On Theta Functions Associated to Indefinite Quadratic Forms

SOLOMON FRIEDBERG*

Department of Mathematics, Harvard University, Cambridge, MA 02138

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We introduce a family of theta functions associated to an indefinite quadratic form, and prove a modular transformation formulas by regarding each such function as a specialization of a symplectic theta function. An eighth root of unity arises in these formulas, and it is expressly given in all cases. The theta functions feature many "translation variables," which are useful for the study of the liftings of modular forms. © 1986 Academic Press, Inc.

INTRODUCTION

We shall study the transformation properties of certain theta functions, constructed from an indefinite quadratic form, with respect to modular substitutions. These theta functions are generalizations of ones first used by Siegel [8] in his fundamental work on quadratic forms, and they have been much studied recently in the context of liftings, i.e., correspondences between automorphic forms on various orthogonal groups (cf. Kudla [5] and the references there). One description of their transformation properties may be found in Shintani [6], where they are derived using the Weil representation. Here, following Siegel [7] (with several modifications), we shall take an alternate approach, regarding them as specializations of a certain "Siegel modular form": the symplectic theta function, $\vartheta$. We then deduce their transformation properties from those of $\vartheta$.

Though this method is related to that of Shintani-Weil (see [3] for the precise connection), it has several advantages. First, it allows one to give transformation formulas for a larger subgroup of modular substitutions than is usual, and with weaker hypotheses on the indefinite quadratic form (compare, e.g., [6, Proposition 1.6] and Theorem 3 below). In addition, it makes natural the insertion of an additional "translation variable" into the theta series. This variable plays a key role in our rewriting of the (3, 1)
theta function as a Poincaré-type series on the entire upper half space in [4], and ought to be useful in other lifting contexts. Finally, a root of unity $\chi'$ arises in the transformation formulas for the indefinite quadratic theta functions, and previously, it could not be given explicitly in all cases (e.g., [6, p. 951, c odd). Combining the specialization with a recent result of Stark [9], which gives the precise transformation formula for $\vartheta$ in an important special situation, we completely determine $\chi'$. Here we must make use of symplectic transformations of $\vartheta$ which do not correspond to modular substitutions of the indefinite quadratic theta series.

Section 1 concerns the symplectic theta function. We recall its transformation formula and Stark's theorem, and then show how to get additional transformation properties by inclusion of an isotropy vector. In Section 2 we define the theta functions associated to an indefinite quadratic form, and deduce their transformation properties from those in Section 1. Section 3 gives the explicit determination of the root of unity $\chi'$. Finally, Section 4 collects remarks on the expansion of the forms at other cusps, character twists, Maass operators, and higher genus cases.

Notation.

\[ e^{ix} = \exp(\pi ix). \]

1. THE SYMPLECTIC THETA FUNCTION

Let $\,^t m$ denote the transpose of a matrix or vector $m$, $B \langle A \rangle = \,^tABA$, and $I = I_n$ be the $n \times n$ identity matrix. We write $\text{Sp}(n, \mathbb{R})$ for the symplectic group

\[ \text{Sp}(n, \mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(2n, \mathbb{R}) \mid \,^tDA - \,^tBC = I; \,^tCA, \,^tDB \text{ symmetric} \right\} \]

and $\mathfrak{H}^{(n)}$ for the Siegel upper half space

\[ \mathfrak{H}^{(n)} = \{ Z \in M(n, \mathbb{C}) \mid \,^tZ = Z, \text{Im}(Z) \text{ positive definite} \}. \]

The symplectic group $\text{Sp}(n, \mathbb{R})$ acts on $\mathfrak{H}^{(n)}$ by linear fractional transformation

\[ M \circ Z = (AZ + B)(CZ + D)^{-1}; \]

we shall be especially concerned with transformations arising from the theta subgroup

\[ \Gamma_\vartheta = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{Z}) \mid \,^tCA, \,^tDB \text{ have even diagonal entries} \right\}. \]

Note $\Gamma_\vartheta = \,^t\Gamma_\vartheta$. 

\[ e^{ix} = \exp(\pi ix). \]
For $u, v$ in $\mathbb{C}^n$ (column vectors), $Z$ in $\mathfrak{S}^{(n)}$, one defines the symplectic theta function by

$$\mathcal{g} \left( Z, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \sum_{m \in \mathbb{Z}^n} e^{\{ Z \langle m + v \rangle - 2imu - ivu \}}. \quad (1)$$

Loosely speaking, $\mathcal{g}$ is a Siegel modular form of weight $1/2$; more precisely, for $M = (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})$ in $\Gamma_\beta$,

$$\mathcal{g} \left( M \circ Z, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \chi(M) \left[ \det(CZ + D) \right]^{1/2} \mathcal{g} \left( Z, \begin{pmatrix} u \\ v \end{pmatrix} \right) \quad (2)$$

(cf. Eichler [2]). Here $\chi$ is an eighth root of unity which depends upon the choice of square root of $\det(CZ + D)$, but which is otherwise independent of $Z$, $u$, and $v$. An expression for $\chi(M)$ as a Gauss sum is well known. As for a more explicit determination, we have the following result of Stark [9].

**Theorem 1 (Stark).** Suppose $M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ is in $\Gamma_\beta$, and $C^{-1}$, $D^{-1}$ exist. Suppose further that for some odd prime $p$, $D$ is of level $p$, i.e., $pD^{-1}$ is in $M(n, Z)$. Then (mod $p$), the symmetric matrix $pD^{-1}C$ has rank $h$, where $|\det D| = p^h$. Let $(pD^{-1}C)^{(h)}$ be a nonsingular (mod $p$) $h \times h$ principal submatrix of $pD^{-1}C$, and $s$ be the signature (the number of positive eigenvalues minus the number of negative eigenvalues) of $C^{-1}D$. Then, with the choice of square root

$$\left[ \det(CZ + D) \right]^{1/2} \overset{\text{def}}{=} |\det C|^{1/2} \{ \det[ -iC^{-1}(CZ + D) ] \}^{1/2},$$

where $|\det C|^{1/2}$ is positive and $\{ \det[ -iC^{-1}(CZ + D) ] \}^{1/2}$ is computed by analytic continuation from the principal value when $Z = -C^{-1}D + iY$, $\chi(M)$ is given by

$$\chi(M) = \varepsilon_p^{-h} \left( \frac{2^h \det[ (pD^{-1}C)^{(h)} ]}{p} \right) e^{\{ s/4 \}},$$

where $\varepsilon_p = 1$ for $p \equiv 1$ (mod 4), $\varepsilon_p = i$ for $p \equiv 3$ (mod 4), and $(\cdot / p)$ is the Legendre symbol.

To include spherical functions in the discussion of the indefinite quadratic theta series below, let us modify (1) and (2) as follows. For $w$ in $\mathbb{C}^n$, $f$ a nonnegative integer, and $Z$, $u$, and $v$ as above, define

$$\mathcal{g} \left( Z, \begin{pmatrix} u \\ v \end{pmatrix}, w, f \right) = \sum_{m \in \mathbb{Z}^n} \left[ ^t wZ(m + v) \right]^{(f)} e^{\{ Z \langle m + v \rangle - 2imu - ivu \}}.$$
Then we have

**Proposition 2.** Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be in $\Gamma_3$, and if $f > 1$, suppose that $w$ satisfies the isotropy condition

$$Z(CZ + D)^{-1}CZ\langle w \rangle = 0. \quad (3)$$

Then

$$\mathcal{F}\left( M \circ Z, M \begin{pmatrix} u \\ v \end{pmatrix}, (AZ + B)^{-1}Zw, f \right)$$

$$= \chi(M)[\det(CZ + D)]^{1/2} \mathcal{F}\left( Z, \begin{pmatrix} u \\ v \end{pmatrix}, w, f \right),$$

where $\chi(M)$ is the eighth root of unity specified in Theorem 1 (and is in particular independent of $w$ and $f$).

**Proof:** Replace $v$ by $v + \sigma w$ in (2) and multiply by $e\{-\sigma^2Z\langle w \rangle\}$. Then, after some calculation, one finds that

$$\sum_{m \in \tilde{Z}^n} e\{M \circ Z\langle m + Cu + Dv \rangle - 2\langle m + Cu + Dv \rangle - \langle Cu + Dv \rangle(Au + Bv)$$

$$+ 2\sigma^2 wZ(CZ + D)^{-1}(m + Cu + Dv) - \sigma^2Z(CZ + D)^{-1}CZ\langle w \rangle \}$$

$$= \chi(M)[\det(CZ + D)]^{1/2} \sum_{m \in \tilde{Z}^n} e\{Z\langle m + v \rangle - 2\mu u - \mu u + 2\sigma^2 wZ(m + v) \}.$$

(4)

For example, the $\sigma^2$ term is $(\pi i$ times)

$$M \circ Z\langle Dw \rangle - 1RD\langle w \rangle - Z\langle w \rangle$$

$$= [\langle D(AZ + B) - B(CZ + D) \rangle(CZ + D)^{-1}D\langle w \rangle - Z\langle w \rangle$$

$$= \langle (DA - BC)Z + 1DB - BD \rangle(CZ + D)^{-1}D\langle w \rangle - Z\langle w \rangle$$

$$= Z(CZ + D)^{-1}D\langle w \rangle - Z\langle w \rangle$$

$$= -Z(CZ + D)^{-1}CZ\langle w \rangle,$$

we have used the conditions for $M$ to be in $\text{Sp}(n, \mathbb{R})$ here. But, by the isotropy condition, the $\sigma^2$ term vanishes identically when $f > 1$. Thus differentiating (4) $f$ times with respect to $\sigma$ and setting $\sigma = 0$ gives the result.

2. **The Indefinite Quadratic Theta Series**

Let $A$ be a free rank $n$ $\mathbb{Z}$-module in $\mathbb{C}^n$, with basis $\lambda_1, \ldots, \lambda_n$, and $L = (\lambda_1 \cdots \lambda_n)$ in $GL(n, \mathbb{C})$. Fix a nondegenerate indefinite quadratic form $Q$
of type \((p, q)\) with respect to \(A\); we view \(Q\) as a symmetric matrix in \(GL(n, \mathbb{C})\) such that

\[
Q_1 = 'LQL
\]

is equivalent to

\[
E_{p,q} = \begin{pmatrix} I_p & -I_q \\ -I_q & I_p \end{pmatrix}
\]

over \(GL(n, \mathbb{R})\). Also, for \(u, v\) in \(\mathbb{C}^n\), we write \((u, v) = 'uQv'.\) Let \(R\) be a majorant of \(Q\), i.e., a symmetric \(n \times n\) complex matrix such that \(RQ^{-1}R = Q\) and \(R<\lambda>\) is positive for all \(\lambda\) in \(A - \{0\}\).

For \(z = x + iy\) in \(\mathcal{S}^{(1)}\), \(u', v', w'\) in \(\mathbb{C}^n\), and \(f\) a nonnegative integer, define the indefinite quadratic theta series

\[
\theta \left( z, \begin{pmatrix} u' \\ v' \end{pmatrix}, w', f \right) = y^{q/2} \sum_{\lambda \in A} [(w', \lambda + v')]^f e\{(xQ + iyR)(\lambda + v') - 2(\lambda, u') - (v', u')\}.
\]

We have

**Theorem 3.** Let \(\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be in \(SL(2, \mathbb{R})\), and suppose that

\[
M = \begin{pmatrix} aI & bQ_1 \\ cQ_1 & dI \end{pmatrix} \quad \text{is in } \Gamma_\beta.
\]

If \(f \geq 1\), suppose also that \(Qw' = Rw'\), and if \(f > 1\), that \(Q<w'> = 0\). Then

\[
\theta \left( \mu \circ z, \mu \begin{pmatrix} u' \\ v' \end{pmatrix}, w', f \right) = \chi'(\mu)(cz + d)^{(p - q)/2 + f} \theta \left( z, \begin{pmatrix} u' \\ v' \end{pmatrix}, w', f \right),
\]

where \(\chi'(\mu)\) is an eighth root of unity which depends upon the choice of argument of \((cz + d)^{(p - q)/2}\) but which is otherwise independent of \(z, u', v', w', f\), and \(R\).

**Proof.** Put

\[
Z = xQ_1 + iy'LRL, \quad u = 'LQu', \quad v = L^{-1}v',
\]

and \(w = L^{-1}w'\). Notice that \(Z\) is in \(\mathcal{S}^{(n)}\), as \('LRL\) is positive definite. Since \(Qw' = Rw'\) if \(f \geq 1\), we see that

\[
y^{-q/2} \theta \left( z, \begin{pmatrix} u' \\ v' \end{pmatrix}, w', f \right) = z^{-f} \theta \left( Z, \begin{pmatrix} u' \\ v' \end{pmatrix}, w, f \right).
\]
We would like to apply Proposition 2. To do so, we need a symplectic matrix which expresses the $\mu$ action on our new variables $Z$, $u$, and $v$. We claim that the diagrams commute, where the vertical arrows are matrix multiplications in the first case and linear fractional transformations in the second, and the horizontal arrows are given by the variable changes (7). Indeed, the first diagram is a straightforward calculation. As for the second, take a matrix $T$ in $GL(n, \mathbb{R})$ such that

$$Q_1 \langle T \rangle = E_{p,q}, \quad (^{1LRL} \langle T \rangle) = I_n.$$ 

Then

$$Z \langle T \rangle = \text{diag}(z, ..., z, -\bar{z}, ..., -\bar{z}),$$

the diagonal matrix with $p$ $z$'s and $q$ $\bar{z}$'s.

If we set

$$M_1 = \begin{pmatrix} aI & bE_{p,q} \\ cE_{p,q} & dI \end{pmatrix},$$

then

$$Z \langle T \rangle \mapsto M_1 \circ (Z \langle T \rangle)$$

clearly commutes. Since

$$Z \langle T \rangle = \begin{pmatrix} ^{1T} & 0 \\ 0 & T^{-1} \end{pmatrix} \circ Z$$
and

\[ M = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1} M_1 \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}, \]

the commutativity of (8) follows.

When \( f > 1 \), \( Q(\omega) = 0 \) implies that the isotropy condition (3) holds. Thus we may apply Proposition 2. Observe that

\[ \det(cQ_1^{-1}Z + dI) = (cz + d)^p(c\bar{z} + d)^q. \]

After some short computations, the result then follows.

**Remarks.**

1. The determination of \( \chi' \), and the relationship between \( \chi' \) and \( \chi \), will be given in Section 3.

2. Note that the set of \( \mu \)'s satisfying the hypothesis (5) of Theorem 3 forms a group, which we write \( \Gamma_\theta \); in fact, multiplication of matrices in \( \Gamma_\theta \) corresponds exactly to multiplication of the related symplectic matrices in \( \Gamma_\gamma \).

3. If instead of the hypothesis \( Q_\omega = R_\omega \) we require \( Q_\omega = -R_\omega \), we can obtain a result similar to Theorem 3. The corresponding transformation formula is slightly different; we must replace \( (cz + d)^f \) by \( (c\bar{z} + d)^f \) in Equation (6). Details are left to the reader.

4. The translation variable \( u' \) here is the one mentioned in the introduction.

5. The majorants of \( Q \) fill out the symmetric space attached to \( SO(Q) \). It is the independence of the transformation formula of Theorem 3 from this choice of \( R \) that makes the lifting of automorphic forms possible; cf. Kudla [5].

Let \( A^* \) be the \( Q \)-dual of \( A \), i.e., \( A^* = Q^{-1}L^{-1}Z^n \). Then we can rephrase the conditions for \( \mu \) to be in \( \Gamma_\theta \) as follows.

**Proposition 4.** \( \mu \) lies in \( \Gamma_\theta \) if and only if

(i) \( a, d \in \mathbb{Z} \),

(ii) \( b(\lambda, \lambda') \in \mathbb{Z} \) for all \( \lambda, \lambda' \in \Lambda \),

(iii) \( c(\lambda^*, \lambda'^*) \in \mathbb{Z} \) for all \( \lambda^*, \lambda'^* \in A^* \),

(iv) \( bd(\lambda, \lambda) \equiv ac(\lambda^*, \lambda^*) \equiv 0 \) (mod 2) for all \( \lambda \in \Lambda, \lambda^* \in A^* \).

**Proof.** Let \( \{ \lambda_i^* \} \) be a \( Q \)-dual basis to \( \{ \lambda_i \} \), so \( (\lambda_i^*, \lambda_j) = \delta_{ij} \). Then the \( (i, j) \)th entry of \( Q_1 \) (resp. \( Q_1^{-1} \)) is \( (\lambda_i, \lambda_j) \) (resp. \( (\lambda_i^*, \lambda_j^*) \)). Thus conditions (i)–(iii) hold if and only if \( M \) is in \( \text{Sp}(n, \mathbb{Z}) \), while (iv) gives the condition for \( M \) to be in \( \Gamma_\theta \).
Note that since $\Gamma_\alpha = \Gamma_\beta$, (iv) is equivalent to

$$(iv') \quad ab(\lambda, \lambda^*) = cd(\lambda^*, \lambda^*) \equiv 0 \pmod{2} \text{ for all } \lambda \in \mathbb{A}, \lambda^* \in \mathbb{A}^*.$$ 

Also, since $\Gamma_\alpha$ is the largest group for which $\mathcal{I}$ is known to have a transformation of the form (2), the conditions of Proposition (4) are minimal in a suitable sense. A more precise result can be proven via Stark’s “level determiner theorem” [10], but we shall not discuss this further here.

Finally, we close this section by remarking that the conditions of Proposition 4 do not force $b$ and $c$ to lie in $\mathbb{Z}$. If one prefers dealing with integral modular substitutions, the following modification will do. Choose any real number $\kappa$ such that $\kappa Q_1$ is an integral matrix whose entries do not have a nontrivial common factor. Then

$$\tilde{\theta} \left( z, \left( \begin{array}{c} u' \\ v' \end{array} \right), w', f \right) = \theta \left( \kappa z, \left( \begin{array}{c} \kappa^{1/2} u' \\ \kappa^{-1/2} v' \end{array} \right), \kappa^{-1/2} w', f \right)$$

has a transformation formula whenever

$$\left( \begin{array}{cc} aI & b \kappa Q_1 \\ c(\kappa Q_1)^{-1} & dI \end{array} \right)$$

lies in $\Gamma_\alpha$; in particular, $b$ and $c$ must be integers.

3. Determination of the Root of Unity $\chi'(\mu)$

By changing variables as explained at the end of the previous section, we may assume without loss of generality that $Q$ is integral—i.e., $(\lambda, \lambda')$ lies in $\mathbb{Z}$ for all $\lambda, \lambda'$ in $\mathbb{A}$, or equivalently, $Q_1$ is in $M(n, \mathbb{Z})$—and $\mu = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)$ is in $SL(2, \mathbb{Z})$. Also, we may assume $c$ is nonzero, since the remaining case is trivial ($\chi'(\mu) = 1$).

By $(cz + d)^{(p-q)/2 + f}$ in Theorem 3 we shall mean

$$[(cz + d)^{1/2}]^{(p-q)[cz + d]}$$

where $-\pi/2 < \arg[(cz + d)^{1/2}] < \pi/2$. Let $M$ be the symplectic matrix of that theorem corresponding to $\mu$, and $\text{sgn}(c) = c/|c|$.

**Proposition 5.**

$$\chi'(\mu) = e^{\{ (q-p) \text{ sgn}(c) / 4 \}} \chi(M).$$

**Proof.** This follows immediately from the proof of Theorem 3, after comparing the choice of square root used in Theorem 1 with the one given above.
Thus the determination of $\chi(M)$ will give that of $\chi'(\mu)$. As a first result in this direction, we have

**Proposition 6.** Suppose $d$ is an odd prime. Then

$$\chi'(\mu) = \varepsilon_d a^n \left( \frac{(2c)^n \det(Q^{-1})}{d} \right).$$

**Proof.** Since $dI$ is of level $d$, this follows from Theorem 1 and Proposition 5.

In fact, formula (9) of Proposition 6 determines $\chi'$ far more generally than it first seems. Namely, recall that $Q$ is called even if $(\lambda, \lambda) \equiv 0 \pmod{2}$ for all $\lambda$ in $A$.

**Proposition 7.** If $d$ is any odd integer, then $\chi'(\mu)$ is given by (9). This determines $\chi'(\mu)$ in the case that $Q$ and $d$ are even as well, since then

$$\chi'(\mu) = \chi' \left( \frac{a}{c} \frac{a+b}{c+d} \right),$$

and $c+d$ is odd.

**Proof.** Let $e$ be an integer, with at least one of $e$ and $Q$ even. Then it is clear that

$$\theta \left( z + e, \frac{u'+ev'}{v'}, w', f \right) = \theta \left( z, \frac{u'}{v'}, w', f \right).$$

This implies that

$$\chi'(\mu) = \chi' \left( \frac{1}{0} \frac{e}{1} \right).$$

Suppose $d$ is odd. Thus $(2c, d) = 1$. By Dirichlet's theorem on primes in progressions, we can find an integer $e$ with $(4c)^n + 1$ dividing $e$, such that $ce + d$ is an odd prime. Then $\chi'(\mu) = \chi' \left( \frac{a}{c} \frac{ae+b}{ce+d} \right)$ is determined by Proposition 6. Further, if we extend the Legendre symbol there to a Dirichlet character of $d$, then we see that the right-hand side of (9) is periodic in $d$, with period dividing $(4c)^n + 1$ (with overkill). Thus, subtracting $ce$, we obtain the first part of the proposition. The second part follows immediately from (10) with $e = 1$.

We turn now to the determination of $\chi'(\mu)$ in the remaining case: $Q$ not even, $d$ even. The idea here is that since the $SL(2, \mathbb{Z})$ inversion $\left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ need not be hyperelliptic, we shall instead make use of the
symplectic inversion \( (\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}) \), together with the even symplectic translations \( (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \). In particular, we shall see that elements of the theta subgroup \( \Gamma_3 \) which are not in the image of \( \Gamma_0 \) arise in the determination of \( \chi'(\mu) \).

More precisely, for any \( L = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \) in \( \Gamma_3 \), we find that

\[
\chi \left( \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} L \right) = \chi(L) \left[ \det(L \circ Z) \right]^{1/2} \left[ \det(CZ + D) \right]^{1/2} \left[ \det(AZ + B)^{-1} \right]^{1/2},
\]

by composing the actions of \( (\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}) \) and \( L \) in \( (2) \); here the square roots are defined as in Theorem 1, by analytic continuations from \( L \circ Z = iY \), \( Z = -C^{-1}D + iY \), and \( Z = -A^{-1}B + iY \), respectively, and the right hand side is (of course) independent of \( Z \) in \( \mathcal{S}_0^{(n)} \). Also, since Proposition 7 gives \( \chi' \) for \( \mu \) in a normal subgroup of index 2 in \( \Gamma_0 \), it suffices to determine the root of unity for a single \( \mu \) in the remaining coset (though one must take care in composing modular substitutions, as above, since \( \chi' \) is also not in general a character). An element in this coset always exists with \( a = 2 \), \( b = 1 \), and we restrict to this choice. Set

\[
N = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad M = \begin{pmatrix} -cQ^{-1} & -dI \\ 2I & Q_1 \end{pmatrix}.
\]

Then the discussion above implies that the evaluation of \( \chi' \) is completed if we determine \( \chi(N) \).

Observe that for any even symmetric matrix \( S \) in \( M(n, \mathbb{Z}) \), the action of \( (\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}) \) on \( \mathcal{S} \) is trivial, and thus

\[
\chi(N) = \chi \left( \begin{pmatrix} -cQ^{-1} & -dI \\ 2I & Q_1 \end{pmatrix} \right).
\]

**Proposition 8.** There exists an even symmetric matrix \( S \) in \( M(n, \mathbb{Z}) \) such that \( 2S \circ Q_1 \) is of odd prime level. Hence \( \chi(N) \) is determined by \((11)\) and Theorem 1.

To prove Proposition 8, we need two elementary lemmas, whose proofs we give for the convenience of the reader. We call two matrices \( A \) and \( B \) integrally equivalent if there is some \( C \) in \( GL(n, \mathbb{Z}) \) (\( \det C = \pm 1 \)) such that \( B = A \langle C \rangle \).

**Lemma 9.** There exists an even symmetric integral matrix \( S' \) such that \( 2S' \circ Q_1 \) is integrally equivalent to either

(a) a diagonal matrix with entries \( \pm 1 \),

or

(b) \( (\begin{smallmatrix} 0 & \xi \end{smallmatrix}) \) where \( U \) is a diagonal matrix with entries \( \pm 1 \), and \( V \) is even with odd determinant.
By symmetric elementary row and column operations, we may assume that only the last $r$ diagonal entries of $Q_1$ are odd, $0 < r \leq n$ ($r$ is greater than zero since $Q_1$ is not even). Adding an appropriate $S$, we reduce to the case where each odd diagonal entry is $\pm 1$. Then symmetric elementary row and column operations allow us to annihilate all off-diagonal entries in the last $r$ rows and columns. This proves part (a) ($r = n$) and the first half of part (b) ($r < n$). Finally, note that in part (b),

$$\det(Q_1) \equiv \det(2S' + Q_1) \equiv \det(V) \pmod{2}.$$ 

Since $d$ is even, $c$ is odd, and since $cQ_1^{-1}$ is integral, $\det(Q_1)$ is odd too. This gives the lemma.

**Lemma 10.** Let $V$ in $M(k, \mathbb{Z})$ be an even symmetric matrix with odd determinant. Then there exists an even symmetric integral matrix $S''$ such that $2S'' + V$ is integrally equivalent to a matrix $V'$ with diagonal entries each 2 or 4, anti-diagonal entries (i.e., $(V')_{ii}$, where $i + j = k + 1$) all 1, and all other entries zero.

**Proof.** By suitable symmetric row and column operations and choice of $S$ we may reduce the last row to $(1, 0, \ldots, 0, f)$, $f = 2$ or 4. Using the 1 occurring, we may reduce all the first row and column entries except the diagonal to 0, and remove any multiples of $f$ introduced by another $S$. Repeating this procedure on the submatrix formed by deleting the first and last rows and columns proves the lemma ($k$ is necessarily even).

**Proof of Proposition 8.** In the first case of Lemma 9, take $S = S'$, since a diagonal matrix of plus and minus ones is of prime level for any prime. In the second case, combining lemmas 9 and 10, we may assume that $Q_1 = (V'_{ii})$ with $V'$ and $U$ as above. By Dirichlet's theorem, there are infinitely many integers $n$ such that $2\text{ diag}(2n, 0, \ldots, 0) + Q_1$ has odd prime level, since

$$\det(2\text{ diag}(2n, 0, \ldots, 0) + Q_1) = +(2^g n - 1)$$

for some particular integer $g$ (independent of $n$). Taking $S = \text{ diag}(2n, 0, \ldots, 0)$ for such an $n$ completes the proof.

In summary, the discussion above determines $\chi'(\mu)$ in all cases.

**Remarks.** (1) It is easy to determine $\chi'(\mu)$ up to sign in the last case above, since $\mu^2$ has odd bottom right hand entry.

(2) The symplectic inversion plus the specialization of Theorem 3 gives the relationship between the theta function associated to lattice $A$ and that associated to lattice $A^*$ (of course, this is just Poisson summation).
(3) Styer [11] has used methods similar to those above to study \( \chi(M) \) for any \( M \) in \( \Gamma_0 \).

(4) In his fundamental paper on modular forms of half integral weight [14], Shimura gives transformation laws for the theta functions associated to positive definite quadratic forms. In the situation above, one may also obtain Proposition 7 by using Shimura's methods, which are in fact related to ours. Note, however, that he does not consider the case of non-even form and even \( d \).

4. MISCELLANEOUS REMARKS

(1) We can calculate the Fourier expansion of an indefinite quadratic theta series at an arbitrary cusp, by writing \( \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \) in \( SL(2) \) (\( c \) nonzero, without loss) as \( \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) = \left( \begin{array}{c} 0 \\ \alpha \gamma \\ \alpha \delta \\ 1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} d \\ \gamma \\ \delta \\ 1 \end{array} \right) \), and computing its action in the corresponding 3 parts (we omit the details). The resulting expansion involves Gauss sums.

(2) The translation variable \( v \) allows one to mix in characters. For example, suppose \( A^* \supset A \), and let \( \psi : \mathbb{Z} \rightarrow \mathbb{C} \), \( \psi : A^*/A \rightarrow \mathbb{C} \) be any functions satisfying

\[
\psi(dv) = \psi'(d) \psi(v)
\]

for all \( v \) in \( A^*/A \) and \( d \) in \( \mathbb{Z} \). Then

\[
\theta(z, \psi, w, f) = \sum_{v \in A^*/A} \psi(v) \, \theta \left( z, \left( \begin{array}{c} 0 \\ v \end{array} \right), w, f \right)
\]

satisfies

\[
\theta(\mu \circ z, \psi, w, f) = \psi'(d) \chi' \left( \mu (cz + d)^{(n - q)/2} \right) \theta(z, \psi, w, f)
\]

whenever \( \mu \) is in \( \Gamma_0 \) and multiplication by \( d \) gives an isomorphism of \( A^*/A \).

(3) One can get additional forms satisfying transformation laws by application of the Maass operators; see Vignéras [13].

(4) By arguments similar to those of this note, one can form theta functions from indefinite quadratic forms with variables \( Z \) in \( \mathbb{S}^{(n)} \), \( n > 1 \), which are Siegel modular forms (special cases are Andrianov–Maloletkin [1] and Tsuyumine [12]; cf. also Stark [9]). One must replace the \( M \) of Theorem 3 by

\[
\left( \begin{array}{cc} a \otimes I & b \otimes Q_1 \\ c \otimes Q_1^{-1} & d \otimes I \end{array} \right)
\]

and determine the root of unity by the results of Stark and Styer.
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REFERENCES