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A Feasible Algorithm for Designing Biorthogonal Bivariate Vector-valued Finitely Supported Wavelets

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Abstract

Wavelet analysis has been developed a new branch for over twenty years. The concept of vector-valued binary wavelets with two-scale dilation factor associated with an orthogonal vector-valued scaling function is introduced. The existence of orthogonal vector-valued wavelets with two-scale is discussed. A necessary and sufficient condition is provided by means of vector-valued multiresolution analysis and paraunitary vector filter bank theory. An algorithm for constructing a sort of orthogonal vector-valued wavelets with compact support is proposed, and their orthogonal properties are investigated.

Keywords :bivariate, time-frequency analysis, filter banks,orthogonal,oblique frames, wavelet frames, wavelets.

1.Introduction

Aside from the straightforward construction of Daubechies' wavelets, only a few, specific construcion of multivariate orthonormal wavelet systems exist presently in the literature. The main advantage of wavelets is their time-frequency localization property. Already they have led to exciting applications in signal analysis [1], fractals [2], image processing [3] and so on. Sampling theorems play a basic role in digital signal processing. They ensure that continuous signals can be processed by their discrete samples. Vector-valued wavelets are a class of generalized multiwavelets [4]. Chen and Cheng [5] introduced the notion of vector-valued wavelets and showed that multiwavelets can begenerated from the component functions in vector-valued wavelets. Vector-valued wavelets and multiwavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms but not necessary for discrete vector-valued wavelet transforms [5]. In real life, Video images are vector-valued signals. Vector-valued wavelet transforms have been recently studied for image coding by W. Li. Chen and Cheng studied orthogonal compactly supported vector-valued wavelets with 2-scale. Inspired by [5-7], we are about to investigate the construction of a class of orthogonal compactly supported vector-valued wavelets with three-scale. Similar to uni-wavelets, it is more complicated and

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meaningful to investigate vector-valued wavelets with 4-scale. Based on an observation in [5,8], another purpose of this article is to introduce the notion of orthogonal vector-valued wavelet packets with three-scale and investigate their properties.

2. Multiresolution analysis

By Z and Z_+ denote all integers and all non-negative integers, respectively. Set v be a constant and $2 \le v \in Z$. By $L^2(R^2, C^v)$, we denote the aggregate of arbitrary vector-valued functions $F(t)$, i.e, $L^2(R^2,C^v) := \{ G(t) = (g_1(t),\ g_2(t), \cdots, g_v(t))^T : g_\iota(t) \in L^2(R^2),\ \iota = 1,2,\cdots,v\}$, where T means the transpose of a vector. For example, video images an -d digital films are examples of vector-valued functions where $g_\iota(t)$ denotes the pixel on the $\iota-$th column at time t. For $G(t) \in L^2(R^2,C^v)$, $\|G\|$ denotes the norm of vector-valued function $G(t)$, i.e., $\|G\| := (\sum_{\iota=1}^{u} \int_{R^2} |g_\iota(t)|^2\ dt)^{1/2}$, and its integration is defined as $\int_{R^2} G(t)dt := (\int_{R^2} g_1(t)dt,\ \int_{R^2} g_2(t)dt,\ \cdots,\ \int_{R^2} g_v(t)dt)^T$ .The Fourier transform of $F(t)$ is defined by $\hat{G}(\omega) := \int_{R^2} G(t)\cdot \mathrm{e}^{-it\omega}\ dt$.

For two vector-valued functions $F, G \in L^2(R^2,C^v)$, their *symbol inner product* is defined by

$$\langle F(\cdot), G(\cdot)\rangle := \int_{R^2} F(t)G(t)^* dt, \tag{1}$$

where * means the transpose and the complex conjugate, and I_u denotes the $v \times v$ identity matrix. A sequence $\{G_l(t)\}_{l\in Z} \subset U \subseteq L^2(R^2,C^u)$ is called an orthonormal set of the subspace U, if the following condition is satisfied

$$\langle G_j(\cdot), G_k(\cdot)\rangle = \delta_{j,k} I_v,\ \ j,k \in Z, \tag{2}$$

where $\delta_{j,k}$ is the Kronecker symbol, i.e., $\delta_{j,k} = 1$ as $j = k$ and $\delta_{j,k} = 0$ otherwise.

Definition 1. We say that $H(t) \in U \subseteq L^2(R^2,C^u)$ is an orthogonal vector-valued function of the subspace U if its translations $\{H(t-v)\}_{v\in Z^2}$ is an orthonormal collection of the subspace Y, i.e.,

$$\langle H(\cdot - n), H(\cdot - v)\rangle = \delta_{n,v} I_u,\ \ n,\ v \in Z^2. \tag{3}$$

Definition 2[5]. A sequence $\{G_v(t)\}_{v\in Z^2} \subset U \subseteq L^2(R^2,C^v)$ is called an orthonormal basis of Y, if it satisfies (2), and for any $F(t) \in U$, there exists a unique sequence of $v \times v$ constant matrices $\{Q_k\}_{k\in z}$ such that

$$F(t) = \sum_{v\in Z^2} Q_v G_v(t). \tag{4}$$

Definition 3[5]. A vector-valued multiresolution analysis of $L^2(R^2,C^v)$ is a nested sequence of closed subspaces $\{Y_l\}_{l\in z}$ such that (i) $Y_l \subset Y_{l+1}, \forall\, l \in Z$; (ii) $\bigcap_{l\in z} Y_l = \{0\}$; $\bigcup_{l\in z} Y_l$ is dense in $L^2(R^2,C^v)$, where 0 is the zero vector of $L^2(R^2,C^v)$; (iii) $G(t) \in Y_0$ if and only if $G(2^l t) \in Y_l$, ; (iv) there is $\hbar(t) \in Y_0$ such that the sequence $\{\hbar(t-v), v \in Z^2\}$ is an orthonormal basis of subspace Y_0.

On the basis of Definition 2 and Definition 3, we obtain $\lambda(t)$ satisfies the following equation

$$\hbar(t) = \sum_{u\in Z^2} P_v \hbar(2t-u), \tag{5}$$

where $\{P_k\}_{k\in Z^2,}$ is a finite supported sequence of $\mu\times\mu$ constant matrices, i.e., $\{P_k\}_{k\in z}$ has only finite non-zero terms, and the others are zero matrices. By taking the Fouries transform for the both sides of (5), and assuming $\hat{\hbar}(\xi)$ is continuous at zero, we have

$$\hat{\hbar}(2\omega)=\mathcal{P}(\xi)\hat{\hbar}(\xi),\ \ \xi\in R^2, \tag{6}$$

$$4\mathcal{P}(\xi)=\sum_{u\in Z^2}P_u\cdot\exp\{-iu\xi\}. \tag{7}$$

Let $W_j\,(j\in Z)$ denote the orthocomplement subspace of Y_j in Y_{j+1} and there exist three vector-valued functions $G_s(t)\in L^2(R^2,C^v),\ s=1,2$ such that their translations and dilations form a Riesz basis of W_j, i.e.,

$$W_j=clos_{L^2(R,C^v)}(\text{span}\{G_s(2^jt-u):\ s=1,2,3;u\in Z^2\}),\ j\in Z. \tag{8}$$

Since $G_s(t)\in W_0\subset Y_1,\ s=1,2,$ there exist three finitely supported sequences $\{B_k^{(s)}\}_{k\in z}$ of $v\times v$ constant matrices such that

$$G_s(t)=\sum_{u\in Z^2}B_u^{(s)}\hbar(2t-u),\ \ s=1,2,3. \tag{9}$$

$$4\mathcal{B}^{(s)}(\xi)=\sum_{u\in Z^2}B_u^{(s)}\exp\{-iu\xi\}. \tag{10}$$

Then, the refinement equation (10) becomes thefollowing

$$\hat{G}_s(2\xi)=\mathcal{B}^{(s)}(\xi)\hat{\hbar}(\xi),\ \ s=1,2,3,\ \ \xi\in R^2. \tag{11}$$

If $\hbar(t)\in L^2(R^2,C^v)$ is an orthogonal vector-valued scaling function, then it follows from (3) that

$$\left\langle\hbar(\cdot),\,\hbar(\cdot-v)\right\rangle=\delta_{0,u}I_v,\qquad v\in Z^2. \tag{12}$$

We say that $G_s(t)\in L^2(R^2,C^v),\ s=1,2,3$ are orthogonal vector-valued wavelet functions associated with the vector-valued scaling function $\hbar(t)$, if they satisfy

$$\left\langle\hbar(\cdot-n),\,G_s(\cdot-v)\right\rangle=O,\quad s=1,2,\quad n,v\in Z, \tag{13}$$

and the family $\{G_s(t-v),s=1,2,3,\ v\in Z\}$ is an orthonomal basis of W_0. Thus we have

$$\left\langle G_r(\cdot),\,G_s(\cdot-n)\right\rangle=\delta_{r,s}\delta_{0,n}I_u,\ \ r,s=1,2,3;\ \ n\in Z. \tag{14}$$

Lemma 1[6]. Let $F(t)\in L^2(R^2,C^v)$. Then $F(t)$ is an orthogonal vector-valued function if and only if

$$\sum_{k\in Z^2}\widehat{F}(\xi+2k\pi)\widehat{F}(\xi+2k\pi)^*=I_u,\quad \omega\in R^2. \tag{15}$$

Lemma 2. If $\hbar(t)\in L^2(R^2,C^v)$, defined by (5), is an orthogonal vector-valued scaling function, then for $\forall\ v\in Z$, we have the following equalities,

$$\sum_{\sigma\in z}P_\sigma(P_{\sigma+4v})^*=4\delta_{0,v}I_u. \tag{16}$$

$$\sum_{\sigma=0}^{3}\mathcal{P}(\xi+\sigma\pi)\mathcal{P}(\xi+\sigma\pi)^*=I_u,\ \ \xi\in R^2. \tag{17}$$

Proof. By substituting equation (5) into the relation (12), for $\forall\ k\in Z$, we obtain that

$$\delta_{0,k}I_u = \langle \hbar(\cdot - k), \hbar(\cdot)\rangle = \sum_{l\in Z^2}\sum_{u\in Z^2}\int_R P_l \hbar(2t-2k-l)\hbar(2t-u)^*(P_u)^* dt$$

$$= \frac{1}{4}\cdot\sum_{l\in Z^2}\sum_{u\in Z^2} P_l \langle \lambda(\cdot-2k-l), \lambda(\cdot - u)\rangle (P_u)^* = \frac{1}{4}\sum_{u\in Z^2} P_u (P_{u+4k})^* .$$

Thus, both Theorem 1 and formulas (16), (23) and (24) provide an approach to design a class of compactly supported orthogonal vector-valued wavelets.

3.Construction of wavelets

In the following, we begin with considering the existence of a class of compactly supported orthogonal vector- valued wavelets.

Theorem 1. Let $\hbar(t)\in L^2(R^2, C^v)$ defined by (5), be an orthogonal vector-valued scaling function. Assume $G_s(t)\in L^2(R^2,C^v)$, $s=1,2,3,$ and $\mathcal{P}(\xi)$ and $\mathcal{B}^{(s)}(\xi)$ are defined by (7) and (10), respectively. Then $G_s(t)$ are orthogonal vector-valued wavelet functions associated with $\lambda(t)$ if and only if

$$\sum_{\sigma=0}^{3}\mathcal{P}(\xi+\sigma\pi)\mathcal{B}^{(s)}(\xi+\sigma\pi)^* = O, \qquad (18)$$

$$\sum_{\sigma=0}^{3}\mathcal{B}^{(r)}(\xi+\sigma\pi)\mathcal{B}^{(s)}(\xi+\sigma\pi)^* = \delta_{r,s}I_u, \qquad (19)$$

where $r,\ s\in 1,2,\ \omega\in R$. or equivalently,

$$\sum_{l\in Z^2} P_l (B^{(s)}_{l+2u})^* = O, \quad s=1,2,3,\ u\in Z^2; \qquad (20)$$

$$\sum_{l\in Z^2} B^{(r)}_l (B^{(s)}_{l+2u})^* = 4\delta_{r,s}\delta_{0,u}I_{v,}\ r,s=1,2,3,\ u\in Z^2. \qquad (21)$$

Proof. Firstly, we prove the necessity. By Lemma 1 and (6), (11) and (13), we have

$$O=\sum_{u\in Z^2}\hat{\hbar}(2\xi+u\pi)\hat{G}_s(2\xi+u\pi)^*$$

$$=\sum_{u\in Z^2}\mathcal{P}(\xi+u\pi)\hat{\hbar}(\xi+u\pi)\cdot$$

$$\cdot\ \hat{\hbar}(\xi+u\pi)^*\mathcal{B}^{(s)}(\xi+u\pi)^*$$

$$=\sum_{\sigma=0}^{3}\mathcal{P}(\xi+\sigma\pi/2)\mathcal{B}^{(s)}(\xi+\sigma\pi/2)^*.$$

It follows from formula (14) and Lemma 1 that

$$\delta_{r,s}I_v=\sum_{u\in Z^2}\hat{G}_r(2\xi+2u\pi)\hat{G}_s(2\xi+2u\pi)^*$$

$$=\sum_{u\in Z^2}\mathcal{B}^{(r)}(\xi+u\pi)\hat{\hbar}(\xi+u\pi)$$

$$\cdot\ \hat{\hbar}(\xi+u\pi)^*\mathcal{B}^{(s)}(\xi+u\pi)^*$$

$$=\sum_{\sigma=0}^{3}\mathcal{B}^{(r)}(\xi+\sigma\pi)\mathcal{B}^{(s)}(\xi+\sigma\pi)^{*}.$$

Next, the sufficiency of the theorem will be proven. From the above calculation, we have

$$\sum_{u\in Z^2}\hat{\hbar}(2\xi+u\pi)\hat{G}_s(2\xi+u\pi)^{*}$$

$$=\sum_{\sigma=0}^{3}\mathcal{B}^{(r)}(\omega+\sigma\pi/2)\mathcal{B}^{(s)}(\omega+\sigma\pi/2)^{*}=\delta_{r,s}I_u.$$

Furthermore

$$\langle\hbar(\cdot),G_s(\cdot-k)\rangle=\frac{1}{\pi^2}\int_{[0,\pi/2]^2}\sum_{u\in Z^2}\hat{\hbar}(2\omega+2u\pi)$$

$$\cdot\,\hat{G}_s(2\xi+2u\pi)^{*}e^{4ik\xi}d\xi=O,\ \ s=1,2,3,\ \ k\in Z^2$$

$$\langle G_r(\cdot),G_s(\cdot-k)\rangle=\frac{1}{\pi^2}\int_{[0,\pi/2]^2}\sum_{u\in Z^2}\hat{G}_r(2\omega+2u\pi)$$

$$\cdot\,\hat{G}_s(2\xi+2u\pi)^{*}\cdot e^{4ik\xi}d\xi=\delta_{0,k}\delta_{r,s}I_{v,}\ \ k\in Z^2$$

Thus, $\hbar(t)$ and $G_s(t),s=1,2,3$ are mutually orthogonal, and $\{G_s(t),\ s=1,2,3\}$ are a family of orthogonal vector-valued functions. This shows the ortho-gonality of $\{G_s(\cdot-v),\ s=1,\ 2,3\}_{v\in z}$. Similar to [7,Proposition 1], we can prove its completeness in W_0.

Theorem 2. Let $\hbar(t)\in L^2(R^2,C^v)$ be a 5-coefficient compactly supported orthogonal vector-valued scaling functions satisfying the following refinement equation:

$$\hbar(t)=P_0\hbar(2t)+P_1\hbar(2t-1)+\cdots+P_4\hbar(2t-4).$$

Assume there exists an integer $\ell,\ 0\le\ell\le4$, such that $(4I_v-P_\ell(P_\ell)^*)^{-1}P_\ell(P_\ell)^*$ is a positive definite matrix. Define $Q_s(s=1,2,3)$ to be two essentially distinct Hermitian matrice, which are all invertible and satisfy

$$(Q_s)^2=[4I_v-P_\ell(P_\ell)^*]^{-1}P_\ell(P_\ell)^*. \tag{22}$$

Define

$$\begin{cases}B_j^{(s)}=Q_sP_j, & j\ne\ell,\\ B_j^{(s)}=-(Q_s)^{-1}P_j, & j=\ell,\end{cases}\quad s=1,2,3,\ \ j\in\{0,1,2,3,4\}. \tag{23}$$

Then $G_s(t)\ (s=1,2,3)$, defined by (24), are orthogonal vector-valued wavelets associated with $\lambda(t)$:

$$G_s(t)=B_0^{(s)}\hbar(2t)+B_1^{(s)}\hbar(2t-1)+\cdots+B_4^{(s)}\hbar(2t-4) \tag{24}$$

Proof. For convenience, let $\ell=1$. By Lemma 2, (20) and (21), it suffices to show that $\{B_0^{(s)},B_1^{(s)},B_2^{(s)},B_3^{(s)},B_4^{(s)},\ \ s=1,2,3\}$ satisfy the following equations:

$$P_0(B_4^{(s)})^*=O,\quad s=1,2,3, \tag{25}$$

$$P_4(B_0^{(s)})^*=O,\quad s=1,2,3, \tag{26}$$

$$P_0(B_0^{(s)})^*+P_1(B_1^{(s)})^*+\cdots+P_4(B_4^{(s)})^*=O, \tag{27}$$

$$B_0^{(r)}(B_4^{(s)})^*=O,\quad r,s\in\{1,2,3\}, \tag{28}$$

$$B_0^{(s)}(B_0^{(s)})^* + B_1^{(s)}(B_1^{(s)})^* + \cdots + B_4^{(s)}(B_4^{(s)})^* = 4I_u. \quad (29)$$

If $\{B_0^{(s)}, B_1^{(s)}, B_2^{(s)}, B_3^{(s)}, s = 1, 2\}$ are given by (23), then equ- -ations (26), (27) and (29) follow from (16). For the proof of (28) and (30), it follows from (16) and (27) that

$$P_0(B_0^{(s)})^* + P_1(B_1^{(s)})^* + P_2(B_2^{(s)})^* + P_3(B_3^{(s)})^* + P_4(B_4^{(s)})^*$$
$$= [P_0(P_0)^* + P_2(P_2)^* + P_3(P_3)^* + P_4(P_4)^*]\, Q_s - P_1(P_1)^*(Q_s)^{-1}$$
$$= (P_1(P_1)^* - P_1(P_1)^*)(B_s)^{-1} = O.$$

$$B_0^s(B_0^s)^* + B_1^s(B_1^s)^* + B_2^s(B_2^s)^* + B_3^s(B_3^s)^* + B_4^s(B_4^s)^*$$
$$= Q_s\{P_1(P_1)^* + [P_1(P_1)^*]^{-1}[4I_u - P_1(P_1)^*]P_1(P_1)^*\}(Q_s)^{-1}$$
$$= Q_s\{P_1(P_1)^* + 4I_u - P_1(P_1)^*\}(Q_s)^{-1} = 4I_u.$$

So, (28), (30) follow. This completes the proof ofThm 2.

Example1. Let $\hbar(t) \in L^2(R^2, C^3)$ be 5-coefficient orthogonal vector-valued scaling function satisfy the following equation:

$$\hbar(t) = P_0\hbar(2t) + P_1\hbar(2t-1) + \cdots + P_4\hbar(2t-4).$$

where $P_3 = P_4 = O,\ \ P_0(P_4)^* = O,\ \ P_0(P_0)^* + P_1(P_1)^* + P_2(P_2)^* + P_3(P_3)^* + P_4(P_4)^* = 4I_3.$

$$P_0 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{2}}{3} & 1 \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{pmatrix},\quad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{6} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} \end{pmatrix},\quad P_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{2}}{3} & -1 \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{pmatrix},$$

Suppose $\ell = 1$. By using (22), we can choose

$$Q_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{53}}{53} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, Q_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{53}}{53} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{4} \end{pmatrix}.$$

By applying formula (24), we get that

$$B_0^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{53}}{106} & \frac{\sqrt{106}}{159} & \frac{\sqrt{53}}{53} \\ 0 & 0 & \frac{\sqrt{6}}{6} \end{pmatrix}, B_1^{(1)} = -\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{\sqrt{106}}{6} & 0 \\ 0 & 0 & \frac{2\sqrt{6}}{3} \end{pmatrix}$$

$$B_2^{(1)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{53}}{106} & \frac{\sqrt{106}}{159} & -\frac{\sqrt{53}}{53} \\ 0 & 0 & \frac{\sqrt{6}}{6} \end{pmatrix},$$

$$B_0^{(2)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{53}}{106} & \frac{\sqrt{106}}{159} & \frac{\sqrt{53}}{53} \\ 0 & 0 & -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

$$B_1^{(2)} = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & -\frac{\sqrt{106}}{6} & 0 \\ 0 & 0 & \frac{2\sqrt{6}}{3} \end{pmatrix},$$

$$B_2^{(2)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{53}}{106} & \frac{\sqrt{106}}{159} & -\frac{\sqrt{53}}{53} \\ 0 & 0 & -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

Applying Theorem 2, we obtain that

$$G_\iota(t) = B_0^{(\iota)}\hbar(2t) + B_1^{(\iota)}\hbar(2t-1) + \cdots + B_4^{(\iota)}\hbar(2t-4), \iota = 1,2$$

are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function

4.Conclusion

A necessary and sufficient condition on the existence of a class of orthogonal vector-valued wavelets is presented.
An algorithm for constructing a class of compactly supported orthogonal vector-valued wavelets is proposed.

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