$ which can be true at only one point of the linear series of moments, or *instants*. If $M(p \wedge \neg Pp \wedge \neg Fp)$ is true at some time, the variable p must itself be true at exactly one instant and may be identified with that instant. Then the formula $L(p \rightarrow \alpha)$ expresses that “it is the case at p that α ”, and so if p and q are both such instance-variables, $L(p \rightarrow Pq)$ asserts that it is true at p that it has been q , i.e., p is later than q , and q is earlier than p .

Systems having variables identified with unique instants or worlds are developed most fully in the book [210], where Prior gives (p. 37) an emphatic statement of his metaphysical propensity:

... I find myself quite unable to take ‘instants’ seriously as individual entities; I cannot *understand* ‘instants’, and the earlier-than relation that is supposed to hold between them, except as logical constructions out of tensed facts. Tense logic is for me, if I may use the phrase, *metaphysically fundamental*, and not just an artificially torn-off fragment of the first-order theory of the earlier-than relation.

4.5. Kanger

A semantics is given by Stig Kanger in his 1957 monograph [132] for a version of modal predicate logic whose atomic formulas are propositional variables and expressions of the form $(x_1, \dots, x_n) \varepsilon y$, where $n \geq 1$ and the x_i and y are individual variables or constants. The language included a list of modal connectives M_1, M_2, \dots

A notion of a *system* is introduced as a pair (r, V) where r is a *frame* and V a *primary valuation*. Here r is a certain kind of sequence of non-empty sets whose elements provide values of individual symbols of various types. V is a binary operation that assigns a truth value $V(r, p)$, belonging to $\{0, 1\}$, to each propositional variable p and frame r , as well as interpreting individual symbols and the symbol ε in each frame in a manner that need not concern us. Then a “secondary” truth valuation $T(r, V, \alpha)$ is inductively specified, allowing each formula α to be defined to be *true in system* (r, V) iff $T(r, V, \alpha) = 1$. For this purpose each modality M_i is assumed to be associated with a class R_i of *quadruples* (r', V', r, V) , and it is declared that

$$T(r, V, M_i \alpha) = 1 \quad \text{iff} \quad T(r', V', \alpha) = 1 \text{ for each } r' \text{ and } V' \text{ such that} \\ R_i(r', V', r, V)$$

(so M_i is a “box” type of modality).

Kanger states the following *soundness* results. The theorems of the Feys–von Wright system T are valid (i.e., true in all systems) iff $R_i(r, V, r, V)$ always holds. S4 is validated iff $R_i(r, V, r, V)$ always holds and so does the condition

$$R_i(r, V, r', V') \text{ and } R_i(r'', V'', r, V) \text{ implies } R_i(r'', V'', r', V').$$

S5 is validated iff the S4 conditions hold along with

$$R_i(r, V, r', V') \text{ and } R_i(r'', V'', r', V') \text{ implies } R_i(r'', V'', r, V).$$

Proofs of these assertions are not provided. (In fact it is readily seen that the given conditions on R_i imply validity for the corresponding logics in each case, but the converses are dubious.) A result is proved that equates the existence of an R_i fulfilling the above definition of $T(r, V, M_i\alpha)$ to the preservation of certain inference rules involving M_i . Kanger says of this that

Similar results in the field of Boolean algebras with operators may be found in *Jónsson and Tarski* [128].

Completeness theorems are not proved, or even stated, for this modal semantics. But there is a completeness proof for the non-modal fragment of the language which has a remarkable aspect. Kanger wishes to have the symbol ε interpreted as the genuine set membership relation, and he applies the (much-overused) adjective *normal* to a primary valuation V which does give this interpretation to ε in every frame. Since his language allows atomic formulas like $x \varepsilon x$, normal systems must have *non-well-founded* sets. He introduces a new set-theoretical principle to ensure that enough such sets exist to give the completeness theorem with respect to normal structures.²⁹

Different definitions of R allow the modelling of different notions of necessity. Kanger (p. 35) defines *set-theoretical necessity* to be the modality given by requiring

$$R_i(r', V', r, V) \text{ iff } V' \text{ is normal with respect to } \varepsilon.$$

This means that M_i gets the reading “in all normal systems”. *Analytic necessity* is modelled by the R_i having

$$R_i(r', V', r, V) \text{ iff } V' = V,$$

and *logical necessity* arises when $R_i(r', V', r, V)$ always holds. Thus “logically necessary” means “true in all systems”, which is reminiscent of the modelling of the S5 necessity connective by Carnap and Bayart (Section 4.3).

There is no doubt much scope for defining other modalities in this way, and Kanger offers one other brief suggestion:

We may, for instance, define ‘geometrical necessity’ in the way we defined set-theoretical necessity except that (roughly speaking) V' shall be normal also with respect to the theoretical constants of geometry.

²⁹ This principle is discussed further in *Aczel 1988* [1, pp. 28–31 and 108].

The paper [131] addresses difficulties raised by Quine (in [211] and other writings) about the possibility of satisfactorily interpreting quantificational modal logic. One such obstacle concerns the principle of *substitutivity of equals*, formalised by the schema

$$x \approx y \rightarrow (\alpha \rightarrow \alpha')$$

where α' is any formula differing from α only in having free occurrences of y in some places where α has free occurrences of x . Taking α to be the valid $\Box(x \approx x)$, this allows derivation of

$$x \approx y \rightarrow \Box(x \approx y),$$

which is arguably invalid. For example, it is an astronomical fact that the Morning Star and the Evening Star are the same object (Venus), but this equality is not a *necessary* truth.

Kanger pointed out that his new semantics for quantification and modality made it possible to “recognize and explain the error in the Morning Star paradox”: the principle of substitutivity of equals is not valid without restriction, but only in the weaker form

$$\Box(x \approx y) \rightarrow (\alpha \rightarrow \alpha').$$

Jaakko Hintikka [119] later expressed the opinion that this discussion by Kanger of the Morning Star paradox will

remain a historical landmark as the first philosophical application of an explicit semantical theory of quantified modal logic.

4.6. Montague

Kanger’s quaternary relation R_i might equally well be viewed as a binary relation $(r', V') R_i (r, V)$ between systems. Such a notion appears in a 1960 paper by Richard Montague [187] which was originally presented to a philosophy conference at the University of California, Los Angeles, in May of 1955. Montague did not initially plan to publish the paper because “it contains no results of any great technical interest”, but eventually changed his mind after the appearance of Kanger’s and Kripke’s ideas.

The aim of the paper is to interpret logical and physical necessity, and the deontic modality “it is obligatory that”, and to relate these to the use of quantifiers. Tarski’s model theory for first-order languages is employed for this purpose: a model is taken to be a structure $\mathcal{M} = (D, R, f)$ where D is a domain of individuals, R a function fixing an interpretation of individual constants and finitary predicates in D in the now-familiar way, and f is an assignment of values in D to individual variables. Montague uses these models to provide a semantics for formulas that are constructible from atomic first-order formulas by using the propositional connectives and \Box , but not quantifiers.³⁰ His approach is to take a relation X between such models, and then inductively define

$$\mathcal{M} \text{ satisfies } \Box\alpha \text{ iff for every model } \mathcal{M}' \text{ such that } \mathcal{M} X \mathcal{M}', \mathcal{M}' \text{ satisfies } \alpha.$$

³⁰ Montague uses several symbols for various kinds of modality, but \Box will suffice here.

His first example shows that the Tarskian semantics for \forall fits this definition. Taking X to be the relation Q_x specified by

$$\mathcal{M} Q_x \mathcal{M}' \text{ iff } D = D', R = R' \text{ and } f \text{ and } f' \text{ agree except on } x$$

gives \Box the interpretation “for all x ”. Thus quantification could be handled by associating a modality with each variable, and Montague suggests that this should dispel Quine’s uneasiness about combining modality with quantification.

The relation

$$\mathcal{M} L \mathcal{M}' \text{ iff } D = D' \text{ and } f = f'$$

gives $\Box\alpha$ the interpretation “it is logically necessary that α ”, meaning that α holds no matter what its individual constants and predicates denote.

To interpret physical necessity, Montague uses the idea that a statement is physically necessary if it is deducible from some set of physical laws specified in advance. This is formalised by fixing a set K of first-order \Box -free sentences and specifying a relation P by

$$\mathcal{M} P \mathcal{M}' \text{ iff } D = D', f = f' \text{ and } \mathcal{M}' \text{ is a model of } K.$$

Similarly, “it is obligatory that α ” is taken to mean that α is deducible from some set of ethical laws specified in advance. This is formalised by fixing a class I of *ideal models*, those in which the constants and predicates mean what they ought to according to these laws. Montague suggests as an example that I could be

the class of models which, in Tarski’s sense, satisfy the ten commandments formulated as declarative, rather than imperative, sentences.

The deontic modality then corresponds to the model-relation E such that

$$\mathcal{M} E \mathcal{M}' \text{ iff } D = D', f = f' \text{ and } \mathcal{M}' \text{ belongs to } I.$$

If a model-relation X fulfills the conditions

$$\begin{aligned} &\text{for all } \mathcal{M} \text{ there exists } \mathcal{M}' \text{ with } \mathcal{M} X \mathcal{M}', \\ &\mathcal{M} X \mathcal{M}' \text{ and } \mathcal{M}' X \mathcal{M}'' \text{ implies } \mathcal{M} X \mathcal{M}'', \\ &\mathcal{M} X \mathcal{M}' \text{ and } \mathcal{M} X \mathcal{M}'' \text{ implies } \mathcal{M}' X \mathcal{M}'', \end{aligned}$$

(the last two mirror Kanger’s conditions) then every S5-theorem is valid, i.e., satisfied by every model. Montague states that the converse is true, and that there is a decision method for the class of formulas valid in this sense.

4.7. Hintikka

If \mathcal{M} is a model for predicate logic, of the kind used by Montague, let $\mu_{\mathcal{M}}$ be the set of all formulas that it satisfies. In Jaakko Hintikka’s approach to semantics, such models \mathcal{M} are in effect replaced by the sets $\mu_{\mathcal{M}}$. These sets can be characterised by their syntactic closure properties, obtained by replacing “ \mathcal{M} satisfies α ” by “ $\alpha \in \mu_{\mathcal{M}}$ ” in the clauses of

the inductive definition of satisfaction of formulas. A *model set* is defined as a set μ of formulas that has certain closure properties, such as

- if α is atomic then not both $\alpha \in \mu$ and $\neg\alpha \in \mu$,
- if $\alpha \wedge \beta \in \mu$, then $\alpha \in \mu$ and $\beta \in \mu$,
- if $\alpha \vee \beta \in \mu$, then $\alpha \in \mu$ or $\beta \in \mu$,
- if $\exists x\alpha \in \mu$, then $\alpha(y/x) \in \mu$ for some variable y ,

that are sufficient to guarantee that μ can be extended to a *maximal* model set which has *all* such closure properties corresponding to the conditions for satisfaction for the truth-functional connectives and the quantifiers.³¹

Hintikka's 1957 article [117] gives a definition of satisfaction for formulas of quantified deontic logic using model sets whose conditions

may be thought of as expressing properties of the set of all statements that are true under some particular state of affairs.

He notes (p. 10) that his treatment derives from a

new general theory of modal logics I have developed.

This general modelling of modalities was published in 1961 [118], where he views a maximal model set as the set of all formulas that hold in some state-description in the sense of Carnap, and says that

a model set is the formal counterpart to a partial description of a possible state of affairs (of a 'possible world'). (It is, however, large enough a description to make sure that the state of affairs in question is really possible.)

The point of the last sentence is that for non-modal quantificational logic, every model set is included in $\mu_{\mathcal{M}}$ for some actual model \mathcal{M} . Hence a set of non-modal formulas is satisfiable in the Tarskian sense if it is included in some model set.

The 1957 article deals with a system that has quantifiable variables ranging over individual acts, and dual modalities for *obligation* and *permission*, with formulas $O\alpha$ and $P\alpha$ being read " α is obligatory" and " α is permissible", respectively. The paper makes very interesting historical reading, especially on pages 11 and 12 where one can almost see the notion of a binary relation between model sets quickening in the author's mind as he grapples with the question of what we mean by saying that α is permitted. His answer is that

³¹ In fact it is assumed that formulas are in a certain normal form, but we can overlook the technicalities here.

we are saying that one could have done α without violating one's obligations. In other words, we are saying that a state of affairs different from the actual one is consistently thinkable, *viz.* a state of affairs in which α is done but in which all the obligations are nevertheless fulfilled.

Thus if the actual state is (partially) represented by a model set μ , then to represent this different and consistently thinkable state we need

another set μ^* related to μ in a certain way. This relation will be expressed by saying that μ^* is *copermissible with* μ .

Hintikka is thus led to formulate the following rules.

If $P\alpha \in \mu$, then there a set μ^* copermissible with μ such that $\alpha \in \mu^*$.

If $O\alpha \in \mu$ and if μ^* is copermissible with μ , then $\alpha \in \mu^*$.

The second rule addresses the requirement that all actual obligations be fulfilled in the state in which a permissible act is done. Then there are two more rules:

If $O\alpha \in \mu^*$ and if μ^* is copermissible with some other set μ , then $\alpha \in \mu^*$.

If $O\alpha \in \mu$ and if μ^* is copermissible with μ , then $O\alpha \in \mu^*$.

Motivation for third rule is as follows.

But not only one must be thought of in μ^* as fulfilling the obligations one has now. Sometimes one is permitted to do something only at the cost of new obligations. These must be thought of as being fulfilled in μ^* in order to be sure that all the obligations one has really are compatible with α 's being done.

The fourth rule is justified because

there seems to be no reason why the actually existing obligations should not also hold in the alternative state of affairs contemplated in μ^* . What is thought of as obligatory in μ must hence also be obligatory in μ^* .

Hintikka is well aware that the relation between μ and μ^* cannot be functional: there may be different acts that are each permissible in μ but cannot or must not be performed together, hence must be done in different states copermissible with μ . Also, μ^* may have its own formulas of the form $P\alpha$, requiring further model sets μ^{**} copermissible with μ^* , and so on. The upshot is that a set λ of formulas is defined to be *satisfiable* iff it is included in some model set which itself belongs to a *collection* of model sets that carries a binary relation (called the relation of copermission) obeying the closure rules for P and O .³² A formula α is *valid* if $\{\neg\alpha\}$ is not satisfiable in this sense.

³² Note that the second rule is a consequence of the third and fourth.

This approach gives a method for demonstrating satisfiability and validity, by starting with a set λ and attempting to build a suitable collection of model sets by repeatedly applying all the closure rules. New sets are added to the collection when the rule for P is applied. The other rules enlarge existing sets. If at some point a violation of the rule of consistency is produced, in the form of a contradictory pair $\alpha, \neg\alpha$ in some set, then the original λ is not satisfiable.

Hintikka gives a striking illustration of the effectiveness of this technique for analysing the subtleties of denotic logic. He demonstrates the invalidity of the principle

$$O\alpha \wedge (\alpha \rightarrow O\beta) \rightarrow O\beta,$$

which Prior had thought was a “quite plain truth”, by observing that its negation is satisfied in the simple collection consisting of the two model sets

$$\{O\alpha, \neg\alpha \vee O\beta, P\neg\beta, \neg\alpha\}, \quad \{O\alpha, \neg\beta, \alpha\}.$$

However the principle can be turned into a valid one by making it obligatory:

$$O[O\alpha \wedge (\alpha \rightarrow O\beta) \rightarrow O\beta].$$

Any attempt to build a satisfying structure for the negation of this formula leads to violation of consistency. Several other applications like this are given, analysing complex principles involving the interchange of quantifiers and deontic modalities.

With the advantage of hindsight we can see that the notion of a collection of model sets with closure rules is reminiscent of the notion of a collection of semantic tableaux used in Kripke’s completeness proofs. Hintikka did not however take up an axiomatic development of his system.

The 1961 paper [118] deals with the necessity (N) and possibility (M) modalities, and here the description of satisfiability is essentially the same, but more crisply presented. A *model system* is defined as a pair (Ω, R) with R being a binary relation of “alternativeness” on Ω , and Ω being a collection of model sets that satisfies the following conditions.

If $M\alpha \in \mu \in \Omega$, then there is in Ω at least one alternative ν to μ such that $\alpha \in \nu$.

If $N\alpha \in \mu \in \Omega$, and if $\nu \in \Omega$ is an alternative to μ , then $\alpha \in \nu$.

If $N\alpha \in \mu \in \Omega$, then $\alpha \in \Omega$.

The first two of these are the same as the first two rules for P and O . The third reflects the requirement that any necessary truth be actually true. Hintikka’s description of the new alternativeness relation is that $\mu R \nu$ when ν is a partial description of

some other state of affairs that could have been realised instead of μ .

A set λ of formulas is *satisfiable* (as before) iff there is such a model system with $\lambda \subseteq \mu$ for some $\mu \in \Omega$, and a formula α is *valid* if $\{\neg\alpha\}$ is not satisfiable. Hintikka states that the valid formulas are precisely the theorems of the logic T. Restricting to transitive model systems gives a characterisation of the theorems of S4, while the symmetric systems determine B and the ones that are both transitive and symmetric determine S5. These assertions apply

to the propositional version of the logics. To prove them would require showing in each case that a deductively consistent formula is a member of some model set that belongs to a model system of the appropriate kind, but again the issue of axioms and proof theory is not taken up. The paper is mainly devoted to a discussion of the problem of combining modalities with quantifiers, and proposes various modifications on the closure properties of Ω depending on whether it is required that whatever exists in a particular state of affairs should do so necessarily.

4.8. The place of Kripke

The earlier efforts to develop the seminal ideas of Kripke semantics have inevitably raised questions of priority. In fact, as the above material is intended to show, the idea of using a binary relation to model modality occurred independently to a number of people, and for different reasons, with Hintikka being the first to explain it in terms of conceivable alternatives to a given state of affairs. Kanger was the first to recognise the relevance of *Jónsson and Tarski 1951* [128] to modal logic,³³ and the first to apply this kind of semantical theory to the resolution of philosophical questions about existence and identity.

But it is only in Kripke's writings that we see such seminal ideas developed into an attractive model theory of sufficient power to fully resolve the long-standing issue of a satisfactory semantics for modality and of sufficient generality to advance the field further. A fundamental point (mentioned in Section 4.1) is that he was the first to propose, and make effective use of, *arbitrary* set-theoretic structures as models. The methods of Hintikka, Kanger and Montague are all variations on the theme of a binary relation between models of the non-modal fragment of the predicate languages they use. Also, they did not present complete axiomatisations of their semantics. Kripke was the first to do this, and by allowing R to be any relation on any set K , he opened the door to all kinds of model constructions, which were rapidly provided by himself and then others. (His models for non-normal logics appear to lack any historical antecedents.) It is due to his innovation that we now have a model *theory* for intensional logics.

As already noted in Section 4.2, Kripke developed his ideas independently. His analysis of S5 was initiated in 1956 when he was still at high-school (he turned 16 years old on November 13th of that year). From the paper *Prior 1956* [204] he learned of the axioms for S5, and began to think of modelling that system by truth tables with missing rows (see Section 4.1). Early in 1957 E. W. Beth sent him his papers on the method of semantic tableaux, which provided Kripke with a technique for proving completeness theorems. By 1958 Kripke had worked out his relational semantics for modal and intuitionistic systems, as announced in the abstract [143] which was received by the editors on 25 August 1958. It was through exploring different conditions connecting tableaux in order to model the different subsystems of S5 that Kripke came to the idea of using a binary relation between worlds as the basis of a model theory.

Kripke had been introduced to Beth by Haskell B. Curry, who wrote to Beth on 24 January 1957 that

³³ As *Føllesdal* [74] emphasises.

I have recently been in communication with a young man in Omaha Nebraska, named Saul Kripke. . . . This young man is a mere boy of 16 years; yet he has read and mastered my Notre Dame Lectures and writes me letters which would do credit to many a professional logician. I have suggested to him that he write you for preprints of your papers which I have already mentioned. These of course will be very difficult for him, but he appears to be a person of extraordinary brilliance, and I have no doubt something will come of it.³⁴

The Notre Dame Lectures of *Curry 1950* [45] presented a number of deductive systems of modal logic, including one equivalent to Lewis’s S4 for which a cut elimination theorem was demonstrated in *Curry 1952* [46]. Other such sources that were influential for Kripke included the McKinsey–Tarski papers and the paper of *Lemmon 1957* [153] which showed how to axiomatize the Lewis systems in the style of Gödel.

In late 1958 Kripke entered Harvard University as an undergraduate, and encountered a philosophical environment that was hostile to modal logic. He was advised to abandon the subject and concentrate on majoring in mathematics. This caused the evident delay in publication of his work until the appearance of the major articles of 1963 and 1965.

Looking back over the intervening decades we see the strong influence of Kripke’s ideas on many areas of mathematical logic, ranging across the foundations of constructive logic and set theory, substructural logics (including relevance logic, linear logic), provability logic, the Kripke–Joyal semantics in topos theory and numerous logics of transition systems in theoretical computer science.

A *proposition* is defined by Kripke in [145] to be a function from worlds to truth values, while in [146] an n -ary predicate letter is modelled as a function from worlds to n -ary relations. Those definitions formed a cornerstone of Montague’s approach to intensional logic,³⁵ and stimulated the substantial development of formal semantics for natural languages in the theories of Montague [190], Cresswell [42], Barwise [4] and others. Kripke’s models, and his intuitive descriptions of them, also stimulated many philosophical and formal investigations of the nature of possible worlds, and the questions of existence and identity that they generate (see [161]).

5. The post-Kripkean boom of the sixties

The 1960s was an extraordinary time for the introduction of new model theories. At the beginning of the decade Abraham Robinson created nonstandard analysis by constructing models of the higher-order theory of the real numbers. Then Paul Cohen’s invention of forcing revolutionized the study of models of set theory, and freed up the log-jam of questions that had been building since the time of Cantor. Kripke related forcing to his models of Heyting’s predicate calculus, and Dana Scott and Robert Solovay re-formulated it as the technique of Boolean-valued models. Scott then replaced “Boolean-valued” by “Heyting-valued” and extended the topological interpretation from intuitionistic predicate

³⁴ Quoted from pp. 290–291 of *de Jongh and van Ulsen* [49].

³⁵ As acknowledged in several places, e.g., footnote 5 of *Montague 1970* [189].

logic to intuitionistic real analysis. F. William Lawvere’s search for categorical axioms for set theory and the foundations of mathematics culminated at the end of the decade in his development, in collaboration with Miles Tierney, of topos theory. This encompassed, in various ways, both classical and intuitionistic higher order logic and set theory, including the models of Kripke, Cohen, Scott, and Solovay, as well as incorporating the sheaf theory of the Grothendieck school of algebraic geometry. Scott’s construction of models for the untyped lambda calculus in 1969 was to open up the discipline of denotational semantics for programming languages, as well as stimulating new investigations in lattice theory and topology, and further links with categorical and intuitionistic logic.

The introduction of Kripke models had a revolutionary impact on modal logic itself. Binary relations are much easier to visualise, construct, and manipulate than operators on Boolean algebras. They fall into many naturally definable classes that can be used to define corresponding logics. Here then were the tools that would enable an exhaustive investigation of the subject, and some important new ideas were developed during this period.

5.1. The Lemmon and Scott collaboration

Pioneers in this investigation were John Lemmon and Dana Scott, who conducted an extensive collaboration. They planned to write a book called *Intensional Logic*, for which Lemmon had drafted some initial chapters when he died in 1966. Scott then made this material available in a mimeographed form which was circulated informally for a number of years, becoming known as the “Lemmon Notes”. Eventually it was edited by Scott’s student Krister Segerberg, and published in 1977 as [156]. Scott also investigated broad issues of intensional logic (individuals and concepts, possible worlds and indices, intensional relations and operators etc.) in discussion with Montague, Kaplan and others. Some of his ideas were presented in the article [218]. His considerable influence on the subject has been disseminated through the publications of Lemmon and Segerberg, and is also reported in *Prior 1967* [209] in relation to tense logic, and in a number of Montague’s papers.

The relationship between modal algebras and model structures was first systematically explored in Lemmon’s two part article [154,155] from 1966. Here a model structure has the form $\mathfrak{S} = (K, R, Q)$, with Q playing the role of the set of non-normal (“queer”) worlds.³⁶ Notably absent is Kripke’s *real* world $G \in K$. Instead a formula α is said to be *valid* in \mathfrak{S} if in all models on \mathfrak{S} , α is *true* (i.e., assigned the value \top) at all points of K .

Associated with \mathfrak{S} is the modal algebra \mathfrak{S}^+ comprising the powerset Boolean algebra $\mathcal{P}(K)$ with the additive operator

$$f(X) = \{x \in K : x \in Q \text{ or } \exists y \in X(xRy)\}$$

to interpret \diamond . Note that $f(\emptyset) = Q$, so f is a *normal* operator iff K has only normal members. Lemmon proved the result that a formula is valid in \mathfrak{S} iff it is satisfied in the algebra \mathfrak{S}^+ with just the element 1 (= K) designated. This follows from the natural correspondence between models Φ on \mathfrak{S} and assignments to propositional variables in \mathfrak{S}^+ , under

³⁶ At the time this work was done Kripke’s [148] had not appeared, but Lemmon had learned about non-normal worlds in conversation with Kripke.

which a variable p is assigned the set $\{x: \Phi(p, x) = \top\} \in \mathfrak{S}^+$. The result itself is an elaboration of Kripke's construction in [145] of the matrix of propositions associated with any model structure. It remains true for S2-like systems if validity in \mathfrak{S} is confined to truth at normal worlds, and also all elements of \mathfrak{S}^+ that include $K - Q$ are designated.

Any finite modal algebra $\mathfrak{A} = (\mathfrak{B}, f)$ is readily shown to be isomorphic to one of the form \mathfrak{S}^+ , with \mathfrak{S} based on the set of atoms of \mathfrak{B} . Combining that observation with McKinsey's finite algebra constructions enabled Lemmon to deduce the completeness of a number of modal logics with respect to validity in their (finite) model structures. For an arbitrary \mathfrak{A} he gave a representation theorem, "due in essentials to Dana Scott", that embeds \mathfrak{A} as a sub-algebra of some \mathfrak{S}^+ . This was done by an extension of Stone's representation of Boolean algebras, basing \mathfrak{S} on the set K of all ultrafilters of \mathfrak{B} , with uRt iff $\{fx: x \in t\} \subseteq u$ for all ultrafilters u, t , while $Q = \{x \in K: f0 \in x\}$. Each $x \in \mathfrak{A}$ is represented in \mathfrak{S}^+ by the set $\{u \in K: x \in u\}$ of ultrafilters containing x , as in Stone's theory.

In the Lemmon Notes there is a model-theoretic analogue of this representation of modal algebras that has played a pivotal role ever since. Out of any normal logic Λ is constructed a model

$$\mathcal{M}^\Lambda = (K^\Lambda, R^\Lambda, \Phi^\Lambda)$$

in which K^Λ is the set of all *maximally Λ -consistent* sets of formulas, with

$$uR^\Lambda t \text{ iff } \{\diamond\alpha: \alpha \in t\} \subseteq u \text{ iff } \{\alpha: \Box\alpha \in u\} \subseteq t,$$

and $\Phi^\Lambda(p, u) = \top$ iff $p \in u$. The key property of this construction is that an arbitrary formula α is true in \mathcal{M}^Λ at u iff $\alpha \in u$. This implies that \mathcal{M}^Λ is a *model of α* , i.e., α is true at all points of \mathcal{M}^Λ , iff α is an Λ -theorem. Thus \mathcal{M}^Λ is a single characteristic model for Λ , now commonly called the *canonical Λ -model*. Moreover, the properties of this model are intimately connected with the proof-theory of Λ . For example, if $(\Box\alpha \rightarrow \alpha)$ is an Λ -theorem for all α , then it follows directly from properties of maximally consistent sets that R^Λ is reflexive. This gives a technique for proving that various logics are characterised by suitable conditions on models, a technique that is explored extensively in [156].

If Scott's representation of modal algebras is applied to the Lindenbaum algebra of Λ , the result is a model structure isomorphic to (K^Λ, R^Λ) . The construction can also be viewed as an adaptation of the method of completeness proof introduced in *Henkin 1949* [110], and first used for modal logic in *Bayart 1958* [5] (see Section 4.3). There were others who independently applied this approach to the relational semantics for modal logic, including David Makinson [166] and Max Cresswell [40], their work being completed in 1965 in both cases. Makinson dealt with propositional systems, while Cresswell's appears to be the first Henkin-style construction of relational models of *quantificational* modal logic. David Kaplan outlined a proof of this kind in his 1966 review [133] of Kripke's [145], explaining that the idea of adapting Henkin's technique to modal systems had been suggested to him by Dana Scott.

Another construction of lasting importance from the Lemmon Notes is a technique for proving the finite model property by forming *quotients* of the model \mathcal{M}^L . To calculate the truth-value of a formula α at points in \mathcal{M}^L we need only know the truth-values of the finitely many subformulas of α . We can regard two members of \mathcal{M}^L as *equivalent* if they assign the same truth-values to all subformulas of α . If there are n such subformulas, then

there will be at most 2^n resulting equivalence classes of elements of \mathcal{M}^A , even though \mathcal{M}^A itself is uncountably large. Identifying equivalent elements allows \mathcal{M}^A to be collapsed to a *finite* quotient model which will falsify α if \mathcal{M}^A does. This process, which has become known as *filtration*,³⁷ was first developed in a more set-theoretic way in *Lemmon 1966* [155, p. 209] as an alternative to McKinsey’s finite algebra construction. In its model-theoretic form it has proven important for completeness proofs as well as for proofs of the finite model property. Some eighteen modal logics were shown to be decidable by this method in [156].

5.2. Bull’s tense algebra

A singular contribution from the 1960s is the algebraic study by Robert Bull, a student of Arthur Prior,³⁸ of logics characterised by linearly ordered structures. Prior had observed that the Diodorean temporal reading of $\Box\alpha$ as “ α is and always will be true” leads, on intuitive grounds, to a logic that includes S4 but not S5. In his 1956 John Locke Lectures at Oxford on *Time and Modality* (published as [205]) he attempted to give a mathematical precision to this reading by interpreting formulas as sets of sequences of truth values. In effect he was dealing with the complex closure algebra $\text{Cm}(\omega, \leq)$, where $\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers viewed as a sequence of moments of time. The question became one of identifying the logic that is characterised by this algebra, or equivalently by the model structure (ω, \leq) . Prior called this logic D.³⁹

In 1957 Lemmon observed that D includes the formula

$$\Box(\Box p \rightarrow \Box q) \vee \Box(\Box q \rightarrow \Box p),$$

which arises from the intuitionistically *in* valid formula $(p \rightarrow q) \vee (q \rightarrow p)$ by applying the translation of McKinsey and Tarski [178]. Lemmon’s formula is therefore not an S4-theorem, and when added as an axiom to S4 produces a system called S4.3. In 1958 Michael Dummett showed that the formula

$$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond \Box p \rightarrow \Box p)$$

also belongs to D, and then Prior pointed out in [208] that this is due to the *discreteness* of the ordering \leq on ω : if time were a continuous ordering then Dummett’s formula would not be valid, but Lemmon’s would. In fact the property used by Prior to invalidate Dummett’s formula was *density* (between any two moments there is a third) rather than continuity in the sense of Dedekind (no “gaps”).

Kripke showed in 1963 that D is exactly the normal logic obtained by adding Dummett’s formula as an axiom to S4.3. His proof, using semantic tableaux, is unpublished.

³⁷ This term was first used in *Seegerberg 1968* [220], where “canonical model” was also introduced.

³⁸ Initially at Christchurch, New Zealand, and then at Manchester, England. Bull was one of two graduate students from New Zealand who studied with Prior at Manchester at the beginning of the 1960s. The other was Max Cresswell, who later became the supervisor of the present author.

³⁹ The letter D later became a label for the system $\text{K} + (\Box p \rightarrow \Diamond p)$, or equivalently $\text{K} + \Diamond \top$, because of its connection with Deontic logic.

Dummett conjectured to Bull that taking time as “continuous” would yield a characterisation of S4.3.⁴⁰ Bull proved this in his 1965 paper [28] which, in addition to giving an algebraic proof of Kripke’s completeness theorem for D, showed that S4.3 is characterised by the complex algebra of the ordering (\mathbb{R}^+, \leq) of the positive real numbers. He noted that \mathbb{R}^+ could be replaced here by the positive rationals, or any linearly ordered set with a subset of order type ω^2 . In particular this shows that propositional modal formulas are incapable of expressing the distinction between dense and continuous time under the relational semantics.

Bull made effective use of Birkhoff’s fundamental decomposition [15] of an abstract algebra into a subdirect product of subdirectly irreducible algebras. Birkhoff had observed that subdirectly irreducible closure algebras are *well-connected* in the sense of McKinsey and Tarski [176] (see Section 3.2). Applying this to Lindenbaum algebras shows that every normal extension of S4 is characterised by well-connected closure algebras, and in the case of extensions of S4.3 the closed $(\mathbf{C}x = x)$ elements of a well-connected algebra are linearly ordered. Bull used this fact, together with the strategy of McKinsey’s finite algebra construction, to build intricate embeddings of finite S4.3-algebras into $\mathbf{Cm}(\mathbb{R}^+, \leq)$ or $\mathbf{Cm}(\omega, \leq)$. He later refined this technique to establish in [29] one of the more celebrated meta-theorems of modal logic:

every normal extension of S4.3 has the finite model property.

Proofs of this result using relational models were subsequently devised by Kit Fine [67] and Håkan Franzén (see [224]). Fine gave a penetrating analysis of finite S4.3 models to establish that there are exactly \aleph_0 normal extensions of S4.3, all of which are finitely axiomatisable and hence decidable. Segerberg [225] proved that in fact every logic extending S4.3 is normal.

The indistinguishability of rational and real time is overcome by passing to the more powerful language of Prior’s *PF*-calculus for tense logic (Section 4.4). A model structure for this language would in principle have the form (K, R_P, R_F) , with R_P and R_F being binary relations on K interpreting the modalities P and F . But for modelling tense logic, with its *interaction* principles $p \rightarrow G P p$ and $p \rightarrow H F p$, the relations R_P and R_F should be mutually inverse. Thus we continue to use structures (K, R) with the understanding that what we really intend is (K, R^{-1}, R) . For linearly ordered structures, the ability of the two modalities to capture properties “in each direction” of the ordering produces formulas that express the Dedekind completeness of \mathbb{R} , a fact that was first realised by Montague and his student Nino Cocchiarella.⁴¹

Bull applied his algebraic methodology in the 1968 paper [30] to give complete axiomatisations of the tense logics characterised by each of the strictly linearly ordered structures $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$. In addition to a common set of axioms for linear orderings without first or last element, for integer time \mathbb{Z} he used the special axiom

$$\Box(Gp \rightarrow p) \rightarrow \Box Gp \vee \Box \neg Gp,$$

⁴⁰ See Chapter II of *Prior 1967* [209] as well as *Bull 1965* [28] for this historical background.

⁴¹ See [209, pp. 57, 72].

where \Box is the S5-modality defined by $\Box\alpha = \alpha \wedge G\alpha \wedge H\alpha$. For rational time \mathbb{Q} this was replaced by the *density* axiom $Fp \rightarrow FFp$. The axiomatisation of real time required the density axiom as well as

$$\Box(Gp \rightarrow PGp) \rightarrow \Box Gp \vee \Box \neg Gp.$$

(The reader may find it instructive to verify that validity of this last formula in any model on $(\mathbb{R}, <)$ depends on the fact that there are no unfilled Dedekind cuts in the real line.) Bull also established that the tense logics of rational and real time have the finite model property, but that the logic of integer time does not.⁴²

This is not quite the end of the story about Diodorean modality. Prior made an interesting observation in [209, p. 203] about the (non-linear) temporal ordering of locations in relativistic spacetime. In the Minkowskian spacetime of *special* relativity theory, this ordering is *directed*: for any two locations x, y there is a third that is in the future of both x and y . This is because any two future light-cones eventually intersect (but not so in *general* relativity, where the effect of gravitation can prevent light-cones overlapping). Directedness causes the Diodorean interpretation of \Box to validate the formula $\Diamond\Box p \rightarrow \Box\Diamond p$, which is itself equivalent in the field of S4 to the formula $\Box\neg\Box p \vee \Box\Diamond p$ that arises by the McKinsey–Tarski translation of the intuitionistically invalid $\neg p \vee \neg\neg p$. Adding $\Diamond\Box p \rightarrow \Box\Diamond p$ to S4 gives the logic S4.2. Both S4.2 and S4.3 were introduced in *Dummett and Lemmon 1959* [54], and shown to have the finite model property in *Bull 1964* [27].

In *Goldblatt 1980* [89] a completeness proof is given to show that S4.2 is exactly the Diodorean logic of n -dimensional Minkowski spacetime for all $n \geq 2$, as well as being the logic of the product structure $(\mathbb{R}, \leq) \times (\mathbb{R}, \leq)$.⁴³ But the problem of axiomatising the PF-calculi characterised by these spacetimes remains open.

5.3. Segerberg’s essay

Krister Segerberg’s dissertation, *An Essay in Classical Modal Logic* [223], provided a comprehensive semantic analysis of whole families of modal logics, as well as developing important new concepts, some of which had been announced in his papers of 1968 [220] and 1970 [222]. These works established some notational and terminological conventions that have been lasting. For instance the term *frame* was used in place of *model structure*, and the Lemmon–Scott satisfaction notation $\models_x^{\mathcal{M}} \alpha$ was used throughout in place of Kripke’s $\Phi(\alpha, x) = \top$, where $\mathcal{M} = (\mathfrak{S}, \Phi)$. Later authors have tended to reduce the use of superscripts and write $\mathcal{M} \models_x \alpha$ instead of $\models_x^{\mathcal{M}} \alpha$. $\mathcal{M} \models \alpha$ then means that α is *true in* \mathcal{M} , i.e., true at all points of \mathcal{M} , and $\mathfrak{S} \models \alpha$ means that α is valid in the frame \mathfrak{S} .

The weakest system discussed in the *Essay* is E, the smallest logic that is closed under the rule *from* $\alpha \leftrightarrow \beta$ *infer* $\Box\alpha \leftrightarrow \Box\beta$. An algebraic semantics for this logic would employ algebras $\mathfrak{A} = (\mathfrak{B}, f)$ having f as a unary function on \mathfrak{B} satisfying no particular conditions. The corresponding “relational” models use *neighbourhood semantics*, the idea of which is attributed to Montague [188] and Scott [218]. Segerberg presents this by the device of

⁴² An error in the proof for rational time is corrected in [31].

⁴³ The latter result was obtained independently by Shehtman [229].

a *neighbourhood frame* $\mathfrak{S} = (K, N)$, where N , the *neighbourhood system*, is a function assigning to each $x \in K$ a collection N_x of subsets of K , called *neighbourhoods of x* .⁴⁴ Writing $\mathcal{M}(\alpha)$ for the “truth set” $\{y \in K : \mathcal{M} \models_y \alpha\}$ interpreting α in \mathcal{M} , the satisfaction clause for \Box in a model \mathcal{M} on such a frame \mathfrak{S} is

$$\mathcal{M} \models_x \Box \alpha \quad \text{iff} \quad \mathcal{M}(\alpha) \in N_x.$$

A topology on K has a naturally associated neighbourhood system in which $X \in N_x$ iff x is interior to X , i.e., $x \in U \subseteq X$ for some open set U . In this case $\mathcal{M}(\Box \alpha)$ is the topological interior of $\mathcal{M}(\alpha)$, and the result is an S4-model. But different logics can be characterised by validity in frames with weaker conditions imposed on their neighbourhoods. A relational frame (K, R) is equivalent to the neighbourhood frame (K, N) having $U \in N_x$ iff $\{y : xRy\} \subseteq U$.

Any neighbourhood frame (K, N) has an associated algebra $(\mathcal{P}(K), f^N)$, where the operation f^N , interpreting \Box on the powerset algebra $\mathcal{P}(K)$, is given by

$$f^N(X) = \{x \in K : X \in N_x\}.$$

Inversely, any function $f : \mathcal{P}(K) \rightarrow \mathcal{P}(K)$ induces the neighbourhood system N^f on K , where

$$X \in N_x^f \quad \text{iff} \quad x \in f(X).$$

Thus, whereas Jónsson and Tarski’s analysis shows that relational semantics corresponds to *completely additive and normal* operators on powerset algebras (see Section 3.3), neighbourhood systems can be used to represent arbitrary operations on such algebras. The relationship between neighbourhood frames and modal algebras has been systematically investigated by Kosta Došen [52].

Filtration (see Section 5.1) was used extensively by Segerberg to prove completeness theorems. This technique can be effective in dealing with logics whose canonical model does not satisfy some desired property, and comes into its own when seeking to axiomatise logics defined by some condition on *finite* frames. For example, Segerberg showed [223, p. 68] that the normal logic K4W, with axioms

$$\begin{aligned} 4 : \Box p &\rightarrow \Box \Box p \\ W : \Box(\Box p \rightarrow p) &\rightarrow \Box p, \end{aligned}$$

is characterised by the class of finite frames (K, R) in which R is transitive and irreflexive, i.e., a strict ordering. (This logic later proved important in studies of the provability interpretation of modality. See Section 7.5.) The basic method was to obtain a falsifying model for a given non-theorem by filtration of the canonical model, and then to “deform” this into a model of the desired kind without affecting the truth value of the formula concerned. This involved an analysis of the way a transitive relations presents itself as an ordered set of connected components, called *clusters*. The method was applied in the *Essay* and the 1970 paper [222] to axiomatise a whole range of logics, including those characterised by the classes of finite partial orderings, finite linear orderings (both irreflexive and reflexive),

⁴⁴ Some authors use a relation $R \subseteq K \times \mathcal{P}(K)$ in place of N , where xRU iff $U \in N_x$.

and the modal and tense logics of the structures (K, R) where K is any of $\omega, \mathbb{Z}, \mathbb{Q}$, and \mathbb{R} , while R is any of $<, >, \leq, \text{ and } \geq$.

The logic characterised by the class of all finite partial orderings is particularly significant. Segerberg proved [223, p. 101] that it is S4Grz, the normal logic axiomatised by adding to S4 the axiom

$$\text{Grz: } \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p.$$

He named this for Andrzej Grzegorzcyk whose 1967 paper [101] added a further insight to the relationship between intuitionistic and modal logic. Grzegorzcyk showed that the formula

$$[(p \rightarrow \Box q) \rightarrow \Box q] \wedge ((\neg p \rightarrow \Box q) \rightarrow \Box q) \rightarrow \Box q$$

is not a theorem of S4 (nor indeed of S5), and when added to S4 gives a system into which the intuitionistic logic IPC can be translated by the Gödel–McKinsey–Tarski procedures. The translation of a propositional formula is an S4-theorem iff it is a theorem of Grzegorzcyk’s stronger logic, which is deductively equivalent to S4Grz.

Segerberg initiated the use of truth-preserving maps between relational models and frames in [220]. Given models \mathcal{M} and \mathcal{M}' on frames $\mathfrak{S} = (K, R)$ and $\mathfrak{S}' = (K', R')$ respectively, a function φ from K onto K' was called a *pseudo-epimorphism* from \mathcal{M} to \mathcal{M}' if

- (i) $x R y$ implies $\varphi(x) R' \varphi(y)$,
- (ii) $\varphi(x) R' \varphi(y)$ implies $\exists z \in K (x R z \ \& \ \varphi(z) = \varphi(y))$, and
- (iii) $\mathcal{M} \models_x p$ iff $\mathcal{M}' \models_{\varphi(x)} p$.

For such a function every formula α has $\mathcal{M} \models_x \alpha$ iff $\mathcal{M}' \models_{\varphi(x)} \alpha$, so if \mathcal{M} is a model of α , then \mathcal{M}' will be also. From this it can be shown that if α is valid in \mathfrak{S} , then the existence of a function from K onto K' satisfying (i) and (ii) implies that α is valid in \mathfrak{S}' as well.⁴⁵

The name “pseudo-epimorphism” was shortened to “p-morphism” in [222,223] and this uninformative term has been very widely adopted, even for functions that are not surjective but, in place of (ii), satisfy

- (ii') $\varphi(x) R' w$ implies $\exists z \in K (x R z \ \& \ \varphi(z) = w)$.

The notion was generalised by Johan van Benthem [263] to that of a “p-relation” between models, which is itself intimately related to the concept of a *bisimulation relation* that has been fundamental to the study of computational processes (see Section 7.2).

There is another explanation of why functions of this type are natural and important in the modal context. Any function $\varphi : K \rightarrow K'$ induces the function $\varphi^+ : \mathcal{P}(K') \rightarrow \mathcal{P}(K)$ in the reverse direction, taking each subset X of K' to its inverse image $\{x \in K : \varphi(x) \in X\}$.

⁴⁵ A surjection between partial orderings that satisfies (i) and (ii) was defined to be *strongly isotone* in *de Jongh and Troelstra 1966* [50], where the notion was used to demonstrate connections between partial orderings and certain algebraic models for intuitionistic propositional logic.

This φ^+ is a Boolean algebra homomorphism. The conditions (i) and (ii') are precisely what is required for it to preserve the operators f_R and $f_{R'}$, and hence be a homomorphism between the modal algebras $\text{Cm}(K', R')$ and $\text{Cm}(K, R)$. If φ is surjective, then φ^+ is injective and so makes $\text{Cm } \mathfrak{S}'$ isomorphic to a subalgebra of $\text{Cm } \mathfrak{S}$. Hence all modal-algebraic equations satisfied by $\text{Cm } \mathfrak{S}$ will be satisfied by $\text{Cm } \mathfrak{S}'$. But a propositional modal formula α can be viewed as a *term* in the language of the algebra $\text{Cm } \mathfrak{S}$, with α being valid in the frame \mathfrak{S} precisely when the algebraic equation “ $\alpha \approx 1$ ” is satisfied by $\text{Cm } \mathfrak{S}$. This gives another perspective on why validity is preserved by surjective p-morphisms.

Of equal importance is the validity-preserving notion of *subframe*. This originated in Kripke's definition in [146] of a model structure (G, K, R) as being *connected* when $K = \{H: GR^*H\}$, where R^* is the reflexive-transitive closure of R . Lemmon adapted this in [155] to the notion of the *connected model structure* \mathfrak{S}_x generated from \mathfrak{S} by an element x , which is the substructure of \mathfrak{S} based on $\{y: xR^*y\}$. He observed that a formula falsified by $\text{Cm } \mathfrak{S}$ must be falsified by $\text{Cm } \mathfrak{S}_x$ for some x . Segerberg showed in [223, p. 36] that a model \mathcal{M} on \mathfrak{S} can be restricted to a model \mathcal{M}_x on \mathfrak{S}_x (the submodel of \mathcal{M} generated by x) in such a way that in general $\mathcal{M}_x \models_y \alpha$ iff $\mathcal{M} \models_y \alpha$. From this it follows that any formula valid in \mathfrak{S} will be valid in \mathfrak{S}_x , and conversely a formula valid in \mathfrak{S}_x for all x in \mathfrak{S} will be valid in \mathfrak{S} itself (as essentially observed by Lemmon). This notion of *point-generated* substructure turned out to be the relational analogue of the notion of *subdirectly irreducible* algebra. Indeed the algebra $\text{Cm } \mathfrak{S}$ is subdirectly irreducible iff \mathfrak{S} is equal to \mathfrak{S}_x for some x , a fact that was first demonstrated by Wim Blok [18, p. 12], [19, Lemma 4.1].

A frame \mathfrak{S} is a *subframe* of frame \mathfrak{S}' if it is a substructure of \mathfrak{S}' that is closed under R' , i.e., if $x \in K$, then $\{y \in K': xR'y\} \subseteq K$ (some authors call this a “generated” subframe even though there is no longer any generator involved). Then the inclusion function $\varphi: K \hookrightarrow K'$ is a p-morphism inducing φ^+ as a *surjective* homomorphism from $\text{Cm } \mathfrak{S}'$ to $\text{Cm } \mathfrak{S}$. Since equations are preserved by surjective homomorphisms, modal-validity is preserved in passing from \mathfrak{S}' to the subframe \mathfrak{S} .

The disjoint union $\coprod_j \mathfrak{S}_j$ of a collection $\{\mathfrak{S}_j: j \in J\}$ of frames also preserves validity. The construction was first applied to modal model theory in Goldblatt 1974 [85] and Fine 1975 [70]. $\coprod_j \mathfrak{S}_j$ is simply the union of a collection of pairwise disjoint copies of the \mathfrak{S}_j s. Each \mathfrak{S}_j is isomorphic to a subframe of $\coprod_j \mathfrak{S}_j$, and so the above properties of subframes guarantee that a formula is valid in $\coprod_j \mathfrak{S}_j$ iff it is valid in every \mathfrak{S}_j .

These observations about morphisms, subframes and disjoint unions form the basis of a theory of *duality* between frames and modal algebras that is discussed in Section 6.5.

6. Metatheory of the seventies and beyond

The semantic analysis of particular logics eventually gave way to investigations of the nature of the relational semantics itself: the strengths and limitations of its techniques, and its relationship to other formalisms, particularly first-order and monadic second-order predicate logic. Some of the questions raised have yet to be answered.

Throughout Section 6 the term “logic” will always mean a *normal* logic.

6.1. Incompleteness

A logic Λ is *sound* with respect to a class \mathcal{C} of frames if every member of \mathcal{C} is a Λ -frame, i.e., validates all Λ -theorems. By definition Λ is sound with respect to the class $Fr(\Lambda)$ of all Λ -frames. In the converse direction, Λ is *complete* with respect to \mathcal{C} if any formula that is valid in all members of \mathcal{C} is a Λ -theorem. For example, every normal logic is complete with respect to $\mathcal{C} = \{\mathfrak{S}^\Lambda\}$, where $\mathfrak{S}^\Lambda = (K^\Lambda, R^\Lambda)$ is the *canonical frame* of Λ as defined in Section 5.1. For if a formula is valid in \mathfrak{S}^Λ , then it is true in the canonical model \mathcal{M}^Λ on \mathfrak{S}^Λ , and so is a Λ -theorem. Whether or not Λ is sound with respect to \mathfrak{S}^Λ is an important issue that will be discussed in Section 6.6.

A logic Λ is *characterised* by a class \mathcal{C} if it is both sound and complete with respect to \mathcal{C} . Λ is *complete* per se if it is complete with respect to some class \mathcal{C} of Λ -frames, in which case it is characterised by that \mathcal{C} , as well as by the class $Fr(\Lambda)$ of all Λ -frames. It is important to recognise that a given logic may be characterised by many different classes. For example, S4 is characterised by each of the class of all quasi-orderings, the class of finite quasi-orderings, and the class of all partial-orderings (but not the *finite* partial-orderings, which characterise S4Grz as we saw in Section 5.3).

Lemmon was sufficiently taken with the power of Kripke semantics to conjecture that every normal logic is characterised by some class of relational frames [156, p. 74]. It turned out that this was as far from the truth as it could be. Wim Blok showed that, in a manner which will be explained below, “most” logics Λ are not characterised by any class of frames, and hence are *incomplete* in the sense that there exist formulas that are valid in all Λ -frames but are not Λ -theorems.

The first example of an incomplete logic was devised by Steve Thomason [253], and is a readily described tense logic in Priors *PF*-language. In addition to a set of postulates for linearly-ordered frames it has the axioms

$$\begin{aligned} Gp &\rightarrow Fp, \\ Pp &\rightarrow P(p \wedge \neg Pp), \\ GFp &\rightarrow FGP. \end{aligned}$$

The first of these is valid in a frame (K, R) only if the “endless time” condition $\forall x \exists y (xRy)$ is satisfied. The second axiom is equivalent to $H(Hp \rightarrow p) \rightarrow Hp$, which is Segerberg’s axiom *W* for the past modality *H*. Its validity entails that *R* is irreflexive. Thus if x_0 is a point in any frame validating the first two axioms, $\{y: x_0Ry\}$ is an irreflexive linear ordering with no last element. Interpreting p as a set such that both it and its complement are unbounded in $\{y: x_0Ry\}$ then gives a model on the frame that falsifies the third axiom at x_0 . In this model the truth-value of p alternates forever over time.

Thus Thomason’s logic is not valid on any frame whatsoever! In other words it is indistinguishable in terms of frame-validity from the *inconsistent* logic in which all formulas are theorems. But it is not itself inconsistent, because it is satisfied by the algebra which consists of all the finite and cofinite subsets of the structure $(\omega, <)$. In this algebra the interpretation of each formula is constrained to cease changing with time.

It proved more difficult to devise incomplete \Box -logics, i.e., propositional logics in a language with just one modality \Box . Unlike tense logic, any consistent normal \Box -logic is

validated by some frame, and in fact by some one-element frame. There are two such structures: \mathfrak{S}_\circ is the one consisting of a single reflexive point, while \mathfrak{S}_\bullet consists of a single irreflexive point. \mathfrak{S}_\circ characterises the normal logic $\Lambda_\circ = \mathbf{K} + (\Box p \leftrightarrow p)$ and \mathfrak{S}_\bullet characterises $\Lambda_\bullet = \mathbf{K} + \Box \perp$, both of which are *maximal* logics in the sense of having no proper consistent extensions. Makinson [169] proved that every consistent normal \Box -logic is either valid in \mathfrak{S}_\circ or valid in \mathfrak{S}_\bullet , and so is a sublogic of one of Λ_\circ and Λ_\bullet .

The first incomplete \Box -logics were found by Thomason [254] and Kit Fine [68], who independently constructed some rather complicated examples. Later van Benthem [265, 266] found some simpler ones. The simplest unearthed to date is the normal logic with axiom

$$\Box(\Box p \leftrightarrow p) \rightarrow \Box p.$$

Lon Berk showed that any frame validating this formula also validates Segerberg's axiom \mathbf{W} , while Roberto Magari showed that \mathbf{W} is not a theorem of the logic. Proofs of these results are presented in *Boolos and Sambin 1985* [24].

The *degree of incompleteness* of a logic Λ was defined by Fine [68] as the number of logics that are valid in exactly the same frames that Λ is. For any class \mathcal{C} , the set $\Lambda_{\mathcal{C}} = \{\alpha: \mathcal{C} \models \alpha\}$ of all formulas validated by \mathcal{C} is, by definition, characterised by \mathcal{C} . If some other logic Λ is valid in all members of \mathcal{C} and no other frames, then Λ must be a proper sublogic of $\Lambda_{\mathcal{C}}$, with both having degree of incompleteness ≥ 2 . The logic \mathbf{K} has degree 1: it is the only logic valid in all frames whatsoever. Any Λ that has degree 1 must be complete, since it must be equal to $\Lambda_{\mathcal{C}}$ where \mathcal{C} is the class of all Λ -frames. Fine asked which cardinals can occur as the degree of incompleteness of some logic, and whether there are any logics other than \mathbf{K} that are “intrinsically complete” in the sense of having degree 1.

Those questions were resolved in a remarkable way by Blok, who proved that any logic Λ containing the axiom $\Box p \rightarrow p$ must have degree of incompleteness 2^{\aleph_0} , so that there are uncountably many different logics which are indistinguishable from Λ by the Kripke relational semantics. The same applies whenever Λ contains the axiom $\Box^n p \leftrightarrow \Box^{n+1} p$ for some natural number n . As just one illustration of this situation, consider the case of Λ_\circ itself. The only connected Λ_\circ -frame is the one-element reflexive frame \mathfrak{S}_\circ (and any other Λ_\circ -frame is just a disjoint union of copies of \mathfrak{S}_\circ). But there are uncountably many other (incomplete) logics whose only connected validating frame is also \mathfrak{S}_\circ .

These results were obtained in 1976–1977, and published in [19]. The report [18] then gave the following complete answer to Fine's two questions: every normal logic is either of degree 1 or of degree 2^{\aleph_0} , and there are 2^{\aleph_0} logics of degree 1. The degree 1 logics all have the finite model property. Moreover Blok provided a semantic characterisation of these degree 1 logics, using the notion of a *splitting logic*. This is a logic Λ_s for which there is some other logic Λ'_s such that every logic Λ has either $\Lambda_s \subseteq \Lambda$ or $\Lambda \subseteq \Lambda'_s$, but not both. Thus the collection of all normal logics is split into the two disjoint collections $\{\Lambda: \Lambda_s \subseteq \Lambda\}$ and $\{\Lambda: \Lambda \subseteq \Lambda'_s\}$. A simple example is given by putting $\Lambda_s = \mathbf{K} + \Diamond \top$ and $\Lambda'_s = \Lambda_\bullet = \mathbf{K} + \Box \perp$. If $\Lambda \not\subseteq \Lambda_\bullet$, then by the maximality of Λ_\bullet , $\Box \perp$ cannot be consistently added to Λ , hence its negation $\Diamond \top$ is a Λ -theorem, showing $\mathbf{K} + \Diamond \top \subseteq \Lambda$.

Let Λ/\mathfrak{S} be the intersection of all logics that are not validated by frame \mathfrak{S} . Then a logic is a splitting logic iff it is equal to the logic Λ/\mathfrak{S} for some finite frame \mathfrak{S} that is

generated from a point and has $\mathfrak{S} \models \Box^n \perp$ for some n . The last condition holds for a finite \mathfrak{S} iff \mathfrak{S} is *circuit-free*, i.e., it includes no sequence of the form $x_1 R x_2 \cdots R x_k R x_1$ for any k . If $\Lambda_s = \Lambda/\mathfrak{S}$ is a splitting logic, then the corresponding Λ'_s is the logic $\{\alpha: \mathfrak{S} \models \alpha\}$ characterised by \mathfrak{S} .

Every splitting logic is of degree 1, and is finitely axiomatisable. A logic Λ is of degree 1 if and only if it is a *join of splitting logics*, i.e., is equal to the least logic that includes the splitting logics Λ/\mathfrak{S} for all \mathfrak{S} in some collection \mathcal{C} of finite generated circuit-free frames. This is the same as requiring that Λ be the least logic not validated by any member of \mathcal{C} .

Blok used algebraic methods, studying *varieties*, or equationally defined classes, of modal algebras rather than normal logics directly. He applied some powerful new techniques, including the splitting notion that had been developed in lattice theory by Ralph McKenzie [173], and an important lemma of Jónsson [124] characterising subdirectly irreducible algebras in congruence distributive varieties.

Blok's resolution of the issue of incompleteness for Kripke semantics was announced in his abstract [17], but his report [18] giving the detailed proofs was not published. Model-theoretic accounts of the results may be found in [34, Chapter 10] and [141, Chapter 7].

The issue of the adequacy of neighbourhood semantics (see Section 5.3) was investigated in a series of papers by Martin Gerson [79–81], who showed that the two logics of Thomason [254] and Fine [68], which are not characterised by their relational frames, are also incomplete with respect to their neighbourhood frames. He then gave examples of normal logics that are complete under the neighbourhood semantics but not complete for any class of relational frames. These possibilities can also be revealingly expressed in terms of algebraic semantics, beginning with the observation that complete and atomic Boolean algebras are, up to isomorphism, the same thing as powerset algebras. As we observed in Section 5.3, relational frames correspond to completely additive and normal operators on powerset algebras, while neighbourhood frames represent arbitrary operations on such algebras. Thus a logic that is incomplete for the relational semantics is one that is not characterised by those of its complete and atomic algebras whose operators are completely additive and normal; while a logic that is incomplete for the neighbourhood semantics is one that is not characterised by complete and atomic algebras at all.

6.2. Decidability and complexity

The finite model property does not give a universal method for proving the decidability of modal logics. Although every finitely axiomatisable logic with the finite model property is decidable, the converse is not true. This was shown by Dov Gabbay, building on some earlier work of Makinson 1969 [167] which had exhibited the first example of a normal logic that lacked the finite model property. Makinson's example is a proper sublogic of S4, but all of its finite algebras satisfy S4 as well.

Gabbay's 1972 paper [75] extended Makinson's idea to produce finitely axiomatisable modal and tense logics that lacked the finite model property, but could still be shown to be decidable by appealing to a powerful result of Michael Rabin [212]. This concerns the decidability of monadic second-order theories of successor functions, and has many applications. For each ordinal n with $2 \leq n \leq \omega$, consider the structure

$$\mathfrak{S}_n = (T_n, \{s_m: m < n\}, \leq, \approx),$$

where T_n is the n -ary branching tree of all finite sequences of elements of the set $[n] = \{m \in \omega : m < n\}$, s_m is the successor function $x \mapsto xm$ on the tree, \leq is the “initial segment” ordering of sequences, and \preceq is their lexicographical ordering induced by the natural ordering $<$ on $[n]$. Rabin proved that the monadic second-order theory S_nS of the structure \mathfrak{S}_n is decidable. To do this he developed a theory of finite-state automata that process infinite labelled trees, and established the decidability of the *emptiness problem* of whether any given automaton accepts at least one tree. The decidability of S_nS was then reduced to this emptiness problem. It was later shown that the decision problem for S_nS is intractable: Albert Meyer [180] proved that no algorithm for deciding if a sentence is in S_nS can run in elementary time, i.e., time bounded by some fixed number of compositions of exponential functions.

Gabbay developed a method of coding Kripke models into the structure \mathfrak{S}_ω and thereby reducing the decidability problem for certain logics to Rabin’s decidability results for $S_\omega S$. The technique is explained in Part 5 of the book *Gabbay 1976* [76], where it is used to establish decidability results for many modal systems.

Gabbay’s method was later used by Cresswell [43] in adapting an incomplete logic from *van Benthem 1979* [266] to construct a decidable modal logic that is finitely axiomatisable but incomplete with respect to Kripke frames (and hence lacks the finite model property). Cresswell’s example is a proper sublogic of the logic characterised by the class of finite strict linear orderings, but the two logics are validated by exactly the same frames.

For any logic Λ , the problem of deciding if a given formula is Λ -provable is the same as the Λ -*validity problem* of deciding if a given formula is true in all models \mathcal{M} such that $\mathcal{M} \models \Lambda$. The Λ -*satisfiability problem* of whether a given formula is true at some point of some Λ -model is equivalent to the validity problem in the sense that α is Λ -satisfiable iff its negation $\neg\alpha$ is not Λ -valid. Thus a deterministic algorithm that solved the validity problem could be used to solve the satisfiability problem, and vice versa. But if nondeterministic algorithms are considered, the two problems may differ as to their computational complexity. The classic example of this concerns the set of non-modal propositional formulas. Satisfiability of any of these can be tested in nondeterministic polynomial time. But the same is not known for validity: to test the validity of a formula with n variables appears to require examination of all 2^n truth-value assignments to these variables.

To discuss this further, recall that NPTIME, or more briefly NP, is (informally) the class of all problems that are solvable by a nondeterministic algorithm whose running time for any execution is bounded above by some polynomial function of the length of the input. co-NP is the class of problems whose complement is in NP. The Λ -satisfiability problem is in NP iff the Λ -validity problem is in co-NP. The satisfiability of non-modal formulas is NP-*hard*, meaning that any problem in NP has a polynomial-time reduction to this problem *Cook 1971* [37]. The Λ -satisfiability problem for any consistent modal logic Λ is therefore also NP-hard. Since non-modal satisfiability itself belongs to NP, it is said to be an NP-*complete* problem.

PSPACE is the class of problems solvable by a deterministic algorithm using an amount of space that is polynomially bounded by the length of the input. PSPACE includes NPTIME and is closed under complementation. It is also known that any nondeterministic polynomially space-bounded algorithm is equivalent to a deterministic one (*Savitch 1970*

[217]). Thus

$$\text{NP} \subseteq \text{PSPACE} = \text{co-PSPACE} = \text{NPSpace}.$$

It is not known if the stated inclusion is proper, but it is widely believed that PSPACE-complete problems are not in NP.

Richard Ladner [150] applied these concepts to determine computational complexities of some of the basic normal modal logics. He showed that the satisfiability problem for each of the logics K, T, and S4 is in PSPACE, by optimising the space requirements of the decision procedures from Kripke's [145]. Hence the provability problems for these logics is in PSPACE as well. He proved further that any problem in PSPACE has a polynomial time reduction⁴⁶ to the provability problem of any normal sublogic of S4. Thus provability for any of these logics is PSPACE-hard, and for K, T, and S4 it is PSPACE-complete. The method used was to reduce to Λ -provability a known PSPACE-complete problem, namely the validity of quantified non-modal propositional formulas.

The logic S5 is more tractable than the sublogics of S4. Ladner showed that S5-satisfiability is in NP, and therefore is NP-complete. The key to this result is that S5 has the *poly-size model property*: any non-theorem is falsifiable in a model whose size is a polynomial in the size of the formula. Edith Spaan [236] extended this to prove that every one of the (\aleph_0 many) extensions of the logic S4.3 has the poly-size model property and has an NP-complete satisfiability problem. On the other hand Joseph Halpern and Yoram Moses [103,104] showed that satisfiability for any logic having at least two S5-modalities is PSPACE-hard.

As to *undecidability*, there must be undecidable logics because there are uncountably many logics altogether but only countably many algorithms. In Thomason's [258] an undecidable modal logic is exhibited that is finitely axiomatisable, and so cannot have the finite model property. This was produced by encoding a presentation of a recursive function with undecidable range into a model of a logic with a large number of temporal modalities, and then reducing this to a logic with one modality by methods that are described below in Section 6.4.

The question of how undecidable a logic can be was answered by Alasdair Urquhart [261] who showed that for any set X of natural numbers there exists a normal modal logic Λ_X such that the decision problem for X is reducible to that of Λ_X . Urquhart used this to construct a logic with the finite model property that has a decidable set of axioms but is undecidable. Spaan [236] showed that there are (uncountably many) undecidable logics that have the poly-size model property.

Undecidability of quantificational modal logic was considered by Kripke [144] in an early application of his model theory from [142]. Whereas the first-order calculus of monadic predicates is decidable, the modal monadic calculus turns out to be undecidable. Kripke showed that the decision problem for provability of non-modal first-order formulas in a binary predicate R , which is known to be undecidable, is reducible to that of modal

⁴⁶ Actually he showed that these reductions are in "log-space": they have a space requirement bounded by a logarithmic function of the length of the input. This implies a polynomial time-bound. Ladner originally proved the reduction result for T and for S4, and subsequently used an argument of S.K. Thomason to extend it to all normal sublogics of S4.

formulas in two monadic predicates P and Q , by replacing $R(x, y)$ by $\diamond(P(x) \wedge Q(y))$. This applies to any modal system which is a sublogic of the quantificational version of S5 of [142] and which obeys certain general rules satisfied by all then known systems and “probably by the vast majority of those that will be proposed in the future”.

6.3. First-order definability

Validity of a modal formula α in a relational frame $\mathfrak{S} = (K, R)$ is an intrinsically *second-order* concept. α is valid when true at all points in all models on \mathfrak{S} . Since a model interprets each propositional variable p in α as a subset of K , this amounts to treating p as a set variable, or a *monadic predicate* variable. Meredith’s U -calculus associates with α a formula $(\alpha)x$ in the first-order language of \mathfrak{S} , with x as its sole free individual variable. If the propositional variables of α are p_1, \dots, p_k , then regarding these as set variables we have that α is valid in \mathfrak{S} iff \mathfrak{S} is a model of the sentence

$$\forall p_1 \cdots \forall p_k \forall x (\alpha)x$$

of the monadic second-order language of a binary predicate, i.e., the second-order language in which all the second-order variables are monadic. This is a simple kind of second-order sentence, technically known as Π_1^1 , with all its second-order quantifiers being universal and at the front.

Some modal formulas express properties that are well-recognised as being second-order in nature. For example, Segerberg’s axiom W is valid in \mathfrak{S} iff R^{-1} is transitive and *well-founded* (see Boolos 1979 [22, p. 82]). However, a substantial reason for the great success of the relational semantics is that many logics were shown to be characterised by frames satisfying simple *first-order* conditions on R , like reflexivity, transitivity, linearity etc. To consider this phenomenon, recall that a class of relational frames is called *elementary* if it is definable in first-order logic, i.e., if it is the class of all models of some set of sentences in the first-order language of a binary predicate R . A *basic elementary* class is one that is defined by a single first-order sentence.⁴⁷ A modal logic is (*basic*) *elementary* if it is characterised by some (basic) elementary class of frames.

The Lemmon Notes provided many examples of basic elementary logics, and formulated a conjecture about the situation, which will now be briefly described. First we say that a modal formula is *positive* if it can be built from propositional variables using only the connectives \wedge , \vee , \diamond , and \Box . If β is any positive formula with variables p_1, \dots, p_k and $m = (m_1, \dots, m_k)$ and $n = (n_1, \dots, n_k)$ are any k -tuples of natural numbers, consider the formula

$$\beta_n^m : \diamond^{m_1} \Box^{n_1} p_1 \wedge \cdots \wedge \diamond^{m_k} \Box^{n_k} p_k \rightarrow \beta.$$

Associated with β_n^m is a certain first-order condition $R\beta_n^m$ on binary relations, which can be read off from the formation of β_n^m itself. The conjecture was that the normal logic axiomatised by adding β_n^m to K is characterised by the basic elementary class of frames

⁴⁷ Some authors use “ Δ -elementary” in place of “elementary”, and “elementary” in place of “basic elementary”.

satisfying $R\beta_n^m$ (see [156, p. 78]). This was confirmed independently by the present author and Henrik Sahlqvist in 1973 (see [85,87] and [215]), but Sahlqvist generalised the result considerably to consider any formula of the type $\Box^n(\alpha \rightarrow \beta)$ where $n \geq 0$, β is positive, and α is constructed from propositional variables and/or their negations using only the connectives $\wedge, \vee, \diamond, \Box$ in such a way that no positive occurrence of a variable is in a subformula that has \wedge, \vee , or \diamond within the scope of a \Box . He proved that the class of frames validating such a formula is definable by an explicit first-order sentence, and that this basic elementary class characterises the normal logic axiomatised by adding the formula to K . The result has been extensively analysed and extended to “polymodal” logics and to equational classes of BAOs in general: see [51,82,126,216].

The simplest formula not covered by Sahlqvist’s scheme is

$$M: \Box \diamond p \rightarrow \diamond \Box p,$$

commonly known as the *McKinsey axiom*.⁴⁸ This is the \Box -version of the formula $GFp \rightarrow FGP$ that figures as an axiom in Thomason’s incomplete tense logic. In the Lemmon Notes a proof was given that the normal logic $S4 + M$ is characterised by the elementary class of all quasi-ordered frames satisfying the condition

$$\forall x \exists y (x R y \wedge \forall z (y R z \rightarrow y = z)).$$

Seegerberg [220] then showed that this logic has the finite model property and is characterised by the finite quasi-orders satisfying this condition. But the status of the logic $K + M$ remained unresolved.

It turned out that the class of all frames validating the McKinsey axiom is not elementary, let alone basic elementary. This was proved in *Goldblatt 1974* [85] (Section 17), which showed further that no elementary class can characterise the logic $K + M$, and indeed any class that does characterise this logic must fail to be closed under ultraproducts. van Benthem [262] gave a Löwenheim–Skolem argument to show that the class of all frames validating M is not even closed under elementary equivalence.⁴⁹ On the other hand Fine [69] proved that the logic $K + M$ is in some respects quite well-behaved: it has the finite model property, so is decidable and is characterised by its (finite) validating frames.

From such examples the question naturally arises of when the collection $Fr(\alpha) = \{\mathfrak{S}: \mathfrak{S} \models \alpha\}$ of all frames validating the formula α is an elementary class. To answer this, note first that the complement of $Fr(\alpha)$ is always closed under ultraproducts. That can be shown directly, or by observing that the complement of $Fr(\alpha)$ is defined by an existential second-order sentence

$$\exists p_1 \dots \exists p_k \exists x \neg(\alpha)x$$

⁴⁸ This something of a misnomer. The system $S4 + \Box \diamond p \wedge \Box \diamond q \rightarrow \Box \diamond (p \wedge q)$ was investigated by McKinsey [175], who called it $S4.1$. Sobociński [234] showed that it is the same as $S4 + (\Box \diamond p \rightarrow \diamond \Box p)$, and renamed it $K1$, since it is not a subsystem of $S4.2$.

⁴⁹ Two structures are elementarily equivalent when they satisfy the same first-order sentences.

of the kind (Σ_1^1) that is always preserved by ultraproducts.⁵⁰ From this it follows by the Keisler–Shelah characterisation of elementary classes⁵¹ that *Fr*(α) is elementary iff it is basic elementary iff it is closed under ultraproducts, as deduced by this author in [85,86]. But then van Benthem discovered a striking strengthening of the result:

Fr(α) is basic elementary iff it is closed under elementary equivalence.

This means that any class of the form *Fr*(α) is quite special: if it is closed under ultrapowers then it must be closed under ultraproducts. van Benthem’s proof was an interesting model-theoretic compactness argument,⁵² but in his published version [264] he used instead a subsequent argument of the present author, namely that there is an injective p-morphism

$$\left(\prod_J \mathfrak{S}_j\right)/F \longrightarrow \left(\coprod_J \mathfrak{S}_j\right)^J/F$$

of any ultraproduct of frames \mathfrak{S}_j into the associated ultrapower of their disjoint union $\coprod_J \mathfrak{S}_j$, and this maps the ultraproduct isomorphically onto a *subframe* of the ultrapower. Since *Fr*(α) is invariably closed under disjoint unions, subframes and isomorphism, the desired result follows immediately from this embedding. But the argument also works for the class *Fr*(Λ) of all frames validating a *set* Λ of formulas, to show that

Fr(Λ) is elementary iff it is closed under elementary equivalence.

The study of the definability of modal formulas in predicate logic was dubbed *Correspondence Theory* by van Benthem in his thesis [263], who gave further expositions of this theory in his works of [268] and [269].

6.4. Thomason’s second-order reduction

A deep investigation of the expressive power of modal semantics was made by Steve Thomason in a series of papers [255,257–259] reporting work, carried out in 1973, that constitutes a tour de force of model-theoretical analysis in combination with coding techniques of the kind used in recursion theory. This confirmed his belief, expressed earlier in [252], that

propositional modal logic (with the usual relational semantics) must be understood as a rather strong fragment of (classical) second-order predicate logic.

A “logic” is taken to consist of a symbolic language together with a semantic interpretation specifying when a formula is *valid* in a *structure*. **M** is the logic given by the language of propositional modal logic with the semantics based on frames (K, R) as structures, while **T** is the propositional tense logic of Prior’s *PF*-language with structures (K, R^{-1}, R) . Each logic determines a *logical consequence* relation $\Gamma \models \alpha$ between sets of formulas Γ and

⁵⁰ Chang and Keisler [35, Corollary 4.1.14].

⁵¹ Ibid., Corollary 6.1.16.

⁵² A discussion of van Benthem’s original proof is presented in Goldblatt 1999 [97].

formulas α , meaning that α is valid in every structure in which all members of Γ are valid. Thomason proved in [252] that the Compactness Theorem fails in **M** for this relation: there is a case of an α which is a logical consequence of some set Γ but not of any finite subset of Γ . In the paper [257] he showed that there is a **T**-formula γ whose set $\{\alpha: \gamma \models \alpha\}$ of logical consequences is not effectively enumerable, and has a high degree of undecidability—technically what is known as a *complete* Π_1^1 set. Moreover γ is *categorical* in the sense that all its connected validating structures are isomorphic. In addition, for $0 \leq m < \omega + \omega$ there is a categorical formula γ_m whose unique validating structure has size \beth_m , where $\beth_0 = \aleph_0$, $\beth_{m+1} = 2^{\beth_m}$, and $\beth_\omega = \lim\{\beth_m: m < \omega\}$. The formula γ describes a structure which encodes presentations of certain recursive functions that define a complete Π_1^1 set. The formulas γ_m describe structures that encode copies of the iterated powersets ω , $\mathcal{P}(\omega)$, $\mathcal{P}(\mathcal{P}(\omega))$, \dots . The proofs of these facts are reminiscent of the arithmetisation procedures and expressibility results involved in Gödel’s incompleteness theorems, and graphically illustrate the expressive power of **T**. The facts themselves are quite contrary to the situation in first-order logic, where the logical consequences of a given sentence are effectively enumerable, and no sentence with an infinite model is categorical.

A logic **L**₁ is said to be *reducible* to a logic **L**₂ if there exists an **L**₂-formula δ and an effective transformation ψ of **L**₁-formulas to **L**₂-formulas such that for every collection $\Gamma \cup \{\alpha\}$ of **L**₁-formulas,

$$\Gamma \models \alpha \quad \text{iff} \quad \{\delta\} \cup \{\psi(\gamma): \gamma \in \Gamma\} \models \psi(\alpha).$$

This definition captures the idea that **L**₁ can be regarded as a fragment of the logic **L**₂, and is motivated by a notion of interpretation of one first-order theory in another that appears in *Shoenfield 1967* [230]. Here δ may be thought of as describing a certain structure, with $\psi(\gamma)$ asserting that γ is valid in that structure. In [255] it is shown that tense logic **T** is reducible to modal logic **M**. The formula δ used for this has the property that for any **T**-structure $\mathfrak{S} = (K, R^{-1}, R)$ there is an **M**-structure \mathfrak{S}' that contains within it definable copies of (K, R) and (K, R^{-1}) in such a way that “*P*” statements about \mathfrak{S} can be interpreted as “ \diamond ” statements about \mathfrak{S}' . Applying this reduction to the results about **T** from [257], Thomason concludes that there is an **M**-formula whose set of logical consequences is a complete Π_1^1 set.

The full monadic second-order theory **S** of a binary predicate is shown to be reducible to **M** in [258]. For this purpose the logic **T**_{*n*} of *n* temporal orderings is introduced. It has *n* pairs of modalities $P_1, F_1, \dots, P_n, F_n$, and structures having *n* binary relations and their inverses to interpret these connectives. It is shown that for $n > 1$, **T**_{*n*} is reducible to **T**_{*n*−1}. Since reducibility is a transitive relation, it follows that each **T**_{*n*} is reducible to **T** (= **T**₁), and hence reducible to **M**. This is then applied to prove the reducibility of **S**. The argument involves defining a **T**₁₅-formula δ with the property that for each frame $\mathfrak{S} = (K, R)$ there is a model of δ with 15 temporal orderings that includes within it definable copies of \mathfrak{S} ; the powerset $\mathcal{P}(K)$; the membership relation from K to $\mathcal{P}(K)$; the set of all (codes for) **S**-formulas, the set of all assignments in K and $\mathcal{P}(K)$ to the individual and set variables of **S**; and the satisfaction relation between **S**-formulas and assignments in \mathfrak{S} as a second-order model. This leads to a reduction of **S** to **T**₁₅, which can then be combined with the reduction of **T**₁₅ to **M** to give the desired result. Thomason concludes that

the logical consequence relation of propositional modal logic (with the Kripke relational semantics) is as complex as it could possibly be.

6.5. Duality and the calculus of class operations

The keystone constructions in the general theory of algebras are homomorphic images, subalgebras, and direct products. The famous Variety Theorem due to Garrett Birkhoff 1935 [14] states that a class of abstract algebras is a *variety*, i.e., is definable by equations, iff it is closed under these three constructions. The standard convention in this subject is to use the letters **H**, **S** and **P** for the operations that assign to each class of algebras its closure under homomorphic images, subalgebras, and direct products, respectively. Thus Birkhoff's theorem states that a class \mathcal{A} of algebras is a variety if and only if $\mathbf{H}\mathcal{A} \subseteq \mathcal{A}$ and $\mathbf{S}\mathcal{A} \subseteq \mathcal{A}$ and $\mathbf{P}\mathcal{A} \subseteq \mathcal{A}$. A refinement due to Tarski [247,248] is that for each class \mathcal{A} of algebras, $\mathbf{H}\mathbf{S}\mathbf{P}\mathcal{A}$ is the smallest variety that includes \mathcal{A} . Hence $\mathbf{H}\mathbf{S}\mathbf{P}\mathcal{A}$ is known as the variety *generated* by \mathcal{A} .

The corresponding constructions for relational modal semantics are subframes, p-morphic images, and disjoint unions. As explained in Section 5.3, a p-morphism $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}'$ induces an algebraic homomorphism $\varphi^+ : \mathbf{Cm}\mathfrak{S}' \rightarrow \mathbf{Cm}\mathfrak{S}$, allowing us to show that if \mathfrak{S} is (isomorphic to) a subframe of \mathfrak{S}' then $\mathbf{Cm}\mathfrak{S}$ is a homomorphic image of $\mathbf{Cm}\mathfrak{S}'$, and if \mathfrak{S}' is a p-morphic image of \mathfrak{S} then $\mathbf{Cm}\mathfrak{S}'$ is (isomorphic to) a subalgebra of $\mathbf{Cm}\mathfrak{S}$. Disjoint unions of structures correspond naturally to direct products of algebras via an isomorphism

$$\mathbf{Cm} \coprod_J \mathfrak{S}_j \cong \prod_J \mathbf{Cm}\mathfrak{S}_j \quad (1)$$

between the complex algebra of a disjoint union and the direct product of the complex algebras of its factors.

The assignments $\mathfrak{S} \mapsto \mathbf{Cm}\mathfrak{S}$ and $\varphi \mapsto \varphi^+$ form a contravariant functor from the category **Frm** of frames and p-morphisms to the category **Malg** of normal modal algebras and homomorphisms. In the reverse direction there is a construction that assigns to each normal BAO \mathfrak{A} a certain relational structure $\mathbf{Cst}\mathfrak{A}$, called the *canonical structure* of \mathfrak{A} , whose points are the ultrafilters of \mathfrak{A} . The complex algebra $\mathbf{Em}\mathfrak{A} = \mathbf{Cm}\mathbf{Cst}\mathfrak{A}$ of this structure is the *canonical embedding algebra* of \mathfrak{A} , and is isomorphic to the perfect extension \mathfrak{A}^σ , as described in Section 3.3. The Jónsson–Tarski representation of \mathfrak{A} amounts to the fact that there is an injective homomorphism $\mathfrak{A} \hookrightarrow \mathbf{Em}\mathfrak{A}$.

When applied to modal algebras, the assignment $\mathfrak{A} \mapsto \mathbf{Cst}\mathfrak{A}$ gives rise to a contravariant functor from **Malg** to **Frm** that takes each homomorphism $\theta : \mathfrak{A}' \rightarrow \mathfrak{A}$ to a p-morphism $\mathbf{Cst}\mathfrak{A}' \rightarrow \mathbf{Cst}\mathfrak{A}$ which maps each ultrafilter of \mathfrak{A}' to its θ -inverse image in \mathfrak{A} . These functors provide a *duality* between frames and modal algebras. It is not however a dual *equivalence*, because we do not in general have \mathfrak{S} isomorphic to $\mathbf{Cst}\mathbf{Cm}\mathfrak{S}$, or \mathfrak{A} isomorphic to $\mathbf{Cm}\mathbf{Cst}\mathfrak{A}$: the assignment $\mathfrak{S} \mapsto \mathbf{Cm}\mathfrak{S}$ increases cardinality, as does $\mathfrak{A} \mapsto \mathbf{Cst}\mathfrak{A}$ for infinite \mathfrak{A} .

The category **Frm** is dually equivalent to the category of *complete and atomic* modal algebras with \sum -preserving homomorphisms (Thomason [256]). To obtain a category of structures equivalent to **Malg** it is necessary to modify the notion of “frame”. A first at-

tempt at this was made by Makinson [168] who defined a *relational model* as a structure (K, R, \mathcal{H}) , where \mathcal{H} is a collection of truth-valuations Φ on (K, R) in Kripkes sense that satisfies certain closure properties. That did not produce a full equivalence between algebras and models. A language independent-approach was taken by Thomason [253] who defined a “first-order semantics” using structures $\mathfrak{S} = (K, R, P)$, where P is a collection of subsets of K that forms a subalgebra of the full complex algebra $\text{Cm}(K, R)$. This subalgebra is taken in place of $\text{Cm}(K, R)$ as the algebra assigned to \mathfrak{S} . Validity in \mathfrak{S} is defined as truth in all models $\mathcal{M} = (\mathfrak{S}, \Phi)$ on \mathfrak{S} satisfying the constraint that the set $\mathcal{M}(p) = \{x: \Phi(p, x) = \top\}$ belongs to P for all variables p .

By imposing suitable restrictions on P , essentially set-theoretic versions of the conditions (i)–(iii) of Section 3.3 that defined the Jónsson-Tarski perfect extensions, a notion of “descriptive” frame (K, R, P) is arrived at. This theory was developed in *Goldblatt 1974* [85], where the descriptive frames were shown to form a category dually equivalent to **Malg**. A topological approach to duality for closure algebras and quasi-orderings was independently investigated by Leo Ésakia [62].

Connections between relational structures and algebras can be conveniently expressed in the “calculus” of class operations. We use the symbols \mathbb{S} , \mathbb{H} , and \mathbb{Ud} for the operations of closing a class of structures under subframes, p-morphic images, and disjoint unions, respectively. \mathbb{Pu} and \mathbb{Pw} are used for closure under ultraproducts and ultrapowers, while

$$\text{Cm } \mathcal{C} = \{\mathfrak{A}: \mathfrak{A} \cong \text{Cm } \mathfrak{S} \text{ for some } \mathfrak{S} \in \mathcal{C}\}$$

is the class of all (isomorphic copies of) complex algebras of structures in the class \mathcal{C} . Then the isomorphism (1) above implies that $\text{Cm } \mathbb{Ud } \mathcal{C} = \mathbb{P } \text{Cm } \mathcal{C}$ for any class \mathcal{C} of frames. Similarly, the representation

$$\left(\prod_j \mathfrak{S}_j\right) / F \longrightarrow \left(\bigsqcup_j \mathfrak{S}_j\right)^J / F$$

from Section 6.3 of an ultraproduct of frames as a subframe of an ultrapower of a disjoint union yields the conclusion that in general

$$\mathbb{Pu } \mathcal{C} \subseteq \mathbb{SPw } \mathbb{Ud } \mathcal{C}.$$

There are numerous properties that can be express in this way using class operations, for example

$$\begin{aligned} \mathbb{SHC} &\subseteq \mathbb{LSC}, & \mathbb{S } \text{Cm } \mathbb{HC} &= \mathbb{S } \text{Cm } \mathcal{C}, \\ \mathbb{S } \mathbb{Ud } \mathcal{C} &= \mathbb{Ud } \mathbb{S } \mathcal{C}, & \mathbb{Pu } \mathbb{SHC} &\subseteq \mathbb{HSPu } \mathcal{C}. \end{aligned}$$

An inventory of such facts may be found in [96,98].

Dual to the formation of the algebra $\text{Em } \mathfrak{A} = \text{Cm } \mathbb{Cst } \mathfrak{A}$ is the association with any structure \mathfrak{S} of its *canonical extension* $\mathbb{Ex } \mathfrak{S} = \mathbb{Cst } \text{Cm } \mathfrak{S}$, a structure whose points are the ultrafilters on the underlying set of \mathfrak{S} (hence $\mathbb{Ex } \mathfrak{S}$ is sometimes called the *ultrafilter extension* of \mathfrak{S}). There is a p-morphism

$$\mathfrak{S}^J / F \twoheadrightarrow \mathbb{Ex } \mathfrak{S}$$

from a suitably chosen ultrapower of any given frame \mathfrak{S} onto $\mathbb{E}x \mathfrak{S}$, yielding the observation that in general

$$\mathbb{E}x \mathcal{C} \subseteq \text{HIPw } \mathcal{C}. \quad (2)$$

The proof of this requires the choice of a sufficiently saturated ultrapower of \mathfrak{S} (see [92, Section 3.6]) and is motivated by a model construction of Fine [70] that is discussed further in the next section.

Duality can be used to bring methods of universal algebra to bear on relational semantics. A notable example is the problem of characterising classes of the form $Fr(\Lambda)$, the class of all frames validating a set Λ of modal formulas. The question of when $Fr(\Lambda)$ is elementary was discussed in Section 6.3. It is natural to ask, conversely, for conditions under which a given elementary class of frames is equal to the class $Fr(\Lambda)$ for some Λ . The following answer was given in *Goldblatt and Thomason 1975* [100], where the $\mathbb{E}x$ construction was first introduced (see also [95, 20.6], [92, 3.7.6(2)]).

If \mathcal{C} is an elementary class of frames, then \mathcal{C} is equal to $Fr(\Lambda)$ for some set Λ of modal formulas if, and only if,

- (i) \mathcal{C} is closed under disjoint unions, p -morphic images and subframes; and
- (ii) the complement of \mathcal{C} is closed under canonical extensions, i.e., $\mathbb{E}x \mathfrak{S} \in \mathcal{C}$ implies $\mathfrak{S} \in \mathcal{C}$.

The proof applies the Birkhoff–Tarski analysis of varieties to the variety generated by $\text{Cm } \mathcal{C}$, and uses the construction for (2) above to show that if \mathcal{C} is elementary and closed under p -morphic images then it is closed under canonical extensions.

Duality theory has been developed by this author for arbitrary relational structures and BAOs by using suitable generalisations of p -morphisms and subframes, called “bounded” morphisms and “inner” substructures [92,96]. This provides algebraic and relational semantics for *polymodal* languages having n -ary connectives which generate formulas $\Box(\alpha_1, \dots, \alpha_n)$ for $n > 1$. Most of the ideas and results we have discussed about completeness, canonicity, elementarity, class operations etc. carry over to this broader context and apply to cylindric algebras, relation algebras and other kinds of BAOs in addition to modal algebras. This reveals that, mathematically, much of modal semantics is just the case $n = 1$ of a broader structural theory of finitary operators on lattices. A survey of this general theory is given in [98].

If Λ is a normal logic, then the class $V(\Lambda)$ of modal algebras that satisfy all Λ -theorems is a variety. Algebraic constructions in $V(\Lambda)$ provide tools for studying metalogical questions about Λ , such as whether it fulfills analogues of the Beth Definability Theorem and the Craig Interpolation Theorem. This is related to amalgamation properties of algebras in $V(\Lambda)$, as has been shown by Larisa Maksimova, whose article [170] gives an account of the subject and further references to the literature.

6.6. Canonicity

A logic is called *canonical* if it is valid in its canonical frame \mathfrak{S}^A , in which case it is characterised by this frame, and so is complete. Almost all proofs that a particular logic is

elementary have consisted of a demonstration that \mathfrak{S}^A satisfies some first-order conditions that imply validity of Λ . Such a proof establishes also that Λ is canonical, a conclusion that is inescapable in view of the following profound results of Kit Fine [70], proven in 1973.

- (i) *If the class $Fr(\Lambda)$ of all Λ -frames is closed under elementary equivalence and characterises Λ (i.e., Λ is complete), then Λ is canonical.*
- (ii) *If Λ is elementary (i.e., characterised by some elementary class), then Λ is canonical.*⁵³

In fact something much stronger was proved. We have been using a language for propositional modal logic that is based on a countably infinite set of variables, but we could consider larger languages by assuming we have available a variable p_ξ for each ordinal ξ . Then for a given ordinal η we can generate the set $Form(\eta)$ of modal formulas having variables from the set $\{p_\xi: \xi < \eta\}$. A logic Λ as originally conceived is a subset of $Form(\omega)$, but it has a manifestation $\Lambda_\eta \subseteq Form(\eta)$ for each η , obtained by closing Λ under uniform substitution in $Form(\eta)$ when $\omega < \eta$, and by putting $\Lambda_\eta = \Lambda \cap Form(\eta)$ when $\eta < \omega$. Then we can define a canonical frame \mathfrak{S}_η^A for each η , based on the maximally Λ_η -consistent subsets of $Form(\eta)$. \mathfrak{S}_η^A is of cardinality $2^{\text{card}\eta}$. If it validates Λ_η , we say that Λ is η -canonical.

Fine proved that under each of the hypotheses given in (i) and (ii), Λ is η -canonical for all ordinals η . He also gave an example of a logic that is η -canonical for all η , and is elementary, but for which $Fr(\Lambda)$ is not closed under elementary equivalence. Thus the converse of (i) is false.

The idea of the proof of (i) was to use disjoint unions to obtain a single model \mathcal{M} that characterised Λ_η and was based on a Λ_η -frame, then to view \mathcal{M} as a first-order model and take a saturated elementary extension of it that could be mapped onto the canonical frame \mathfrak{S}_η^A by a p-morphism. This was the first application of saturated models to modal logic, and it motivated the construction for result (2) of the previous section. The proof of (ii) combined it with an additional ultraproduct construction.

Canonicity of a logic Λ is intimately connected with the question of whether satisfaction of Λ is preserved by perfect extensions $\text{Em } \mathfrak{A} = \text{Cm } \text{Cst } \mathfrak{A}$ of algebras or canonical extensions $\text{Ex } \mathfrak{S} = \text{Cst } \text{Cm } \mathfrak{S}$ of frames. van Benthem [267] refined the proof of Fine’s result (ii) above to show that

if a logic Λ is elementary, then the class $Fr(\Lambda)$ of all Λ -frames is closed under canonical extensions, i.e., $\mathfrak{S} \models \Lambda$ implies $\text{Ex } \mathfrak{S} \models \Lambda$.

Another way to describe this conclusion is to say that if $\text{Alg}(\Lambda)$ is the variety (equational class) of all modal algebras satisfying Λ , then in general $\text{Cm } \mathfrak{S} \in \text{Alg}(\Lambda)$ implies $\text{Cm } \text{Ex } \mathfrak{S} \in \text{Alg}(\Lambda)$. But $\text{Cm } \text{Ex } \mathfrak{S} = \text{Em } \text{Cm } \mathfrak{S}$, so the conclusion says that $\text{Alg}(\Lambda)$ con-

⁵³ At the time, (i) was not recognised as a consequence of (ii). However, as explained at the end of Section 6.3, it was later discovered that closure of $Fr(\Lambda)$ under elementary equivalence implies the ostensibly stronger assertion that $Fr(\Lambda)$ is elementary. So (ii) does imply (i).

tains the canonical embedding algebras of all its full complex algebras. This can then be strengthened, by applying duality theory, to show that $Alg(\Lambda)$ contains the algebra $Em \mathfrak{A}$ for any of its members \mathfrak{A} [92, Theorem 3.5.5]). Actually, to conclude that $Alg(\Lambda)$ is closed under canonical embedding algebras it is enough to know that Λ is valid in the canonical frame $\mathfrak{S}_\kappa^\Lambda$ for all infinite cardinals κ . This follows by duality from the fact that $\mathfrak{S}_\kappa^\Lambda$ is isomorphic to the canonical structure $Cst \mathfrak{A}_\kappa$, where \mathfrak{A}_κ is the free algebra in $Alg(\Lambda)$ on κ -many generators, together with the fact that each member of $Alg(\Lambda)$ is a homomorphic image of some such free algebra.

Ultimately this analysis can be generalised to any kind of Boolean algebra with operators, to yield the following result:

if \mathcal{C} is any class of relational structures of the same type that is closed under ultraproducts, then the variety of BAOs generated by the class of algebras $Cm \mathcal{C}$ is closed under canonical embedding algebras.

This theorem was first formulated in *Goldblatt 1989* [92, Theorem 3.6.7], with a proof that used the important result of Jónsson [124] on subdirectly irreducible algebras in congruence-distributive varieties and an obscure diagonal construction on ultraproducts. An entirely different argument was given in [94] and analysed further in [96]. It used the fact (2) from the previous section, i.e., $Ex \mathcal{C} \subseteq HPPw \mathcal{C}$, and another formula,

$$Cst H S P Cm \mathcal{C} \subseteq SLUdPu \mathcal{C},$$

which shows how the canonical structures of algebras from the variety generated by $Cm \mathcal{C}$ can themselves be built from members of \mathcal{C} . When \mathcal{C} is closed under ultraproducts, so that $Pu \mathcal{C} = \mathcal{C}$, this takes the form

$$\mathfrak{A} \in H S P Cm \mathcal{C} \text{ implies } Cst \mathfrak{A} \in SLUd \mathcal{C},$$

showing how canonical structures mediate between the dual operations on algebras and structures. This result in turn depends on another fundamental fact,

$$Pu Ub \mathcal{C} \subseteq Ub Pu \mathcal{C},$$

which states that the ultraproduct operation commutes with *bounded* unions. A structure \mathfrak{S} is the bounded union of a collection $\{\mathfrak{S}_j : j \in J\}$ if the \mathfrak{S}_j s are all inner substructures (subframes) of \mathfrak{S} and their union is \mathfrak{S} itself. This notion is dual to that of *subdirect* product, and indeed in the situation just described there is a subdirect product representation

$$Cm \mathfrak{S} \twoheadrightarrow \prod_J Cm \mathfrak{S}_j$$

of $Cm \mathfrak{S}$ induced by the surjections $Cm \mathfrak{S} \twoheadrightarrow Cm \mathfrak{S}_j$ (see [98, Section 4.5]).

The first example of non-canonicity in the modal context occurs in *Kripke 1967* [149], where it is stated that Dummett's Diodorean axiom

$$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond \Box p \rightarrow \Box p)$$

is not preserved by the Jónsson–Tarski representation of modal algebras. The McKinsey axiom $\Box \Diamond p \rightarrow \Diamond \Box p$ was shown not to be canonical in *Goldblatt 1991* [93].

The formulas of Sahlqvist (see Section 6.3) define logics Λ for which the class $Fr(\Lambda)$ is elementary and includes all the canonical frames $\mathfrak{S}_\eta^\Lambda$. These formulas have been generalized by Maarten de Rijke and Yde Venema [51], who defined *Sahlqvist equations* for any type of BAO and showed that the structures \mathfrak{S} whose complex algebras $\text{Cm } \mathfrak{S}$ satisfy such an equation form a basic elementary class. Jónsson [126] has refined the techniques of [128] to develop an elegant algebraic proof that varieties of BAOs defined by Sahlqvist equations are closed under canonical embedding algebras.

The converse of Fine's theorem (ii) is a perplexing open question. If true, it would provide a satisfactory explanation of the observed connections between intensional and first-order logic. But it is not known whether a logic that is ω -canonical, or η -canonical for all η , must be elementary. No counter-example has been found among the numerous canonically closed varieties of modal algebras, cylindric algebras and relation algebras that have been investigated over the years. If it should turn out that every ω -canonical logic is elementary, then it would follow from Fine's work that every ω -canonical logic is η -canonical for all $\eta > \omega$. This is not known. All that is known is that there are logics that are η -canonical for all $\eta < \omega$, but not ω -canonical.

One approach to the problem would be to show that if Λ is valid in $\mathfrak{S}_\omega^\Lambda$, then it is valid in every structure elementarily equivalent to $\mathfrak{S}_\omega^\Lambda$, which would imply that it is characterised by the elementary class of all models of the first-order theory of $\mathfrak{S}_\omega^\Lambda$. The appropriateness of that approach is demonstrated by the following result from of this author [95, (11.3.1)] and [96, (4.15)]:

if a logic Λ is characterised by some elementary class, then it is characterised by the elementary class of structures elementarily equivalent to $\mathfrak{S}_\omega^\Lambda$.

This is a strengthening of Fine's theorem. There are further strengthenings available in the references just cited, and also some results in [99] about the first-order equivalence of the various canonical structures $\mathfrak{S}_\eta^\Lambda$, but the full relationship between canonicity and elementarity remains a matter of conjecture.

7. Some mathematical modalities

The seed of relational semantics sown in the 1950s has grown into a tree with many branches. The most notable new dimension of activity beyond that already described has been the application of relational modal semantics to a range of formalisms of computational and mathematical interest. This final section will briefly survey some studies of this kind, providing a sketch of the key ideas and a guide to the literature.

7.1. Dynamic logic of programs

Dynamic logic was invented by Vaughan Pratt, who described its origins in [201] as follows.

In the spring of 1974 I was teaching a class on the semantics and axiomatics of programming languages. At the suggestion of one of the students, R. Moore, I considered applying modal logic to a formal treatment of a construct due to C.A.R. Hoare, “ $p\{a\}q$ ”, which expresses the notion that if p holds before executing program a , then q holds afterwards. Although I was skeptical at first, a weekend with Hughes and Cresswell convinced me that a most harmonious union between modal logic and programs was possible. The union promised to be of interest to computer scientists because of the power and mathematical elegance of the treatment. It also seemed likely to interest modal logicians because it made a well-motivated and potentially very fruitful connection between modal logic and Tarski’s calculus of binary relations.⁵⁴

Pratt’s idea was to assign a box-modality $[\pi]$ to each program π , with the formula $[\pi]\alpha$ being read “after π , α ”. Then Hoare’s construct⁵⁵ $p\{\pi\}q$ can be defined as $p \rightarrow [\pi]q$, but more complex assertions about program correctness and termination can be formalised by combining $[\pi]$ with other connectives, including modalities for other programs. The connective $[\pi]$ is interpreted, not as an accessibility relation between possible worlds, but as a transition relation R_π between “possible execution states”, with $x R_\pi y$ when there is an execution of π that starts in state x and terminates in state y . The dual modality $\langle \pi \rangle \alpha$, definable as $\neg[\pi]\neg\alpha$, asserts that there is an execution of π that terminates with α true. In particular, $\langle \pi \rangle \top$ asserts that there exists a terminating execution of program π .

Pratt’s first paper in 1976 [199] describes a predicate language with modalities for a class of programs generated from basic assignments and tests by a number of operations, including *alternation* $\pi \cup \pi'$ and *composition* $\pi; \pi'$. The interpreting relations for programs satisfy appropriate conditions, including $R_{\pi \cup \pi'} = R_\pi \cup R_{\pi'}$ and $R_{\pi; \pi'} = R_\pi \circ R_{\pi'}$. A complete axiomatisation was presented for the language of these “loop-free” programs, and then the class of *regular* programs was defined by adding the *iteration* construct π^* , with interpretation $R_{\pi^*} =$ reflexive transitive closure of R_π . The universal quantifier $\forall x$ was identified with a modality $[x \leftarrow \text{RANDOM}]$ corresponding to a random assignment to the variable x .

The purely propositional fragment of this language was isolated by Michael Fischer and Richard Ladner [71,72] who defined the system PDL of *propositional dynamic logic of regular programs*. Its programs are generated from some set of atomic commands by the operations of alternation, composition and iteration. A Kripke model for PDL assigns a binary relation to each atomic program, and then interprets complex programs by the above conditions on $R_{\pi \cup \pi'}$, $R_{\pi; \pi'}$ and R_{π^*} . Fischer and Ladner proved that this semantically defined logic has the finite model property by a version of the filtration construction. That method produces a falsifying model for a given non-theorem α whose size is exponential in the length of α . The result was used to establish an upper bound of *nondeterministic exponential time* for the complexity of the satisfiability problem: there is a nondeterministic algorithm for deciding PDL-satisfiability that runs in a time bounded above by an exponential function c^n of the length n of the formula concerned (for some constant c). They also gave a lower bound of *deterministic exponential time* for the complexity of this problem:

⁵⁴ The “weekend” reference is of course to the classic text of Hughes and Cresswell [121].

⁵⁵ See Hoare 1969 [120].

there is a constant $d > 1$ such that no deterministic algorithm can decide the satisfiability question for all formulas in time less than d^n . The technique used was to construct a PDL-formula that encodes the computations of a certain kind of Turing machine that was known to require exponential running time. The gap between these upper and lower bounds was closed by Pratt [202], who used Hintikka's model sets and tableaux methods to give a deterministic exponential time algorithm for deciding satisfiability/validity in PDL.

A finite axiomatisation of PDL was proposed in *Seegerberg 1977* [226], the most notable feature being the *induction* axiom

$$p \rightarrow ([\pi^*](p \rightarrow [\pi]p) \rightarrow [\pi^*]p).$$

The first proof of completeness for PDL was published by Rohit Parikh [193], with other proofs being attributed to Gabbay, Seegerberg [227] and Pratt.⁵⁶ The first extensive study of quantificational dynamic logic was made in David Harel's 1978 dissertation under Pratt's supervision, published as [105].

Many variants of dynamic logic have been studied by varying the modelling, the set of formulas, and the set of programs having associated modalities. *Deterministic* programs are modelled by requiring R_π to be a functional relation. Program predicates may be used to express computational behaviour of particular programs, such as **loop**(π), meaning that some execution of π fails to terminate, and **repeat**(π), meaning that π can be repeatedly executed infinitely many times. PDL programs can be viewed as regular sets of sequences of basic commands, but allowing *context-free* sets of sequences as programs results in a stronger logic that is Π_1^1 -complete and hence highly undecidable [108].

Dynamic *algebras* were introduced by Dexter Kozen and Pratt in 1979 and their structure and representations investigated in a number of papers.⁵⁷ They comprise a “Kleene algebra” that abstracts the algebra of regular expressions and acts as a collection of operators on a Boolean algebra. Concrete models are provided by the complex algebras of Kripke models for PDL. But the relationship between the operators interpreting π and π^* in the algebra of a Kripke model is not equationally expressible, and there are dynamic algebras that belong to the equational class generated by the algebras of Kripke models but are not themselves representable in such models.

Process logic was introduced in *Pratt 1979* [200] by interpreting a program, not as a relation between states, but as the set of possible state-sequences that can be generated by executing the program. In addition to “after”, he proposed the following modalities

throughout π , α : α holds at every state of any sequence generated in executing π .

during π , α : every π -computation has α true at some point.

π *preserves* α : in every π -computation, once α becomes true it remains so thereafter.

Parikh [194] developed a decidable system of second-order process logic that subsumed Pratt's, and allowed quantification over states and state-sequences. Then Nishimura [191] combined PDL with some temporal connectives to devise a system extending Parikh's.

⁵⁶ More background on the beginnings of dynamic logic is provided in *Goldblatt 1986* [91].

⁵⁷ See *Kozen and Tiuryn 1990* [140] for references.

All of these were subsumed by the powerful system of process logic of Harel, Kozen and Parikh [107] which was shown to be decidable by reduction to the second-order decidability results of Rabin [212].

The 1984 article [106] surveys the first decade of dynamic logic, and there is a further review in the 1990 book [140].

7.2. Hennessy–Milner logic

Matthew Hennessy and Robin Milner [114,115] applied modal logic to *process algebra* in a manner that is reminiscent of the Kripke modelling of PDL. They used a modal language to express assertions about transitions between processes in such a way that two processes prove to be “observationally equivalent” just when they satisfy the same modal properties.

A *process* is viewed as an agent that interacts with its environment by performing observable actions which cause it to change its state. Processes are identified with their states, so an observation changes a process into a new process. The notation $\langle p, p' \rangle \in R_i$ means that process p can become p' by performing, or participating in, the observation i . Thus R_i is a binary relation on a given set P of processes, and we envisage a collection $\{R_i; i \in I\}$ of such observation relations corresponding to a set I of “types of observation”. A particular pair $\langle p, p' \rangle \in R_i$ represents a singular observation, and is also viewed as an “experiment” performed by the observer on process p . (In subsequent literature the notation $p \xrightarrow{i} p'$ became standard in place of $\langle p, p' \rangle \in R_i$.)

The Hennessy–Milner modal language has no propositional variables, but constructs formulas from the constant \top by the truth-functional connectives and the modalities $\langle i \rangle$ for $i \in I$. The box modality $[i]$ is defined to be $\neg \langle i \rangle \neg$. The relation $p \models \alpha$, meaning “process p satisfies formula α ”, is defined inductively, with

$$p \models \langle i \rangle \alpha \quad \text{iff} \quad \text{for some } i\text{-experiment } \langle p, p' \rangle, \quad p' \models \alpha.$$

Two processes are regarded as equivalent if there is no observable action that either can perform to distinguish them. Informally this means that to each observable action that one can perform there is an action that the other can perform which leads to an equivalent outcome, so each process can “simulate” the other. Spelling this out,

p is equivalent to q if, and only if,

- (i) for every result p' of an experiment on p , there is an equivalent result q' of an experiment on q ;
and
- (ii) for every result q' of an experiment on q , there is an equivalent result p' of an experiment on p

(Milner 1980 [182], p. 41). As a definition of equivalence this appears to be circular, since the word “equivalence” occurs on both sides of the “if and only if”. To formalise the idea, a sequence of equivalence relations \sim_n for $n \geq 0$ is defined on P . For each relation $S \subseteq P \times P$, define a relation $E(S)$ by putting $\langle p, q \rangle \in E(S)$ if for every $i \in I$,

- (1) $\langle p, p' \rangle \in R_i$ implies, for some q' , $\langle q, q' \rangle \in R_i$ and $\langle p', q' \rangle \in S$; and

(2) $\langle q, q' \rangle \in R_i$ implies, for some $p', \langle p, p' \rangle \in R_i$ and $\langle p', q' \rangle \in S$.

Put $p \sim_0 q$ for all $p, q \in P$, and inductively $p \sim_{n+1} q$ if $\langle p, q \rangle \in E(\sim_n)$. Then p and q are defined to be *observationally equivalent*, written $p \sim q$, if $p \sim_n q$ for every n .

Now a relation $R \subseteq P \times P$ is *image-finite* if the set $\{p' : \langle p, p' \rangle \in R\}$ is finite for each $p \in P$. Hennessy and Milner gave a logical characterisation of observational equivalence by showing that if each R_i is image-finite, two processes are equivalent iff they satisfy the same formulas:

$$p \sim q \quad \text{iff} \quad \text{for all formulas } \alpha, \quad p \models \alpha \quad \text{iff} \quad q \models \alpha. \quad (*)$$

Note that the operator E on relations is *monotonic*: $R \subseteq S$ implies $E(R) \subseteq E(S)$. This property implies, by induction, that $\sim_{n+1} \subseteq \sim_n$, and so iteration of E generates a *decreasing* chain of relations

$$\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \dots \supseteq \sim_n \supseteq \dots$$

Let $\sim_\omega = \bigcap \{\sim_n : n \geq 0\}$ be the intersection of the chain. Then in the image-finite case, \sim_ω is the *largest fixed point* of the operator E , i.e., putting $S = \sim_\omega$ gives the largest solution to the equation $S = E(S)$ (see [115, Theorem 2.1]). In that case $\langle p, q \rangle \in S$ iff $\langle p, q \rangle \in E(S)$, legitimizing the circular definition of equivalence.

The monotonicity of E alone is enough to guarantee that E has a largest fixed point (see Section 7.4), but in the absence of image-finiteness this fixed point need not be the relation \sim_ω . It may be a proper subrelation of \sim_ω that can only be reached by iterating E transfinitely often. Consequently this largest fixed point has become the general definition of the observational-equivalence relation \sim , and it is only in the image-finite case that \sim is identified with \sim_ω .

This analysis indicates that standard induction on natural numbers n (applied to the relations \sim_n) may not be effective as a method for proving equivalence of processes. Instead, as was first realised by David Park,⁵⁸ a new kind of proof rule is called for, based on the notion of a *bisimulation*. This is a relation $S \subseteq P \times P$ satisfying $S \subseteq E(S)$, i.e., $\langle p, q \rangle \in S$ implies (1) and (2) hold. The union of any collection of bisimulations is a bisimulation, and so there is a largest bisimulation—the union of all of them—which turns out to be the same as the largest fixed point of E . In other words, the observational relation \sim is the largest bisimulation on any structure $(P, \{R_i : i \in I\})$. It is an equivalence relation in the mathematical sense (reflexive, symmetric and transitive) and is known as *bisimulation equivalence* or *bisimilarity* (Milner 1989 [184]). It

admits an elegant proof technique; to show $p \sim q$, it is necessary and sufficient to find some bisimulation containing the pair $\langle p, q \rangle$

(Milner 1983 [183], p. 283). In the general setting, when \sim is not equal to \sim_ω , the same modal-logical characterisation of bisimilarity as (*) above can be obtained by expanding

⁵⁸ Information from Robin Milner, personal communication.

the class of formulas to allow formation of the conjunction $\bigwedge_{j \in J} \alpha_j$ for any set $\{\alpha_j : j \in J\}$ (possibly infinite) of formulas.

The term “bisimulation” was first used in *Park 1981* [196] for a relation of mutual simulation between states of two automata, with motivation from an earlier notion of simulation of programs from *Milner 1971* [181]. Park showed that if two deterministic automata are related by a bisimulation, then they accept the same set of inputs. The concept and its use was systematically developed in *Milner 1983* [183]. It is closely related to the notion of “p-relation” of van Benthem [263] mentioned in Section 5.3. Segerberg’s p-morphisms are essentially bisimulations (between Kripke models) that are total and functional.

Process algebra is now a substantial field, with many concepts and constructions for building processes, and many important variations on the notion of observational equivalence or bisimilarity (see [11]). For any given family of transition systems, i.e., systems of observation relations, we can seek to devise modalities that generate formulas giving a logical characterisation of the bisimilarity relations for those systems in the manner of (*). This programme has been carried out for many cases. Logics for more recently developed theories of “mobile” and “message-passing” processes are discussed in *Milner et al. 1993* [185] and *Hennessey and Liu 1995* [113]. They provide modalities that formalise complex structural assertions, for example the formula $\langle c!x \rangle \alpha$ expressing “it is possible to output some value v on channel c and thereby evolve to a state in which $\alpha[v/x]$ is true”.

Axiomatisations of various modal process logics may be found, inter alia, in *Stirling 1987* [237] and *Larsen 1990* [151]. Other work on modal aspects of process algebra is collected in [198].

7.3. Temporal logic for concurrency

In 1977 Amir Pnueli proposed to use temporal logic to formalising reasoning about the behaviour of *concurrent* programs involving a number of processors acting in parallel and sharing a memory environment, so that each can alter the values of variables used by the others (see [197]). This is particularly relevant to the specification and analysis of *reactive* programs, like operating systems and systems for airline reservation or process control, that repeatedly interact with their environment and are not expected to terminate. As such a program runs, each success state is obtained by one processor being chosen to execute one instruction. Thus from an initial state x_0 , many different sequences x_0, x_1, \dots of states may be generated depending on which processors get chosen to act at each step.

Pnueli observed that temporal modalities could be used to formulate computationally significant properties of execution sequences, such as *fair scheduling* (no processor is delayed forever), freedom from *deadlock* (when none can act), and many others. He used Prior’s future-tense modality G (and its dual F), but with the Diodorean reading of “at all future states including the present”, as well as a connective X with the reading “at the next state”. The latter had first been introduced to tense logic for discrete time by Dana Scott (see *Prior 1967* [209, p. 66]). Programs do not appear in the syntax in this approach. Instead, temporal formulas describe properties of a particular execution sequence of a single (concurrent) program.

The paper *Gabbay, Pnueli, Shelah and Stavi 1980* [78] added a binary connective U to this formalism, with $\alpha U \beta$ meaning “ α until β ”, i.e., “ β will be true, and α will be true at

all times *until* β is”. This connective and its past-tense version α *since* β had been studied by Hans Kamp in his 1968 thesis [130]. He showed that they form an *expressively complete* set of connectives in the sense that for models in which time is a complete linear ordering, all tense-logical connectives can be defined in terms of them. Gabbay et al. adapted this to show that U by itself plays a similar role for the future-tense logic of state sequences. They gave an axiomatisation for this extended logic, which they called DUX, and proved that it is decidable. By way of illustration of the expressive completeness of U , they noted that $F\alpha$ can be defined as $\top U\alpha$, and then $G\alpha$ as $\neg F\neg\alpha$, while $X\alpha$ can be defined as $\perp U\alpha$. DUX is now more commonly known as PLTL (propositional linear temporal logic).

Since there are many different execution sequences with a given starting state any particular sequence is just one “branch” or “path” of the “tree” of all possible future states. Considering the tree as a whole gives rise to some interesting new modalities that can formalise reasoning about future behaviour. This line was pursued by Mordechai Ben-Ari, Amir Pnueli and Zohar Manna [9], defining a system \mathcal{UB} (the *unified* system of *branching* time), which combined G and X with the symbols \forall , \exists for quantification over paths to produce the following modal forms:

- $\forall G\alpha$: along all future paths, α is true at all states.
- $\exists G\alpha$: along some path, α is true at all states.
- $\forall X\alpha$: along all paths, α is true at the next state.

Dual modalities were defined by writing $\exists F$ for $\neg\forall G\neg$, $\forall F$ for $\neg\exists G\neg$, and $\exists X$ for $\neg\forall X\neg$. The logic \mathcal{UB} was shown to be finitely axiomatisable and have the finite model property, using semantic tableaux methods. It was also stated that, in contrast to PLTL, no temporal language for branching time with a finite number of modalities could be expressively complete, this theorem being credited to Gabbay.

The *until* connective U was added to \mathcal{UB} by Edmund Clarke and Allen Emerson [36] to define the system CTL of Computation Tree Logic, which was axiomatised and shown to have the finite model property by Emerson and Joseph Halpern [58]. CTL has the limitation that the path quantifiers \forall and \exists are tied to a single linear-time state quantifier (modality) as in the forms $\forall G$, $\exists F$, or a single instance of U as in $\exists(\alpha U\beta)$ etc. It does not allow a combination like $\exists GF\alpha$, expressing “there is a path along which α is true infinitely often”, a property of relevance to fair scheduling conditions. Emerson and Halpern [57,59] devised a new system CTL* that allows such formations. It distinguishes between *state formulas*, which are true or false at each state, and *path formulas*, which are true or false of each path. The path formulas include the state formulas and both categories are closed under the truth-functional connectives. If α , β are path formulas then $\alpha U\beta$, $G\alpha$ and $X\alpha$ are path formulas, while $\forall\alpha$ and $\exists\alpha$ are state formulas. $\forall\alpha$ (respectively $\exists\alpha$) is true at state s iff α is true of all (respectively some) paths that start at s .

In addition to being more expressive than CTL, CTL* is more complex. Whereas CTL and PDL are decidable by algorithms that run in deterministic exponential time, the complexity of CTL* is that of deterministic *doubly* exponential time. The lower bound here was established by Moshe Vardi and Larry Stockmeyer [270], and the upper bound by Emerson and Charanjit Jutla [60,61]. Methods from tree automata theory are used to prove decidability results in this context. Models can be viewed as infinite branching trees, or at least can be “unravalled” into such tree structures. Associated with each formula α is an

automaton A_α that accepts a tree model iff it satisfies α at its root. Thus the satisfiability problem for many logics can be reduced to the *emptiness* problem for automata on infinite trees that was shown to be decidable in *Rabin 1969* [212] (see Section 6.2). This technique was first developed in the 1980 Masters thesis of Robert Streett (see [241]) who used it to prove the decidability of PDL with the **repeat** construct.

The logic CTL^* was defined semantically, and a sound and complete axiomatisation of it was hard to find. Eventually one was provided by Mark Reynolds [214].

A property of paths not expressible in linear time logic, or even in CTL^* , is that a formula be true at every *even* state along the path (and possibly at others). Sets of sequences that have this property can be generated by formal grammars, or characterised by finite-state automata that process infinite strings. Pierre Wolper [275] showed that any regular grammar gives rise to a temporal connective creating formulas that are true just of paths generated by that grammar in a certain way. He also showed that the linear time connectives G , F , X and U can each be expressed by such a grammar, and dubbed this formalism ETL for “*extended temporal logic*”. The idea can be applied to branching time systems, and leads to a logic ECTL^* into which CTL^* can be translated (see *Thomas 1989* [251]).

Surveys of computational temporal logic, and its various applications to reasoning about programs, are given in [55] and [238].

A different kind of use of modalities of the branching-time type was made by Glynn Winskel [274] in constructing *powerdomains*. These structures arise in the denotational semantics of programs, and are intended to provide domain-theoretic analogues of powersets. In dynamic logic a non-deterministic program is modelled as a binary transition relation R on a set S of possible program states. Alternatively this can be viewed as a function from S to its powerset $\mathcal{P}(S)$, taking each state $x \in S$ to the set $\{y : xRy\}$ of states that can be reached by different possible executions of the program. Analogously, given a domain \mathcal{D} , a non-deterministic program may be modelled as a function from \mathcal{D} to its powerdomain.

There are several different powerdomain constructions, and Winskel shows how to build them out of formulas of some modal languages associated with \mathcal{D} . This involves tree-like models of the languages that represent certain computations. For the “Smyth” powerdomain a modality \Box is used that is read “inevitably”. $\Box\alpha$ has the same meaning in these models as the CTL-modality $\forall F\alpha$, i.e., along every future path there is a state at which α holds. The construction of the “Hoare” powerdomain uses \Diamond , for “possibly”, with $\Diamond\alpha$ meaning that there is a future path with α true somewhere, i.e., $\exists F\alpha$. For the “Plotkin” powerdomain, both of these modalities are involved.

7.4. The modal μ -calculus

Mathematics and computer science abound with concepts and objects that are defined recursively, or self-referentially. Many of these have an elegant formulation as special *fixed points* of certain operations. The μ -calculus L_μ of Kozen [136,137] admits formulas that are interpreted as fixed points, and is expressively more powerful than any of the modal program logics considered above.

Let $\Theta : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ be an operation on the powerset of a set S . Tarski applied the term “*fixpoint*” to any subset T of S such that $\Theta(T) = T$. If Θ is *monotonic* in the sense that $T \subseteq T'$ implies $\Theta(T) \subseteq \Theta(T')$, then Θ has a *least* fixpoint $\mu\Theta$ and a *greatest* fixpoint

$\nu\Theta$, given by

$$\begin{aligned} \mu\Theta &= \bigcap \{T \subseteq S: \Theta(T) \subseteq T\}, \\ \nu\Theta &= \bigcup \{T \subseteq S: T \subseteq \Theta(T)\}. \end{aligned}$$

The fact that Θ has a fixpoint was first shown by Tarski and B. Knaster in 1927. In 1939 Tarski generalised this to any monotonic function on a complete lattice, showing that its fixpoints also form a complete lattice, with greatest and least elements specified by the lattice versions of the definitions just given (see *Tarski* [249] for this historical background).

Pratt [203] introduced the idea of using a “minimisation” operator in a PDL-like context, but interpreted μ as a *least root* operator rather than a least fixpoint one. He developed a language of terms intended to denote elements of a Boolean algebra, with a term of the form $\mu Q.\tau(Q)$ interpreted as the least solution of the equation “ $\tau(Q) = 0$ ”. A syntactic restriction was imposed on τ to ensure that at least one solution exists. A translation of PDL into the resulting calculus was given, and the system was shown to have the finite model property by a refinement of the McKinsey method. A deterministic exponential time algorithm was given for the problem of deciding satisfiability terms.

Pratt’s work provided the inspiration for Kozen’s development of the calculus $L\mu$, whose language is generated from some collection Π of atomic programs (or action labels) π . $L\mu$ -formulas are constructed from propositional variables using the truth-functional connectives, the modalities $[\pi]$ and $\langle\pi\rangle$ for $\pi \in \Pi$, and the constructions $\mu p.\alpha$ and $\nu p.\alpha$, where p is a propositional variable and α is a formula. The operations μp and νp function like quantifiers, binding occurrences of p in α . $\mu p.\alpha$ and $\nu p.\alpha$ are only allowed to be formed when α is *positive* in the sense that all free occurrences of p in α are within the scope of an *even* number of negations \neg . This condition is satisfied for instance by any formula constructed from variables using only \top , \perp , \wedge , \vee , $[\pi]$, $\langle\pi\rangle$, μp and νp . The “binder” ν is definable in terms of μ by taking $\nu p.\alpha$ as $\neg\mu p.\neg\alpha(\neg p/p)$. Vice versa, μ could be defined in terms of ν .

An $L\mu$ model $\mathcal{M} = (S, \{\overset{\pi}{\rightarrow}: \pi \in \Pi\}, \Phi)$ is just like a Kripke model for dynamic logic, or a labelled transition system for Hennessy–Milner logic augmented by a valuation Φ to interpret the variables p . \mathcal{M} gives each formula α the interpretation $\mathcal{M}(\alpha) = \{x \in S: \mathcal{M} \models_x \alpha\}$. If α contains the variable p , then varying the interpretation of p causes the interpretation of α to vary, and in this way α induces an operation on $\mathcal{P}(S)$. To make this precise, for $T \subseteq S$ let $\mathcal{M}_{p:=T}$ be the model that is identical to \mathcal{M} except in interpreting p as T , i.e., $\mathcal{M}_{p:=T}(p) = T$. Then the operation induced by α on $\mathcal{P}(S)$ relative to \mathcal{M} is the function

$$\Theta_\alpha^{\mathcal{M}}: T \longmapsto \mathcal{M}_{p:=T}(\alpha).$$

If α is positive, then Θ_α is monotonic. Assuming inductively that Θ_α has been specified, $\mathcal{M}(\mu p.\alpha)$ and $\mathcal{M}(\nu p.\alpha)$ are defined to be the least and greatest fixpoints $\mu\Theta_\alpha^{\mathcal{M}}$ and $\nu\Theta_\alpha^{\mathcal{M}}$ given by the Tarski-Knaster Theorem.

The meaning of $\mu p.\alpha$ and $\nu p.\alpha$ for particular α can be hard to fathom, but it helps to think of them as solutions of the equation “ $p = \alpha$ ” and repeatedly replace p by α in α itself. It turns out that $\mu p.(\alpha \vee \langle\pi\rangle p)$ has the same interpretation in a model as the PDL-formula $\langle\pi^*\rangle\alpha$, while $\nu p.(\alpha \wedge [\pi]p)$ has the same meaning as $[\pi^*]\alpha$. Also $\mu p.\langle\pi\rangle p$ is true

at x_0 iff there is an *infinite* sequence $x_0 \xrightarrow{\pi} x_1 \xrightarrow{\pi} \dots$ in \mathcal{M} , which is the condition for truth of the formula **repeat**(π). Using these observations it can be shown that the logic PDL with the **repeat** construct has a simple translation into the μ -calculus.

A CTL-model can be viewed as an L_μ -model with a single transition relation $\xrightarrow{\pi}$, and with a *path* being a sequence $x_0 \xrightarrow{\pi} x_1 \xrightarrow{\pi} \dots$ in the model. CTL translates into L_μ by translating $\exists(\alpha U \beta)$ as $\mu p. \beta \vee (\alpha \wedge \langle \pi \rangle p)$ and $\forall(\alpha U \beta)$ as $\mu p. \beta \vee (\alpha \wedge [\pi] p \wedge \langle \pi \rangle \top)$. The L_μ -formula $\nu p. \alpha \wedge [\pi][\pi] p$ means “along all paths, α is true at every even state”, a property expressible in ECTL* but not CTL*. Mads Dam [47] has constructed algorithms for translating both CTL* and ECTL* into L_μ .

Kozen proposed a finite axiomatisation of L_μ which, for the binder μ , has the axiom schema

$$\alpha(\mu p. \alpha / p) \rightarrow \mu p. \alpha$$

and the inference rule:

$$\text{from } \alpha(\beta / p) \rightarrow \beta \text{ infer } (\mu p. \alpha) \rightarrow \beta \text{ if } p \text{ is not free in } \beta.$$

Validity of the axiom follows from the fact that $T = \mu \Theta_\alpha^{\mathcal{M}}$ is a solution of the “inequality” $\Theta(T) \subseteq T$, and soundness of the rule is due to $\mu \Theta_\alpha^{\mathcal{M}}$ being the least such solution. Kozen was able to prove the completeness of a limited fragment of L_μ for which he also showed the finite model property and an exponential time decision procedure. The full L_μ was proved decidable by Kosen and Parikh [139] by reduction to Rabins *SnS*. Streett and Emerson [242,243] used tree automata to improve this to a deterministic triple-exponential time decision algorithm and establish the full finite model property. Emerson and Jutla [60,61] sharpened the complexity result further to a deterministic exponential time algorithm, which is the best possible result since it is the lower bound for PDL and therefore for the μ -calculus. Kozen [138] gave a different proof of the finite model property using techniques from the theory of well-quasi orders, and proved a completeness theorem for L_μ using an infinitary rule of inference.

The problem of whether L_μ is complete for Kozen’s originally proposed axiomatisation proved challenging, and remained open for some time. It was eventually solved in the affirmative by Igor Walukiewicz [272,273].

The formalism of the μ -calculus originates in some unpublished notes of Jaco de Bakker and Dana Scott from 1969. Kozen’s inference rule derives from the Fixpoint Induction rule of *Park 1969* [195]. Another early independent formulation of a modal program logic with a greatest and least fixpoint operators appears in *Emerson and Clarke 1980* [56]. For a recent survey of the field of modal μ -calculi, see *Bradfield and Stirling* [26].

7.5. Solovay on provability in arithmetic as a modality

Let PA be the first-order system of Peano Arithmetic that is the subject of Gödel’s incompleteness theorems, and let $PA \vdash \sigma$ signify that sentence σ is provable in PA . Gödel showed that this notion can be “arithmetised” and expressed in the language of PA itself. There is a PA -formula $Bew(v)$ with one free variable v such that in general $PA \vdash \sigma$ iff the sentence $Bew(\ulcorner \sigma \urcorner)$ is *true* (i.e., true of the standard PA -model $(\omega, +, \cdot, 0, 1)$). Here $\ulcorner \sigma \urcorner$

is the numeral for the Gödel number of σ . Now all *PA*-provable sentences are true, so for every σ the sentence

$$\text{Bew}(\ulcorner \sigma \urcorner) \rightarrow \sigma$$

is true. But it is not always *PA-provable*, a fact which is a manifestation of the first incompleteness theorem. Gödel gave an example of this in 1933 [84] in observing that if the modality “provable” is taken to mean provable in *PA* then some principles of S4 do not hold:

For example, $B(Bp \rightarrow p)$ never holds for that notion, that is it holds for no system S that contains arithmetic. For otherwise, for example, $B(0 \neq 0) \rightarrow 0 \neq 0$ and therefore also $\neg B(0 \neq 0)$ would be provable in S , that is, the consistency of S would be provable in S .

Provability in S of the consistency of S would contradict the second incompleteness theorem.

The question therefore arises as to which modal principles do hold if \Box is read as “*PA*-provable”. To make this precise, define a *realisation* to be a function ϕ assigning to each propositional variable p some *PA*-sentence p^ϕ . This extends inductively to all modal formulas by taking \top^ϕ to be $(0 = 0)$, realising the non-modal connectives as themselves, and defining

$$(\Box\alpha)^\phi := \text{Bew}(\ulcorner \alpha^\phi \urcorner).$$

A modal formula α is *PA-valid* if $PA \vdash \alpha^\phi$ for every realisation ϕ . The question becomes that of determining which modal formulas are *PA*-valid.

The set of all *PA*-valid formulas is a normal logic, known as G (for Gödel).⁵⁹ To show that it is normal it is necessary to verify that the following hold in general:

$$PA \vdash \text{Bew}(\ulcorner \sigma \rightarrow \sigma' \urcorner) \rightarrow \text{Bew}(\ulcorner \sigma \urcorner) \rightarrow \text{Bew}(\ulcorner \sigma' \urcorner);$$

$$\text{If } PA \vdash \sigma, \text{ then } PA \vdash \text{Bew}(\ulcorner \sigma \urcorner).$$

These results were distilled by Martin Löb in 1955 [160] from properties of *Bew* that were established in *Hilbert and Bernays 1939* [116]. Löb then proved

$$PA \vdash \text{Bew}(\ulcorner \sigma \urcorner) \rightarrow \text{Bew}(\ulcorner \text{Bew}(\ulcorner \sigma \urcorner) \urcorner),$$

which shows that $\Box p \rightarrow \Box\Box p$ is *PA*-valid and hence a G-theorem. However the other S4-axiom $\Box p \rightarrow p$ is not *PA*-valid, and indeed not even the formula $\Box\perp \rightarrow \perp$ is a G-theorem, since $(\Box\perp \rightarrow \perp)^\phi$ is

$$\text{Bew}(\ulcorner 0 \neq 0 \urcorner) \rightarrow 0 \neq 0,$$

which is not *PA*-provable by Gödel’s reasoning above.

Robert Solovay [235] demonstrated that G is identical to Segerbergs logic K4W, discussed in Section 5.3, which is characterised by the class of finite strictly ordered (i.e.,

⁵⁹ Also known as GL for Gödel–Löb.

transitive and irreflexive) Kripke frames.⁶⁰ The validity of the axiom W, i.e.,

$$\Box(\Box p \rightarrow p) \rightarrow \Box p,$$

follows from an answer given in [160] to a question raised by Leon Henkin in 1952 about the status of sentences that assert their own provability. Any PA-formula $F(v)$ has *fixed points*: sentences σ for which

$$PA \vdash \sigma \leftrightarrow F(\ulcorner \sigma \urcorner)$$

(this is usually called the *Diagonalisation Lemma*). A fixed point of $Bew(v)$ has

$$PA \vdash \sigma \leftrightarrow Bew(\ulcorner \sigma \urcorner)$$

so is equivalent to the assertion of its own provability. Must it in fact be provable?⁶¹ Löb answered this in the affirmative by proving that

$$\text{if } PA \vdash Bew(\ulcorner \sigma \urcorner) \rightarrow \sigma, \quad \text{then } PA \vdash \sigma.$$

Equivalently, if $Bew(\ulcorner Bew(\ulcorner \sigma \urcorner) \rightarrow \sigma \urcorner)$ is true then so is $Bew(\ulcorner \sigma \urcorner)$, i.e., the sentence

$$Bew(\ulcorner Bew(\ulcorner \sigma \urcorner) \rightarrow \sigma \urcorner) \rightarrow Bew(\ulcorner \sigma \urcorner)$$

is true. But more strongly it can be shown that this sentence is PA-provable for any σ , including $\sigma = \alpha^\phi$, giving the PA-validity of W.

Solovay's completeness theorem for G is a remarkable application of the machinery of arithmetisation and recursive functions to show that any finite strictly ordered frame (K, R) can be "embedded into Peano Arithmetic". A recursive function $h : \omega \rightarrow K$ is defined that is in fact constant, but which cannot be proven to be constant in PA. Each element x of K is represented by a sentence σ_x expressing " $\lim_{n \rightarrow \infty} h(n) = x$ ". This sentence is consistent with PA, i.e., $PA \not\vdash \neg \sigma_x$. The construction has a flavour of self-referential paradox similar to that of Gödel's incompleteness proof, because the sentences σ_x are used to define the function h itself. But that is resolved by some version of diagonalisation.⁶² The structure of the ordering R is represented in PA by the fact that if $x R y$ then

$$PA \vdash \sigma_x \rightarrow \neg Bew(\ulcorner \neg \sigma_y \urcorner),$$

and if not $x R y$ then

$$PA \vdash \sigma_x \rightarrow Bew(\ulcorner \neg \sigma_y \urcorner).$$

Any model \mathcal{M} on this frame determines a realisation ϕ by putting

$$p^\phi = \bigvee \{ \sigma_x : \mathcal{M} \models_x p \}.$$

Then the truth conditions in \mathcal{M} are PA-representable by the fact that for any modal formula α ,

⁶⁰ The axiom 4: $\Box p \rightarrow \Box \Box p$ is not needed in formulating this logic. It is deducible from W, as was shown independently by several people, including de Jongh, Kripke and Sambin.

⁶¹ This is a generalisation of Henkin's question: see Smoryński 1991 [233] for discussion.

⁶² Solovay's argument used Kleene's Recursion Theorem on fixed points in the enumeration of partial recursive functions.

if $\mathcal{M} \models_x \alpha$ then $PA \vdash \sigma_x \rightarrow \alpha^\phi$; while
 if $\mathcal{M} \not\models_x \alpha$ then $PA \vdash \sigma_x \rightarrow \neg \alpha^\phi$ and so $PA \vdash \alpha^\phi \rightarrow \neg \sigma_x$.

Since $PA \not\vdash \neg \sigma_x$, the last case gives $PA \not\vdash \alpha^\phi$, showing α is not PA -valid. Therefore any PA -valid formula must be true in all models on finite strictly ordered frames, and therefore be a G-theorem.

A modal formula α is called ω -valid if α^ϕ is true for all realisations ϕ . The set G^* of all ω -valid formulas is a logic that includes G, but also includes $\Box p \rightarrow p$, since $Bew(\ulcorner \sigma \urcorner) \rightarrow \sigma$ is always true. However Gödel's example shows that $Bew(\ulcorner Bew(\ulcorner \perp^\phi \urcorner) \urcorner) \rightarrow \perp^\phi$ is not true, so G^* does not contain $\Box(\Box p \rightarrow p)$, and therefore is not a normal logic. Solovay extended his analysis of G to prove that G^* can be axiomatised by taking all theorems of G and instances of $\Box \alpha \rightarrow \alpha$ as axioms, and detachment as the only rule of inference.

Another natural reading of \Box in this context is “true and provable”, formalised by modifying the definition of realisation to

$$(\Box \alpha)^\phi := \alpha^\phi \wedge Bew(\ulcorner \alpha^\phi \urcorner).$$

The fact that “provable” implies “true” might make it seem that “true and provable” has the same status as “provable”, but this is not so because of the existence of true but unprovable sentences of PA . In general, $Bew(\ulcorner \sigma \urcorner)$ is PA -provable iff $\sigma \wedge Bew(\ulcorner \sigma \urcorner)$ is PA -provable, and the two are equivalent in the sense that

$$Bew(\ulcorner \sigma \urcorner) \leftrightarrow \sigma \wedge Bew(\ulcorner \sigma \urcorner)$$

is true, but this equivalence is not itself PA -provable unless σ is, by Löb's theorem.

The modal logic of formulas PA -valid under this modified realisation turns out to be the system S4Grz characterised by finite partial orderings (see Section 5.3). This was proved in Goldblatt 1978 [88] by showing that replacing $\Box \alpha$ by $\alpha \wedge \Box \alpha$ gives a proof-invariant translation of S4Grz into G, and then applying Solovay's theorem for G.⁶³ Since the intuitionistic propositional calculus IPC can be translated into S4Grz (by the result of Grzegorzczuk mentioned in Section 5.3), these translations can be composed to obtain a translation $\alpha \mapsto \alpha^\tau$ of propositional formulas into modal formulas such that α is provable in IPC iff α^τ is PA -valid. In fact α^τ is PA -valid iff it is ω -valid [88, Theorem 5].

Research into the modal logic of provability since the 1970s has contributed much to our understanding of the phenomena of self-reference and diagonalisation that underly the incompleteness of PA and other systems. An account of the origins of the subject has been given by George Boolos and Giovanni Sambin [25], and extensive expositions are provided in the books of Boolos [22,23] and Craig Smoryński [232]. The most recent survey is that of Giorgi Japaridze and Dick de Jongh [123].

7.6. Grothendieck topology as intuitionistic modality

By composing his semantic analysis of S4 with the McKinsey–Tarski translation of IPC into S4, Kripke [147] derived a relational model theory for intuitionistic logic based on

⁶³ The result was independently found by A. Kuznetsov and A. Muzavitski (*Abstracts of Reports of the Fourth All-Union Conference on Mathematical Logic*, Kishiniev, 1976, p. 73, in Russian).

structures $\mathfrak{S} = (K, R)$ in which R is a quasi-ordering, i.e., reflexive and transitive. He interpreted the members of K informally as “evidential situations” temporally ordered by R . His paper presented a semantics for *predicate* logic, proving completeness by the method of tableaux⁶⁴. It also showed that attention can be confined to structures that are partially ordered, i.e., antisymmetric as well. By identifying elements $x, y \in K$ whenever xRy and yRx we pass to a partially ordered quotient \mathfrak{S}' which validates the same intuitionistic formulas as \mathfrak{S} . More strongly, any model on \mathfrak{S} has an equivalent model on \mathfrak{S}' . This contrasts with the modal semantics on these structures: it can happen that \mathfrak{S}' validates the modal axiom Grz while \mathfrak{S} does not (see Section 5.3).

Seegerberg [221] studied the propositional fragment of this model theory, using only partially ordered frames from the outset. He constructed canonical models and applied the filtration method to prove the finite model property for a number of logics, including some that are weaker than or independent of IPC. The fact that IPC is characterised by the finite partially ordered frames, which also characterise S4Grz under the modal semantics, provides a clear picture of why IPC translates into S4Grz and not just S4.

Here is a brief description of the relational models for IPC. Given a partial ordering $\mathfrak{S} = (K, \leq)$, a subset X of K will be called *increasing* if it is closed “upwards” under the ordering, i.e., whenever $x \in X$ and $x \leq y$, then $y \in X$. The definition of a model $\mathcal{M} = (\mathfrak{S}, \Phi)$ requires that the set $\{x \in K : \Phi(p, x) = \top\}$ be increasing for all propositional variables p . Formally this requirement is dictated by the modal translation of p as $\Box p$, while informally it conveys the idea that once p is established as true in a given evidential situation then it remains true in the future. The truth conditions for implication and negation are

$$\begin{aligned} \mathcal{M} \models_x \alpha \rightarrow \beta & \text{ iff for all } y \geq x, \text{ if } \mathcal{M} \models_y \alpha \text{ then } \mathcal{M} \models_y \beta, \\ \mathcal{M} \models_x \neg\alpha & \text{ iff for all } y \geq x, \text{ not } \mathcal{M} \models_y \alpha. \end{aligned}$$

The modelling of \wedge and \vee is as for classical logic. By induction it is demonstrable that for each formula α the set $\mathcal{M}(\alpha) = \{x \in K : \mathcal{M} \models_x \alpha\}$ is increasing.

The topological and algebraic modellings of IPC from Section 3.2 are in evidence here. The increasing sets form a topology on K , and the associated Heyting algebra of open sets satisfies a formula α iff α is valid in \mathfrak{S} , i.e., iff $\mathcal{M}(\alpha) = K$ for all models \mathcal{M} on \mathfrak{S} . At the same time α is valid in \mathfrak{S} iff it is satisfied by the Brouwerian algebra of *closed* subsets of this space, with the least element \emptyset of the algebra being designated. This follows from properties of the set

$$\overline{\mathcal{M}(\alpha)} = \{x \in K : \text{not } \mathcal{M} \models_x \alpha\}$$

of points at which α fails to hold in model \mathcal{M} . $\overline{\mathcal{M}(\alpha)}$ is closed, being the complement of the open set $\mathcal{M}(\alpha)$, and takes the designated value \emptyset iff α is true in the model \mathcal{M} . These “falsity sets” can be reconstructed by applying the Brouwerian operations that correspond to the propositional connectives:

$$\overline{\mathcal{M}(\alpha \wedge \beta)} = \overline{\mathcal{M}(\alpha)} \cup \overline{\mathcal{M}(\beta)}$$

⁶⁴ An extension of intuitionistic predicate logic that is incomplete for Kripke’s semantics was found by Hiroakira Ono [192], and an incomplete extension of intuitionistic propositional logic was obtained by Valentin Shehtman [228].

$$\begin{aligned} \overline{\mathcal{M}}(\alpha \vee \beta) &= \overline{\mathcal{M}}(\alpha) \cap \overline{\mathcal{M}}(\beta) \\ \overline{\mathcal{M}}(\alpha \rightarrow \beta) &= \overline{\mathcal{M}}(\alpha) \div \overline{\mathcal{M}}(\beta) \\ \overline{\mathcal{M}}(\neg\alpha) &= \overline{\mathcal{M}}(\alpha) \div K. \end{aligned}$$

This analysis accounts for the dual nature of the Brouwerian algebraic semantics.

Modal systems based on intuitionistic logic typically take \Box and \Diamond as independent connectives that are not interdefinable using \neg . Logics of this kind, using one or both of \Box and \Diamond , have been studied by a number of authors, for a variety of philosophical and technical motivations, beginning with a paper published by F.B. Fitch in 1948 [73]. The history of much of this work is reviewed in the 1994 dissertation of Alex Simpson [231, Section 3.3]. Here we will consider another system which has a particular mathematical significance associated with topos theory.

A topos is a category \mathcal{E} that may be thought of, roughly speaking, as a model of intuitionistic higher order logic or set theory. It includes a special entity Ω , the *object of truth values*, with morphisms

$$\cap, \cup, \Rightarrow : \Omega \times \Omega \rightarrow \Omega, \quad \neg : \Omega \rightarrow \Omega \tag{3}$$

satisfying categorical formulations of the laws of Heyting algebra. A “global element” of Ω is a morphism of the form $1 \rightarrow \Omega$, where 1 is the terminal object of \mathcal{E} . In the category **Set** of all sets and functions 1 is a one-element set and morphisms $1 \rightarrow X$ correspond precisely to actual elements of the set X . Thus global elements of Ω in a topos are also called *truth values*. The morphisms (3) induce operations on the collection $\mathcal{E}(1, \Omega)$ of truth values that make it into a Heyting algebra, which is just the two-element Boolean algebra in the case of **Set**. But for each topological space S there exists a topos in which $\mathcal{E}(1, \Omega)$ is (isomorphic to) the Heyting algebra $\mathcal{O}(S)$ of open subsets of S .

Grothendieck generalised the notion of a topology on a set to that of a topology on a category, by generalising the notion of an open covering of a set. He used this as a basis on which to formulate sheaf theory. Lawvere and Tierney showed that the theory could be developed axiomatically by starting with a topos \mathcal{E} having a morphism $j : \Omega \rightarrow \Omega$, called a *topology on \mathcal{E}* , satisfying properties that allow the construction of a certain sub-topos of “ j -sheaves”. The pair (\mathcal{E}, j) will be called a *site*. The axioms for j are categorical versions of the requirement that an operation on a lattice be

- multiplicative:* $j(x \cdot y) = jx \cdot jy$,
- idempotent:* $j(jx) = jx$, and
- inflationary:* $x \leq jx$.

In the 1970 address at which he first announced this new theory Lawvere [152] stated that

A Grothendieck “topology” appears most naturally as a modal operator of the nature “it is locally the case that”.

Intuitively, a property holds *locally* at a point x of a topological space if it holds at all points “near” to x , or throughout some neighbourhood of x . Alternatively, a property holds locally of an object if it is covered by open sets for each of which the property holds. For

example a locally constant function is one whose domain is covered by open sets on each of which the function is constant.

Define a *local operator*⁶⁵ on a Heyting algebra \mathfrak{H} to be any operation j that is multiplicative, idempotent and inflationary, and call the pair $\mathfrak{A} = (\mathfrak{H}, j)$ a *local algebra*. The general theory of these algebras has been studied by Donald Macnab [164,165], who showed that local operators can be alternatively defined by the single equation

$$(x \Rightarrow jy) = (jx \Rightarrow jy).$$

Any local algebra is a candidate for modelling a modal logic based on the intuitionistic calculus IPC. Since j is multiplicative and has $j1 = 1$, this will be a *normal* logic when \Box is interpreted as j , but there has been some uncertainty as to whether a modality modelled by j is of universal or existential character. Note that a local operator has a mixture of the properties of topological interior and closure operators. It fulfills all of the axioms of an interior operator except $ix \leq x$, satisfying instead the inflationary condition which is possessed by closure operators. But topological closure operators are additive ($\mathbf{C}(x + y) = \mathbf{C}x + \mathbf{C}y$), a property not required of j .

Let \mathcal{J} be the set of all modal propositional formulas satisfied by all local algebras with 1 designated. The proof theory and semantics (algebraic, relational, neighbourhood, topos-theoretic) of this logic was investigated in *Goldblatt 1981* [90] where the symbol ∇ was used in place of \Box . It was shown that \mathcal{J} can be axiomatised by adding to the axioms and rules for IPC the three axioms

$$\begin{aligned} \nabla(p \rightarrow q) &\rightarrow (\nabla p \rightarrow \nabla q), \\ \nabla\nabla p &\rightarrow \nabla p, \\ p &\rightarrow \nabla p. \end{aligned}$$

The last axiom allows derivation of the rule *from α infer $\nabla\alpha$* . There are a number of alternative axiomatisations of \mathcal{J} , one of which is to add to IPC the axioms

$$\begin{aligned} (p \rightarrow q) &\rightarrow (\nabla p \rightarrow \nabla q), \\ \nabla\nabla p &\rightarrow \nabla p, \\ \nabla\top. \end{aligned}$$

As Macnab's characterisation of local operators suggests, \mathcal{J} can also be specified by the single axiom

$$(p \rightarrow \nabla q) \leftrightarrow (\nabla p \rightarrow \nabla q).$$

In the presence of classical Boolean logic, the middle axiom $\nabla\nabla p \rightarrow \nabla p$ in the first group is deducible from the other two, and the logic becomes the rather uninteresting system $\mathbf{K}+(p \rightarrow \nabla p)$ whose only connected validating frames are the two one-element frames \mathfrak{S}_\bullet and \mathfrak{S}_\circ (see Section 6.1). But in the absence of the law of excluded middle we have a modal logic with many interesting models. In particular it has relational models based on

⁶⁵ Also known in the literature as a "nucleus".

structures $\mathfrak{S} = (K, \leq, <)$ which refine the Kripke semantics for IPC. Here \leq is a partial ordering of K and $<$ is a binary relation interpreting ∇ as a universal quantifier in the familiar way:

$$\mathcal{M} \models_x \nabla \alpha \quad \text{iff} \quad \mathcal{M} \models_y \alpha \text{ for all } y \text{ such that } x < y.$$

To ensure that $\mathcal{M}(\nabla \alpha)$ is \leq -increasing it is required that $x \leq y < z$ implies $x < z$. The logic \mathcal{J} is characterised by the class of such frames in which $<$ is a subrelation of \leq that is *dense* in the sense that $x < y$ implies $\exists z(x < z < y)$. There is a canonical frame $\mathfrak{S}^{\mathcal{J}}$ of this kind that characterises \mathcal{J} , and the logic also has the finite model property with respect to such frames. In addition there is a characterisation of \mathcal{J} by *neighbourhood* frames (K, \leq, N) (see Section 5.3), where N_x is a filter in the lattice of \leq -increasing subsets of K , and the following conditions hold:

$$\begin{aligned} x \leq y \text{ implies } N_x &\subseteq N_y, \\ \{y: x \leq y\} &\in N_x, \\ \{y: U \in N_y\} \in N_x &\text{ implies } U \in N_x. \end{aligned}$$

If $\nabla \alpha$ is *defined* to be the formula $\neg\neg\alpha$, then the axioms of \mathcal{J} become theorems of IPC. Lawvere [152] observed that

There is a standard Grothendieck topology on any topos, namely double negation, which is more appropriately put into words as “it is cofinally the case that”.

Now if Y and Z are subsets of a partially ordered set (K, \leq) , then Z is *cofinal with* Y if every element of Y has an element of Z greater than it, i.e.,

$$\forall y \in Y \exists z \in Z \quad y \leq z.$$

The Kripke modelling of IPC has

$$\mathcal{M} \models_x \neg\neg\alpha \quad \text{iff} \quad \mathcal{M}(\alpha) \text{ is cofinal with } \{y: x \leq y\},$$

which explains Lawvere’s interpretation of double negation as a modality. On the algebraic level, putting $j(x) = \neg\neg x$ in a Heyting algebra \mathfrak{H} defines a local operator whose set $\{x: \neg\neg x = x\}$ of fixpoints is a *Boolean* subalgebra of \mathfrak{H} . On the categorical level, putting $j = \neg \circ \neg$ defines a topology on any topos \mathcal{E} for which the associated subtopos $\mathcal{E}_{\neg\neg}$ of sheaves is a model of classical Boolean logic. These constructions are mathematical manifestations of the *double-negation translation* of classical propositional calculus into IPC, originating in a 1925 paper of Kolmogorov [135], which works by inserting $\neg\neg$ in front of each subformula.

For any partially-ordered set $\mathfrak{S} = (K, \leq)$ there is a topos $\mathcal{E}_{\mathfrak{S}}$ whose objects are certain “set-valued functors” $(P, \leq) \rightarrow \mathbf{Set}$, and whose algebra $\mathcal{E}_{\mathfrak{S}}(1, \Omega)$ of truth values is isomorphic to the Heyting algebra of all increasing subsets of \mathfrak{S} . In the case that \mathfrak{S} is an appropriate set of “forcing conditions”, the topos $(\mathcal{E}_{\mathfrak{S}})_{\neg\neg}$ of “double-negation sheaves” becomes a model showing that the continuum hypothesis (for example) is independent of the axioms for topos theory including classical logic (see *Tierney 1972* [260]).

If $j: \Omega \rightarrow \Omega$ is a Lawvere–Tierney topology on topos \mathcal{E} , then the site (\mathcal{E}, j) can be used to interpret modal formulas as truth values $1 \rightarrow \Omega$ in \mathcal{E} . The morphism j induces a local operator $f \mapsto j \circ f$ on the Heyting algebra $\mathcal{E}(1, \Omega)$ of truth values in \mathcal{E} . If a formula is satisfied by the resulting local algebra then it is said to be *valid in the site* (\mathcal{E}, j) .

The modal formulas that are valid in all sites are precisely the \mathcal{J} -theorems. This is shown in Goldblatt 1981 [90] by the construction out of any \mathcal{J} -frame $\mathfrak{G} = (P, \leq, <)$ of a particular site $(\mathcal{E}_{\mathfrak{G}}, j_{\mathfrak{G}})$ that validates exactly the same modal formulas as does \mathfrak{G} . $\mathcal{E}_{\mathfrak{G}}$ is the topos of functors $(P, \leq) \rightarrow \mathbf{Set}$ as above. The relation $<$ is used to define $j_{\mathfrak{G}}$. Applying this construction to the canonical frame $\mathfrak{G}^{\mathcal{J}}$ produces a *canonical site* that characterises the logic \mathcal{J} .

It is possible to study topoi from a logical perspective, building these categories out of the syntactic and proof-theoretic machinery of formal languages of types. By including a \mathcal{J} -style modality in these languages the Lawvere–Tierney sheaf categories can be constructed in such a way. This approach to the theory of sheaves and topoi has been developed by John Bell [8].

There have been several independently motivated introductions of versions of the system \mathcal{J} . A Gentzen-style calculus studied by Haskell Curry in 1952 [46] for proof-theoretic purposes has rules for a possibility modality \diamond that gives a variant of \mathcal{J} when \diamond is identified with ∇ . Recently the logic has re-emerged in a different guise as the Propositional Lax Logic (PLL) of Matt Fairtlough and Michael Mandler [65,66]. This is a system based on intuitionistic logic that is intended to formalise reasoning about the behaviour of hardware devices, like circuits, subject to certain “constraints”. A modality \bigcirc is used, with $\bigcirc\alpha$ having the intuitive interpretation “for some constraint c , α holds under c ”. This appears to be an *existential* reading of the modality, but the authors suggest that \bigcirc “has a flavour both of possibility and necessity”. Their proposed axioms are

$$\begin{aligned} (p \rightarrow q) &\rightarrow (\bigcirc p \rightarrow \bigcirc q), \\ \bigcirc \bigcirc p &\rightarrow \bigcirc p, \\ p &\rightarrow \bigcirc p, \end{aligned}$$

showing that the system is indeed a version of \mathcal{J} with \bigcirc in place of ∇ . They give a relational semantics for PLL using structures (K, \leq, R) with R being a quasi-ordered sub-relation of \leq . The connective \bigcirc is interpreted by the *universal-existential* clause

$$\mathcal{M} \models_x \bigcirc \alpha \quad \text{iff} \quad \text{for all } y \geq x \text{ there exists } z \text{ such that } yRz \text{ and } \mathcal{M} \models_z \alpha.$$

It is shown that (K, \leq, R) validates the same formulas as the *neighbourhood* \mathcal{J} -frame (K, \leq, N) of the above kind, where a \leq -increasing set U is a neighbourhood of x (i.e., $U \in N_x$) iff

$$\text{for all } y \geq x \text{ there exists } z \text{ such that } yRz \text{ and } z \in U.$$

In other words, $U \in N_x$ iff U is R -cofinal with $\{y: x \leq y\}$.

Yet another manifestation of \mathcal{J} is the *CL-logic* of Nick Benton, Gavin Bierman and Valeria de Paiva [10]. This is designed to analyse a typed lambda calculus, due to Eugenio Moggi [186], which gives a denotational semantics for programs using a constructor T that produces a *type of computations*. The denotation of a program computing values of

type A is itself an element of the type TA . The CL-logic is an intuitionistic propositional calculus corresponding to this type system, and has a “curious possibility-like modality \diamond ” corresponding to the type constructor T . The axioms given for \diamond are

$$\begin{aligned} \diamond p &\rightarrow ((p \rightarrow \diamond q) \rightarrow \diamond q), \\ p &\rightarrow \diamond p, \end{aligned}$$

again equivalent to the axiomatisation of \mathcal{J} when \diamond is identified with ∇ .

Double negation constitutes just one way of combining non-modal connectives to define a modality fulfilling the \mathcal{J} axioms. Other possibilities are to define $\nabla\alpha$ to be any of $\beta \vee \alpha$, $\beta \rightarrow \alpha$, or $(\beta \rightarrow \alpha) \rightarrow \alpha$, where β is some fixed (but arbitrary) formula. Peter Aczel [2] has studied the interpretation of $\nabla\alpha$ as the second-order formula $\forall p((\alpha \rightarrow p) \rightarrow p)$, where the variable p ranges over all propositions. He calls this the “Russell–Prawitz modality” because of its relevance to certain definitions of the connectives \wedge , \vee , \neg , \exists in terms of \rightarrow and \forall that were introduced by Bertrand Russell and later shown by Dag Prawitz to be derivable as equivalences in second-order intuitionistic logic.

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