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On Some Nonparametric Tests for Profile Analysis of Several Multivariate Samples*

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For profile analysis of independent samples from several multivariate populations, a nonparametric analog of the hypothesis of parallelism of population profiles is formulated. A class of asymptotically distribution-free statistics is offered to test this hypothesis. These are based on generalized U statistics and are in some sense modifications of statistics offered previously by one of the authors for testing the homogeneity hypothesis. Consistency of these statistics is established for suitable alternatives and also asymptotic power is investigated.

1. INTRODUCTION

When independent random samples are obtained from each of k p-variate populations, the parametric statistical inference assumes the model that the population distributions are p-variate normal with *common* unknown nonsingular covariance matrices with possible differences only in locations. If $\boldsymbol{\mu}_i' = (\mu_i^{(1)},..., \mu_i^{(p)})$ denotes the mean of the ith population, then the hypothesis of homogeneity of populations is equivalent to

$$H_0\colon \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_k . \tag{1.1}$$

There are several well-known (see, e.g., [1, 4, 6, 9]) MANOVA tests available for this purpose. Although these are optimal in some sense (see, e.g., [5, p. 298]) no single test is uniquely optimal.

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Furthermore, the hypothesis of parallelism of population profiles is stated as

$$H_1 : \mu_i^{(1)} - \mu_1^{(1)} = \cdots = \mu_i^{(p)} - \mu_1^{(p)}, \qquad i = 2,..., k. \tag{1.2}$$

H_1 may also be interpreted as the hypothesis of no interaction between p variables and k populations. If H_1 is acceptable in the sense that the population profiles could be assumed to be parallel, then one might be interested in testing that the profiles are identical, given that they are parallel. In other words, one then wants to test H_2 which is H_0/H_1 (to be read as H_0 given H_1). Classical parametric tests for H_1 and H_2 in profile analysis are discussed in statistical literature (see, e.g., [4, 6]).

Now the need for discarding the stringent assumption of normality and for developing suitable nonparametric procedures has been recognized for quite some time. Accordingly, such nonparametric tests have been offered for the hypothesis of homogeneity by several workers (see, [2, 8, 10, 11]). The main objective of this paper is to extend some of these techniques to profile analysis of several samples.

In Section 2 we begin with notation and preliminary results; in Section 3 we develop appropriate nonparametric analogs of various hypotheses that are relevant in profile analysis and suitable statistics are presented as test criteria. The consistency of these is discussed in Section 4 and the asymptotic powers are then obtained in Section 5. The paper concludes with some remarks in Section 6.

2. Notation and Preliminary Results

Let $\mathbf{X}_{ij}' = (X_{ij}^{(1)},..., X_{ij}^{(p)})$, $j = 1,..., n_i$ be independent random vectors from the ith population with nonsingular continuous c.d.f. F_i, $i = 1,..., k$. The hypothesis of homogeneity of these k populations is then

$$H_0 : F_1 = \cdots = F_k = F \text{ (say)}. \tag{2.1}$$

Nonparametric tests for H_0 have been presented independently by Bhapkar [2] and Suguira [10] based on the technique of generalized U statistics.

As in [2], let

$$U_i^{(\alpha)} = \left(\prod_{j=1}^{k} n_j\right)^{-1} \sum_{t_1=1}^{n_1} \cdots \sum_{t_k=1}^{n_k} \phi_i^{(\alpha)}(\mathbf{X}_{1t_1},..., \mathbf{X}_{kt_k}), \tag{2.2}$$

$\alpha = 1,..., p$ and $i = 1,..., k$, and

$$\mathbf{U}_i' = (U_i^{(1)},..., U_i^{(p)}), \qquad \mathbf{U}' = (\mathbf{U}_1',..., \mathbf{U}_k').$$

We assume here

$$\phi_i^{(\alpha)}(\mathbf{x}_1,..., \mathbf{x}_k) = \phi(r_i^{(\alpha)}), \tag{2.3}$$

where $r_i^{(\alpha)}$ is the rank of $x_i^{(\alpha)}$ among $\{x_j^{(\alpha)}, j = 1,..., k\}$. In view of continuity assumption, with probability one there are not ties. Note that the functions considered by Bhapkar [2] and Suguira [10] are special cases of functions satisfying (2.3).

Let $\mathbf{F}' = (F_1, ..., F_k)$ and define $\eta_i^{(\alpha)}(\mathbf{F}) = E(U_i^{(\alpha)}) = E\phi_i^{(\alpha)}(\mathbf{X}_1, ..., \mathbf{X}_k)$, where $\mathbf{X}_i$'s represent independent random vectors with c.d.f. F_i's, respectively. Then we have

$$\eta_i^{(\alpha)}(\mathbf{F}) = \sum_{j=1}^{k} \phi(j)\, P[R_i^{(\alpha)} = j] = \sum_{j=1}^{k} \phi(j)\, \nu_{ij}^{(\alpha)}(\mathbf{F}); \tag{2.4}$$

here $R_i^{(\alpha)}$ is the rank of $X_i^{(\alpha)}$ among $\{X_j^{(\alpha)}, j = 1,..., k\}$ and

$$\nu_{ij}^{(\alpha)}(\mathbf{F}) = P[R_i^{(\alpha)} = j], \tag{2.5}$$

with the probabilities computed under $\mathbf{F}$.

Now if $n_i \to \infty$ in such a way that $n_i/N \to p_i$, where $N = \Sigma_i n_i$, $0 < p_i < 1$, $i = 1,..., k$, then, as in [2],

$$N^{1/2}(\mathbf{U}_\mathbf{n} - \boldsymbol{\eta}(\mathbf{F})) \xrightarrow{\mathscr{L}} \mathscr{N}(\mathbf{0}, \mathbf{T}(\mathbf{F})), \tag{2.6}$$

for any $\mathbf{F}$. Here the subscript $\mathbf{n}$ denotes the vector of sample sizes on which $\mathbf{U}$ is based, $\mathscr{L}$ denotes convergence in distribution, and $\mathscr{N}$ denotes the normal vector of appropriate dimensions. Let

$$\phi = \frac{1}{k} \sum_{j=1}^{k} \phi(j). \tag{2.7}$$

It was shown in [2] that under H_0 (2.1),

$$\boldsymbol{\eta}(\mathbf{F}) = \boldsymbol{\eta}(F) = \phi \mathbf{j}, \qquad \mathbf{T}(\mathbf{F}) = \mathbf{T}(F) = \boldsymbol{\Sigma} \otimes \mathscr{P}(F), \tag{2.8}$$

where $\mathbf{A} \otimes \mathbf{B} = [a_{ij}\mathbf{B}]$, and $\boldsymbol{\Sigma} = [\sigma_{ij}]$ is given by

$$\boldsymbol{\Sigma} = \frac{\mu}{(k-1)^2} \{q\mathbf{J} + k^2\boldsymbol{\Delta} - k\mathbf{q}\mathbf{j}' - k\mathbf{j}\mathbf{q}'\}, \tag{2.9}$$

with $\mathbf{J} = [1]_{k \times k}$, $\boldsymbol{\Delta}$ = diagonal $(p_i^{-1}, i = 1,..., k)$, $q = \Sigma_i p_i^{-1}$, and $\mathbf{q}' = (p_1^{-1}, ..., p_k^{-1})$. Also $\mathscr{P}$ is a matrix of correlation coefficients $\rho_{\alpha\beta}$ between $\phi_i^{(\alpha)}(\mathbf{X}_1, ..., \mathbf{X}_k)$ and $\phi_i^{(\beta)}(\mathbf{Y}_1, ..., \mathbf{Y}_k)$, where $\mathbf{X}$'s and $\mathbf{Y}$'s are independent with common c.d.f. F except that $\mathbf{X}_i = \mathbf{Y}_i$, and

$$\mu = E[\psi^2(X_i^{(\alpha)})] - [E(\psi(X_i^{(\alpha)})]^2,$$

where (2.10)

$$\psi(x_i^{(\alpha)}) = E\{\phi_i^{(\alpha)}(\mathbf{X}_1, ..., \mathbf{X}_k) \mid \mathbf{X}_i = \mathbf{x}_i\}.$$

Under the slightly weaker conditions assumed in [2], μ could depend on F; however, under the somewhat stronger condition (2.3) assumed in this paper, μ is distribution-free, as shown in the lemma below. Incidentally, we may note here that the explicit statistics worked out in [2] for some specific functions indeed satisfy (2.3).

LEMMA 2.1. *For functions $\phi^{(\alpha)}$ satisfying* (2.3), *if H_0 holds,*

$$\mu = \sum_{l=1}^{k} \sum_{m=1}^{k} \phi(l)\,\phi(m) \binom{k-1}{l-1}\binom{k-1}{m-1} B(l+m-1, 2k-l-m+1) - \phi^2,$$

where ϕ is given by (2.7).

Proof. From (2.10) we have

$$\psi(x_i^{(\alpha)}) = \sum_{j=1}^{k} \phi(j)\, P[x_i^{(\alpha)} \text{ has rank } j \text{ among } X_1^{(\alpha)},\ldots, X_k^{(\alpha)} \text{ given } \mathbf{X}_i = \mathbf{x}_i]$$

$$= \sum_{j=1}^{k} \phi(j) \binom{k-1}{j-1} [F^{(\alpha)}(x_i^{(\alpha)})]^{j-1} [1 - F^{(\alpha)}(x_i^{(\alpha)})]^{k-j},$$

where $F^{(\alpha)}$ is the common c.d.f. under H_0 of $X^{(\alpha)}$. Thus

$$E\psi(X_i^{(\alpha)}) = \sum_{j=1}^{k} \phi(j) \binom{k-1}{j-1} B(j, k-j+1) = \phi;$$

also,

$$[\psi(x_i^{(\alpha)})]^2 = \sum_{l=1}^{k} \sum_{m=1}^{k} \phi(l)\,\phi(m) \binom{k-1}{l-1}\binom{k-1}{m-1}$$

$$\times [F^{(\alpha)}(x_i^{(\alpha)})]^{l+m-2} [1 - F^{(\alpha)}(x_i^{(\alpha)})]^{2k-l-m}$$

and the lemma follows.

It has been shown in [2] that, if the common F is nonsingular, in the sense that no set of p-dimensional Lebesgue measure zero contains the whole probability mass, then $\mathscr{P}(F)$ is nonsingular.

Let $\hat{\mathscr{P}}$ be a matrix of consistent estimators, as in [2], of $\mathscr{P}$ and partition $\mathbf{U}'$ as $(\mathbf{U}_1', \mathbf{U}_0')$. If $\mathbf{\Sigma}_{11}$ is the cofactor of σ_{11} in $\mathbf{\Sigma}$, define

$$T_0 = N(\mathbf{U}_0' - \phi\mathbf{j}')(\mathbf{\Sigma}_{11}^{-1} \otimes \hat{\mathscr{P}}^{-1})(\mathbf{U}_0 - \phi\mathbf{j}); \tag{2.11}$$

then it was shown in [2] that

$$T_0 = \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\mathbf{U}_i - \bar{\mathbf{U}})' \hat{\mathscr{P}}^{-1}(\mathbf{U}_i - \bar{\mathbf{U}}), \tag{2.12}$$

where $p_i = n_i/N$ and $\bar{\mathbf{U}} = \Sigma_i p_i \mathbf{U}_i$ and, moreover, it has a limiting $\chi^2(p(k-1))$ distribution under H_0. Explicit statistics denoted by V, B, L, and W were offered as possible nonparametric test criteria (for the hypothesis H_0) in [2].

Suguira [10] considered the class of functions

$$\phi_i^{(\alpha)}(x_1, \ldots, x_k) = \frac{(j-1)_r}{(k-1)_r} - \frac{(k-j)_s}{(k-1)_s}, \tag{2.13}$$

where j is the rank of $x_i^{(\alpha)}$ among $\{x_l^{(\alpha)},\ l = 1, \ldots, k\}$, and $(a)_r = a!/(a-r)!$. His statistic is essentially the same as (2.12) except that he uses somewhat different estimates for $\mathscr{P}$. We may note here, however, that his estimates are consistent only under H_0, while those in [2] are valid for any $\mathbf{F}$ and hence the latter are to be preferred.

3. Nonparametric Test for Parallelism of Profiles

First, we want to formulate an appropriate nonparametric analog of the hypothesis H_1 of parallelism of profiles. In the parametric case the profiles are defined in terms of population means and, hence, H_1 takes the form (1.2). In the more general nonparametric case we give the following definition:

Definition 3.1. The populations $F_1, F_2, \ldots, F_k$ are said to have parallel profiles if $\mathbf{F}' = (F_1, \ldots, F_k)$ satisfies

$$H_1 : \nu_{ij}^{(1)}(\mathbf{F}) = \cdots = \nu_{ij}^{(p)}(\mathbf{F}), \qquad i, j = 1, \ldots, k, \tag{3.1}$$

where $\nu_{ij}^{(\alpha)}(\mathbf{F})$ is defined by (2.5).

One might wonder whether (1.2) and (3.1) are equivalent in some sense under the normality assumption The answer is no, except possibly the special case where the variances $\sigma_{\alpha\alpha}$ of $X^{(\alpha)}$ are the same for all $\alpha = 1, \ldots, p$. We prove here only the weaker statement:

Lemma 3.1. *If $\mathbf{X}_1, \ldots, \mathbf{X}_k$ are independent $\mathscr{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}), \ldots, \mathscr{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma})$ respectively, and the diagonal elements of $\boldsymbol{\Sigma}$ are equal, then* (1.2) *implies* (3.1).

Proof. Note that

$$\begin{aligned}\nu_{ij}^{(\alpha)}(\mathbf{F}) = P[R_i^{(\alpha)} = j] &= \sum_c P[\text{Each of } \{X_{i_l}^{(\alpha)},\ l = 1, \ldots, j-1\} < X_i^{(\alpha)} \\ &\qquad < \text{Each of } \{X_{i_m}^{(\alpha)},\ m = j+1, \ldots, k\}] \\ &= \sum_c P[\text{Each of } \{Y_{i_l}^{(\alpha)} + \mu_{i_l}^{(\alpha)} - \mu_i^{(\alpha)}\} < Y_i^{(\alpha)} \\ &\qquad < \text{Each of } \{Y_{i_m}^{(\alpha)} + \mu_{i_m}^{(\alpha)} - \mu_i^{(\alpha)}\}];\end{aligned} \tag{3.2}$$

here Σ_c denote the sum over $\binom{k-1}{j-1}$ combinations of subscripts i_l, $l = 1,..., j-1$ chosen out of $k-1$ distinct subscripts i_l, $l = 1,..., k$ (except j) (denoting integers $1,..., k$ except i).

Now $Y_i^{(\alpha)}$ for $i = 1,..., k$ are independent and identical normal variables for each α. If condition (1.2) is satisfied, we see from (3.2) that $\nu_{ij}^{(\alpha)}(\mathbf{F})$ does not depend on α and hence, (3.1) is satisfied.

In fact, normality as such is not used at all except for the fact that $\boldsymbol{\mu}_i$ are location parameters. By using essentially the same argument we have thus proved the

THEGREM 3.1. *Suppose $\mathbf{X}_1,..., \mathbf{X}_k$ are independent with c.d.f.*

$$F_i(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\mu}_i), \qquad i = 1,..., k \tag{3.3}$$

for some continuous F, and assume that the marginal c.d.f.'s $F^{(\alpha)}$, $\alpha = 1,..., p$, of F are identical, then condition (1.2) *implies condition* (3.1).

In order to test H_1 we now propose the statistic

$$\begin{aligned} T_1 &= \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\mathbf{U}_i - \bar{\mathbf{U}})' [\hat{\mathscr{P}}^{-1} - \hat{\gamma}\hat{\mathscr{P}}^{-1}\mathbf{J}\hat{\mathscr{P}}^{-1}](\mathbf{U}_i - \bar{\mathbf{U}}) \\ &= T_0 - T_2, \end{aligned} \tag{3.4}$$

where T_0 is the statistic (2.12) for H_0,

$$T_2 = \frac{N(k-1)^2\hat{\gamma}}{\mu k^2} \sum_{i=1}^{k} p_i(\mathbf{U}_i - \bar{\mathbf{U}})' \hat{\mathscr{P}}^{-1}\mathbf{J}\hat{\mathscr{P}}^{-1}(\mathbf{U}_i - \bar{\mathbf{U}}), \tag{3.5}$$

and $\hat{\gamma} = 1/\mathbf{j}'\hat{\mathscr{P}}^{-1}\mathbf{j}$. T_1 is to be regarded as a large-sample $\chi^2((p-1)(k-1))$ criterion for H_1, while T_2 is regarded as a $\chi^2(k-1)$ criterion for testing H_0, assuming H_1, i.e., for testing the "pure" differences among the populations after eliminating from T_0 the interaction contribution, if any.

It may be noted here that if $\mathbf{P}$ is any $(p-1) \times p$ matrix of rank $p-1$ satisfying $\mathbf{Pj} = 0$, then

$$\mathscr{P}^{-1} - \gamma\mathscr{P}^{-1}\mathbf{J}\mathscr{P}^{-1} = \mathbf{P}'(\mathbf{P}\mathscr{P}\mathbf{P}')^{-1}\mathbf{P},$$

where $\mathscr{P}$ is a positive definite matrix and $\gamma = 1/\mathbf{j}'\mathscr{P}^{-1}\mathbf{j}$. Since $\mathscr{P}$ is a nonsingular correlation matrix, it is positive definite, and so is $\hat{\mathscr{P}}$ with probability tending to one as $n_i \to \infty$. Thus, we may also express T_1 as

$$T_1 = \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\mathbf{U}_i - \bar{\mathbf{U}})' \mathbf{P}'(\mathbf{P}\hat{\mathscr{P}}\mathbf{P}')^{-1} \mathbf{P}(\mathbf{U}_i - \bar{\mathbf{U}}). \tag{3.6}$$

It is straightforward to show that, *if H_0 holds*, $T_1 \xrightarrow{\mathscr{L}} \chi^2((p-1)(k-1))$ and $T_2 \xrightarrow{\mathscr{L}} \chi^2(k-1)$; this will also follow from Theorem 5.1 established in

Section 5. However, what we would like to have *if possible* is the stated limiting distribution of T_1 *under* H_1 alone. This does not seem to be possible by the present approach (and perhaps by any other approach) without discarding the relative simple form of the statistic. Note in (2.6) that, in general, the limiting covariance matrix $\mathbf{T}$ is a $pk \times pk$ matrix of functionals depending on $\mathbf{F}$. It is only under H_0 that $\mathbf{T}$ had the structure $\mathbf{\Sigma} \otimes \mathscr{P}$, where $\mathbf{\Sigma}$ is known, and now $\mathscr{P}$ is a $p \times p$ matrix of functional depending on common F. Discarding the Kronecker product structure would make it necessary to estimate all terms of $\mathbf{T}$, thus making the computation much more involved. However, as we shall show in Section 5, the use of concept of "local alternatives" to H_0 still makes it possible to justify the use of statistic T_1 for testing H_1.

4. Consistency of T_0, T_1, and T_2

Now in order to study consistency of these criteria for testing the respective hypotheses we need the lemma:

LEMMA 4.1. *Suppose* $N^{1/2}(\mathbf{Y}_\mathbf{n} - \boldsymbol{\xi}) \to^{\mathscr{L}} \mathscr{N}(0, \boldsymbol{\Psi})$ *and* $\mathbf{A}$ *is positive definite. Then the quadratic form* $Q_\mathbf{n} = N(\mathbf{Y}_\mathbf{n} - \boldsymbol{\delta})' \mathbf{A}(\mathbf{Y}_\mathbf{n} - \boldsymbol{\delta}) \to^{P} \infty$, *in the sense that*

$$P[Q_\mathbf{n} > c] \to 1 \qquad as \quad n_i \to \infty, \text{ for all } i$$

for every fixed c, if and only if $\boldsymbol{\xi} \neq \boldsymbol{\delta}$.

The proof is straightforward and hence is deleted. We insert now the subscript $\mathbf{n}$ while studying asymptotic properties.

THEOREM 4.1. *Let* $T_{0,\mathbf{n}}$, $T_{1,\mathbf{n}}$, *and* $T_{2,\mathbf{n}}$ *be defined as* (2.12), (3.4), *and* (3.5) *for functions* $\phi_i^{(\alpha)}$ *satisfying* (2.3). *If* $n_i \to \infty$ *in such a way that* $n_i/N \to p_i$, $0 < p_i < 1$, *then*

(i) $T_{0,\mathbf{n}} \to^P \infty$ *iff* $\mathbf{F} \notin \{\mathbf{F} \mid \Sigma_j \phi(j) \nu_{ij}^{(\alpha)}(\mathbf{F})$ *is independent of i and* α, $i = 1,\ldots, k$, $\alpha = 1,\ldots, p\}$,

(ii) $T_{1,\mathbf{n}} \to^P \infty$ *iff* $\mathbf{F} \notin \{\mathbf{F} \mid \Sigma_j \phi(j) \nu_{ij}^{(\alpha)}(\mathbf{F})$ *is independent of* $\alpha = 1,\ldots, p$ *for each* $i = 1,\ldots, k\}$

and, if $\mathscr{P}^{-1} = [\rho^{\alpha\beta}]$, *then*

(iii) $T_{2,\mathbf{n}} \to^P \infty$ *iff* $\mathbf{F} \notin \{\mathbf{F} \mid \Sigma_j \phi(j) \Sigma_{\alpha,\beta} \rho^{\alpha\beta} \nu_{ij}^{(\alpha)}(\mathbf{F})$ *is independent of i*$\}$.

Proof. Letting $\mathbf{U}' = (\mathbf{U}_1', \mathbf{U}_0')$, as in (2.11), it follows from (2.6) and Lemma 4.1 that

$$N(\mathbf{U}_{0,\mathbf{n}} - \phi \mathbf{j})' [\mathbf{\Sigma}_{11}^{-1} \otimes \mathscr{P}^{-1}](\mathbf{U}_{0,\mathbf{n}} - \phi \mathbf{j}) \overset{P}{\not\to} \infty \tag{4.1}$$

only if $\eta_i(\mathbf{F}) = \phi\mathbf{j}$ for $i = 2,..., k$. Since $\Sigma_{i=1}^{k} \eta(\mathbf{F}) = k\phi\mathbf{j}$, (4.1) holds iff $\eta_i^{(\alpha)}(\mathbf{F}) = \phi$ for all $i = 1,..., k$, and $\alpha = 1,..., p$. If we replace $\mathscr{P}$ by consistent estimators, we see that $T_{0,n}$ and the quadratic form in (4.1) have the same limiting distribution in view of (2.11) and (2.12). This establishes (i).

Now using the argument in [2] for reducing (2.11) to (2.12), from (3.6) we have

$$T_{1,n} = N[\mathbf{U}_{0,n} - \phi\mathbf{j})' [\mathbf{\Sigma}_{11}^{-1} \otimes \mathbf{P}'(\mathbf{P}\hat{\mathscr{P}}\mathbf{P}')^{-1} \mathbf{P}](\mathbf{U}_{0,n} - \phi\mathbf{j})$$
$$= N[(\mathbf{I}_{k-1} \otimes \mathbf{P})(\mathbf{U}_{0,n} - \phi\mathbf{j})]' [\mathbf{\Sigma}_{11}^{-1} \otimes (\mathbf{P}\hat{\mathscr{P}}\mathbf{P}')^{-1}][(\mathbf{I}_{k-1} \otimes \mathbf{P})(\mathbf{U}_{0,n} - \phi\mathbf{j})].$$

Again from (2.6) and Lemma 4.1 it follows that $T_{1,n} \not\to^{P} \infty$ only if

$$(\mathbf{I}_{k-1} \otimes \mathbf{P}) \begin{pmatrix} \eta_2 \\ \eta_k \end{pmatrix} = \phi(\mathbf{I}_{k-1} \otimes \mathbf{P})\mathbf{j} = \mathbf{0},$$

i.e., $\mathbf{P}\eta_i(\mathbf{F}) = \mathbf{0}$, $i = 2,..., k$. Since $\Sigma_i \eta_i(\mathbf{F}) = k\phi\mathbf{j}$, the condition for $T_{1,n} \not\to^{P} \infty$ is that $\mathbf{P}\eta_i(\mathbf{F}) = \mathbf{0}$ for $i = 1,..., k$, which is equivalent to the condition that $\eta_i(\mathbf{F}) \propto \mathbf{j}$; this establishes (ii). The proof of (iii) is easy to obtain along similar lines.

Remark. We thus note here that the tests T_0, T_1 designed for H_0, H_1, respectively, are consistent only against alternatives to the hypotheses "effectively" being tested, viz.

$$H_{0\phi} : \sum_{j=1}^{k} \phi(j)\, \nu_{ij}^{(\alpha)}(\mathbf{F}) \text{ is independent of } i \text{ and } \alpha$$

and

$$H_{1\phi} : \sum_{j=1}^{k} \phi(j)\, \nu_{ij}^{(\alpha)}(\mathbf{F}) \text{ is independent of } \alpha, \text{ for every } i,$$

depending on the function ϕ used for T's. Of course, this undesirable feature of nonparametric tests is usually unavoidable, e.g., the Mann–Whitney test, Sign test, and Kruskal–Wallis tests all suffer from a similar disadvantage.

Note also that if H_1 is accepted, i.e., $\nu_{ij}^{(\alpha)}$ is independent of α, then $T_{2,n} \to^{P} \infty$ unless $\Sigma_j \phi(j)\nu_{ij}(\mathbf{F})$ is independent of i, which is precisely the condition for $T_{1,n} \not\to^{P} \infty$ assuming H_1.

5. Asymptotic Distributions

In the previous section we found the class of *fixed* alternatives $\mathbf{F}' = (F_1,..., F_k)$ for which the tests are consistent, i.e., for which the power of the respective test tends to 1 as $n_i \to \infty$. We shall now find the limiting distributions of T_0, T_1 and T_2 under the sequence of Pitman location alternatives

$$H_N: F_{iN}(\mathbf{x}) = F(\mathbf{x} - N^{-1/2}\,\boldsymbol{\delta}_i), \qquad i = 1,\dots, k \tag{5.1}$$

where the $\boldsymbol{\delta}_i$'s are not all equal, and $\Sigma_i \boldsymbol{\delta}_i = \mathbf{0}$.

THEOREM 5.1. *Consider the sequence $\{H_N\}$ of distributions $\{\mathbf{F}_N\}$ given by* (5.1) *and assume that $F^{(\alpha)}$ is differentiable and has a bounded derivative $f^{(\alpha)}$ almost everywhere, $\alpha = 1,\dots, p$. Suppose further that there exist functions $g^{(\alpha)}$ such that for sufficiently small h*

$$\left| \frac{F^{(\alpha)}(x+h) - F^{(\alpha)}(x)}{h} \right| \leqslant g^{(\alpha)}(x)$$

for almost all x, and $\int_{-\infty}^{\infty} g^{(\alpha)}(x)\, dF^{(\alpha)}(x) < \infty$. Then as $n_i \to \infty$, so that $n_i/N \to p_i$, $0 < p_i < 1$,

$$N^{1/2}(\mathbf{U}_n - \phi\mathbf{j}) \xrightarrow{\mathscr{L}} \mathscr{N}(\boldsymbol{\gamma}(F), \mathbf{T}(F)), \tag{5.2}$$

where $\mathbf{T}(F)$ *is given by* (2.8) *and*

$$\gamma_i^{(\alpha)}(F) = k\delta_i^{(\alpha)} q^{(\alpha)}(\phi, F),\ \boldsymbol{\gamma}_i' = (\gamma_i^{(1)},\dots, \gamma_i^{(p)}),\ \boldsymbol{\gamma}' = (\boldsymbol{\gamma}_1',\dots, \boldsymbol{\gamma}_k'),$$

$$\begin{aligned} q^{(\alpha)}(\phi, F) = {} & \sum_{j=1}^{k} \phi(j) \\ & \times \left[\binom{k-2}{j-2} a^{(\alpha)}(j-2, k-j, F) - \binom{k-2}{j-1} \right. \\ & \left. \times\, a^{(\alpha)}(j-1, k-j-1, F) \right] \end{aligned} \tag{5.3}$$

$$a^{(\alpha)}(b, c, F) = \int_{-\infty}^{\infty} [F^{(\alpha)}(y)]^b\, [1 - F^{(\alpha)}(y)]^c f^{(\alpha)}(y)\, dF^{(\alpha)}(y).$$

Proof. The details of the proof consist mainly in showing that for large N

$$\begin{aligned} E_{H_N}(\mathbf{U}_n) &= E_{H_0}(\mathbf{U}_n) + N^{-1/2}\boldsymbol{\gamma} + \mathbf{o}(N^{-1/2}) \\ NV_{H_N}(\mathbf{U}_n) &= NV_{H_0}(\mathbf{U}_n) + \mathbf{O}(N^{-1/2}); \end{aligned} \tag{5.4}$$

the result then follows from (2.8). We shall only sketch the proof of the first part of (5.4); the proof of the second part follows by straightforward though lengthy verification of all possible terms and hence will be deleted.

We note that

$$E_{H_N} U_n^{(\alpha)} = E_{H_N} \phi_i^{(\alpha)}(\mathbf{X}_1,\dots, \mathbf{X}_k) = \sum_{j=1}^{k} \phi(j)\, P_{H_N}[R_i^{(\alpha)} = j],$$

and, using the notation of (3.2),

$$P_{H_N}[R_i^{(\alpha)} = j] = \sum_c \int_{-\infty}^{\infty} \prod_{l=1}^{j-1} [F^{(\alpha)}(y + N^{-1/2}\epsilon_{il}^{(\alpha)})]$$

$$\times \prod_{m=1}^{k-j} [1 - F^{(\alpha)}(y + N^{-1/2}\epsilon_{im}^{(\alpha)})] \, dF^{(\alpha)}(y),$$

where $\epsilon_{il}^{(\alpha)} = \delta_i^{(\alpha)} - \delta_{il}^{(\alpha)}$. Then, in view of our assumptions,

$$\begin{aligned} P_{H_N}&[R_i^{(\alpha)} = j] \\ &= P_{H_0}[R_i^{(\alpha)} = j] \\ &\quad + N^{-1/2} \sum_c \left\{ \left(\sum_{l=1}^{j-1} \epsilon_{il}^{(\alpha)} \right) \int_{-\infty}^{\infty} [F^{(\alpha)}(y)]^{j-2} [1 - F^{(\alpha)}(y)]^{k-j} f^{(\alpha)}(y) \, dF^{(\alpha)}(y) \right. \\ &\quad \left. - \left(\sum_{m=1}^{k-j} \epsilon_{im}^{(\alpha)} \right) \int_{\infty}^{\infty} [F^{(\alpha)}(y)]^{j-1} [1 - F^{(\alpha)}(y)]^{k-j-1} f^{(\alpha)}(y) \, dF^{(\alpha)}(y) \right\} \\ &\quad + o(N^{-1/2}) \\ &= P_{H_0}[R_i^{(\alpha)} = j] + N^{-1/2} \left(\sum_{l=1}^{k} \epsilon_{il}^{(\alpha)} \right) \binom{k-2}{j-2} a^{(\alpha)}(j-2, k-j, F) \\ &\quad - N^{-1/2} \left(\sum_{m=1}^{k} \epsilon_{im}^{(\alpha)} \right) \binom{k-2}{j-1} a^{(\alpha)}(j-1, k-j-1, F) + o(N^{-1/2}) \\ &= P_{H_0}[R_i^{(\alpha)} = j] + kN^{-1/2} \delta_i^{(\alpha)} \left[\binom{k-2}{j-2} a^{(\alpha)}(j-2, k-j, F) \right. \\ &\quad \left. + \binom{k-2}{j-1} a^{(\alpha)}(j-1, k-j-1, F) \right] + o(N^{-1/2}), \end{aligned}$$

which establishes the first assertion of (5.4).

THEOREM 5.2. *Assume the conditions of Theorem* 5.1. *Let*

$$T_n = \frac{N(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i (\mathbf{U}_{in} - \bar{\mathbf{U}}_n)' \, \mathbf{Q}'(\mathbf{Q}\mathscr{P}\mathbf{Q}')^{-1} \, \mathbf{Q}(\mathbf{U}_{in} - \bar{\mathbf{U}}_n), \tag{5.5}$$

where $\mathbf{Q}$ *is any* $q \times p$ *matrix of rank* q. *Then under* $\{H_N\}$ *as* $n_i \to \infty$

$$T_n \xrightarrow{\mathscr{L}} \chi^2(q(k-1), \lambda_{\mathbf{Q}}(\phi, \boldsymbol{\delta}, F)),$$

and the noncentrality parameter is

$$\lambda_Q(\phi, \boldsymbol{\delta}, F) = \frac{(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}})' \, \mathbf{Q}'(\mathbf{Q}\mathscr{P}\mathbf{Q}')^{-1} \, \mathbf{Q}(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}}) \quad (5.6)$$

when

$$\bar{\boldsymbol{\gamma}} = \Sigma_i p_i \boldsymbol{\gamma}_i .$$

Proof. It can be seen, by argument as in [2] taking $\mathbf{U}' = (\mathbf{U}_1', \mathbf{U}_0')$,

$$T_n = N[(\mathbf{I}_{k-1} \otimes \mathbf{Q})(\mathbf{U}_{0,n} - \phi\mathbf{j})]' \, (\boldsymbol{\Sigma}_{11}^{-1} \otimes \mathscr{P}^{-1})[(\mathbf{I}_{k-1} \otimes \mathbf{Q})(\mathbf{U}_{0,n} - \phi\mathbf{j})].$$

In view of (5.2),

$$N^{1/2}(\mathbf{I}_{k-1} \otimes \mathbf{Q})(\mathbf{U}_{0,n} - \phi\mathbf{j}) \xrightarrow{\mathscr{L}} \mathscr{N}(\mathbf{I}_{k-1} \otimes \mathbf{Q}) \, \boldsymbol{\gamma}(F), \boldsymbol{\Sigma}_{11} \otimes \mathbf{Q}\mathscr{P}(F) \, \mathbf{Q}'),$$

and hence, the theorem follows in view of the fact that the noncentrality parameter is

$$\boldsymbol{\gamma}_0'(F)(\mathbf{I}_{k-1} \otimes \mathbf{Q})' \, (\boldsymbol{\Sigma}_{11}^{-1} \otimes (\mathbf{Q}\mathscr{P}(F) \, \mathbf{Q}')^{-1})(\mathbf{I}_{k-1} \otimes \mathbf{Q}) \, \boldsymbol{\gamma}_0(F) \quad (5.7)$$

with $\boldsymbol{\gamma}' = (\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_0')$, and thus reduces to (5.6).

Now we replace $\mathscr{P}$ by the consistent estimators $\hat{\mathscr{P}}$ and in (5.5) take $\mathbf{Q}$ respectively equal to $\mathbf{I}_p$, $\mathbf{P}$ as in (3.6) and $\mathbf{j}'\hat{\mathscr{P}}^{-1}$. Then we have the following corollary the Theorem 5.2:

COROLLARY 5.1. *Assume the conditions of Theorem* 5.1, *and let* $T_{0,n}$, $T_{1,n}$, $T_{2,n}$ *be defined by* (2.12), (3.4), *and* (3.5), *respectively. Then*

$$\begin{aligned} T_{0,n} &\xrightarrow{\mathscr{L}} \chi^2(p(k-1), \lambda_0(\phi, \boldsymbol{\delta}, F)), \\ T_{1,n} &\xrightarrow{\mathscr{L}} \chi^2((p-1)(k-1), \lambda_1(\phi, \boldsymbol{\delta}, F)) \\ T_{2,n} &\xrightarrow{\mathscr{L}} \chi^2((k-1), \lambda_2(\phi, \boldsymbol{\delta}, F)), \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} \lambda_0(\phi, \boldsymbol{\delta}, F) &= \frac{(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}})' \, \mathscr{P}^{-1}(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}}), \\ \lambda_1(\phi, \boldsymbol{\delta}, F) &= \frac{(k-1)^2}{\mu k^2} \sum_{i=1}^{k} p_i(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}})' \, [\mathscr{P}^{-1} - \gamma\mathscr{P}^{-1}\mathbf{J}\mathscr{P}^{-1}](\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}}), \end{aligned} \quad (5.9)$$

and

$$\lambda_2(\phi, \boldsymbol{\delta}, F) = \lambda_0(\phi, \boldsymbol{\delta}, F) - \lambda_1(\phi, \boldsymbol{\delta}, F).$$

Now we are in a position to identify the sequences $\{H_N\}$ of distributions $\{\mathbf{F}_N\}$ for which the criteria have limiting null distributions.

THEOREM 5.3. *Assume conditions of Theorem 5.1 and suppose that* $q^{(\alpha)}(\phi, F) \neq 0$. *Then*

$$\text{(i)} \quad T_{0,n} \xrightarrow{\mathscr{L}} \chi^2(p(k-1)) \text{ iff } H_0 \text{ holds;}$$

furthermore, if $F^{(\alpha)} = F^{(\beta)}$ *for all* $\alpha \neq \beta$, *then*

$$\text{(ii)} \quad T_{1,n} \xrightarrow{\mathscr{L}} \chi^2((p-1)(k-1)) \text{ iff } \delta_i^{(1)} = \cdots = \delta_i^{(p)}, \quad i = 1, \ldots, k$$

and (5.10)

$$\text{(iii)} \quad T_{2,n} \xrightarrow{\mathscr{L}} \chi^2(k-1) \text{ iff } H_0 \text{ holds, assuming } \delta_i^{(\alpha)} = \delta_i^{(\beta)} \text{ for all } \alpha \neq \beta.$$

Proof. (i) From (5.9) we see that λ_0 vanishes only if $\boldsymbol{\gamma}_i = \bar{\boldsymbol{\gamma}}$ for all i, and hence only if $\boldsymbol{\delta}_i$'s are all equal in view of (5.3).

(ii) Expressing λ_1 in (5.9) as the noncentrality parameter in (5.7) with $\mathbf{Q} = \mathbf{P}$ (as before, of rank $p - 1$ & $\mathbf{Pj} = \mathbf{0}$) we see that $\lambda_1 = 0$ only if

$$(\mathbf{I}_{k-1} \otimes \mathbf{P})\, \boldsymbol{\gamma}_0(F) = \mathbf{0}$$

i.e., if $\mathbf{P}\boldsymbol{\gamma}_i(F) = \mathbf{0}$, $i = 1, \ldots, k$. However, from (5.3) we see that

$$\sum_{i=1}^{k} \gamma_i^{(\alpha)} = kq^{(\alpha)} \sum_{i=1}^{k} \delta_i^{(\alpha)} = 0,$$

and hence $\lambda_1 = 0$ only if $\mathbf{P}\boldsymbol{\gamma}_i(F) = \mathbf{0}$, $i = 1, \ldots, k$.

If now we assume $F^{(\alpha)} = F^{(\beta)}$ for all $\alpha \neq \beta$, then in (5.3), $q^{(\alpha)} = q^{(\beta)}$ for all α, β, and then $\mathbf{P}\boldsymbol{\gamma}_i = \mathbf{0}$ implies $\delta_i^{(\alpha)} = \delta_i^{(\beta)}$ for all α, β and each i. Thus (ii) is established. (iii) can be proved along similar lines.

Finally, in this section, we present the form of q's for some specific ϕ functions corresponding to the statistics referred to in Section 2:

$$\begin{aligned} q^{(\alpha)}(\phi_V, F) &= -a^{(\alpha)}(0, k-2, F) \\ q^{(\alpha)}(\phi_B, F) &= a^{(\alpha)}(k-2, 0, F) \\ q^{(\alpha)}(\phi_L, F) &= a^{(\alpha)}(0, k-2, F) + a^{(\alpha)}(k-2, 0, F) \\ q^{(\alpha)}(\phi_W, F) &= \int_{\infty}^{\infty} f^{(\alpha)}(y)\, dF^{(\alpha)}(y). \end{aligned} \tag{5.11}$$

6. CONCLUDING REMARKS

We have thus esyablished, first of all, consistency of the three tests T_0, T_1, and T_2 for a specific ϕ function against alternatives to H_0, H_1, and H_2, respectively, in the *direction* of the specific ϕ function used. Next we have obtained their asymptotic powers for *local* alternatives to H_0, and have established that if all marginals of F are identical and the location parameters $N^{-1/2}\delta_i^{(\alpha)}$ are the same for all α, then T_1 is asymptotically $\chi^2((p-1)(k-1))$. Note from Theorem 3.1 that H_1 in terms of condition (3.1) is satisfied in such a case.

Computer programs for T_0 and T_1 have been written for specific functions ϕ_V, ϕ_B, ϕ_L and the multivariate version (see [10, 11]) of the Kruskal–Wallis H statistic. (It has been noted (see, e.g., [10]) that W statistics (i.e., T's using ϕ_W) have the same limiting properties as H.) Also, simulation studies have been carried out to investigate χ^2 approximations under H_0 and powers under some alternatives to H_0 (some satisfying H_1) for three different distributions and several covariance structures. These studies [7] are being presented in another paper [3] and these seem to indicate that, apart from the partial justification provided for the test T_1 for the hypothesis H_1, there is also reasonable empirical justification to believe that the concept of local alternatives to H_0 in the direction of H_1 might indeed provide the way out of the theoretical hurdle encountered earlier.

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