Uncancellative factorizations of Baer-local formations

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Received 28 August 2001
Communicated by Michel Broué

Abstract

This paper describes all possible uncancellative factorizations of one-generated Baer-local formations, and hence answers an open problem recently proposed by A.N. Skiba in 1997. © 2003 Elsevier Inc. All rights reserved.

Keywords: Groups; Formations; Baer-local formations; Factorizations

1. Introduction

A variety \( \mathcal{F} \) of groups is called a Cross variety if \( \mathcal{F} \) is generated by a finite group. It was pointed out by P.M. Neumann in 1964 that if the product \( \mathcal{F} = \mathcal{M} \mathcal{H} \) of two non-trivial varieties \( \mathcal{M} \) and \( \mathcal{H} \) of groups is a Cross variety, then the variety \( \mathcal{H} \) is abelian. This observation was further analyzed by A.L. Shmel’kin in [7] and he obtained the following important characterization for decomposable Cross varieties.

**Theorem.** The product \( \mathcal{M} \mathcal{H} \) of non-trivial varieties of groups is a Cross variety if and only if \( \mathcal{M} \) is nilpotent, \( \mathcal{H} \) is abelian and \( \mathcal{M}, \mathcal{H} \) have coprime exponents.
In fact, the above theorem of Shmel’kin [7] actually describes all possible factorizations of the Cross varieties. Indeed, by [6, Theorem 24.34], we know that every nilpotent variety is indecomposable, that is, it can not be written as a product of two non-trivial varieties.

Furthermore, the theorem of Shmel’kin can be extended to the formation of finite groups.

Recall that a formation \( F \) is a class of finite groups which is closed under homomorphic images and also every finite group \( G \) has a smallest normal subgroup (denoted by \( G^F \)) with quotient in \( F \). A non-empty formation \( F \) is called local (or saturated) if every finite group \( G \) has a smallest normal subgroup (denoted by \( G^F \)) with quotient in \( F \). A Baer-local formation is a non-empty formation \( F \) which contains every finite group \( G \) such that \( G/\phi(R(G)) \in F \), where \( R(G) \) denotes the soluble radical of \( G \), that is, \( R(G) \) is the product of all its soluble normal subgroups. If \( G \) is a finite group and \( F \) is the intersection of all local formations containing \( G \), then \( F \) is called the local formation generated by \( G \). Formations of this type are called one-generated local formations. The one-generated Baer-local formations can be defined analogously. The one-generated Baer-local formations are particularly important because many problems on Baer-local formations can be naturally reduced to the study of one-generated Baer-local formations.

The product \( MH \) of formations \( M \) and \( H \) is the class \( \{ G \mid G^M H \in M \} \), where \( G^M H \) is the intersection of all normal subgroups \( N \) such that \( G/N \in H \). A factorization \( F_1F_2\cdots F_t \) of a formation \( F \) is called uncancellative if \( F \neq F_1\cdots F_{i-1}F_{i+1}\cdots F_t \) for all \( i = 1, 2, \ldots, t \), where \( F_i, i = 1, 2, \ldots, t \), are formations.

In the papers \([10,11]\), A.N. Skiba considered the one-generated local formation \( F_0 \). He gave conditions for \( F_0 = M_0H_0 \), where \( M_0 \) and \( H_0 \) are non-trivial formations. Later on, he considered all uncancellative factorizations of all one-generated local formations in \([12, \text{Theorem 3.2.6}]\). The theorem of Shmel’kin was then derived as one of the corollaries of his result. At the end of his monograph (see \([12, \text{Chapter 3}]\)), A.N. Skiba also proposed the following interesting open problem (see \([12, \text{Problem 3.5.21, p. 150}]\)): Is it possible to describe all uncancellative factorizations of the one-generated Baer-local formations?

An initial result of the above problem is the following theorem of Skiba: Let \( F = MH \) be a one-generated Baer-local formation such that \( F \neq H \). Then \( M \) is also a Baer-local formation (see \([12, \text{Section 3.5}]\)).

This result was essentially modified by W. Guo in \([4]\) in which he proved that in this situation \( M \) is a local formation. However, in the papers \([4,12]\), A.N. Skiba and W. Guo were unable to describe what kind of conditions would lead the one-generated Baer-local formation \( F \) to be expressible as an uncancellative factorization. In this paper, we shall give such conditions. This result generalizes the above theorems of Shmel’kin \([7]\), Skiba \([12]\), and Guo \([4]\), and also gives an affirmative answer to the above open problem proposed by A.N. Skiba \([12]\). The proof of this result is rather complicated. In Section 2, we first cite some useful results from the literature. In Section 3, we shall provide some technical lemmas and in Section 4, we shall concentrate on the main result. We shall divide the proof into several steps. The reader is referred to the monographs of A.N. Skiba \([12]\) and W. Guo \([5]\) for terminologies and notations not mentioned in this paper.
2. Preliminaries

All groups considered in this paper are finite groups. Also, all wreath products considered are regular.

For the sake of easy reference, we first cite some basic concepts and notations from the monographs [3,5].

We call a set $\mathcal{X}$ of groups a class of groups if $\mathcal{X}$ contains a group $G$, then it also contains all groups which are isomorphic to $G$. When a group $G \in \mathcal{X}$, we call $G$ a $\mathcal{X}$-group. We denote by $\langle \mathcal{X} \rangle$ the intersection of all classes of groups which contains the set $\mathcal{X}$ of groups. A class $\mathcal{F}$ of groups is called a formation if $\mathcal{F}$ is closed under homomorphic images and also every group $G$ has a smallest normal subgroup whose respective quotient is in $\mathcal{F}$. We call such smallest normal subgroup of a group $G$ the $\mathcal{F}$-residual of $G$, and is denoted by $G^F$. In other words, $G^F = \bigcap \{N \triangleleft G \mid G/N \in \mathcal{F}\}$. We call a formation s-closed if every subgroup of $G$ belongs to $\mathcal{F}$ whenever $G \in \mathcal{F}$.

Now, we let $\mathcal{X}$ be a set of groups. Then, it is well known that the intersection of all formations containing $\mathcal{X}$ is still a formation. We shall call such a formation the formation generated by $\mathcal{X}$, and is usually denoted by $\text{form } \mathcal{X}$. If $\mathcal{X} = \{G\}$, where $G$ is a group, then $\text{form } \mathcal{X} = \text{form } G$ is called the formation generated by one element $G$, in brevity, we call $G$ a one-generated formation.

A function $f$ which maps every prime number $p$ to some formation of groups $f(p)$ is called a formation function. The symbol $\mathcal{L}(f)$ denotes the set of all groups such that either $G = 1$ or $G \neq 1$ and $G/CG(H/K) \in f(p)$, for every chief factor $Gi/Gi_{−1}$ of $G$ and for every $p \in \pi(Gi/Gi_{−1})$. For a formation $\mathcal{F}$, if there exists a formation function $f$ such that $\mathcal{F} = LF(f)$, then $\mathcal{F}$ is called a local formation. In this case, $f$ is called a local formation function of $\mathcal{F}$ or a screen of $\mathcal{F}$. It is known that a formation $\mathcal{F}$ is local if and only if $G \in \mathcal{F}$ when $G/\phi(G) \in \mathcal{F}$.

We also need the following concepts due to R. Baer and L.A. Shemetkov on Baer-local formations.

For a set $\mathcal{X}$ of groups, we let $C(\mathcal{X})$ be the class of all simple groups $A$ such that $A \cong H/K$ for some composition factor $H/K$ of some group $G \in \mathcal{X}$. In particular, for a group $G$, $C(G)$ denotes the class of all simple groups $A$ such that $A$ is isomorphic to some composition factor of $G$. A function $f$ which associates with every elementary group $H$ to some (probably empty) formation $f(H)$ is called a Baer function [3, p. 370] or is called a composition screen (L.A. Shemetkov [8]) if $f(A) = f(B)$ for any two elementary groups $A$ and $B$ with $C(A) = C(B)$.

The symbol $\mathcal{B}(f)$ denotes the class of all groups $G$ such that $G \in \mathcal{B}(f)$ if and only if either $G = 1$, or $G \neq 1$ and $G/CG(H/K) \in f(H/K)$ for each chief factor $H/K$ of $G$. Let $\mathcal{F}$ be a formation. If there exists a Baer function $f$ such that $\mathcal{F} = \mathcal{B}(f)$, then we call $f$ a composition screen of the formation $\mathcal{F}$. According to Baer, (see [3, Theorem IV, 4.17]), a non-empty formation $\mathcal{F}$ can be expressed by $\mathcal{F} = \mathcal{B}(f)$ for some composition screen $f$ if and only if $\mathcal{F}$ contains every group $G$ with quotient $G/\phi(R(G)) \in \mathcal{F}$, that is, $\mathcal{F}$ is a Baer-local formation. A formation $\mathcal{F}$ is called a soluble (abelian, nilpotent, metanilpotent) formation if every group in $\mathcal{F}$ is a soluble (abelian, nilpotent, metanilpotent, respectively) group. Thus, by definition, it is easy to see that a soluble local formation is a Baer-local.
formation. Local formations are mainly due with soluble groups. For groups which are not necessarily soluble, the Baer-local formations are more applicable.

Let $A$ be a simple group and $G$ a group. Then, following A.N. Skiba and L.A. Shemetkov in [13], we use the symbol $C^A(G)$ to mean the intersection of all centralizers of all chief factors of $G$ whose composition factors are isomorphic to $A$ (if $G$ has no chief factors of this kind, we just write $C^A(G) = G$). If $A = \mathbb{Z}_p$ is a group of prime order $p$, then the subgroup $C^A(G)$ is also denoted by $C^p(G)$. We use $E(A)$ to denote the class of all groups such that $G \equiv 1$ or $C(G) \equiv (A)$. The product of all normal subgroups of $G$ belonging to $E(A)$ is denoted by $GE(A)$.

Now, let $G$, $A$, and $B$ be groups, $\mathcal{X}$ a class of groups and $\mathcal{F}$ a formation. Then we cite the following notations:

- $|G|$: the order of $G$.
- $\pi(G)$: the set of all prime divisors of $|G|$.
- $\exp(G)$: the exponent of $G$.
- $c(G)$: the nilpotent class of $G$.
- $G^n$: the direct product of $n$ copies of $G$.
- $[A]B$: the semidirect product of $A$ and $B$.
- $A \wr B$: the (regular) wreath product of $A$ by $B$.
- $H(\mathcal{X}) = \{G \mid G \text{ is an epimorphic image of an } \mathcal{X}-\text{group}\}$.
- $R_0(\mathcal{X}) = \{G \mid \text{there are normal subgroups } N_1, \ldots, N_t (t \geq 2) \text{ of } G \text{ such that } \bigcap_{i=1}^t N_i = 1 \text{ and } G/N_i \in \mathcal{X}, i = 1, 2, \ldots, t\}$.
- $\mathcal{F} = \mathcal{F} \cap G$.
- $N$: the class of all nilpotent groups.
- $\mathcal{A}$: the class of all abelian groups.
- $\mathcal{S}$: the class of all soluble groups.
- $\mathcal{N}^2$: the class of all metanilpotent groups.
- $\text{iform } \mathcal{X}$: the local formation generated by $\mathcal{X}$, that is, the intersection of all local formations containing $\mathcal{X}$.
- $\text{cform } \mathcal{X}$: the Baer-local formation generated by $\mathcal{X}$, that is, the intersection of all Baer-local formations containing $\mathcal{X}$.

We cite here some known results as lemmas which will be useful later on in our paper.

**Lemma 2.1** [5, Lemma 2.1.3]. Let $N \trianglelefteq G$ and $\mathcal{F}$ a nonempty formation. Then $(G/N)^{\mathcal{F}} = G^F/N/N$.

**Lemma 2.2** [5, Lemma 2.3.2]. If $G$ is a primitive group and $G$ has two distinct minimal normal subgroups $N$ and $R$, then $C_G(N) = R$.

**Lemma 2.3** [4, Lemma 8]. If $A \in \text{form } G$, then the following statements hold.

1. $\exp(A) \leq \exp(G)$.
2. The order of every chief factor of $A$ does not exceed the maximal order of chief factors of $G$. 
Lemma 2.4 ([3, Lemmas A.18.2 and A.18.5] or [12, Lemma 3.1.9]). Let $G = A \wr B = \langle K \rangle B$, where $K$ is the base group of $G$. Assume that $A_1$ is the first copy of $A$ in $K$ and $A_1$ has an unique minimal normal subgroup $R_1$ such that $R_1 \nsubseteq Z(A_1)$. Then $R = \bigcap_{b \in B} R_1^b$ is the unique minimal normal subgroup of $G$ such that $R \nsubseteq Z(G)$ and $G/R \cong (A/R_1) \wr B$.

Lemma 2.5 [12, Lemma 3.1.7]. Let $A$ be a group of prime order $p$. Then $c(A \wr A^n) \geq n + 1$, where $n$ is a natural number and $A^n$ is the direct product of $n$ copies of $A$.

Lemma 2.6 [12, Corollary 3.3.7]. Let $p$ be a prime. Then $N_p = MH$, where $N$ is a metanilpotent normal subgroup of $G$. Then $M \in lform G$.

Lemma 2.7 ([1] or [5, Theorem 4.5.19]). If $G$ is a finite soluble group, then the number of local subformations of the formation $lform G$ is finite.

Lemma 2.8 [5, Theorem 4.2.1]. Let $X$ be a class of groups. Then $form X = HR_0X$.

Lemma 2.10 [12, Corollary 3.1.19]. Let $F = MH$ be a factorization of a local formation $F$ such that $M$ is also local formation. Then $F$ is one-generated local formation if and only if the following statements hold:

1. $|\pi(H)| \subseteq |\pi(M)|$ and $|\pi(M)| > 1$;
2. $M$ is a metanilpotent local formation and $H$ is one-generated formation;
3. For all primes $p$, $\pi(m(p)) \cap \pi(H) = \emptyset$, where $m$ is the minimal local screen of $M$;
4. If $M$ is not nilpotent formation, then $H$ is abelian.

Lemma 2.11 [5, Lemma 1.7.11]. If $H/K$ is a pd-chief factor of a group $G$, where $p$ is a prime, then $O_p(G/C_G(H/K)) = 1$.

Lemma 2.12 [5, Theorem 4.3.13 and Theorem 4.3.15]. Let $F$ be a local formation such that $F \nsubseteq N^2$. Then $F$ has a subformation $F_1$ such that $F_1 = lform G$, where $G$ is a group with an unique minimal normal subgroup $P$ such that $P = G^{N^2}$ and one of the following conditions is satisfied:

1. $P = G^N$ and $P$ is a non-abelian group.
2. $G = [P]H$, where $P = C_G(P)$ is a $p$-group for some prime $p$, $H = [Q]N$ such that $Q$ is the unique minimal normal subgroup of $H$, $Q = C_H(Q)$ is a $q$-group for some prime $q \neq p$, and $N$ is a nilpotent group.

Lemma 2.13 [12, Lemma 3.5.20]. Let $R$ be a normal subgroup of $G$ and $R$ a elementary abelian $p$-group. Then $G \in form(Z_p \wr (G/R))$, where $Z_p$ is a group of order $p$. 
The following lemma follows as a direct consequence of the concept of Baer-local formations.

**Lemma 2.14.** Let $\mathcal{F} = BLF(f)$ be a Baer-local formation and $G$ a group. Then $G \in \mathcal{F}$ if and only if $G/C^A(G) \in f(A)$ for all $A \in C(G)$.

### 3. Technical lemmas

We now use $G_{cp}$ to denote the class of all groups $G$ such that $C_G(H/K) = G$ for every $p$-chief factor $H/K$ of $G$.

Recall that a class of groups is called a Fitting class if the following conditions are satisfied:

1. If $G \in \mathcal{F}$ and $N \unlhd P G$, then $N \in \mathcal{F}$.
2. If $A, B \unlhd P G$ and $A, B \in \mathcal{F}$, then $AB \in \mathcal{F}$.

We first prove the following result.

**Lemma 3.1.** The class $G_{cp}$ is a Fitting class.

**Proof.** In order to prove that the class $G_{cp}$ is a Fitting class, we need to verify that conditions (1) and (2) of Fitting class hold. We first let $1 \neq N \trianglelefteq P G \in G_{cp}$. Then, we consider the following chain:

$$1 = N_0 \subset N_1 \subset \cdots \subset N_t = N$$

where $N_i/N_{i-1}$ is a chief factor of $G$, $i = 1, 2, \ldots, t$. If $N_i/N_{i-1}$ is a $p$-factor, then $C_G(N_i/N_{i-1}) = G$, and hence $C_N(N_i/N_{i-1}) = C_G(N_i/N_{i-1}) \cap N = N$. Now, by using the well-known Jordan–Hölder theorem (cf. [3, Theorem A,3.2]), we see that $N \in G_{cp}$. This shows that condition (1) above holds.

To verify that condition (2) above also holds, we let $A, B \subseteq G$ with $A, B \in G_{cp}$. Without loss of generality, we may put $G = AB$. If $A \cap B = 1$, then, evidently, $G = A \times B \in G_{cp}$. If $A \cap B \neq 1$, then we can let $L$ be a minimal normal subgroup of $G$ contained in $A \cap B$. Thus, by induction, we see that $G/L \in G_{cp}$. Hence, if $L$ is not a $p$-group, then $G \in G_{cp}$.

On the other hand, if $L$ is a $p$-group. Then $L$ may be considered as a simple $F_pG$-module over $F_p$, where $F_p$ is a field with $p$ elements. By [3, Lemma B, 7.1], $L = L_1 \times \cdots \times L_t$, where $L_1, \ldots, L_t$ are the minimal normal subgroups of $A$. Since $A \in G_{cp}$, we have $C_A(L_i) = A$ for all $i = 1, 2, \ldots, t$. Hence $C_A(L) = A$. Analogously, we have $C_B(L) = B$, and hence $C_G(L) = G$. Thus $G \in G_{cp}$. This shows that $G_{cp}$ satisfies condition (2) above. Hence the class $G_{cp}$ is indeed a Fitting class. 

We also need to prove the following lemmas which will be useful in the sequel.

**Lemma 3.2.** Let $G$ be a group and $N$ a normal subgroup of $G$. If $A$ is a simple group such that $A \notin C(N)$, then $C^A(G)/N = C^A(G/N)$. 


Proof. Let $H/K$ be any chief factor of $G$ such that $A \in C(H/K)$. Then, by the well-known Jordan–Hölder theorem, we see that there is a chief factor $T/L$ of $G$ such that $N \subseteq L$ and $H/K$ is $G$-isomorphic to $T/L$. Hence $C_G(H/K) = C_G(T/L)$. However, it is clear that $N \subseteq C_G(T/L)$. Therefore, $C^G(A) \subseteq N$, and hence $C^G(A)/N = C^G(A/N)$. 

Lemma 3.3. Let $G$ be a group and $A$ a simple non-abelian group. If $H/K$ is a chief factor of $G$ such that $A \in C(H/K)$. Then $C(Soc(G/C(H/K))) = (A)$. 

Proof. Let $M$ be a maximal subgroup of $G$ such that $MH = G$ and $K \subseteq M$, and let $C = C_G(H/K)$. Then the factor $H/K$ is $G$-isomorphic to the factor $HMG/MG$, where $MG$ is the maximal normal subgroup of $G$ contained in $M$, and hence $C = C_G(H/K) = C_G(HMG/MG)$. 

Without loss of generality, we may put $MG = 1$. If $H$ is the unique minimal normal subgroup of $G$, then $C = 1$ since $H$ is a non-abelian group, and hence $A \in C(Soc(G/C)) = C(Soc(G)).$ On the other hand, if $G$ has two minimal normal subgroups $H$ and $L$. Then, by Lemma 2.2, we have $L = C_G(H)$. If the factor group $G/L$ has a minimal normal subgroup $T/L \neq LH/L$, then $T/L \subseteq C_G(LH/L)/L = C_G(H)/L = L/L$. This contradiction shows that $LH/L = Soc(G/L) = Soc(G/C)$. Hence $(A) = C(Soc(G/C))$. 

Lemma 3.4. Let $A$ be a simple group and $G$ a group with $N_1, N_2, \ldots, N_t \subseteq G$ such that $\bigcap_{i=1}^{t} N_i = 1$. If $A \notin C(Soc(G/N_i))$ for all $i = 1, 2, \ldots, t$, then $A \notin C(Soc(G))$. 

Proof. Assume that $A \in C(Soc(G))$ and $L$ is a minimal normal subgroup of $G$ such that $A \in C(L)$. Since $L \notin \bigcap_{i=1}^{t} N_i$, there is an index $i$ such that $L \notin N_i$. But then we have $LN_i/N_i \subseteq Soc(G/N_i)$ and so $A \in C(Soc(G/N_i))$. This contradiction completes the proof. 

By Lemmas 3.3, 3.4, and 2.11, we obtain the following corollaries.

Corollary 3.5. Let $A$ be a simple non-abelian group and $A \in C(G)$. Then $(A) = C(Soc(G/C^A(G))).$ 

Corollary 3.6. Let $p$ be a prime and $G$ a group. Then $O_p(G/C^p(G)) = 1$. 

Now, let $A$ be a simple group. Let $E(A')$ be the class of all groups $G$ such that $A \notin C(G)$. Then, $E(A')$ is clearly a Fitting class and a formation.

If $G$ is a group and $\mathcal{F}$ is a Fitting class, then $G_\mathcal{F}$ denotes the product of all normal subgroups of $G$ which belongs to $\mathcal{F}$. We have the following interesting results.

Lemma 3.7. Let $A \in C(G)$. Then following statements hold.

1. If $A$ is a non-abelian group, then $C^A(G) = G_{E(A')}$. 
2. If $A$ is a group of prime order $p$, then $C^A(G) = G_{G_{E_p}}$. 

Proof. (1) Let $A$ be a non-abelian group. Then $(A) = C(Soc(G/C^A(G)))$ by Corollary 3.5. Thus $G_{E(A)} \subseteq C^A(G)$. On the other hand, if $G_{E(A)} = K \leq H \subseteq C^A(G)$, where $H/K$ is a chief factor of $G$, then $A \leq C(H/K)$, and hence $C^A(G) \subseteq C_G(H/K)$. It follows that $H/K$ is an abelian group, and hence $A$ is abelian. This contradiction shows that $G_{E(A)} = C^A(G)$.

(2) Now let $A = Z_p$ be a group of prime order $p$. Let $N = G_{Z_p}$. We prove that $N \leq C^p(G)$. Let $L$ be a minimal normal subgroup of $G$ such that $L \leq N$. If $Z_p \not\leq C(L)$. Then, by Lemma 3.2, we have $C^p(G)/L = C^p(G/L)$. However, by induction, we have $N/L \leq C^p(G/L)$. Hence $N \leq C^p(G)$. Now let $L$ be a $p$-group. Then, by [3, Lemma B; 7.1], $L \approx L_1 \times \cdots \times L_t$, where $L_i$ is a minimal normal subgroup of $N$, $i = 1, \ldots, t$. Since $N \leq G_{Z_p}$, we have $N \leq \bigcap_{i=1}^t C_N(L_i)$, and hence $N \leq C_G(L)$. Besides, by induction, we have $N/L \leq C^p(G/L)$. Let $C^p(G)/L = K/L$. Then $C^p(G) = K \cap C_G(L)$. This proves that $N \leq C^p(G)$. The inverse inclusion is evident. Therefore $G_{Z_p} = C^p(G)$. □

The following corollary follows trivially from Lemma 3.7.

Corollary 3.8. Let $G$ be a group and $A$ a simple group. Then $C^A(G) \cap N = C^A(N)$ for every normal subgroup $N$ of $G$.

Lemma 3.9 [4, Lemma 1]. Let $\mathcal{F} = \mathcal{MH}$, where $\mathcal{M} = BLF(m)$ for some inner composition screen $m$ and $\mathcal{H}$ is a formation such that $C(H) \cap A \subseteq C(\mathcal{M})$. If $N_\pi(\mathcal{M}) \subseteq \mathcal{M}$, then $\mathcal{F} = BLF(f)$, where

$$f(A) = \begin{cases} 
   m(Z_p)\mathcal{H}, & \text{if } A = Z_p \in C(\mathcal{M}) \cap A, \\
   \mathcal{F}, & \text{if } A \text{ is a simple non-abelian group}, \\
   \phi, & \text{if } A = Z_p \text{ is an abelian group and } A \notin C(\mathcal{M}).
\end{cases}$$

Let $\{f_i \mid i \in I\}$ be a set of composition screens. Then we define $\bigcap_{i \in I} f_i$ to be the composition screen $f$ by $f(H) = \bigcap_{i \in I} f_i(H)$ for all elementary groups $H$.

A composition screen $f$ is called inner if $f(A) \subseteq BLF(f)$ for every elementary group $A$. It is easy to see that every Baer-local formation has at least one inner composition screen. This is because if $f$ is an arbitrary composition screen of $\mathcal{F}$, then the composition screen $f_1$ satisfying $f_1(A) = f(A) \cap \mathcal{F}$ for every elementary group $A$ is an inner composition screen of $\mathcal{F}$.

We now assign a partial order in the set of all composition screens by defining $f_1 \leq f_2$ if and only if $f_1(A) \leq f_2(A)$ for every elementary group $A$.

If $\{f_i \mid i \in I\}$ is the set of all composition screens of a Baer-local formation $\mathcal{F}$, then it is clear that $f = \bigcap_{i \in I} f_i$ is still a composition screen of $\mathcal{F}$, which is called the minimal composition screen of $\mathcal{F}$.

It is also clear that the minimal composition screen of $\mathcal{F}$ is an inner screen.

By [8, Theorem 3.2], every Baer-local formation $\mathcal{F}$ has an unique maximal inner composition screen $f$, and $f$ satisfies the following properties:

1. $f(A) = \mathcal{F}$, if $A$ is a simple non-abelian group;
2. $f(A) = N_p f(A)$, if $A$ is an elementary abelian $p$-group.
Lemma 3.10. Let $\mathcal{F} = \text{cf} \mathcal{X}$ and $f$ the minimal composition screen of the formation $\mathcal{F}$. Then $f(A) = \text{form}(G/C\mathcal{X}(G) \mid G \in \mathcal{X})$ for $A \in C(\mathcal{X})$ and $f(A) = \emptyset$ for each simple group $A \notin C(\mathcal{X})$. Besides, if $h$ is a composition screen such that $\mathcal{F} = \text{BLF}(h)$, then

$$f(A) = \text{form}(G \mid G \in h(A) \cap \mathcal{F} \text{ and } C(\text{Soc}(G)) = (A)),$$

for every non-abelian group $A \in C(\mathcal{F})$, and

$$f(A) = \text{form}(G \mid G \in h(A) \cap \mathcal{F}, \ G_{E(A)} = 1),$$

for every abelian group $A \in C(\mathcal{F})$.

Lemma 3.11. Let $\mathcal{F} = \mathcal{F}^{\mathcal{F}}$ be a Fitting formation and $\mathcal{X}$ a non-empty class of groups such that $G \in \text{form} \mathcal{X}$. If $G_{\mathcal{F}} = 1$, then

$$G \in \text{form}(A/A_{\mathcal{F}} \mid A \in \mathcal{X}).$$

Proof. Let $\mathcal{X}_1 = H\mathcal{X}$. Assume that $G \in \mathcal{X}_1$. Then $G \cong T/L$ for some normal subgroup $L$ of some group $T \in \mathcal{X}$. Since $\mathcal{F} = \mathcal{F}^{\mathcal{F}}$ is a formation, $T_{\mathcal{F}}L/L \cong T_{\mathcal{F}}/L \cap T_{\mathcal{F}} \in \mathcal{F}$. However, since $G_{\mathcal{F}} = 1$, we have $T_{\mathcal{F}} \subseteq L$. It follows that

$$G \in H(T/T_{\mathcal{F}} \mid T \in \mathcal{X}) \subseteq \text{form}(A/A_{\mathcal{F}} \mid A \in \mathcal{X}).$$

Now let $G \notin \mathcal{X}_1$. Then by [5, Theorem 4.2.11], there is a group $H \in \text{form} \mathcal{X} = \text{form} \mathcal{X}_1$ with normal subgroups $N, M, N_1, M_1, \ldots, N_t, M_t$ ($t \geq 2$) such that the following statements hold:

1. $H/N \cong G$ and $M/N = \text{Soc}(H/N)$;
2. $N_1 \cap N_2 \cap \cdots \cap N_t = 1$ and $H/N_i$ is a $\mathcal{X}_1$-group with a unique minimal normal subgroup $M_i/N_i$, $i = 1, 2, \ldots, t$;
3. $L_i = N_1 \cap \cdots \cap N_{i-1} \cap M_i \cap N_{i+1} \cap \cdots \cap N_t$ is a minimal normal subgroup of $H$ such that $L_iN_i = M_i$ and $L_i \nsubseteq N_i$, $i = 1, 2, \ldots, t$.

Now, by (1) and (2) above, we have $G \in \text{form}(H/N_1, \ldots, H/N_t)$. By (3), we also have $L_i \cong L_iN_i/N_i = M_i/N_i \cong L_iN/N = M/N$. However, since $(H/N)_{\mathcal{F}} \cong G_{\mathcal{F}} = 1$, we have $(H/N_i)_{\mathcal{F}} = 1$ for all $i = 1, 2, \ldots, t$. Thus, we conclude that

$$H/N_i \in \text{form}(A/A_{\mathcal{F}} \mid A \in \mathcal{X})$$

and therefore

$$G \in \text{form}(A/A_{\mathcal{F}} \mid A \in \mathcal{X}).$$

The lemma is now proved. $\square$
Lemma 3.12 [8, Lemma 3.11]. Let \( f \) be a inner composition screen. Then \( \mathcal{N}_p f(Z_p) \subseteq \text{BLF}(f) \).

Lemma 3.13 [4, Lemma 5]. Let \( \mathcal{F} = \text{BLF}(f) \), where \( f \) is the minimal composition screen of the formation \( \mathcal{F} \). Then \( \mathcal{F} \) is a one-generated Baer-local formation if and only if the set \( C(\mathcal{F}) \) contains only a finite number of pairwise non-isomorphic groups and \( f(A) \) is a one-generated formation for each \( A \in C(\mathcal{F}) \).

Lemma 3.14 [4, Theorem]. Let \( \mathcal{F} = \text{MH} \) be a one-generated Baer-local formation. If \( \mathcal{H} \neq \mathcal{F} \), then \( \mathcal{M} \) is a local formation.

The following lemma is technical.

Lemma 3.15. Let \( \mathcal{F} = \mathcal{MH} \), where \( \mathcal{F}, \mathcal{M} \) and \( \mathcal{H} \) are formations, and every simple group in \( \mathcal{M} \) is abelian. Let \( A \in \mathcal{M}, B \in \mathcal{H} \), and \( m \) be a positive integer. Also, we suppose that for every positive integer \( i \), the group \( A \) has a normal subgroup \( A_i \) such that the factor group \( A/A_i \) is simple. Assume that every group of the sequence

\[
G_1 = (A/A_1) \rtimes B, \quad G_2 = (A/A_2) \rtimes G_1, \quad \ldots, \quad G_t = (A/A_t) \rtimes G_{t-1}, \quad \ldots
\]

belongs to the formation \( \mathcal{H} \). Then there are a prime \( p \in \pi(M) \) and a group \( T \in \mathcal{H} \) such that one of its subgroups \( M \) is a cyclic group of order \( p^n \) and each minimal normal subgroup of \( T \) is a \( p \)-group.

Proof. Since \( |A| < \infty \), there exists a prime \( p \) and an infinite sequence of indices \( i_1, i_2, \ldots, i_n, \ldots \) such that \( p \in \pi(A/A_{i_j}) \), where \( A_{i_j} \in \{ A_i \mid i = 1, 2, \ldots \} \) for all \( j = 1, 2, \ldots \). Let \( Z \) be a group of order \( p \) and

\[
T_1 = Z, \quad T_2 = Z \triangleleft T_1, \quad \ldots, \quad T_{n+1} = Z \triangleleft T_n, \quad \ldots
\]

We will show that for each \( i \) there is an index \( j \) such that the group \( T_i \) is isomorphic to some subgroup of \( G_j \). If \( i = 1 \), then the result is trivial.

Let \( i > 1 \), and let \( j \) be an index such that the group \( T_{i-1} \) is isomorphic to some subgroup of \( G_j \). Then there is an index \( j > 1 \) such that \( p \in \pi(A/A_j) \), i.e., \( Z \) is isomorphic to some subgroup of \( A/A_i \). This shows that \( T_i = Z \triangleleft T_{i-1} \) is isomorphic to some subgroup of \( (A/A_i) \rtimes G_{i-1} \), by [3, Lemma A, 18.2].

Hence, for every natural number \( i \), there exists a natural number \( j \) such that the group \( T_i \) is isomorphic to some subgroup of \( G_j \). Now let \( P \) be any \( p \)-group and \( l \) the length of its composition series. Then, by applying [3, Theorem A, 18.9] and by using induction, we see that the group \( P \) is isomorphic to some subgroup of \( G_j \in \mathcal{H} \).

Hence there exists a group \( T \in \mathcal{H} \) such that one of its subgroup \( M \) is a cyclic group of order \( p^n \) and each minimal normal subgroup of \( T \) is a \( p \)-group. This finishes the proof. \( \square \)

Lemma 3.16. Let \( n \) be a positive integer, \( q \) be a prime number and \( T = M \times \cdots \times M \) be the direct product of \( n \) copies of a non-abelian group \( M \) such that \( q \) does not divide \( |M| \).
Also, let $F_q$ be a field with $q$ elements. Then there exists a simple $F_q T$-module $V$ such that $\dim_{F_q} V \geq 2^n$.

**Proof.** Let $\overline{F}_q$ be the algebraic closure of $F_q$. Since $M$ is non-abelian, there exists at least as simple $F_q M$-module $R$ with $(T : F_q) \geq 2$. Let $D$ be the outer tensor product (cf. [2, Section 43]) of $n$ copies of the module $R$. Then, we can easily show that $D$ is still a simple $F_q (M)$-module which $(D : F_q) \geq 2^n$ (see [2, Exercise 2, p. 189]). Now, by [2, p. 71] formula (12.20), we see that there exists a simple $F_q [M_n]$-module $V$ such that $D$ is a direct summand of the module $V^{\overline{F}_q}$ (Ref. [2, Exercise 8, p. 206]). This leads to $(V : F_q) \geq 2^n$. Thus, the lemma is proved. $\blacksquare$

4. The main result

We now use all the lemmas in Sections 2 and 3 to prove our main result. We first let $H$ be a class of groups. Then, we use $H/Op(H)$ to denote the formation form $(G/Op(G) | G \in H)$. The following theorem gives an answer to the problem proposed by A.N. Skiba in his monograph (see [12, Problem 3.5.21]).

The proof is long.

**Theorem 4.1.** Let

$\mathcal{F} = \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$ (1)

be a cancellative factorization of a Baer-local formation $\mathcal{F}$ and $\mathcal{H} = \mathcal{M}_2 \cdots \mathcal{M}_t$. Then $\mathcal{F}$ is one-generated formation if and only if the following conditions are satisfied:

1. $\mathcal{M}_1$ is a metanilpotent one-generated local formation and $C(\mathcal{H}) \cap A \subseteq \mathcal{M}_1$;
2. If the formation $\mathcal{M}_1$ is not primary (i.e., $|\pi(\mathcal{M})| > 1$), then $\mathcal{H}$ is a one-generated formation; besides, if $\mathcal{M}_1$ is not nilpotent, then $\mathcal{H}$ is abelian;
3. For all groups $A \in \mathcal{M}_1$ and $B \in \mathcal{H}$, the groups $A/F(A)$ and $B$ have coprime exponents;
4. If $\mathcal{M}_1 = N_p$ for some prime $p$, then $\mathcal{H}/Op(\mathcal{H})$ is a one-generated formation;
5. $t \leq 3$, and if $t = 3$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are nilpotent, $\mathcal{M}_3$ is abelian and the exponents of $\mathcal{M}_2$ and $\mathcal{M}_3$ are coprime.

**Proof.** (Necessity) Let $\mathcal{F} = cform G$ for some group $G$, and let $|G| = m$. We use $f$ to denote the minimal composition screen of the formation $\mathcal{F}$. And let $\mathcal{M} = \mathcal{M}_1$. Since the proof is quite complicated, we now divide the proof of the necessity part into several parts.

1. We first show that every simple group $T$ in $\mathcal{M}$ is abelian.

Assume that our assertion is false and let $T$ be a simple non-abelian group in $\mathcal{M}$. Let $B$ be a non-identity group in $\mathcal{H}$ and $D = T : (B^m) = [K](B^m)$, where $K$ is the base group of $D$. Then by [3, Proposition A,18.5] or [12, Lemma 3.1.9], we know that $D$ has an unique
minimal normal subgroup which coincides with $K$. Clearly $D \in \mathcal{F}$. Now, by Lemma 3.10, we have

$$D \cong D/C_D(K) \in f(K) = \text{form}(G/C^T(G)).$$

but $|K| = |T|^{|B|^m} > m$. This contradicts Lemma 2.3. Hence, every simple group in $\mathcal{M}$ must be abelian.

(II) We next prove that the formation $\mathcal{M}$ must be soluble, that is, every group in $\mathcal{M}$ is a soluble group.

Assume that the assertion does not hold. Then we can let $A$ be a group of minimal order in $\mathcal{M}\setminus \mathcal{S}$. In this case, $A$ has a unique minimal normal subgroup $P = A^S$. Since every simple group in $\mathcal{M}$ is abelian, it is clear that $P \neq A$ and $P$ is itself non-abelian.

Let $B$ be a non-identity group in $\mathcal{H}$ and $D = A \wr (B^m) = [K](B^m)$, where $K$ is the base group of $D$. Assume that the $\mathcal{H}$-residual $D^\mathcal{H}$ of the group $D$ is contained subdirectly in $K$. Because $A \in \mathcal{M}$, we have $D^\mathcal{H} \in \mathcal{M}$, and so $D \in \mathcal{F}$. Let $A_1$ be the first copy of $A$ in $K$, and let $L_1 \simeq P$ be the unique minimal normal subgroup of $A_1$. Then, by Lemma 2.4, we know that $\prod_{b \in B^m} L_1^b$ is the unique minimal normal subgroup of $D$. Hence, we derive that

$$D/C^F(D) \cong D \in f(F) = \text{form}(G/C^F(G))$$

where $F = C(P)$. However, we can easily see that $|\prod_{b \in B^m} L_1^b| > m = |G|$, which is impossible by our Lemma 2.3. This shows that $D^\mathcal{H}$ is not contained subdirectly in $K$. Hence, by [3, Lemma A.18.2], $A$ has a maximal normal subgroup $M$ such that $(A/M) : ((B^m) : K) \in \mathcal{H}$. Analogously, we can show that there is a maximal normal subgroup $H$ of $A$ such that $(A/H) : ((A/M) : (B^m)) \in \mathcal{H}$, and so on. By using Lemma 3.15, we can see that there exists a prime number $p$ and a group $T \in \mathcal{H}$ such that $T$ has a cyclic subgroup with order $p^m$ and every minimal normal subgroup of $T$ must be a $p$-group. We also observe that for every group $X$ and for every group $Z_p$ of prime order $p$, we have $Z(Z_p \wr X) \neq 1$, where $Z(Z_p \wr X)$ is the center of $Z_p \wr X$. (In fact, let $G = Z_p \wr X = [K]X$, where $K$ is the base group of $G$. Also let $\Delta = \{(a, a, \ldots, a) : a \in Z_p\}$. Then, we can easily see that $1 \neq \Delta \subseteq Z(Z_p \wr X)$.) Thus, we can see that there exists a group $Z_p$ of order $p$ in $\mathcal{H}$ (cf. [5, Lemma 2.4.1] or [9, Lemma 3.32]).

It is clear that $Z_p \in \mathcal{F}$. Assume that there exists an abelian group $Z_q$ with prime order $q \neq p$ in $C(\mathcal{F})$. Then $\mathcal{N}_{[p,q]} \subseteq \mathcal{F}$. It is straightforward to verify that for every prime $r \in \pi(\mathcal{F})$, we have

$$\mathcal{N}_r = (\mathcal{N}_r \cap \mathcal{M})(\mathcal{N}_r \cap \mathcal{H}).$$

Hence if $r \in \pi(\mathcal{F})$, then either $\mathcal{N}_r \subseteq \mathcal{M}$ or $\mathcal{N}_r \subseteq \mathcal{H}$ by Lemma 2.6. We now consider the following possible cases:
(i) $Z_p \in \mathcal{M}$ and $N_{p} \subseteq \mathcal{H}$. Let $H$ be a cyclic group of order $q^m$ and let $D = Z_p \wr H = [K]H$, where $K$ is the base group of the wreath product $D$. Then $K = C^q(D)$ (cf. [14, Lemma 2]) and obviously, $D \in \mathcal{F}$. Hence

$$D/C^q(D) = D/K \cong H \in f(Z_p) = \text{form}(G/C^q(G)).$$

This contradicts Lemma 2.3 since $\exp(H) = q^m > m = |G|$.

(ii) $Z_p, Z_q \notin \mathcal{M}$ and $N_{[p,q]} \subseteq \mathcal{H}$. Let $P$ be any simple group in $\mathcal{M}$. Then, as proved by step (I) above, we see that $P$ is an abelian $r$-group, where $r$ is a prime number different from $q$. In this case, we just return to case (i).

(iii) $N_{[p,q]} \subseteq \mathcal{M}$. We consider the wreath product $D = Z_q \wr T = [K]T$, where $K$ is the base group of the wreath product $D$. Since $C^q(D) = K$, the group $D/C^q(D)$ contains a cyclic subgroup of order $p^m$. However, we have

$$D/D^q(D) \in f(Z_q) = \text{form}(G/C^q(G)).$$

This contradicts Lemma 2.3 and hence this case is impossible. We thus conclude that the order of every abelian group in $C(\mathcal{F})$ must be a prime $p$.

We show that $D = (T^\mathcal{H}) \wr (T/T^\mathcal{H}) \in \mathcal{F}$ for every group $T \in \mathcal{F}$. For this purpose, we let $T^\mathcal{H}/M$ be a chief factor of $T$. Since $T^\mathcal{H} \in \mathcal{M}$, $T^\mathcal{H}/M$ is a $p$-group. By Lemma 2.13, $T/M \in \text{form}(Z_p \wr T^\mathcal{H})$. This leads to $Z_p \wr (T/T^\mathcal{H}) \notin \mathcal{H}$, and hence, by [3, Lemma A, 18.2], the $\mathcal{H}$-residual $D^\mathcal{H}$ of $D$ is contained subdirectly in the base group of $D$. However, since $T \in \mathcal{F}$, we have $T^\mathcal{H} \in \mathcal{M}$. It follows that $D^\mathcal{H} \in \mathcal{M}$ and hence $D \in \mathcal{F}$.

We now show that $\mathcal{H} = N_p^\mathcal{H}$. Suppose on the contrary that $N_p^\mathcal{H} \not\subseteq \mathcal{H}$. Then, we can let $E$ be a group of minimal order in $N_p^\mathcal{H}\setminus\mathcal{H}$. In this case, $E$ has an unique minimal normal subgroup $R = E^\mathcal{H}$. Since there exists a group of order $p$ in $\mathcal{H}$, we have $R \neq E$. Thus, by repeating the above arguments, we can show that $E \in \text{form}(Z_p \wr (E/R))$, where $Z_p$ is a group of order $p$. Consequently, $Z_p \wr (E/R) \notin \mathcal{H}$. In this case, we can easily see that the $\mathcal{H}$-residual $T^\mathcal{H}$ of $T = A \wr (E/R)$ is contained subdirectly in the base group of $T$. Since $A \in \mathcal{M}$, we have $T \in \mathcal{M} \mathcal{H} = \mathcal{F}$. Clearly, $T^\mathcal{H} \neq 1$. Hence if $D = T^\mathcal{H} = T_1 \times \cdots \times T_m$, where $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_m \supseteq T$, then $D^\mathcal{H} \neq 1$ and $D \in \mathcal{F}$.

From above, we can further derive that $H = D^\mathcal{H} \wr (D/D^\mathcal{H}) \in \mathcal{F}$. Let $t = |D/D^\mathcal{H}|$. Then, it is clear that $D^\mathcal{H} \leq T_1^\mathcal{H} \times \cdots \times T_m^\mathcal{H}$ and $T_i/T_i^\mathcal{H} \neq 1$. This shows that $t > m$. Thus, by Lemma 2.4, for every minimal normal subgroup $K$ of $H$, we have $|K| \geq t > m$.

It is not difficult to see that every composition factor of every minimal normal subgroup $L$ of the group $H$ is isomorphic to $F$, i.e., $L$ is a non-abelian group and $|L| > m$. Let $M$ be the greatest normal subgroup of $H$ with the property $L \not\subseteq M$. Then $H/M$ has an unique minimal normal subgroup $LM/M$ which is isomorphic to $L$ and hence $|LM/M| > m$. But $H/M \in \mathcal{F}$, we see that $H/M$ is a homomorphic image of

$$H/C^F(M) \in f(F) = \text{form}(G/C^F(G)).$$

This contradicts Lemma 2.3. Thus, we have $N_p^\mathcal{H} \subseteq \mathcal{H}$. On the other hand, the inclusion $\mathcal{H} \subseteq N_p^\mathcal{H}$ obviously holds. Hence $\mathcal{H} = N_p^\mathcal{H}$. 

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Now we claim that $F = H$. Assume that $F \nsubseteq H$ and let $E$ be a group of minimal order in $F \setminus H$. Then $E$ has an unique minimal normal subgroup $R = E/\Phi$. Let $R = A_1 \times \cdots \times A_t$, where $A_1 \cong A_2 \cong \cdots \cong A_t$ is a simple group. Because $E \in F = MH$, we have $A_1 \in M$. Hence, by our step (I), we have $A_1 \cong Z_p$. It follows that $E \in N_pH = H$. This contradiction shows that $F \subseteq H$. Because the inclusion $H \subseteq F$ is obvious and hence $F = H$, this proves our claim. However, by the hypothesis of our theorem, we have $F \neq M_2 \cdots M_t = H$. This contradiction shows that $M$ is a soluble formation. On the other hand, by Lemma 3.14, $H$ has an unique minimal normal subgroup $E$. Hence, by our step (I), we have $A_1 \cong Z_p$. It follows that $E \in N_pH = H$. This contradiction shows that $M$ is a soluble formation. Therefore, by Lemma 3.14, $M$ is a local formation. This means that $M$ is a soluble local formation so that $N_{\pi(M)} \subseteq M$. We now denote the minimal composition screen of $M$ by $m_1$.

(III) Let us prove that $C(H) \cap A \subseteq C(M)$.

Assume that there exists a group $Z_p$ of prime order $p$ such that $Z_p \in C(H) \setminus C(M)$. Since $H \subseteq F$, we have $N_p \subseteq F$ and $Z_p \notin M$. Hence if $A$ is a simple group in $M$, then $|A| = q \neq p$. Let $B$ be a cyclic group of order $p^m$ and $D = A \ast B = [K]B$, where $K$ is the base group of the wreath product $D$. Clearly, $D \in F$. We hence obtain that

$$D/C^s(D) = D/K \cong B \in f(A) = \text{form}(G/C^s(G)).$$

This again contradicts Lemma 2.3. Therefore, $C(H) \cap A \subseteq C(M) \subseteq M$. By using Lemma 3.9, we conclude that $F = BLF(h)$ such that $h(Z_p) = m_1(Z_p)H$ for all $Z_p \in C(M) \cap A$.

(IV) To prove that if $p$ and $q$ are two different primes such that $Z_q$ is a group of order $q$ in $m_1(Z_p)$, then $q \notin \pi(H)$. Moreover, in this case, the formation $H$ is abelian.

Assume that there exist two different primes $p$ and $q$ such that $Z_q \in m_1(Z_p)$ and $q \in \pi(H)$. Also, let $H$ be a group in $H$ with $q \in \pi(H)$. Let $D = H^m$ and $T = Z_q \lhd D$. Then, it is clear that $T \in h(Z_p) = m_1(Z_p)H$. Since $O_p(T) = 1$, by Lemma 3.10, we have

$$T \in f(Z_p) = \text{form}(G/C^p(G)).$$

If $L$ is a subgroup of order $q$ of $H$, then by Lemma 2.5, $Z_q \lhd (L^m)$ is a nilpotent group whose nilpotent class $c \geq m + 1 = |G| + 1$. This contradicts Lemma 2.3. Hence, if $p$ and $q$ are different primes and $Z_q$ is a group of order $q$ in $m(Z_p)$, then $q \notin \pi(H)$. In this situation, the formation $H$ is abelian. In fact, if this is not true, then we can let $M$ be a non-abelian group in $H$. Write $T = M^m$. Then, we have $q \notin \pi(T)$. In this case, we let $F_q$ be a field with $q$ elements. Thus, by Lemma 3.16, there exists a simple $F_qT$-module $L$ such that $\dim_{F_q} L \geq 2^m$. Since $M \in H$ and $L$ is an elementary abelian $q$-group, we have $R = [L]T \in m_1(Z_p)H$. However, by $m_1(Z_p)H = h(Z_p)$, and by $R \in F$, we have $R/O_p(R) \in h(Z_p) \cap F$. Consequently, by Lemma 3.10, we have $R/O_p(R) \in f(Z_p)$. Also, we can observe that the group $R/O_p(R)$ has the chief factor $L_{O_p(R)}/O_p(R)$ with order $\geq q^{2m}$. This contradicts Lemma 2.3. Hence $H$ is an abelian formation.
(V) We now show that $\mathcal{M}$ is a metanilpotent formation.

Assume that $\mathcal{M} \not\subseteq N^2$. Then by Lemma 2.12, $\mathcal{M}$ has a subformation $\mathcal{H}_1$ such that $\mathcal{H}_1 = \text{Iform} \, A$, where $A$ is a group such that $A = [P]|\{Q|N\}$. $P = C_A(P)$ is a $p$-group and is an unique minimal normal subgroup of $A$. $Q = C_{[Q|N]}(Q)$ is a $q$-group and is the unique minimal normal subgroup of $\{Q|N\}$, and $N$ is a non-identity nilpotent group. By Lemma 2.11, we have $O_q(N) = 1$. It is clear that $F_q(A) = C^q(A) = PQ$. Since $A \in \mathcal{M}$, we have $A/C^q(A) \simeq A/PQ \simeq N \in m_1(Z_q)$. Consequently, there is a group $Z_r$ of prime order $r \neq q$ such that $Z_r \in m_1(Z_q)$. Hence, by (IV), we see that $\mathcal{H}$ is abelian and $\pi(N) \cap \pi(\mathcal{H}) = \emptyset$. Let $T = A \wr (B^m)$, where $B$ is some non-identity group in $\mathcal{H}$. Let $K$ be the base group of $T$ and let $A_1$ be the first copy of $A$ in $K$. Let $P_1 \equiv P$ be the unique minimal normal subgroup of $A_1$ and $Q_1 \equiv Q$ be the subgroup of $A_1$ such that $Q_1 P_1 / P_1$ is the unique minimal normal subgroup of $A_1 / P_1$. Then, in virtue of Lemma 2.4, $R = \prod_{b \in B^m} P_1^b$ is the unique minimal normal subgroup of $T$ such that $T/R \simeq E = ([Q|N]) \wr (B^m)$.

Let $M_1$ be the first copy of $\{Q|N\}$ in the base group of $E$. Let $Q_2$ be the unique minimal normal subgroup of $M_1$. Then by Lemma 2.4 again, we see that $L = \prod_{b \in B^m} Q_2^b$ is the unique minimal normal subgroup of $E$.

Assume that $T \not\in \mathcal{F}$. Because $A \in \mathcal{M}$, the subgroup $T^\mathcal{H}$ is not contained subdirectly in $K$. This implies that $A_1$ has a maximal normal subgroup $M$ such that $(A_1/M) \wr (B^m) \in \mathcal{H}$, and thereby, $\pi(N) \cap \pi(\mathcal{H}) \neq \emptyset$. This contradiction shows that $T \in \mathcal{F}$. Hence, we obtain that $T/C^p(T) \in f(Z_p)$. Since $P$ has a complement in $A$, $R$ has a complement in $T$ as well. This leads to $R = C_T(R)$, and so $C^p(T) = R$. Thus we deduce that

$$E \simeq T/R \simeq T/C^p(T) \in f(Z_p) = \text{form}(G/C^p(G)).$$

But since the group $T/R$ has a minimal normal subgroup of order $|L| > m$, this contradicts Lemma 2.3. Hence $\mathcal{M} \subseteq N^2$, that is, $\mathcal{M}$ is indeed a metanilpotent formation.

(VI) We further show that $\mathcal{M}$ is also a one-generated local formation.

In fact, by Lemma 2.7, every local subformation of $\mathcal{M}$ must be $s$-closed. This means that if $G \in \mathcal{M}$, then $G^\mathcal{H} \in \mathcal{M}$, and hence $G \in \mathcal{F}$. It follows that $\mathcal{M} \subseteq \mathcal{F}$. By Lemma 2.8, the set of all local subformations of $\mathcal{M}$ is finite. Now consider the chain

$$(1) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{M},$$

where $\mathcal{F}_i$ is a maximal local subformation in $\mathcal{F}_{i+1}$, $i = 0, 1, \ldots, n - 1$. Let $H_i \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$, $i = 1, 2, \ldots, n$. Then, we can see that

$$\mathcal{M} = \text{Iform}(H_1, \ldots, H_n) = \text{Iform}(H_1 \times \cdots \times H_n)$$

is a one-generated local formation. This proves condition (1) of our theorem.
(VII) To verify condition (3) of our theorem.

We let $A \in \mathcal{M}$, $B \in \mathcal{H}$. Since $\mathcal{M} = \mathcal{M}_1 \subseteq N^2$, $C^p(A) = F_p(A) \supseteq F(G)$, and $A/F_p(A) \in N$. Since $O_p(A/F_p(A)) = 1$, $p \notin \pi(A/F_p(A))$. However, $A/F_p(A) \in m_1(Z_p)$ since $A \in \mathcal{M}$. By our step (IV), we see that $\pi(A/F_p(A)) \cap \pi(B) = \emptyset$. Since $A/F(A) = A/(F_{p1}(A) \cap \cdots \cap F_{pt}(A))$, where $p_i \in \pi(A)$, we obtain that $\pi(A/F(A)) \cap \pi(B) = \emptyset$. Thus, condition (3) of our theorem is satisfies.

(VIII) Proof of condition (2) of our theorem.

We assume that $\mathcal{M}$ is non-nilpotent. Then there is a prime $p \in \pi(M)$ such that

$(1) \nsubseteq m_1(Z_p)$ and clearly $|\pi(M)| > 1$. Since $\mathcal{M}$ is metanilpotent, $m_1(Z_p) \subseteq N$ (cf. [5, Theorem 3.1.20]). Now, by Lemma 3.10, we see that $m_1(Z_p) \cap N_p = 1$, and hence $m_1(Z_p)$ contains a group of prime order $q \neq p$. Thus, by our step (IV) above, we know that $\mathcal{H}$ in this case is an abelian formation. Suppose that there exists $t \in \pi(\mathcal{H}) \setminus \pi(M)$. Then, we have $Z_t \in \mathcal{H} \subseteq \mathcal{F}$ and so $N_t \subseteq \mathcal{F}$. Thus, we deduce that $N_t = (N_t \cap \mathcal{M})(N_t \cap \mathcal{H}) = (N_t \cap \mathcal{H})$, and consequently, $N_t \subseteq \mathcal{H}$, a contradiction. Hence, we have $\pi(\mathcal{H}) \subseteq \pi(M)$. Since $\mathcal{F} = \mathcal{MH}$, $\mathcal{F}$ is soluble. Let $\mathcal{F}_1 = \text{form}(G/C^p(G) \mid p \in \pi(G))$. Then, it is clear that $\mathcal{F}_1$ is a soluble one-generated formation. Thus, by Lemma 2.8, the set of all subformations of $\mathcal{F}_1$ is finite. Same as step (VI), we can similarly show that every subformation of $\mathcal{F}_1$ is also one-generated. Now we show that $\mathcal{H} \subseteq \mathcal{F}_1$. For this purpose, we let $A$ be a group of minimal order in $\mathcal{H} \setminus \mathcal{F}_1$. Then $A$ has an unique minimal normal subgroup, and consequently, $A$ is a cyclic $p$-group for some prime $p$. Let $q \in \pi(M) \setminus \{p\}$ and $T = Z_q \triangleleft A$. Then, $T = \mathcal{M}H$, and hence $T/C^q(T) \cong A \in f(Z_q) \subseteq \mathcal{F}_1$, a contradiction. This shows that $\mathcal{H} \subseteq \mathcal{F}_1$, and hence, similar to the proof of step (VI), we can prove that $\mathcal{H}$ is also a one-generated formation. Assume that $|\pi(M)| > 1$. Then, same as above, we can show that $\mathcal{H}$ is a one-generated formation.

(IX) We now prove condition (4) of our theorem.

Let $\mathcal{M} = N_p$ for some prime number $p$. Let us show that $\mathcal{H}/O_p(\mathcal{H}) = \text{form}(A/O_p(A) \mid A \in \mathcal{H})$ is a one-generated formation. For this purpose, we let $T = Z_p \triangleleft (A/O_p(A))$. Then $T \in \mathcal{F}$, and so we can deduce that

$$A/O_p(A) \cong T/C^p(T) \in f(Z_p) = \text{form}(G/C^p(G)).$$

Consequently, we obtain $\mathcal{H}/O_p(\mathcal{H}) \subseteq f(Z_p)$. On the other hand, because $G \in \mathcal{F} = \mathcal{M}H = N_p \mathcal{H}$, we have $G^H \in N_p$. This leads to $G/O_p(G) \in \mathcal{H}/O_p(\mathcal{H})$, and thereby $G/O_p(G) \in \mathcal{H}/O_p(\mathcal{H})$. However, since $O_p(G) \subseteq C^p(G)$, we have $G/C^p(G) \in \mathcal{H}/O_p(\mathcal{H})$, and hence $f(Z_p) \subseteq \mathcal{H}/O_p(\mathcal{H})$. This proves that $\mathcal{H}/O_p(\mathcal{H}) = \text{form}(G/C^p(G))$ is indeed a one-generated formation.

(X) Finally, we prove condition (5) of our theorem.

We first prove that $t \leq 3$. Assume, on the contrary, that $t \geq 4$. Let $\mathcal{M} = M_1M_2$ and $\mathcal{H} = M_3 \cdots M_t$. If $\mathcal{F} = \mathcal{H} = \mathcal{M}_4 \cdots \mathcal{M}_t$, then $\mathcal{H} \subseteq M_2M_3 \cdots M_t \subseteq \mathcal{F}$, and hence
Thus, the contradiction shows that $A$ is an abelian formation, and $B$ is a metanilpotent one-generated local formation, and if $M \nsubseteq N$, then $H$ is an abelian formation. Assume that $M = N_p$ for some prime $p$. Then, $M_1$ is clearly a metanilpotent local formation, and hence $M_1$ is $s$-closed by Lemma 2.7. However, since $M = M_1 M_2$, we have $M_1 \subseteq M = N_p$. Obviously, $M_1 \neq (1)$ by our hypothesis. Therefore $M_1 = N_p$. It follows that $M_1 M_2 = M_1 = N_p$, and so $F = M_1 M_2 \cdots M_3$. This contradicts the uncancellativity of $F$. Thus, $|\pi(M)| > 1$. If $M$ is nilpotent, then there are groups $A \in M_1$ and $B \in M_2$ such that $|A| = p$ and $|B| = q$ with $q \neq p$. Let $T = A \cdot B = [K] B$, where $K$ is the base group of the wreath product $T$. Then $K$ is a Sylow $p$-subgroup of $T$ and $B$ is a Sylow $q$-subgroup of $T$. Because $T \in M_1 M_2 = M$ and $M$ is nilpotent, we have $B \subseteq C_T(K) = K$. This contradicts $C_T(K) = K$. Hence $M \nsubseteq N$, and by condition (2) of our theorem, we know that $H$ is an abelian formation. Let $A \in M_1$ and $B \in M_1 \cdots M_4$ be non-identity groups. Then $A \cdot B \in H$. However, since $H$ is a nilpotent formation, $B \subseteq A \cdot B$ which contradicts the fact that every normal subgroup of $A \cdot B$ has a non-trivial intersection with the base group of $A \cdot B$ (cf. [5, p. 23, Exercise 2]). Thus, the contradiction shows that $t \leq 3$.

Now assume that $t = 3$. Then, in this case, we have $F \neq M_3$. Thus, by the result above, we may conclude that both $M_1$ and $M_1 M_2$ are metanilpotent one-generated local formations. If $M_1 \nsubseteq N$, then $M_2 M_3$ is abelian. But from our previous arguments, we have already shown that this case is impossible. Hence $M_1$ must be nilpotent. We still need to show that the formation $M_2$ is also nilpotent. Assume that this is not the case. Then we can let $A$ be a group of minimal order in $M_2 \setminus N$. Thus, $A$ contains a unique minimal normal subgroup $R = A^N$. Since $M_1 M_2 \subseteq N^2$, we have $M_2 \subseteq S$ which is the formation of all soluble groups. This implies that every composition factor of $A$ is abelian. However, by condition (1) of our theorem, we know that $C(M_2) \setminus A \subseteq M_1$ and therefore every composition factor of $A$ is in $M_1$. As a consequence, we may choose a prime number $p$ such that $p \in \pi(M_1)$ and $O_p(A) = 1$. Let $P$ be a simple and faithful $F_p A$-module over the field $F_p$, where $F_p$ is a field of $p$ elements. Also, we let $D = [P] A$. Then, we have $F(D) = F$, and thereby, $D$ is not metanilpotent. However, we have $D \in M_1 M_2 \subseteq N^2$. This contradiction shows that $M_2$ is a nilpotent formation.

Assume that $M_1 = N_p$ for some prime $p$. Then, by condition (1) of our theorem, we see that $C(M_2) \subseteq M_1$, and hence $M_2 \subseteq N_p$. It follows that $F = M_1 M_2 M_3 = M_1 M_3$, a contradiction. Hence $M_1 \neq N_p$ for all primes $p$. Let $Z_q \in M_2$. Then by condition (1) again, we have $Z_q \in M_1$. Since $M_1 \neq N_p$ for all primes $p$, there exists $p \in \pi(M_1)$ such that $p \neq q$. Let $T = Z_p \cdot Z_q$. Then $T \in M_1 M_2$ and hence $M_1 M_2$ is not nilpotent. Thus by condition (2) of the theorem, $M_3$ is abelian.

At last, we let $A$ be an arbitrary $p$-group in $M_2$ for some prime $p$ and $B$ an arbitrary group in $M_3$. Let $Z_p \cdot Z_q$ be a group of prime order $q$ contained in $M_1$ with $q \neq p$. Write $T = Z_p \cdot A = [K] A$, where $K$ is the base group of $T$. Then $T \in M_1 M_2 \subseteq N^2$, $K = F(T)$, and $A \cong T/F(T)$. By using the same arguments as the proof of condition (3) of the theorem, we may conclude that the groups $A$ and $B$ have the coprime exponents. Thus, the exponents of the formations $M_2$ and $M_3$ are coprime.
Thus, by Lemma 2.10, generated Baer-local formation. (3), we have 
\[ \pi(\mu(p)) = \emptyset \]
for all prime \( p \). By invoking Lemma 3.11, we deduce that
\[ \frac{G}{Cp(G)} \] is a non-abelian simple group. This implies that
\[ \pi(p) = \emptyset \] for all prime \( p \).

Suppose that \( M \not\subseteq N \). Then by condition (2), \( H \) is abelian, and hence \( F = MH \) is soluble since \( M \) is metanilpotent. Because every soluble Baer-local formation is clearly a local formation, we see that \( F \) is a local formation. In order to prove that \( F \) is one-generated, we only need to verify that the conditions of Lemma 2.10 are all satisfied.

Now, since \( M \not\subseteq N \) and \( H \) is abelian, by condition (1) of our theorem, we have \( \pi(H) = \pi(M) \cap A \subseteq M_1 \) and \( \pi(M_1) > 1 \). Since \( M \not\subseteq N \), \( M \neq N_p \) for any prime \( p \). Hence, by our condition (2), \( H \) is a one-generated formation. Now we show that \( \pi(m(p)) \cap \pi(H) = \emptyset \) for all prime \( p \). In fact, since \( m(p) = \text{form}(G/Fp(G)) \) for all \( p \in \pi(M) \) and \( m(p) = \emptyset \) for \( p \neq \pi(M) \) (cf. [5, Theorem 3.1.15 and Definition 2.4.1], by our condition (3), we have \( \pi(m(p)) \cap \pi(H) = \emptyset \). Therefore \( F \) satisfies all the conditions of Lemma 2.10. Thus, by Lemma 2.10, \( F = \text{form} G = \text{cform} G \) for some group \( G \), that is, \( F \) is a one-generated Baer-local formation.

Now, we assume that \( M \subseteq N \). Since \( F(G) \subseteq C^A(G) \), we have \( G^H \subseteq C^A(G) \) for all groups \( G \in F \) and all simple groups \( A \in C(F) \). Hence by Lemma 3.10, \( F \) has the minimal composition screen \( f \) such that \( f(A) \subseteq H \) for all simple groups \( A \in C(F) \).

By using Lemma 3.13 and the above facts, we only need to prove that for every \( A \in C(F) \), the formation \( f(A) \) is one-generated. We first assume that \( H = \text{form} T \) and \( A \) is a group of prime order \( p \).

Let \( X_p = \{ G \mid G \in H \text{ and } O_p(G) = 1 \} \). Then by Lemma 3.11, \( X_p \subseteq \text{form}(T/O_p(T)) \). Hence form \( X_p = \text{form}(T/O_p(T)) \). We will prove that \( f(A) = \text{form} X_p \).

Let \( X_p = \text{form}(T/O_p(T)) \). We will prove that \( f(A) = \text{form} X_p \). Since \( f(A) = \text{form}(G/Cp(G)) \) for all \( G \in F \) by Lemma 3.10 and \( G^H \subseteq F(G) \subseteq C^p(G) \), we have \( G/Cp(G) \in H \), and so \( O_p(G/Cp(G)) = 1 \) by Lemma 2.11. Hence, we have \( f(A) \subseteq \text{form} X_p \). Let \( D = A \circ G \) where \( G \in X_p \). Then, we can easily see that \( C^p(D) = K \), where \( K \) is the base group of the wreath product \( D \). Obviously, \( D \in F \), and hence, we have
\[ G \cong D/C^p(D) \in f(A) \].

This shows that \( f(A) = \text{form}(T/O_p(T)) \) is a one-generated formation.

We now prove that \( f(A) \) is one-generated when \( A \) is a non-abelian group. In fact, by Lemmas 3.7 and 3.10, we have \( f(A) = \text{form}(G/C^A(G)) \) for all \( G \in F \).

By invoking Lemma 3.11, we deduce that \( (G/C^A(G))/G/C^A(G))_{E(A)} \cong G/C^A(G) \) in \( \text{form}(T/T_{E(A)}) \). Thus, we have \( (G/C^A(G)) \cap F \subseteq \text{form}(T/T_{E(A)}) \) and hence \( f(A) \subseteq \text{form}(T/T_{E(A)}) \). On the other hand, we have \( T \in H \subseteq F \), and so \( T/C^A(T) = T/T_{E(A)} \in f(A) \). Therefore \( f(A) = \text{form}(T/T_{E(A)}) \). For the case that \( H/O_p(H) = \text{form}(T/O_p(T)) \), by Lemma 3.10, we see that if \( C(G) = (A_1, A_2, \ldots, A_t) \), then \( C(M) = (A_1, A_2, \ldots, A_t) \). Since \( H \) is a one-generated formation, \( C(M) \) contains only finite number of pairwise non-isomorphic simple groups. This implies that
\[ \pi(M) \] contains only finite number of pairwise non-isomorphic simple groups.
form\( (T/O_p(T) \mid T \in \mathcal{H}) = \text{form} T_1 \), we can also prove that \( f(A) = \text{form}(T/O_p(T) \mid T \in \mathcal{H}) = \text{form} T_1 \), analogously. Thus, the proof of the theorem is completed.

Acknowledgment

The authors are grateful to the referee for the valuable comments contributed to this paper.

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