Calderón–Zygmund theory for nonlinear elliptic problems with irregular obstacles

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Abstract

We consider a nonhomogeneous elliptic problem with an irregular obstacle involving a discontinuous nonlinearity over an irregular domain in divergence form of $p$-Laplacian type, to establish the global Calderón–Zygmund estimate by proving that the gradient of the weak solution is as integrable as both the gradient of the obstacle and the nonhomogeneous term under the BMO smallness of the nonlinearity and sufficient flatness of the boundary in the Reifenberg sense.

Keywords: Irregular obstacle; Calderón–Zygmund estimate; Discontinuous nonlinearity; $p$-Laplacian; BMO; Reifenberg domain

1. Introduction

The obstacle problem has been a classical one that arises from the mathematical study of variational inequalities and free boundary problems in the area of partial differential equations and their applications, since it was introduced for a membrane and for a plate as one of the simplest unilateral problems from the classical linear elasticity theory. This problem is also naturally

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involved in the study of minimal surfaces and the capacity of a set in potential theory as geometrical problems. Its main purpose is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. Applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, stopping time optimal control problem for Brownian motion, phase transitions, groundwater hydrology, financial mathematics, etc. We refer to [9,17,18,20,28] for a further discussion on the obstacle problem and its applications.

There have been research activities on the regularity theory of obstacle problems. $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity for a general class of elliptic and parabolic obstacle problems was obtained by Choe in [11,12]. In [13] Eleuteri obtained Hölder’s continuity for minimizers of the integral functionals with obstacle under standard growth conditions of p-type. This result was extended under non-standard growth conditions in [15] by Eleuteri and Habermann. In [2] Bögelein, Duzzar and Mingione considered elliptic and parabolic variational problems involving divergence form of $p$-Laplacian type with discontinuous obstacles to establish the Calderón–Zygmund theory for solutions, by proving that the (spatial) gradient of solutions is as integrable as the gradient of the assigned obstacles. In [3], Bögelein and Scheven established the self-improving property of integrability for the spatial gradient of solutions to parabolic variational inequalities satisfying an obstacle constraint and involving possibly degenerate respectively singular operators in divergence form. In [14] Eleuteri and Habermann obtained estimates of Calderón–Zygmund type for one-sided obstacle problems considering local minimizers of quasi-convex integral functionals with $p(x)$ growth.

This work is a natural extension of the local Calderón–Zygmund theory in [2] to a global one. Here we allow the nonlinearity to be discontinuous with respect to the (spatial) variable in a bounded domain whose boundary can go beyond the Lipschitz category. The regularity of a solution and its gradient for variational inequality is deeply related to those of the obstacle function and the nonhomogeneous term. Considering the coincidence set where a solution is equal to the obstacle function, we know that a solution cannot be more regular than the obstacle function. In this paper we want to answer as to what are minimal regularity requirements on the nonlinearity and what is the lowest level of geometric assumption on the boundary under which the gradient of the obstacle function and the nonhomogeneous term provide the gradient of a solution with the same regularity in the setting of Lebesgue spaces. Motivated the earlier work [8] where a local Calderón–Zygmund theory was obtained without an obstacle, we assume a smallness in bounded mean oscillation (BMO) on the nonlinearity with respect to the (spatial) variable. When it comes to a minimal geometric assumption we impose a Reifenberg flatness which turns out to be an appropriate one for nonlinear perturbation results, as in [6,21,27]. This is a sort of minimal regularity of the boundary guaranteeing the main results of the geometric analysis continue to hold true. In particular, $C^{1}$-smooth or Lipschitz continuous boundaries belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the Van Koch snowflake, see [30].

It is worth mentioning that our work is influenced by the contents in [2] such as comparison principle for obstacle problems, comparison maps, an existence and regularity. The main approach in [2] is the so-called maximal function-free technique which was introduced in [1] and later employed in many papers, for instance, [8,14] and references therein. This approach is an appropriate substitute for maximal function technique when maximal function technique does not work. This is the case that the problem scales differently in space and time for the parabolic case of $p$-Laplacian type and is forced to use the intrinsic geometry of Dibenedetto. On the other hand, the problem under consideration is concerned with a stationary obstacle problem for the
elliptic case and has a scaling invariance property. We therefore, here, use the maximal function approach which was used in earlier works [5,6,8,21,22,27].

This paper is organized as follows. In the next section we state some background, notation and the main result. In Section 3 we introduce analytic and geometric tools which will be employed such as maximal functions, covering lemma and comparison principle for obstacle problems. In Section 4 we discuss local and global comparison estimates from improved higher regularity and weak compactness method, to find a Vitali type covering result. Finally, in the last section we establish a global Calderón–Zygmund theory for nonlinear elliptic problems with irregular obstacles.

2. Preliminaries and results

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with $n \geq 2$ and $1 < p < \infty$ be a fixed real number. We then consider the convex admissible set

$$\mathcal{A} = \{ \phi \in W^{1,p}_0(\Omega) : \phi \geq \psi \text{ a.e. in } \Omega \} \quad (2.1)$$

with

$$\psi \in W^{1,p}(\Omega) \quad \text{and} \quad \psi \leq 0 \text{ a.e. on } \partial \Omega. \quad (2.2)$$

We are interested in functions $u \in \mathcal{A}$ satisfying the following variational inequality

$$\int_{\Omega} a(Du, x) \cdot D(\phi - u) \geq \int_{\Omega} |F|^{p-2} F \cdot D(\phi - u) \, dx \quad (2.3)$$

for all $\phi \in \mathcal{A}$, where $F \in L^p(\Omega, \mathbb{R}^n)$.

We call such a function $u$ to be a weak solution to the variational inequality (2.3). A given vector-valued function

$$a = a(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is a Carathéodory function, namely, measurable in $x$ and differentiable in $\xi$. Assume, moreover, the following boundedness and ellipticity conditions:

$$|a(\xi, x)| + |\xi| D_\xi a(\xi, x) \leq A|\xi|^{p-1} \quad (2.4)$$

and

$$D_\xi a(\xi, x) \eta \cdot \eta \geq \mu |\xi|^{p-2}|\eta|^2 \quad (2.5)$$

for all $x, \xi, \eta \in \mathbb{R}^n$ and for some constants $0 < \mu \leq 1 \leq A$.

We point out that the primary structure conditions (2.4)–(2.5) imply the following monotonicity condition:
\[(a(\xi, x) - a(\eta, x)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p \quad \text{if } p \geq 2, \quad (2.6)\]
\[(a(\xi, x) - a(\eta, x)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \quad \text{if } 1 < p < 2. \quad (2.7)\]

Here \(\gamma\) is a positive constant depending only on \(\mu, p,\) and \(n\). Hereafter we employ the letter \(c\) to denote any constants that can be explicitly computed in terms of \(n,\) the geometric assumption on \(\Omega, p, q, \mu,\) and \(\Lambda,\) and so \(c\) might vary from line to line.

**Lemma 2.1.** There exists a unique weak solution \(u \in A\) to the variational inequality (2.3) with the estimate

\[\|Du\|_{L^p(\Omega)} \leq c(\|F\|_{L^p(\Omega)} + \|D\psi\|_{L^p(\Omega)}) \quad (2.8)\]

**Proof.** Given a small \(\epsilon > 0,\) let \(u_\epsilon \in W^{1,p}_0(\Omega)\) be the unique weak solution of the following Dirichlet problem:

\[
\begin{aligned}
-\text{div} a(Du_\epsilon, x) &= \frac{1}{\epsilon} (\psi - u_\epsilon)^+)^{p-1} - \text{div}(|F|^{p-2} F) \quad \text{in } \Omega, \\
u_\epsilon &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]

Using the primary conditions (2.4)–(2.5) and Young’s inequality, it follows that \(\{u_\epsilon\}_{\epsilon > 0}\) is uniformly bounded in \(W^{1,p}_0(\Omega).\) Then there exists a subsequence, again labeled with \(\epsilon,\) and a function \(u \in W^{1,p}_0(\Omega)\) such that

\[u_\epsilon \rightharpoonup u \quad \text{in } L^p(\Omega) \quad \text{and} \quad Du_\epsilon \rightharpoonup Du \quad \text{in } L^p(\Omega).
\]

We then can show that \(u \geq \psi\) and \(u\) satisfies the original variational inequality (2.3) including the estimate (2.8) and the uniqueness of solutions. \(\Box\)

The aim of this work is to find the minimal condition on the nonlinearity \(a(\xi, x)\) and a lower level of geometric assumption on \(\partial\Omega\) under which for each \(q \in (1, \infty)\),

\[F \text{ and } D\psi \in L^{pq}(\Omega, \mathbb{R}^n) \quad \Rightarrow \quad Du \in L^{pq}(\Omega, \mathbb{R}^n)
\]

with the estimate

\[\|Du\|_{L^{pq}(\Omega)} \leq c(\|F\|_{L^{pq}(\Omega)} + \|D\psi\|_{L^{pq}(\Omega)}).
\]

In order to measure the oscillation of \(a(\xi, x)\) in the variable \(x\) over the ball \(B_\rho(y)\), we define the function

\[\beta(a, B_\rho(y))(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|a(\xi, x) - \bar{a}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}}, \quad (2.9)\]
where \( \overline{a}_{B_\rho(y)} \) is the integral average of \( a(\xi, \cdot) \) over \( B_\rho(y) \), as is defined by

\[
\overline{a}_{B_\rho(y)}(\xi) = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} a(\xi, x) \, dx.
\]

\[ (2.10) \]

**Definition 2.2.** \( a(\xi, x) \) is \((\delta, R)\)-vanishing if we have

\[
\sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} |\beta(a, B_\rho(y)) (x)| \, dx \leq \delta.
\]

**Definition 2.3.** \( \Omega \) is \((\delta, R)\)-Reifenberg flat if for every \( x \in \partial \Omega \) and every \( \rho \in (0, R] \), there exists a coordinate system \( \{y_1, \ldots, y_n\} \), which can depend on \( \rho \) and \( x \) so that \( x = 0 \) in this coordinate system and that

\[
B_\rho(0) \cap \{y_n > \delta \rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta \rho\}.
\]

This geometric condition prescribes that under all scales the boundary can be trapped between two hyper-planes, depending on the scale chosen. The domain can go beyond Lipschitz category, not necessarily given by graphs.

The following lemma shows that the obstacle problem under consideration has the invariance properties under scaling and normalization.

**Lemma 2.4.** \( u \in \mathcal{A} \) is the weak solution to the variational inequality (2.3). Assume that \( a \) is \((\delta, R)\)-vanishing and \( \Omega \) is \((\delta, R)\)-Reifenberg flat. Fix \( \lambda \geq 1 \) and \( 0 < r < 1 \). We define the rescaled maps

\[
\tilde{a} = \frac{a(\lambda \xi, rx)}{\lambda^{p-1}}, \quad \tilde{u}(x) = \frac{u(rx)}{\lambda^r}, \quad \tilde{F}(x) = \frac{F(rx)}{\lambda}, \quad \tilde{\psi}(x) = \frac{\psi(rx)}{\lambda^r}
\]

and

\[
\tilde{\Omega} = \{(1/r)x: x \in \Omega\}.
\]

Then we have

1. \( \tilde{a} \) satisfies (2.4) and (2.5) with the same constants \( \mu, \Lambda \).
2. \( \tilde{a} \) is \((\delta, \frac{R}{r})\)-vanishing.
3. \( \tilde{\Omega} \) is \((\delta, \frac{R}{r})\)-Reifenberg flat.
4. \( \tilde{u} \in \tilde{\mathcal{A}} = \{\phi \in W^{1,p}_0(\tilde{\Omega}): \phi \geq \tilde{\psi} \text{ a.e. in } \tilde{\Omega}\} \) is the weak solution to the following variational inequality

\[
\int_{\tilde{\Omega}} \tilde{a}(D \tilde{u}, x) \cdot D(\phi - \tilde{u}) \, dx \geq \int_{\tilde{\Omega}} |\tilde{F}|^{p-2} \tilde{F} \cdot D(\phi - \tilde{u}) \, dx, \quad \forall \phi \in \tilde{\mathcal{A}}.
\]
Theorem 2.5. For any given $q \in (1, \infty)$, assume that $F \in L^{pq}(\Omega; \mathbb{R}^n)$ and $D\psi \in L^{pq}(\Omega; \mathbb{R}^n)$. Then there exists a constant $\delta = \delta(\mu, \Lambda, n, p, q) > 0$ such that if $a(\xi, x)$ is $(\delta, R)$-vanishing and $\Omega$ is $(\delta, R)$-Reifenberg flat, then the weak solution $u$ satisfies $Du \in L^{pq}(\Omega; \mathbb{R}^n)$ with the estimate

$$\|Du\|_{L^{pq}(\Omega)} \leq c \left( \|D\psi\|_{L^{pq}(\Omega)} + \|F\|_{L^{pq}(\Omega)} \right),$$

where $c$ is a positive constant depending on $n, p, q, \mu, \Lambda$, and $|\Omega|$.

Remark 2.6. As a consequence of the main result, we have Hölder’s regularity. More precisely, if we take $q$ with $pq > n$, then it follows directly from Sobolev’s inequality that $u \in C^{0,1-n/pq}(\Omega)$.

3. Analytic and geometric tools

We will prove the main theorem using the maximal function, some classical measure theory, a Vitali type covering lemma, and a comparison principle for the obstacle problems.

Definition 3.1. The Hardy–Littlewood maximal function $Mf$ of a locally integrable function $f$ is a function such that

$$(Mf)(x) = \sup_{\rho > 0} \int_{B_\rho(x)} |f(y)| \, dy.$$ 

If $f$ is not defined outside a bounded domain $\Omega$,

$$M\Omega f = M(f \chi_\Omega)$$

for the standard characteristic function $\chi$ on $\Omega$.

Lemma 3.2. (See [29].) If $f \in L^t(\mathbb{R}^n)$ for $1 < t \leq \infty$, then $Mf \in L^t(\mathbb{R}^n)$ and for some $c = c(n, t) > 0$,

$$\frac{1}{c} \|f\|_{L^t(\mathbb{R}^n)} \leq \|Mf\|_{L^t(\mathbb{R}^n)} \leq c \|f\|_{L^t(\mathbb{R}^n)}. \quad (3.1)$$

If $f \in L^1(\mathbb{R}^n)$, then for some $c = c(n) > 0$,

$$\left| \left\{ x \in \mathbb{R}^n : (Mf)(x) > \lambda \right\} \right| \leq \frac{c}{\lambda} \int |f(x)| \, dx. \quad (3.2)$$

Lemma 3.3. (See [5].) Let $C$ and $D$ be measurable sets with $C \subset D \subset \Omega$. Assume that $\Omega$ is $(\delta, 1)$-Reifenberg flat for some small $\delta > 0$. Assume further that there exists a small $\epsilon > 0$ such that

$$|C| \leq \epsilon |B_1|$$
and that

for \( x \in \Omega \) and \( r \in (0, 1) \) with \( |C \cap B_r(x)| \geq \epsilon |B_r(x)| \), \( B_r(x) \cap \Omega \subset D \).

Then we have

\[ |C| \leq \epsilon_1 |D|, \]

where \( \epsilon_1 = \left( \frac{20}{1-\delta} \right)^n \epsilon \).

**Lemma 3.4.** (See [10].) Assume that \( f \) is a nonnegative and measurable function in \( \mathbb{R}^n \). Assume further that \( f \) has a compact support in a bounded subset \( \Omega \) of \( \mathbb{R}^n \). Let \( \theta > 0 \) and \( m > 1 \) be constants. Then for \( 0 < t < \infty \), we have

\[ f \in L^t(\Omega) \iff S = \sum_{k \geq 1} m^{kt} \left| \left\{ x \in \Omega : f(x) > \theta m^k \right\} \right| < \infty \]

and

\[ \frac{1}{c} S \leq \| f \|_{L^t(\Omega)} \leq c (|\Omega| + S), \]

where \( c > 0 \) is a constant depending only on \( \theta, m, \) and \( t \).

**Lemma 3.5.** Suppose that \( \psi, v \in W^{1,p}(\Omega) \) satisfy

\[
\begin{cases}
-\text{div} a(D\psi, x) \leq -\text{div} a(Dv, x) & \text{in } \Omega, \\
\psi \leq v & \text{on } \partial \Omega,
\end{cases}
\]

where (2.4) and (2.5) are assumed. Then there holds \( \psi \leq v \) a.e. in \( \Omega \).

**Proof.** Let \( \varphi \in W^{1,p}_0(\Omega) \) and \( \varphi \geq 0 \) a.e. in \( \Omega \). Then we have

\[
\int_{\Omega} (a(D\psi, x) - a(Dv, x)) \cdot D\varphi \leq 0. \tag{3.3}
\]

Since \( \psi \leq v \) on \( \partial \Omega \), we have \( (\psi - v)^+ \in W^{1,p}_0(\Omega) \) and so we can take \( \varphi = (\psi - v)^+ \). Then it follows from (3.3) that

\[
\int_{\Omega} (a(D\psi, x) - a(Dv, x)) \cdot D((\psi - v)^+) \, dx \leq 0,
\]

which we rewrite as

\[
\int_{\Omega \cap \{\psi > v\}} (a(D\psi, x) - a(Dv, x)) \cdot D(\psi - v) \, dx \leq 0. \tag{3.4}
\]
If $p \geq 2$, from (2.6) and (3.4) we have
\[
\int_{\Omega} |D((\psi - v)^+)|^p \, dx = \int_{\Omega \cap \{\psi > v\}} |D(\psi - v)|^p \, dx \leq \frac{1}{\gamma} \int_{\Omega \cap \{\psi > v\}} (a(D\psi, x) - a(Dv, x)) \cdot D(\psi - v) \, dx \leq 0.
\]

Hence $\psi \leq v$ a.e. in $\Omega$.

If $1 < p < 2$, using Young’s inequality for $\epsilon > 0$, (2.7) and (3.4) it follows that
\[
\int_{\Omega} |D((\psi - v)^+)|^p \, dx = \int_{\Omega \cap \{\psi > v\}} |D\psi - Dv|^p \, dx \\
= \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^{p(2-p)} \left[ (|D\psi| + |Dv|)^{p(p-2)} |D(\psi - v)|^p \right] \, dx \\
\leq \epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^p \, dx \\
+ c(\epsilon) \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^{p-2} |D(\psi - v)|^2 \, dx \\
\leq c\epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi|^p + |Dv|^p) \, dx \\
+ c(\epsilon) \int_{\Omega \cap \{\psi > v\}} (a(D\psi, x) - a(Dv, x)) \cdot (D\psi - Dv) \, dx \\
\leq c\epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi|^p + |Dv|^p) \, dx.
\]

By letting $\epsilon \to 0$, we have
\[
\int_{\Omega} |D((\psi - v)^+)|^p \, dx \leq 0.
\]

Therefore, $\psi \leq v$ a.e. in $\Omega$. \qed

4. Gradient estimates for irregular obstacle problems

We start with interior comparison estimates. To do this, we assume that

\[ B_6 \subset \Omega, \quad (4.1) \]
\[
\begin{align*}
\sup_{0 < \rho \leq 5} \int_{B_\rho} \beta(a; B_\rho) \, dx & \leq \delta, \tag{4.2} \\
\int_{B_5} |Du|^p \, dx & \leq 1, \quad \int_{B_5} |F|^p \, dx \leq \delta^p, \quad \int_{B_5} |D\psi|^p \, dx \leq \delta^p. \tag{4.3}
\end{align*}
\]

Under these assumptions (4.1)–(4.3), we compare \( u \in W^{1,p}_0(\Omega) \) to the unique weak solution \( k \in W^{1,p}(B_5) \) of

\[
\begin{align*}
\begin{cases}
-\text{div} \, a(Dk, x) = -\text{div} \, a(D\psi, x) & \text{in } B_5, \\
k = u & \text{on } \partial B_5.
\end{cases} \tag{4.4}
\end{align*}
\]

We then compare \( k \in W^{1,p}(B_5) \) to the unique weak solution \( w \in W^{1,p}(B_5) \) of

\[
\begin{align*}
\begin{cases}
-\text{div} \, a(Dw, x) = 0 & \text{in } B_5, \\
w = k & \text{on } \partial B_5.
\end{cases} \tag{4.5}
\end{align*}
\]

The limiting problem is

\[
\begin{align*}
\begin{cases}
-\text{div} \, \bar{a}_{B_4}(Dv) = 0 & \text{in } B_4, \\
v = w & \text{on } \partial B_4.
\end{cases} \tag{4.6}
\end{align*}
\]

The following is \( L^p \) estimate for (4.4). This estimate also can be applied for (4.5) and (4.6).

**Lemma 4.1.** Let \( k \) be the weak solution of (4.4). Then we have

\[
\int_{B_5} |Dk|^p \, dx \leq c \left( \int_{B_5} |Du|^p \, dx + \int_{B_5} |D\psi|^p \, dx \right). \tag{4.7}
\]

**Proof.** We take \( k - u \in W^{1,p}_0(B_5) \) as a test function in the weak formulation of (4.4). Then we have

\[
\int_{B_5} a(Dk, x) \, Dk \, dx = \int_{B_5} a(Dk, x) \, Du \, dx + \int_{B_5} a(D\psi, x) \, D(k - u) \, dx. \tag{4.8}
\]

In view of (2.4), (2.6) and (2.7), we estimate the left-hand side of (4.8) as follows:

\[
\int_{B_5} a(Dk, x) \, Dk \, dx \geq \gamma \int_{B_5} |Dk|^p \, dx, \tag{4.9}
\]

since \( a(0, x) = 0 \).
Using (2.4) and Young’s inequality with $\epsilon$, we estimate the right-hand side of (4.8) as follows:

\[
\begin{align*}
\int_{B_5} a(Dk, x) Du\, dx + \int_{B_5} a(D\psi, x) D(k - u)\, dx & \\
& \leq \Lambda \int_{B_5} |Dk|^{p-1} |Du| + |D\psi|^{p-1} |Dk| + |D\psi|^{p-1} |Du|\, dx \\
& \leq c\epsilon \int_{B_5} |Dk|^p\, dx + c(\epsilon) \left( \int_{B_5} |Du|^p\, dx + \int_{B_5} |D\psi|^p\, dx \right). \tag{4.10}
\end{align*}
\]

We combine (4.8)–(4.10), and then take $\epsilon$ so small, in order to derive the conclusion (4.7).

From a direct consequence of Lemma 4.1, we have $L^p$ estimates for (4.5) and (4.6) as follows:

\[
\begin{align*}
\int_{B_5} |Dw|^p\, dx & \leq c \int_{B_5} |Dk|^p\, dx \tag{4.11} \\
\int_{B_4} |Dv|^p\, dx & \leq c \int_{B_4} |Dw|^p\, dx. \tag{4.12}
\end{align*}
\]

Therefore, under the assumptions (4.1)–(4.6) we have

\[
\begin{align*}
\int_{B_4} |Dv|^p\, dx + \int_{B_5} |Dw|^p\, dx + \int_{B_5} |Dk|^p\, dx & \leq c. \tag{4.13}
\end{align*}
\]

We have the following higher integrability result for (4.5)–(4.6).

**Lemma 4.2.** Let $w$ be the weak solution of the problem (4.5) with the assumptions (4.1)–(4.4). Then there exists a small positive constant $\epsilon_0 = \epsilon_0(n, p, \mu, \Lambda)$ such that $|Dw| \in L^{p+\epsilon_0}(B_4)$ with the uniform bound

\[
\int_{B_4} |Dw|^{p+\epsilon_0}\, dx \leq c.
\]

**Proof.** According to the well-known improved regularity for the homogeneous problem (4.5), we find

\[
\left( \int_{B_4} |Dw|^{p+\epsilon_0}\, dx \right)^{\frac{1}{p+\epsilon_0}} \leq c \left( 1 + \int_{B_5} |Dw|^p\, dx \right). \tag{4.14}
\]
for some small positive constant \( \epsilon_0 = \epsilon_0(n, p, \mu, A) \) (see [19]). Then the conclusion follows from (4.13) and (4.14). \( \square \)

The following Lipschitz regularity for the limiting problem (4.6) is crucial for the required \( W^{1,p} \) regularity for the obstacle problem under consideration.

**Lemma 4.3.** Let \( v \) be the weak solution of the problem (4.6) with the assumptions (4.1)–(4.5). Then there exists a positive constant \( N_0 = N_0(n, p, \mu, A) \) such that

\[
\|Dv\|_{L^\infty(B_3)}^p \leq N_0.
\]

**Proof.** Note that the nonlinearity for (4.6) is independent of \( x \)-variable, to see that

\[
\|Dv\|_{L^\infty(B_3)}^p \leq c \int_{B_4} |Dv|^p \, dx. \tag{4.15}
\]

By this estimate (4.15) and (4.13), we complete the proof (see [6,16]). \( \square \)

We are now ready to prove the following interior comparison estimate.

**Lemma 4.4.** Let \( u \) be a weak solution to the variational inequality (2.3). Then for any \( \epsilon > 0 \), there is a small \( \delta = \delta(\epsilon, \mu, A, n, p) > 0 \) such that if the assumptions (4.1)–(4.3) hold, then there exists a weak solution \( v \in W^{1,p}(B_4) \) of (4.6) such that

\[
\int_{B_4} |D(u - v)|^p \, dx \leq \epsilon p. \tag{4.16}
\]

**Proof.** Let \( k \) be the weak solution of (4.4). Since \( k = u \geq \psi \) a.e. on \( \partial B_5 \), it follows from Lemma 3.5 that \( k \geq \psi \) a.e. in \( B_5 \). We next extend \( k \) to \( \Omega \setminus B_5 \) by \( u \) so that \( k \in \mathcal{A} \) and \( k - u = 0 \) in \( \Omega \setminus B_5 \). Then the variational inequality (2.3) when \( \phi = k \) implies that

\[
\int_{B_5} a(Du, x) \cdot D(k - u) \, dx \geq \int_{B_5} |F|^{p-2} F \cdot D(k - u) \, dx. \tag{4.17}
\]

This inequality (4.17) and (4.8) imply that

\[
\int_{B_5} (a(Dk, x) - a(Du, x)) \cdot D(k - u) \, dx \leq \int_{B_5} (a(D\psi, x) - |F|^{p-2} F) \cdot D(k - u) \, dx. \tag{4.18}
\]
We first estimate the left-hand side of (4.18). If $p \geq 2$, it follows from (2.6) that
\begin{equation}
\gamma \int_{B_5} |D(u - k)|^p \, dx \leq \int_{B_5} (a(Dk, x) - a(Du, x)) \cdot D(k - u) \, dx.
\end{equation}

If $1 < p < 2$, using Young’s inequality with $\tau$, (2.7) and (4.13), we estimate as follows:
\begin{align*}
\int_{B_5} |D(u - k)|^p \, dx &= \int_{B_5} \left( |Du| + |Dk| \right)^{p-2} \left[ |Du| + |Dk| \right]^{p-2} |D(u - k)|^p \, dx \\
&\leq \tau \int_{B_5} (|Du| + |Dk|)^p \, dx + c(\tau) \int_{B_5} (|Du| + |Dk|)^{p-2} |D(u - k)|^2 \, dx \\
&\leq c\tau + c(\tau) \frac{1}{\gamma} \int_{B_5} (a(Dk, x) - a(Du, x)) \cdot D(k - u) \, dx,
\end{align*}

which implies that for any $\tau > 0$,
\begin{equation}
\gamma \int_{B_5} |D(u - k)|^p \, dx \leq c\tau + c(\tau) \int_{B_5} (a(Dk, x) - a(Du, x)) \cdot D(k - u) \, dx.
\end{equation}

Combining (4.19) and (4.20), we find that
\begin{equation}
\gamma \int_{B_5} |D(u - k)|^p \, dx \leq c\tau + c(\tau) \int_{B_5} (a(Dk, x) - a(Du, x)) \cdot D(k - u) \, dx.
\end{equation}

We next estimate the right-hand side of (4.18). Using (2.4), Young’s inequality with $\sigma > 0$ and (4.3), we have
\begin{align*}
\int_{B_5} (a(D\psi, x) - |F|^{p-2} F) \cdot D(k - u) \, dx \\
&\leq \Lambda \int_{B_5} (|D\psi|^{p-1} + |F|^{p-1}) |D(k - u)| \, dx \\
&\leq c\sigma \int_{B_5} |D(u - k)|^p \, dx + c(\sigma) \int_{B_5} (|D\psi|^p + |F|^p) \, dx \\
&\leq c\sigma \int_{B_5} |D(u - k)|^p \, dx + c(\sigma) \delta^p.
\end{align*}
From (4.18), (4.21) and (4.22), we discover
$$\int_{B_5} \left| D(u - k) \right|^p \, dx \leq c\tau + \sigma c(\tau) \int_{B_5} |D(u - k)|^p \, dx + c(\tau)c(\sigma)\delta^p.$$ 

We then take $\tau, \sigma$ so small, respectively, in order to discover
$$\int_{B_5} \left| D(u - k) \right|^p \, dx \leq c\delta^{\sigma_1}, \quad (4.23)$$
for some $\sigma_1 = \sigma_1(\mu, \Lambda, n, p) > 0$.

We now let $w$ be the weak solution of the problem (4.5). Take a test function $\psi = k - w \in W_{0}^{1,p}(B_5)$ for (4.4) and (4.5) to find
$$\int_{B_5} \left( a(Dk, x) - a(Dw, x) \right) \cdot (D(k - w)) \, dx = \int_{B_5} a(D\psi, x) \cdot D(k - w) \, dx. \quad (4.24)$$

In the same way we have estimated (4.18), one can derive from (4.24) that
$$\int_{B_5} \left| D(k - w) \right|^p \, dx \leq c\delta^{\sigma_2}, \quad (4.25)$$
for some positive number $\sigma_2 = \sigma_2(\mu, \Lambda, n, p)$.

We next consider the weak solution $v$ of the problem (4.6). Take a test function $\psi = w - v \in W_{0}^{1,p}(B_4)$ for (4.5) and (4.6), to find that
$$\int_{B_4} \left( a(Dw, x) - a_{B_4}(Dv) \right) \cdot (D(w - v)) \, dx = 0,$$
which we write as follows:
$$\int_{B_4} \left( a_{B_4}(Dw) - a_{B_4}(Dv) \right) \cdot (Dw - Dv) \, dx$$
$$\quad = \int_{B_4} \left( a_{B_4}(Dw) - a(Dw, x) \right) \cdot (Dw - Dv) \, dx. \quad (4.26)$$

In view of (4.21), we estimate the left-hand side of (4.26) as follows:
$$\gamma \int_{B_4} |D(w - v)|^p \, dx \leq c\tau + c(\tau) \int_{B_4} \left( a_{B_4}(Dw) - a_{B_4}(Dv) \right) \cdot (D(w - v)) \, dx. \quad (4.27)$$

Recalling (2.9) and using Lemma 4.2 and the smallness condition (4.2), we estimate the right-hand side of (4.26) as follows:
\[
\int_{B_4} (\bar{a}_{B_4}(Dw) - a(Dw, x)) \cdot (Dw - Dv) \, dx \\
\leq \int_{B_4} |\bar{a}_{B_4}(Dw) - a(Dw, x)| |Dw - Dv| \, dx \\
\leq \int_{B_4} \beta(\bar{a}_{B_4}, B_4)|Dw|^{p-1} |Dw - Dv| \, dx \\
\leq \sigma \int_{B_4} |Dw - Dv|^p \, dx + c(\sigma) \int_{B_4} \beta^{\frac{p}{p-1}} |Dw|^p \, dx \\
\leq \sigma \int_{B_4} |Dw - Dv|^p \, dx + c(\sigma) \left( \int_{B_4} \beta^{\frac{p(p+\epsilon_0)}{p-1}} \, dx \right)^{\frac{p}{p-1}} \left( \int_{B_4} |Dw|^{p+\epsilon_0} \, dx \right)^{\frac{p}{p+\epsilon_0}} \\
\leq \sigma \int_{B_4} |Dw - Dv|^p \, dx + c(\sigma) \delta^{\frac{p}{p-1}}.
\]

That is, we find that
\[
\int_{B_4} (\bar{a}_{B_4}(Dw) - a(Dw, x)) \cdot (Dw - Dv) \, dx \leq \sigma \int_{B_4} |Dw - Dv|^p \, dx + c(\sigma) \delta^{\frac{p}{p-1}}. \tag{4.28}
\]

Then it follows from (4.26), (4.27) and (4.28) that for some universal constant \(\sigma_3 = \sigma_3(\mu, \Lambda, n, p) > 0\)
\[
\int_{B_4} |Dw - Dv|^p \, dx \leq c \delta^{\sigma_3}. \tag{4.29}
\]

We now combine (4.23), (4.25) and (4.29), to derive that for some universal constant \(\sigma_4 = \sigma_4(\mu, \Lambda, n, p) > 0\)
\[
\int_{B_4} |Du - Dv|^p \, dx \leq c \delta^{\sigma_4},
\]
from which we take \(\delta > 0\) so small that have the conclusion (4.16). This completes the proof. \(\Box\)

We next extend the interior comparison estimate in Lemma 4.4 to find a boundary version. To do this, we introduce the following notations:
\[
\Omega_\rho = B_\rho \cap \Omega, \quad B_\rho^+ = \{x \in B_\rho: x_n > 0\}
\]
and
\[
\partial_w \Omega_\rho = B_\rho \cap \partial \Omega, \quad T_\rho = \{x \in B_\rho: x_n = 0\}.
\]
We now assume that
\[ B_\rho^+ \subset \Omega_\rho \subset B_\rho \cap \{ x_n > -2\rho \delta \}, \quad \forall \rho \in [1, 6], \] (4.30)
and
\[ \sup_{0 < \rho \leq 6} \int_{B_\rho^+} |B(a, B_\rho^+) (x)| \, dx \leq \delta, \] (4.31)

and
\[
\int_{\Omega_5} |Du|^p \, dx \leq 1, \quad \int_{\Omega_5} |F|^p \, dx \leq \delta^p, \quad \int_{\Omega_5} |D\psi|^p \, dx \leq \delta^p.
\] (4.32)

Under these assumptions (4.30)–(4.32) we consider the following problems:

\[
\begin{aligned}
&\begin{cases}
  -\text{div} \ a(Dk, x) = -\text{div} \ a(D\psi, x) \quad \text{in } \Omega_5, \\
  k = u \quad \text{on } \partial \Omega_5,
\end{cases} \\
&\begin{cases}
  -\text{div} \ a(Dw, x) = 0 \quad \text{in } \Omega_5, \\
  w = k \quad \text{on } \partial \Omega_5,
\end{cases} \\
&\begin{cases}
  -\text{div} \ a_B^{\pm}(Dh) = 0 \quad \text{in } \Omega_4, \\
  h = w \quad \text{on } \partial \Omega_4,
\end{cases}
\end{aligned}
\] (4.33–4.35)

and

\[
\begin{aligned}
&\begin{cases}
  -\text{div} \ a_B^{\pm}(Dv) = 0 \quad \text{in } B_4^+, \\
  v = 0 \quad \text{on } T_4.
\end{cases}
\end{aligned}
\] (4.36)

We can now obtain the following uniform boundedness in \( L^p \) for \( Dk, Dw \) and \( Dh \) in almost exactly the same way that we obtained their counterparts in the proof of Lemma 4.1:

\[
\int_{\Omega_4} |Dh|^p \, dx + \int_{\Omega_5} |Dw|^p \, dx + \int_{\Omega_5} |Dk|^p \, dx \leq c.
\] (4.37)

Returning to the Reifenberg flatness conditions, see Definition 2.3, one can derive

\[ |B_\rho(x_0)| \leq c(\delta, n)|\Omega_\rho(x_0)|, \quad \forall x_0 \in \partial_u \Omega_\rho \text{ and } \forall \rho \in (0, 6]. \]

Thanks to this measure density condition, the Reifenberg domains are \( W^{1,t}_\text{ext} \)-extension domains, \( 1 \leq t \leq \infty \) and the usual extension theorem, Sobolev’s inequality and Poincaré’s inequality hold true on the Reifenberg domains, see [5,21,23,24,27] and the references therein. Moreover, this density condition guarantees a quantified higher integrability of the gradient of a weak solution of the homogeneous problem (4.34), see [19,25,26] and the references therein. Then using the
$L^p$-uniform boundedness assumption (4.37) we observe that the homogeneous problem (4.34) has the following improved higher regularity with the uniform bound

$$\int_{\Omega^4} |Dw|^{p+\sigma^*} \, dx \leq c,$$

where $\sigma^* = \sigma^*(n, p, \mu, A)$ is a positive small universal constant.

We need the following Lipschitz regularity for a limiting problem (4.36).

**Lemma 4.5.** (See [6,16].) Let $v \in W^{1,p}(B^+_{4})$ be a weak solution of (4.36). Then we have

$$\|Dv\|^p_{L^\infty(B^+_3)} \leq c \int_{B^+_4} |Dv|^p \, dx. \tag{4.38}$$

In addition, if $v_0$ is the zero extension of $v$ from $B^+_4$ to $\Omega^4$, then we find

$$\|Dv_0\|^p_{L^\infty(\Omega_3)} = \|Dv\|^p_{L^\infty(B^+_3)} \leq c \int_{B^+_4} |Dv|^p \, dx. \tag{4.39}$$

**Lemma 4.6.** For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if $B^+_4 \subset \Omega^4 \subset B_4 \cap \{x_n > -8\delta\}$ and $h \in W^{1,p}(\Omega^4)$ is a weak solution of

$$\begin{cases}
-\text{div} \, a_{B^+_4}(Dh) = 0 & \text{in } \Omega^4, \\
h = 0 & \text{on } \partial_w \Omega^4,
\end{cases} \tag{4.40}$$

with

$$\int_{\Omega^4} |Dh|^p \, dx \leq 1,$$

then there exists a weak solution $v \in W^{1,p}(B^+_4)$ of (4.36) such that

$$\int_{B^+_4} |Dv|^p \, dx \leq 1 \quad \text{and} \quad \int_{B^+_4} |h - v|^p \, dx \leq \epsilon^p.$$ 

**Proof.** We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{h_k\}_{k=1}^\infty$ and $\{\Omega^k_4\}_{k=1}^\infty$ such that $h_k \in W^{1,p}(\Omega^k_4)$ is a weak solution of

$$\begin{cases}
-\text{div} \, a_{B^+_4}(Dh_k) = 0 & \text{in } \Omega^k_4, \\
h_k = 0 & \text{on } \partial_w \Omega^k_4,
\end{cases} \tag{4.41}$$

and

$$\int_{\Omega^k_4} |Dh_k|^p \, dx \leq 1 \quad \text{and} \quad \int_{B^+_4} |h_k - v|^p \, dx \leq \epsilon^p.$$ 

By contradiction, we reach the desired result.
with

$$B_4^+ \subset \Omega_4^k \subset B_4 \cap \left\{ x_n > -\frac{8}{k} \right\},$$  \hspace{1cm} (4.41)$$

and

$$\int_{\Omega_4^k} |Dh_k|^p \, dx \leq 1, \hspace{1cm} (4.42)$$

but for any weak solution \( v \in W^{1,p}(B_4^+) \) of

$$\begin{cases}
-\text{div} \, \mathbf{a}_{B_4^+}(Dv) = 0 & \text{in } B_4^+, \\
v = 0 & \text{on } T_4,
\end{cases}$$  \hspace{1cm} (4.43)

with

$$\int_{B_4^+} |Dv|^p \, dx \leq 1, \hspace{1cm} (4.44)$$

we have

$$\int_{B_4^+} |h_k - v|^p \, dx > \epsilon_0^p. \hspace{1cm} (4.45)$$

We extend \( h_k \) by zero from \( \Omega_4^k \) to \( B_4 \) and denote it by \( h_k \) also. Then by Poincaré’s inequality and (4.42), we have \( \|h_k\|_{W^{1,p}(B_4)} \leq c \). That is, \( \{h_k\}_{k=1}^\infty \) is uniformly bounded in \( W^{1,p}(B_4^+) \). Therefore, there exists a subsequence, which we still denote by \( \{h_k\} \), and \( h_\infty \in W^{1,p}(B_4^+) \) such that

$$h_k \rightharpoonup h_\infty \ \text{weakly in } W^{1,p}(B_4^+) \ \text{and} \ h_k \rightarrow h_\infty \ \text{strongly in } L^p(B_4^+). \hspace{1cm} (4.46)$$

Then we observe from (4.40), (4.41) and (4.46) that \( h_\infty \) is a weak solution of

$$\begin{cases}
-\text{div} \, \mathbf{a}_{B_4^+}(Dh_\infty) = 0 & \text{in } B_4^+, \\
h_\infty = 0 & \text{on } T_4.
\end{cases}$$  \hspace{1cm} (4.47)

It follows from (4.41), (4.42) and weak lower semicontinuity property that

$$\int_{B_4^+} |Dh_\infty|^p \, dx \leq \liminf_{k \to \infty} \int_{B_4^+} |Dh_k|^p \, dx \leq 1. \hspace{1cm} (4.48)$$

We then reach a contradiction to (4.45) from (4.46). This completes the proof. \( \square \)
Lemma 4.7. Let \( u \) be the weak solution to the variational inequality (2.3). Then for any \( \epsilon > 0 \), there is a small \( \delta = \delta(\epsilon) > 0 \) such that if (4.30), (4.31) and (4.32) hold, then there exists a weak solution \( v \in W^{1,p}(B_4^+) \) of (4.36) such that

\[
\|Dv_0\|_{L^\infty(\Omega_3)} \leq c
\]  

(4.49)

and

\[
\int_{\Omega_2} |D(u - v_0)|^p \leq \epsilon^p,
\]  

(4.50)

where \( v_0 \) is the zero extension of \( v \) from \( B_4^+ \) to \( B_4 \).

**Proof.** Let \( k \in W^{1,p}(\Omega_5) \) be the weak solution of (4.33), and then \( w \in W^{1,p}(\Omega_5) \) be the weak solution of (4.34), and then \( h \in W^{1,p}(\Omega_4) \) be the weak solution of (4.35). Then we can derive in a similar way as in the proof of Lemma 4.4 that

\[
\int_{\Omega_4} |Du - Dh|^p \, dx \leq c \delta_{\sigma_5},
\]  

(4.51)

where \( \sigma_5 = \sigma_5(n, p, \mu, \Lambda) \) is a small positive constant.

From (4.37) and Lemma 4.6 we see that there is a weak solution \( v \in W^{1,p}(B_4^+) \) of (4.36) such that

\[
\int_{B_4^+} |Dv|^p \, dx \leq c
\]  

(4.52)

and

\[
\int_{B_3^+} |h - v|^p \, dx \leq c_* \epsilon^p,
\]  

(4.53)

where \( c_* \) is to be determined small in a universal way. We next let \( v_0 \) be the zero extension of \( v \) from \( B_4^+ \) to \( B_4 \). Then the Lipschitz bound (4.49) follows from Lemma 4.5 and (4.52).

A direct computation shows that \( v_0 \) is a weak solution of

\[
\begin{aligned}
-\text{div} \tilde{a}_{B_4^+} (Dv_0) &= D_n g^n \quad \text{in } \Omega_4, \\
v_0 &= 0 \quad \text{on } \partial_w \Omega_4,
\end{aligned}
\]  

(4.54)

where

\[
g^n = \begin{cases} 0 & \text{if } x_n > 0, \\ \tilde{a}_{B_4^+} (Dv(x', 0)) & \text{if } x_n < 0. \end{cases}
\]  

(4.55)
Choose a cutoff function $\eta \in C_0^\infty (B_3)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_2$ and $|D\eta| \leq 2$. We test the problems (4.35) and (4.54) by $\varphi = \eta^p (h - v_0)$ $\in W_0^{1,p}(\Omega_3)$, to discover

$$\int_{\Omega_3} (\tilde{a}_{B_3^+} (Dh) - \tilde{a}_{B_3^+} (Dv_0)) \cdot D(\eta^p (h - v_0)) \, dx = \int_{\Omega_3} g^p D_n (\eta^p (h - v_0)) \, dx,$$

from which we perform standard $L^p$ estimate by making use of (2.4)–(2.7), to derive

$$\int_{\Omega_2} |D(h - v_0)|^p \, dx \leq c \left( \delta + \int_{\Omega_3} (|h - v_0|^p + |g^p|^{\frac{p}{p-1}}) \, dx \right). \tag{4.56}$$

We estimate the right-hand side of (4.56) as follows:

$$\int_{\Omega_3} |h - v_0|^p \, dx \leq \int_{B_3^+} |h - v|^p \, dx + \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} |h|^p \, dx \leq c_* \epsilon^p + \frac{1}{|\Omega_3|} \left( \int_{\Omega_3} |h|^{\frac{np}{p-1}} \, dx \right)^{\frac{p-1}{n}} |\Omega_3 \setminus B_3^+|^\frac{n}{p} \leq c_* \epsilon^p + c \delta \frac{p}{n} \int_{\Omega_3} |Dh|^p \, dx \leq c_* \epsilon^p + c \delta \frac{p}{n}. \tag{4.57}$$

Here in the first line we have used (4.30) and the fact that $v_0 = 0$ in $\Omega_4 \setminus B_4^+$. In the second line we have used (4.53), (4.30) and Hölder’s inequality. In the third line we have used (4.30) and Sobolev’s inequality, assuming $1 < p < n$, otherwise $h$ is of class $C^{1-\frac{n}{p}}$ or BMO. In the last line we have used (4.37).

$$\int_{\Omega_3} |g^p|^{\frac{p}{p-1}} \, dx \leq \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} \left| \tilde{a}_{B_3^+} (Dv(x', 0)) \right|^\frac{p}{p-1} \, dx \leq c \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} |Dv(x', 0)|^p \, dx \leq c \frac{|\Omega_3 \setminus B_3^+|}{|B_3^+|} \leq c \delta, \tag{4.58}$$

where we have used (4.55), (2.4), (4.49), and (4.30). Combining (4.56), (4.57) and (4.58), we deduce
\[ \int_{\Omega} \vert D(h - v_0) \vert^p \, dx \leq c \left( c_\ast \epsilon^p + c \delta^{\sigma_6} \right) \quad (4.59) \]

for some positive constant \( \sigma_6 = \sigma_6(n, p, \mu, \Lambda) \). But then (4.51) and (4.59) imply

\[ \int_{\Omega} \vert D(u - v_0) \vert^p \, dx \leq c \left( c_\ast \epsilon^p + c \delta^{\sigma_7} \right) \quad (4.60) \]

for some positive constant \( \sigma_7 = \sigma_7(n, p, \mu, \Lambda) \). Finally, taking \( c_\ast \) small enough, and then \( \delta \), in order to arrive at the conclusion

\[ \int_{\Omega} \vert D(u - v_0) \vert^p \, dx \leq \epsilon^p. \] \( \square \)

**Lemma 4.8.** Given a vector-valued function \( F \in L^p(\Omega, \mathbb{R}^n) \), let \( u \in W^{1,p}_0(\Omega) \) be the weak solution of the variational problem (2.3). Then, there exists a universal constant \( N = N(\mu, \Lambda, n, p) > 1 \) such that for each \( 0 < \epsilon < 1 \) one can select a small \( \delta = \delta(\epsilon) > 0 \) such that if \( a \) is \( (\delta, 42) \)-vanishing, \( \Omega \) is \( (\delta, 42) \)-Reifenberg flat, and \( B_r(y) \) with \( y \in \Omega \) and \( r \in (0, 1) \) satisfies

\[ \left\{ x \in \Omega : |D|^{p} > N^p \right\} \cap B_r(y) \geq \epsilon |B_r(y)| \] \( (4.61) \)

for such a small \( \delta \), then we have

\[ B_r(y) \cap \Omega = \Omega_r(y) \subset \left\{ x \in \Omega : |D|^{p} > 1 \right\} \cup \left\{ x \in \Omega : \mathcal{M}(\| F \|^p) > \delta^p \right\} \]

\[ \cup \left\{ x \in \Omega : \mathcal{M}(\| D\psi \|^p) > \delta^p \right\}. \] \( (4.62) \)

**Proof.** We argue by contradiction. If \( B_r(y) \) satisfies (4.61) and the claim (4.62) is false, then there exists a point \( y_1 \in \Omega_{r'}(y) = B_r(y) \cap \Omega \) such that for every \( \rho > 0 \),

\[ \frac{1}{|B_{\rho}(y_1)|} \int_{\Omega_{\rho}(y_1)} |Dv|^p \, dx \leq 1, \]

\[ \frac{1}{|B_{\rho}(y_1)|} \int_{\Omega_{\rho}(y_1)} |F|^p \, dx \leq \delta^p, \quad \frac{1}{|B_{\rho}(y_1)|} \int_{\Omega_{\rho}(y_1)} |D\psi|^p \, dx \leq \delta^p. \] \( (4.63) \)

We first consider the interior case that \( B_{6\rho}(y) \subset \Omega \). Since \( B_{5r}(y) \subset \Omega_{6r}(y_1) \), it follows from (4.63) that

\[ \int_{B_{5r}(y)} |Dv|^p \, dx \leq \frac{1}{|B_{5r}(y)|} \int_{\Omega_{6r}(y_1)} |Dv|^p \, dx \]

\[ \leq \frac{|B_{6\rho}(y)|}{|B_{5r}(y)|} \frac{1}{|B_{6\rho}(y_1)|} \int_{\Omega_{6r}(y_1)} |Dv|^p \, dx \leq \left( \frac{6}{5} \right)^n < 2^n. \] \( (4.64) \)
Likewise, we find
\[
\int_{\Omega_{5r}(y)} |F|^p \, dx \leq 2^n \delta^p, \quad \int_{\Omega_{5r}(y)} |D\psi|^p \, dx \leq 2^n \delta^p. \tag{4.65}
\]

Without loss of generality we assume \( y = 0 \). We then consider the rescaled maps
\[
\bar{a}(\xi, x) = \frac{a(2^{\frac{n}{p}} \xi, rx)}{2^\frac{n(p-1)}{p}}, \quad \bar{\Omega} = \left\{ \frac{1}{r^2} x : x \in \Omega \right\}, \tag{4.66}
\]
and
\[
\bar{u}(x) = \frac{u(rx)}{2^{\frac{n}{p}} r}, \quad \bar{F} = \frac{F(rx)}{2^{\frac{n}{p}} r}, \quad \bar{\psi}(x) = \frac{\psi(rx)}{2^{\frac{n}{p}} r}, \tag{4.67}
\]
with \( x \in B_{\delta} \subset \bar{\Omega} \) and \( \xi \in \mathbb{R}^n \). Because of Lemma 2.4 and (4.64)–(4.67), we are in the setting of Lemma 4.4. This lemma and Lemma 4.3 imply, after scaling back, that there exists \( v \in W^{1, p}(B_{4r}) \) such that
\[
\|Dv\|_{L^\infty(B_{3r})} \leq N_0 \tag{4.68}
\]
for some positive constant \( N_0 = N_0(\mu, \Lambda, n, p) \), and
\[
\int_{B_{4r}} |D(u - v)|^p \, dx \leq \epsilon_0 \epsilon, \tag{4.69}
\]
where \( \epsilon_0 \) is to be determined in a universal way as below. Now we let
\[
N_1 = \max\{2N_0, 2^{\frac{n}{p}}\},
\]
then
\[
\left\{ x \in B_r : M(|Du|^p) > N_1^p \right\} \subset \left\{ x \in B_r : M_{B_{4r}}(|D(u - v)|^p) > N_0^p \right\}. \tag{4.70}
\]
By (4.70), (3.2) in Lemma 3.2 and (4.69), we conclude that
\[
\frac{1}{|B_r|} \left| \left\{ x \in B_r : M(|Du|^p) > N_1^p \right\} \right| \leq \frac{1}{|B_r|} \left| \left\{ x \in B_r : M_{B_{4r}}(|D(u - v)|^p) > N_0^p \right\} \right| 
\leq c \int_{B_r} |D(u - v)|^p \, dx 
\leq (\epsilon \epsilon_0) \epsilon < \epsilon,
\]
from the choice of a sufficiently small \( \epsilon_0 \). Then we arrive at a contradiction to (4.61).
We next consider the boundary case when $B_{6r}(y) \not\subset \Omega$. In this case, there is a boundary point $y_0 \in \partial \Omega \cap B_{6r}(y)$. From the Reifenberg flatness condition and small $BMO$ condition, we assume that there exists a new coordinate system, modulo reorientation of the axes and translation, depending on $y_0$ and $r$, whose variables we denote by $z$ such that in this new coordinate system the origin is $y_0 + \delta_0 n_0$ for some small $\delta_0 > 0$ and some inward unit normal $-n_0$, $y = z_0$, $y_1 = z_1$,

$$B^+_\rho \subset \Omega_\rho \subset \{z \in B_\rho: z_n > -2\rho \delta\}, \quad \forall \rho \in [7r, 42r]$$

(4.71) and

$$\sup_{0 < \rho \leq 42} \int_{B^{+}_\rho} |\beta(a, B^+_\rho)(z)| \, dz \leq \delta. \quad (4.72)$$

Then it follows from (4.69) and (4.63) that

$$\int_{\Omega_{35r}} |Du|^p \, dz \leq 2 \left(\frac{42}{35}\right)^n \int_{\Omega_{42r}(z_1)} |Du|^p \, dz \leq 2 \left(\frac{6}{5}\right)^n < 2^{n+1}$$

(4.73) and

$$\int_{\Omega_{35r}} |F|^p \, dz \leq 2^{n+1} \delta^p, \quad \int_{\Omega_{35r}} |D\psi|^p \, dz \leq 2^{n+1} \delta^p. \quad (4.74)$$

As for the interior case, we apply Lemma 2.4 by taking $\rho = 7r$ and $\lambda = 2^{n+1} \frac{p}{n}$, and then use (4.71)–(4.74), to observe that we are in the hypotheses of Lemma 4.7, which yields that there exists a function $v_0 \in W^{1,p}(\Omega_{28r})$ with the properties

$$\int_{\Omega_{7r}} |D(u - v_0)|^p \, dz \leq \epsilon^{**} \epsilon$$

for $\epsilon^{**}$ as selected below, and

$$\|Dv_0\|_{L^\infty(\Omega_{21r})} \leq N_2,$$

where $N_2$ is a universal constant depending on $\mu$, $\Lambda$, $n$ and $p$. Setting

$$N_3 = \max\{2N_2, 2^{n+1}\},$$

we conclude, as in the interior case, that

$$\frac{1}{|B_{7r}|} \{z \in \Omega: M(|Du|^p) > N_3^p \} \cap B_{7r} \leq c \epsilon^{**} \epsilon,$$

which implies, needless to say, that
for some universal constant $c^{**}$ depending on $\mu$, $\Lambda$, $n$ and $p$. Then if $c^{**}\epsilon^{**} < 1$, we reach a contradiction. Now we set $N = \max\{N_1, N_3\}$ to complete the proof. \(\square\)

5. Global Calderón–Zygmund theory for obstacle problems

In this section, we prove Theorem 2.5. This proof is based on the Vitali type covering lemma and the Hardy–Littlewood maximal function.

Proof of Theorem 2.5. Fix $N, \epsilon \in (0, 1)$ and the corresponding $\delta \in (0, \frac{1}{8})$ given by Lemma 4.8. Our strategy is to derive

$$\|Du\|_{L^p(\Omega, \mathbb{R}^n)} \leq c$$

under the assumptions

$$\|F\|_{L^p(\Omega, \mathbb{R}^n)} \leq \delta, \quad \|D\psi\|_{L^p(\Omega, \mathbb{R}^n)} \leq \delta.$$  \hspace{1cm} (5.2)

Then a direct computation with Lemma 3.4 and (5.2) shows

$$\sum_{k=1}^{\infty} N^{pqk} \left| \left\{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{pk} \right\} \right| \leq c \frac{1}{\delta^p} \|\mathcal{M}(|F|)\|_{L^{pq}(\Omega)}^{pq} \leq c$$ \hspace{1cm} (5.3)

and

$$\sum_{k=1}^{\infty} N^{pqk} \left| \left\{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{pk} \right\} \right| \leq c.$$ \hspace{1cm} (5.4)

We now set

$$C = \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > N^p \right\}$$

and

$$D = \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > 1 \right\} \cup \left\{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p \right\} \cup \left\{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p \right\}.$$  \hspace{1cm} (5.5)

Then it follows from Lemma 3.2, standard $L^p$ estimate (2.8) and (5.2) that

$$|C| \leq c(n, p)\|Du\|_{L^p(\Omega)}^p \leq c\|F\|_{L^p(\Omega)}^p + \|D\psi\|_{L^p(\Omega)}^p \leq c\delta^p < \epsilon|B_1|$$

from a choice of $\delta$ corresponding to $\epsilon$. Then it is clear from (5.5) and Lemma 4.8 that we are under the hypotheses of Lemma 3.3. Consequently, we get
\[ \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > N^p \right\} \leq \epsilon_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > 1 \right\} + \epsilon_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p \right\} \right] \\
+ \epsilon_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p \right\} \right]. \] (5.6)

We iterate the estimate (5.6) for \( k \geq 2 \), to find

\[ \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > N^{p^k} \right\} \leq \epsilon^k_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > 1 \right\} + \sum_{i=1}^{k} \epsilon^i_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)} \right\} \right] \\
+ \sum_{i=1}^{k} \epsilon^i_1 \left[ \left\{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)} \right\} \right]. \] (5.7)

Then in view of Lemma 3.4, (5.2), (5.3)–(5.4) and (5.7), we compute as follows:

\[
\| \mathcal{M}(|Du|^p) \|_{L^q(\Omega)}^q \\
\leq c \left( |\Omega| + \sum_{k=1}^{\infty} N^{pqk} \left[ \left\{ x \in \Omega : \mathcal{M}(|Du|^p) > N^{p^k} \right\} \right] \right) \\
\leq c \left( 1 + \sum_{i=1}^{\infty} (N^{pq} \epsilon_1)^i \sum_{k=i}^{\infty} N^{pq(k-i)} \left[ \left\{ x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)} \right\} \right] \right) \\
+ c \sum_{i=1}^{\infty} (N^{pq} \epsilon_1)^i \sum_{k=i}^{\infty} N^{pq(k-i)} \left[ \left\{ x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)} \right\} \right] \\
\leq c \left( 1 + \sum_{k=1}^{\infty} (N^{pq} \epsilon_1)^k \right).
\]

Taking \( \epsilon \) so small, in order to get

\[ N^{pq} \epsilon_1 = N^{pq} \left( \frac{20}{1 - \delta} \right)^n \epsilon \leq N^{pq} \left( \frac{160}{7} \right)^n \epsilon < 1, \]

we conclude that \( \| \mathcal{M}(|Du|^p) \|_{L^q(\Omega)} \leq c \). But then, by (3.2) in Lemma 3.2, we arrive at the claim (5.1).

Now we need to drop the a priori assumptions (5.2). To do this, we consider the normalized functions

\[ \tilde{u} = \frac{u}{\| \|_{L^p(\Omega)} + \| D\psi \|_{L^p(\Omega)}} \],
\[ \tilde{F} = \frac{F}{\| \|_{L^p(\Omega)} + \| D\psi \|_{L^p(\Omega)}} \].
and

\[ \tilde{\psi} = \frac{1}{\delta} \left( \| F \|_{L^{pq}(\Omega)} + \| D\psi \|_{L^{pq}(\Omega)} \right). \]

Clearly, we have

\[ \| \tilde{F} \|_{L^{pq}(\Omega, \mathbb{R}^n)} \leq \delta, \quad \| D\tilde{\psi} \|_{L^{pq}(\Omega, \mathbb{R}^n)} \leq \delta. \]

As a consequence, we conclude

\[ \| D\tilde{u} \|_{L^{pq}(\Omega, \mathbb{R}^n)} \leq c, \]

from which we finally obtain the required estimate

\[ \| Du \|_{L^{pq}(\Omega, \mathbb{R}^n)} \leq c \left( \| F \|_{L^{pq}(\Omega, \mathbb{R}^n)} + \| D\psi \|_{L^{pq}(\Omega, \mathbb{R}^n)} \right). \]

**Remark 5.1.** The result of Theorem 2.5 covers the linear elliptic obstacle problem, i.e. \( p = 2 \) and \( a(\xi, x) = A(x)\xi \) for some positive definite \( n \times n \) matrix \( A(x) \). In this case, the result in the absence of obstacles was proved in [7] where a measurable dependence is considered with respect to one variable by allowing \( A(x) \) to be measurable in one variable. Thanks to the result in [7], one can treat the linear elliptic obstacle problem with the methods proposed here to derive the natural Calderón–Zygmund theory.

**Remark 5.2.** There is a possible generalization of Theorem 2.5 to the linear parabolic obstacle problems with measurable coefficients. A recent work in [4] features the parabolic version of the work in [7]. Indeed, making use of the maximal function operator a natural global Calderón–Zygmund theory was established in [4] for the linear parabolic problem without obstacles. We then use the result in [4], adopt the parabolic settings offered by the authors of [2] for the obstacle problems, and follow the present approach here, to be able to prove the natural integrability result for the linear parabolic obstacle problems.

**Remark 5.3.** In the nonlinear parabolic case, there is no scale invariance, since, even in the absence of obstacles, the related problem is degenerate when \( p > 2 \) and singular when \( 1 < p < 2 \), respectively, and scales differently in space and time. Therefore, the maximal function approach cannot be applied to find the parabolic version of Theorem 2.5. In this respect, it is suggested to follow the so-called maximal function-free technique which has been introduced in [1], in order to overcome such a lack of scaling. On the other hand, once the interior analog of Theorem 2.5 for the parabolic case has been established in [2] and its global version for the elliptic case is presented in the present work, it is expected that by means of the maximal function-free technique, one can use the settings and results in [2], and partially some comparison estimates near the boundary adapted from the present work, to be finally able to find the parabolic version of Theorem 2.5.
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