Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 722–728
Applied Mathematics Letters
www.elsevier.com/locate/aml

A C^6 approximating subdivision scheme

Shahid S. Siddiqi*, Nadeem Ahmad

Department of Mathematics, Punjab University, Lahore 54590, Pakistan

Received 19 December 2006; received in revised form 17 July 2007; accepted 27 July 2007

Abstract

The approximating subdivision scheme, recently developed by Shahid S. Siddiqi and Nadeem Ahmad [An approximating C^4 stationary subdivision scheme, European Journal of Scientific Research 15 (1) (2006) 97–102], is extended. It is proved that the new scheme generates C^6 curves. Its limit function has a support on $[-6, 5]$. The smoothness of the new scheme is shown using the Laurent polynomial method, and the usefulness of the scheme is illustrated in the examples. The Hölder exponent for the scheme is calculated. It can be observed that the method developed generates curves satisfying the variation diminishing property.

Keywords: Approximating subdivision scheme; Mask; Convergence; Smoothness; Laurent polynomial

1. Introduction

The importance of developing new subdivision schemes for curve designing cannot be denied. The subdivision schemes have been appreciated in many fields such as computer animation, computer graphics and computer aided geometric design due to its efficient and simple characteristics. The subdivision scheme defines a curve out of an initial control polygon or a surface out of an initial control mesh by subdividing them according to some refining rules recursively.

In the field of subdivision, de Rham [2] and Chaikin [1] are regarded as the pioneers. They developed the corner cutting schemes, but the scheme developed by Chaikin became much more popular. Hassan and Dodgson [7] introduced a three-point ternary approximating subdivision scheme that generates a C^3 curve. Zheng et al. [15] developed a three-point ternary interpolatory subdivision scheme that generates C^1 curves. Siddiqi and Ahmad [11] introduced a three-point approximating subdivision scheme that generates C^2 curves. It is to be noted that examples considered in this work indicate that the scheme developed by Siddiqi and Ahmad [11] behaves better than that introduced by Hassan and Dodgson [7] as the curves obtained are more consistent with the control polygon. Deslauriers and Dubuc [3] introduced a four-point C^1 interpolatory subdivision scheme. Almost simultaneously, but independently, Dyn et al. [6] developed a general form of four-point C^1 interpolatory subdivision with w as tension parameter, and for a particular value of parameter $w = \frac{1}{16}$ the scheme behaves exactly as that of Deslauriers and Dubuc [3]. Tang et al. [12] proved the scheme developed by Dyn et al. [6] to be C^1 using the Laurent polynomial.

* Corresponding author.
E-mail addresses: shahidsiddiqiprof@yahoo.co.uk (S.S. Siddiqi), nadeemahmadap@yahoo.co.uk (N. Ahmad).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.07.028

Hassan et al. [8] presented a four-point ternary interpolatory subdivision scheme with tension parameter. The scheme was proved to be C^2 for a certain range of tension parameter. Zhang et al. [14] developed a four-point approximating subdivision scheme for a quadrilateral net. The limiting curve was claimed to be C^3. Dyn et al. [5] presented a four-point approximating subdivision scheme that generates C^2 curves and is close to the interpolatory case. Siddiqi and Ahmad [9] developed a four-point approximating subdivision scheme that generates C^4 curves. Weissman [13] described a six-point binary interpolatory subdivision scheme. Siddiqi and Ahmad [10] proved the scheme developed by Weissman [13] to be C^2 using the Laurent polynomial.

In this work a new six-point approximating subdivision scheme is introduced and defined as follows:

$$\begin{aligned} f_{2i}^{k+1} &= af_{i-2}^{k} + bf_{i-1}^{k} + cf_{i}^{k} + df_{i+1}^{k} + ef_{i+2}^{k} + gf_{i+3}^{k}, \\ f_{2i+1}^{k+1} &= gf_{i-2}^{k} + ef_{i-1}^{k} + df_{i}^{k} + cf_{i+1}^{k} + bf_{i+2}^{k} + af_{i+3}^{k}. \end{aligned} \tag{1.1}$$

where $\{f_i^0\}$ is a set of initial control points with $a = \frac{1}{120}w^5$, $b = \frac{1}{120}(1 + 5w + 10w^2 + 10w^3 + 5w^4 - 5w^5)$, $c = \frac{1}{60}(13 + 25w + 10w^2 - 10w^3 - 10w^4 + 5w^5)$, $d = \frac{1}{60}(32 - 30w^2 + 15w^4 - 5w^5)$, $e = \frac{1}{120}(26 - 50w + 20w^2 + 20w^3 - 20w^4 + 5w^5)$ and $g = \frac{1}{120}(w - 1)^5$. It may be noted that the polynomials a, b, c, d, e and g are quintic B-spline basis functions. It is also proved that the scheme gives C^6 limit functions for $w = \frac{1}{4}$ with support on $[-6, 5]$. The behaviour of the scheme can be observed in the examples considered in Section 4.

2. Preliminaries

A binary univariate subdivision scheme is defined in terms of a mask consisting of a finite set of non-zero coefficients $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$. The scheme is given by

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}.$$

Approximating subdivision schemes does not retain the points of stage k as a subset of the points of stage $k + 1$. Thus the general form of an approximating subdivision scheme is

$$\begin{aligned} f_{2i}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{2j} f_i^k, \\ f_{2i+1}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{2j+1} f_{i-j}^k. \end{aligned}$$

The formal definition and the notion of convergence of the subdivision scheme are as follows:

Definition 1. A subdivision scheme S is uniformly convergent if for any initial data $f^0 = \{f_i : i \in \mathbb{Z}\}$, there exists a continuous function f such that for any closed interval $I \subset R$, f satisfies

$$\lim_{k \to \infty} \sup_{i \in 2^k I} |f_i^k - f(2^{-k} i)| = 0.$$

Obviously $f = S^\infty f^0$. For each scheme S with mask $\mathbf{a}$, $a(z)$ is defined as

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i.$$

Since the schemes under consideration have masks of finite support, the corresponding symbols are Laurent polynomials, namely polynomials in positive and negative powers of the variables.

Theorem 1 (*Dyn [4]*). *Let S be a convergent subdivision scheme with a mask* $\mathbf{a}$. *Then*

$$\sum_{j \in \mathbb{Z}} a_{2j} = \sum_{j \in \mathbb{Z}} a_{2j+1} = 1. \tag{2.1}$$

It follows from Theorem 1 that the symbol of a convergent subdivision scheme satisfies

$$a(-1) = 0 \quad \text{and} \quad a(1) = 2.$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of a Laurent polynomial

$$a_1(z) = \frac{2z}{(1+z)} a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 2 (*Dyn [4]*). *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (2.1). Then there exists a subdivision scheme S_1 with the property*

$$\mathrm{d}f^k = S_1 \mathrm{d}f^{k-1},$$

where $f^k = S^k f^0$ and $\mathrm{d}f^k = \{(\mathrm{d}f^k)_i = 2^k \left(f_{i+1}^k - f_i^k\right) : i \in \mathbb{Z}\}$.

The convergence of S can be determined by analyzing the subdivision scheme $\frac{1}{2}S_1$.

Theorem 3 (*Dyn [4]*). *S is a uniformly convergent subdivision scheme if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0, that is*

$$\lim_{k\to\infty} \left(\frac{1}{2}S_1\right)^k f^0 = 0. \tag{2.2}$$

A scheme S_1 satisfying (2.2) for all initial data f^0 is termed contractive. By Theorem 3, checking of the convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether $\|(\frac{1}{2}S_1)^L\|_\infty < 1$, for some $L \in \mathbb{Z}^+$. There are two rules for computing the values at the next refinement level, one with the even coefficients of the mask and one with odd coefficients of the mask. The norm is defined as

$$\|S\| = \max\left\{\sum_i |a_{2i}|, \sum_i |a_{2i+1}|\right\},$$

and

$$\left\|\left(\frac{1}{2}S\right)^L\right\|_\infty = \max\left\{\sum_\beta |a^{[L]}_{\gamma+2^L\beta}| : \gamma = 0, 1, \ldots, 2^L - 1\right\},$$

where

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{2^j})$$

Theorem 4 (*Dyn [4]*). *Let $a(z) = \frac{(1+z)^m}{2^m} b(z)$. If S_b is convergent, then $S_a^\infty f^0 \in C^m(R)$ for any initial data f^0.*

3. Analysis of the scheme

Theorem 5. *The scheme defined in the Section 1 converges and has smoothness C^6.*

Proof. The scheme defined in Section 1 for $w = \frac{1}{4}$, can be written as

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{122\,880} f_{i-2}^k + \frac{3119}{122\,880} f_{i-1}^k + \frac{6719}{20\,480} f_i^k + \frac{31\,927}{61\,440} f_{i+1}^k + \frac{15349}{122\,880} f_{i+2}^k + \frac{81}{40\,960} f_{i+3}^k, \\ f_{2i+1}^{k+1} &= \frac{81}{40\,960} f_{i-2}^k + \frac{15\,349}{122\,880} f_{i-1}^k + \frac{31\,927}{61\,440} f_i^k + \frac{6719}{20\,480} f_{i+1}^k + \frac{3119}{122\,880} f_{i+2}^k + \frac{1}{122\,880} f_{i+3}^k. \end{aligned} \tag{3.1}$$

The Laurent polynomial $a(z)$ for the mask of the scheme can be written as

$$a(z) = \frac{1}{122\,880}z^{-6} + \frac{81}{40\,960}z^{-5} + \frac{3119}{122\,880}z^{-4} + \frac{15\,349}{122\,880}z^{-3} + \frac{6719}{20\,480}z^{-2} + \frac{31\,927}{61\,440}z^{-1} + \frac{31\,927}{61\,440} + \frac{6719}{20\,480}z + \frac{15\,349}{122\,880}z^{2} + \frac{3119}{122\,880}z^{3} + \frac{81}{40\,960}z^{4} + \frac{1}{122\,880}z^{5}.$$

In order to prove the smoothness of this scheme to be C^6 by the Laurent polynomial method, let

$$b^{[m,L]}(z) = \frac{1}{2^L}a_m^{[L]}(z), \quad m = 1, 2, \ldots, L$$

where

$$a_m(z) = \frac{2z}{1+z}a_{m-1} = \left(\frac{2z}{1+z}\right)^m a(z)$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{2^j}).$$

With a choice of $m = 1$ and $L = 1$, the above gives

$$b^{[1,1]}(z) = \frac{1}{2}a_1(z) = \frac{1}{122\,880}z^{-5} + \frac{121}{61\,440}z^{-4} + \frac{959}{40\,960}z^{-3} + \frac{1559}{15\,360}z^{-2} + \frac{13\,921}{61\,440}z^{-1} + \frac{3001}{10\,240} + \frac{13\,921}{61\,440}z + \frac{1559}{15\,360}z^{2} + \frac{959}{40\,960}z^{3} + \frac{121}{61\,440}z^{4} + \frac{1}{122\,880}z^{5}.$$

The norm of subdivision $\frac{1}{2}S_1$ is

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\left\{\sum_\beta |b^{[1,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1,$$

and therefore $\frac{1}{2}S_1$ is contractive by Theorem 3, and so is convergent.

In order to prove the six-point scheme to be C^1, consider $m = 2$ and $L = 1$; the Laurent polynomial gives

$$b^{[2,1]}(z) = \frac{1}{2}a_2(z) = \frac{1}{61\,440}z^{-4} + \frac{241}{61\,440}z^{-3} + \frac{659}{15\,360}z^{-2} + \frac{2459}{15\,360}z^{-1} + \frac{3001}{10\,240} + \frac{3001}{10\,240}z + \frac{2459}{15\,360}z^{2} + \frac{659}{15\,360}z^{3} + \frac{241}{61\,440}z^{4} + \frac{1}{61\,440}z^{5}$$

$$\left\|\frac{1}{2}S_2\right\|_\infty = \max\left\{\sum_\beta |b^{[2,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_2$ is contractive. Consequently, by Theorem 4, S_1 is convergent and $S \in C^1$.

In order to prove the six-point scheme to be C^2, consider $m = 3$ and $L = 1$; the Laurent polynomial gives

$$b^{[3,1]}(z) = \frac{1}{2}a_3(z) = \frac{1}{30\,720}z^{-3} + \frac{1}{128}z^{-2} + \frac{599}{7680}z^{-1} + \frac{31}{128} + \frac{1761}{5120}z + \frac{31}{128}z^{2} + \frac{599}{7680}z^{3} + \frac{1}{128}z^{4} + \frac{1}{30\,720}z^{5}.$$

The norm of subdivision $\frac{1}{2}S_3$ is

$$\left\|\frac{1}{2}S_3\right\|_\infty = \max\left\{\sum_\beta |b^{[3,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\}$$

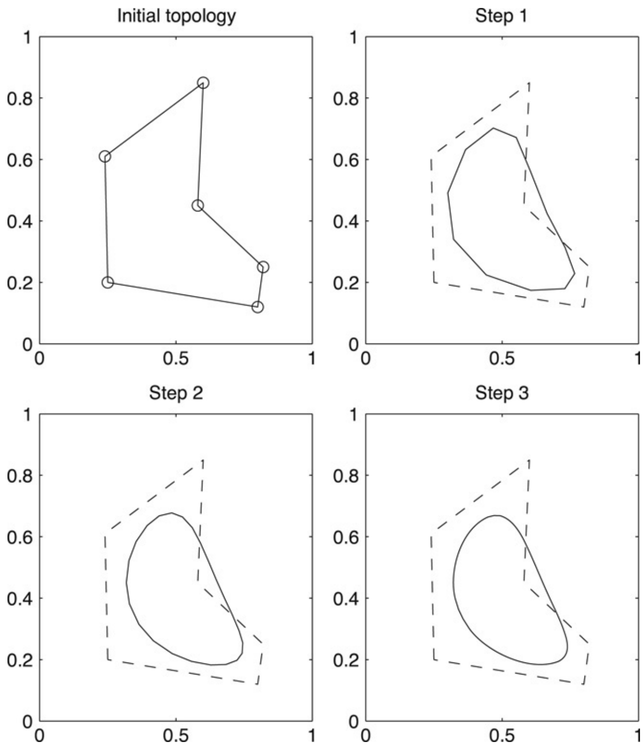

Fig. 1. New six-point scheme with three subdivision steps.

$$= \max\left\{\frac{95}{128}, \frac{1}{2}\right\} = \frac{95}{128} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_3$ is contractive. Consequently, by Theorem 4, S_2 is convergent and $S \in C^2$.

In order to prove the six-point scheme to be C^3, consider $m = 4$ and $L = 1$; the Laurent polynomial gives

$$b^{[4,1]}(z) = \frac{1}{2}a_4(z) = \frac{1}{15\,360}z^{-2} + \frac{239}{15\,360}z^{-1} + \frac{719}{5120} + \frac{1761}{5120}z + \frac{1761}{5120}z^2 + \frac{719}{5120}z^3 + \frac{239}{15\,360}z^4 + \frac{1}{15\,360}z^5.$$

The norm of subdivision $\frac{1}{2}S_4$ is

$$\left\|\frac{1}{2}S_4\right\|_\infty = \max\left\{\sum_\beta |b^{[4,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_4$ is contractive. Consequently, by Theorem 4, S_3 is convergent and $S \in C^3$.

In order to prove the six-point scheme to be C^4, consider $m = 5$ and $L = 1$; the Laurent polynomial gives

$$b^{[5,1]}(z) = \frac{1}{2}a_5(z) = \frac{1}{7680}z^{-1} + \frac{119}{3840} + \frac{1919}{7680}z + \frac{841}{1920}z^2 + \frac{1919}{7680}z^3 + \frac{119}{3480}z^4 + \frac{1}{7680}z^5.$$

The norm of subdivision $\frac{1}{2}S_5$ is

$$\left\|\frac{1}{2}S_5\right\|_\infty = \max\left\{\sum_\beta |b^{[5,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\}$$

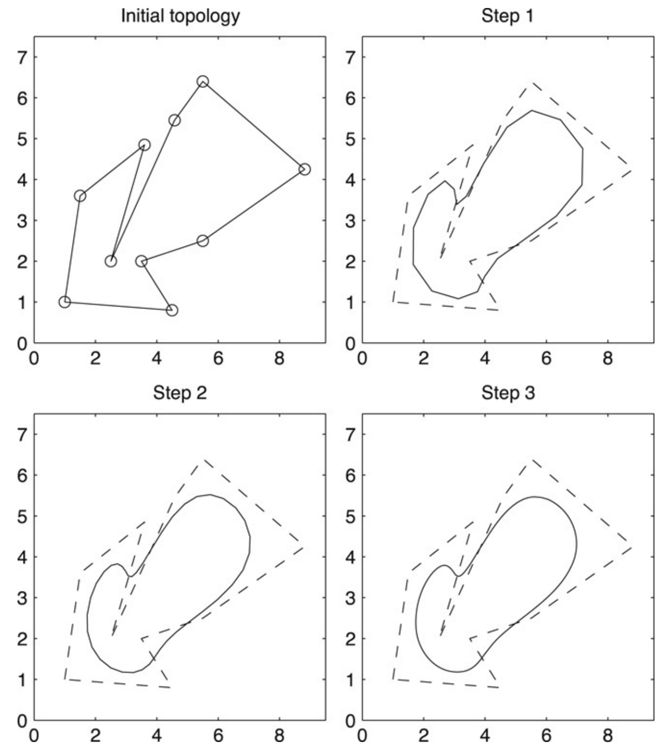

Fig. 2. New six-point scheme with three subdivision steps.

$$= \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_5$ is contractive. Consequently, by Theorem 4, S_4 is convergent and $S \in C^4$.

In order to prove the six-point scheme to be C^5, consider $m = 6$ and $L = 1$; the Laurent polynomial gives

$$b^{[6,1]}(z) = \frac{1}{2}a_6(z) = \frac{1}{3840} + \frac{79}{1280}z + \frac{841}{1920}z^2 + \frac{841}{1920}z^3 + \frac{79}{1280}z^4 + \frac{1}{3840}z^5.$$

The norm of subdivision $\frac{1}{2}S_6$ is

$$\left\|\frac{1}{2}S_6\right\|_\infty = \max\left\{\sum_\beta |b^{[6,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\}$$
$$= \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_6$ is contractive. Consequently, by Theorem 4, S_5 is convergent and $S \in C^5$.

In order to prove the six-point scheme to be C^6, consider $m = 7$ and $L = 1$; the Laurent polynomial gives

$$b^{[7,1]}(z) = \frac{1}{2}a_7(z) = \frac{1}{1920}z + \frac{59}{480}z^2 + \frac{241}{320}z^3 + \frac{59}{480}z^4 + \frac{1}{1920}z^5.$$

The norm of subdivision $\frac{1}{2}S_7$ is

$$\left\|\frac{1}{2}S_7\right\|_\infty = \max\left\{\sum_\beta |b^{[7,1]}_{\gamma+2\beta}| : \gamma = 0, 1\right\}$$
$$= \max\left\{\frac{59}{240}, \frac{181}{240}\right\} = \frac{181}{240} < 1,$$

and therefore the subdivision scheme $\frac{1}{2}S_7$ is contractive. Consequently, by Theorem 4, S_6 is convergent and $S \in C^6$. □

It can be observed from (3.1) that the support of the limiting function is $[-6, 5]$.

4. Examples

Two examples are considered and are depicted in Figs. 1 and 2 after three subdivisions. It can be observed that the scheme developed in this work generates pleasing curves.

5. Conclusion

A new six-point approximating subdivision scheme is introduced. The scheme is analyzed, using the Laurent polynomial method. The scheme is proved to be C^6 for $w = \frac{1}{4}$ with limit function supported on $[-6, 5]$; however it is to be noted that three times continuous differentiability gives curvature continuity which is the most required by a designer. It can be observed from Figs. 1 and 2 that the curves generated by the scheme are smooth and satisfy the variation diminishing property.

References

[1] G.M. Chaikin, An algorithm for high speed curve generation, Comput. Graph. Image Process. 3 (4) (1974) 346–349.
[2] G. de Rham, Sur une courbe plane, J. Math. Pures Appl. 35 (1956) 25–42.
[3] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation processes, Constr. Approx. 5 (1989) 49–68.
[4] N. Dyn, in: Armin Iske, Ewald Quak, Michael S. Floater (Eds.), Tutorials on Multiresolution in Geometric Modelling summer school lecture notes series: Mathematics and visualization, Springer, ISBN: 3-540-43639-1, 2002.
[5] Nira Dyn, Michael S. Floater, Kai Hormann, A C^2 four-points subdivision scheme with fourth order accuracy and its extensions, in: Mathematical Methods for Curves and Surfaces: Tromsø2004, Nashboro Press, Brentwood, 2005, pp. 145–156.
[6] Nira Dyn, J.A. Gregory, D. Levin, A 4-points interpolatory subdivision scheme for curve design, Comput. Aided Geom. Design 4 (1987) 257–268.
[7] M.F. Hassan, N.A. Dodgson, Ternary and three-point univariate subdivision schemes, in: Albert Cohen, Jean-Louis Merrien, Larry L. Schumaker (Eds.), Curve and Surface Fitting: Sant-Malo 2002, Nashboro Press, Brentwood, 2003, pp. 199–208.
[8] M.F. Hassan, I.P. Ivrissimitzis, N.A. Dodgson, M.A. Sabin, An interpolating 4-points C^2 ternary stationary subdivision scheme, Comput. Aided Geom. Design 19 (2002) 1–18.
[9] Shahid S. Siddiqi, Nadeem Ahmad, An approximating C^4 stationary subdivision scheme, Eur. J. Sci. Res. 15 (1) (2006) 97–102.
[10] Shahid S. Siddiqi, Nadeem Ahmad, A proof of the smoothness of 6-point interpolatory scheme, Int. J. Comput. Math. 83 (5–6) (2006) 503–509.
[11] Shahid S. Siddiqi, Nadeem Ahmad, A new 3-point approximating C^2 subdivision scheme, Appl. Math. Lett. 20 (2007) 707–711.
[12] Youchun Tang, Kwan Pyo Ko, Byung-Gook Lee, A new proof of the smoothness of 4-point Deslauriers–Dubuc Scheme, J. Appl. Math. Comput. 18 (2005) 553–562.
[13] A. Weissman, A 6-point interpolatory subdivision scheme for curve design, Master's Thesis, Tel-Aviv University, 1989.
[14] Limei Zhang, Zhongxuan Luo, Weizhong Liu, An approximating C^3 stationary subdivision scheme, J. Inf. Comput. Sci. 1 (1) (2004) 157–161.
[15] Hongchan Zheng, Zhenglin Ye, Zuoping Chen, Hongxing Zhao, A controllable ternary interpolatory subdivision scheme, Int. J. CAD/CAM 5 (1) (2005).