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p -Adic Schrödinger-type operator with point interactions

S. Albeverio ^{a,b,c}, S. Kuzhel ^{d,*}, S. Torba ^d^a *Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany*^b *SFB 611, Bonn, BiBoS, Bielefeld–Bonn, Germany*^c *CERFIM, Locarno and USI, Switzerland*^d *Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereshchenkovskaya 3, 01601 Kiev, Ukraine*

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Abstract

A p -adic Schrödinger-type operator $D^\alpha + V_Y$ is studied. D^α ($\alpha > 0$) is the operator of fractional differentiation and $V_Y = \sum_{i,j=1}^n b_{ij} \langle \delta_{x_j}, \cdot \rangle \delta_{x_i}$ ($b_{ij} \in \mathbb{C}$) is a singular potential containing the Dirac delta functions δ_x concentrated on a set of points $Y = \{x_1, \dots, x_n\}$ of the field of p -adic numbers \mathbb{Q}_p . It is shown that such a problem is well posed for $\alpha > 1/2$ and the singular perturbation V_Y is form-bounded for $\alpha > 1$. In the latter case, the spectral analysis of η -self-adjoint operator realizations of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$ is carried out.

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1. Introduction

The non-Archimedean analysis based on p -adic numbers has a long history and a quite exhaustive presentation of its applications in stochastics, psychology, the theory of dynamical systems, and other areas can be found in [16–18,27]. A strong impetus to the development of p -adic analysis was given by the hypothesis about a possible p -adic structure of physical space–time at sub-Planck distances ($\leq 10^{-33}$ cm) [27]. This idea gave rise to many publications (see the surveys in [18,27]). Whatever form the p -adic models may take in the future, it has become clear that finding p -adic counterparts for all basic structures of the standard mathematical physics is an interesting task. In particular, the concept of p -adic Schrödinger-type operator was first introduced and studied by V.S. Vladimirov and I.V. Volovich [26].

In the present paper we are going to continue the investigation of p -adic Schrödinger-type operators with point interactions started by A. Kochubei [18].

* Corresponding author.

E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), kuzhel@imath.kiev.ua (S. Kuzhel), sergiy.torba@gmail.com (S. Torba).

In ‘usual’ mathematical physics Schrödinger operators with point interactions are well studied and they are used in quantum mechanics to obtain Hamiltonians describing realistic physical systems but having the important property of being exactly solvable, i.e., that all eigenfunctions, spectrum, and scattering matrix can be calculated [1,4].

Since there exists a p -adic analysis based on the mappings from \mathbb{Q}_p into \mathbb{Q}_p and an analysis connected with the mapping \mathbb{Q}_p into the field of complex numbers \mathbb{C} , there exist two types of p -adic physical models. The present paper deals with the mapping $\mathbb{Q}_p \rightarrow \mathbb{C}$, i.e., complex-valued functions defined on \mathbb{Q}_p will be considered. In this case the operation of differentiation is not defined and the operator of fractional differentiation D^α of order α ($\alpha > 0$) plays a corresponding role [18,26,27]. In particular, p -adic Schrödinger-type operators with potentials $V(x) : \mathbb{Q}_p \rightarrow \mathbb{C}$ are defined as $D^\alpha + V(x)$.

The definition of D^α is given in the framework of the p -adic distribution theory with the help of Schwartz-type distributions $\mathcal{D}'(\mathbb{Q}_p)$. One of remarkable features of this theory is that any distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ with point support $\text{supp } f = \{x\}$ coincides with the Dirac delta function at the point x multiplied by a constant $c \in \mathbb{C}$, i.e., $f = c\delta_x$.

For this reason, it is natural to consider the expression $D^\alpha + V_Y$ where the singular potential $V_Y = \sum_{i,j=1}^n b_{ij} \langle \delta_{x_j}, \cdot \rangle \delta_{x_i}$ ($b_{ij} \in \mathbb{C}$) contains the Dirac delta functions δ_x concentrated on points x_k of the set $Y = \{x_1, \dots, x_n\} \subset \mathbb{Q}_p$ as a p -adic analogue of the Schrödinger operator with point interactions.

Since D^α is a p -adic pseudo-differential operator the expression $D^\alpha + V_Y$ gives an example of pseudo-differential operators with point interactions. In the ‘usual’ (Archimedean) theory, expressions of such (and more general) type have been studied in [5].

Obviously the domain of definition $\mathcal{D}(D^\alpha)$ of the unperturbed operator D^α need not contains functions continuous on \mathbb{Q}_p and, in general, may happen that the singular potential V_Y is not well defined on $\mathcal{D}(D^\alpha)$.

In Section 2, together with a presentation of some elements of p -adic analysis needed for reading the paper, we discuss the problem of characterizing $\mathcal{D}(D^\alpha)$ and study in detail the solutions of the equation $D^\alpha - \lambda I = \delta_x$.

Section 3 deals with the spectral analysis of operator realizations of $D^\alpha + V_Y$ ($\alpha > 1$) in $L_2(\mathbb{Q}_p)$. We do not restrict ourselves only to the self-adjoint case and also consider η -self-adjoint operators. The investigation of such operators is motivated by an intensive development of pseudo-Hermitian (\mathcal{PT} -symmetric) quantum mechanics in the last few years [10,14,22,25,28].

Among self-adjoint extensions of the symmetric operator A_{sym} associated with $D^\alpha + V_Y$ ($\alpha > 1$), we pay a special attention to the Friedrichs extension A_F . Since A_F is the ‘hard’ extension of A_{sym} (see [8] for the terminology) and the singular potential V_Y is form bounded the hypothesis that the discrete spectrum of A_F depends on the geometrical structure of Y looks likely. In this way we discuss the connection between the minimal distance $p^{\gamma_{\min}}$ between elements of Y and an infinite sequence of points of the discrete spectrum (type-1 part of discrete spectrum).

We will use the following notations: $\mathcal{D}(A)$ and $\ker A$ denote the domain and the null-space of a linear operator A , respectively. $A \upharpoonright_X$ means the restriction of A onto a set X .

2. Fractional differential operator

2.1. Elements of p -adic analysis

Basically we shall use the same notations as in [27]. Let p be a prime number. The field \mathbb{Q}_p of p -adic numbers is the completion of the field of rational numbers \mathbb{Q} with respect to p -adic norm $|\cdot|_p$, which is defined as follows: $|0|_p = 0$; $|x|_p = p^{-\gamma}$ if a rational number $x \neq 0$ has the form $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and integers m and n are not divisible by p . The p -adic norm $|\cdot|_p$ satisfies the strong triangle inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$. Moreover, $|x + y|_p = \max(|x|_p, |y|_p)$ if $|x|_p \neq |y|_p$.

Any p -adic number $x \neq 0$ can uniquely be presented as a series

$$x = p^\gamma \sum_{i=0}^{+\infty} x^i p^i, \quad x^i = 0, 1, \dots, p - 1, \quad x^0 > 0, \tag{2.1}$$

convergent in the p -adic norm (the canonical presentation of x).

The canonical presentation (2.1) enables one to determine the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$ by the rule: $\{x\}_p = 0$ if $x = 0$ or $\gamma(x) \geq 0$; $\{x\}_p = p^{\gamma(x)} \sum_{i=0}^{-\gamma(x)-1} x^i p^i$ if $\gamma(x) < 0$.

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq p^\gamma\}$ the ball of radius p^γ with the center at a point $a \in \mathbb{Q}_p$ and set $B_\gamma(0) = B_\gamma$. The ring \mathbb{Z}_p of p -adic integers is the ball B_0 ($\mathbb{Z}_p = B_0$).

A complex-valued function f defined on \mathbb{Q}_p is called *locally-constant* if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x)$ such that $f(x + x') = f(x), \forall x' \in B_{l(x)}$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the linear space of locally constant functions on \mathbb{Q}_p with compact supports. For any test function $\phi \in \mathcal{D}(\mathbb{Q}_p)$ there exists $l \in \mathbb{Z}$ such that $\phi(x + x') = \phi(x), x' \in B_l, x \in \mathbb{Q}_p$. The largest of such numbers $l = l(\phi)$ is called the *parameter of constancy* of ϕ . The characteristic function $\Omega(|x|_p) = 1$ if $|x|_p \leq 1$ and $\Omega(|x|_p) = 0$ if $|x|_p > 1$ of the ball B_0 is an example of test functions with parameter of constancy 1.

In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, let us consider the subspace $\mathcal{D}_\gamma^l \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in the ball B_γ and the parameter of constancy $\geq l$. The convergence $\phi_n \rightarrow 0$ in $\mathcal{D}(\mathbb{Q}_p)$ means that: $\phi_k \in \mathcal{D}_\gamma^l$, where the indices l and γ do not depend on k and ϕ_k tends uniformly to zero. This convergence determines the Schwartz topology in $\mathcal{D}(\mathbb{Q}_p)$.

Denote by $\mathcal{D}'(\mathbb{Q}_p)$ the set of all linear functionals (Schwartz-type distributions) on $\mathcal{D}(\mathbb{Q}_p)$. In contrast to distributions on \mathbb{R} , any linear functional $\mathcal{D}(\mathbb{Q}_p) \rightarrow \mathbb{C}$ is automatically continuous. The action of a functional f upon a test function ϕ will be denoted as $\langle f, \phi \rangle$.

It follows from the definition of $\mathcal{D}(\mathbb{Q}_p)$ that any test function $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is continuous on \mathbb{Q}_p . This means that the Dirac delta function $\langle \delta_x, \phi \rangle = \phi(x)$ is well defined for any point $x \in \mathbb{Q}_p$.

On \mathbb{Q}_p there exists the Haar measure, i.e., a positive measure $d_p x$ invariant under shifts $d_p(x + a) = d_p x$ and normalized by the equality $\int_{|x|_p \leq 1} d_p x = 1$.

Denote by $L_2(\mathbb{Q}_p)$ the set of measurable functions f on \mathbb{Q}_p satisfying the condition $\int_{\mathbb{Q}_p} |f(x)|^2 d_p x < \infty$. The set $L_2(\mathbb{Q}_p)$ is a Hilbert space with the scalar product $(f, g) = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} d_p x$.

The Fourier transform of $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is defined by the formula

$$F[\phi](\xi) = \tilde{\phi}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) \phi(x) d_p x, \quad \xi \in \mathbb{Q}_p,$$

where $\chi_p(\xi x) = e^{2\pi i \{\xi x\}_p}$ is an additive character of the field \mathbb{Q}_p for any $\xi \in \mathbb{Q}_p$. The Fourier transform $F[\cdot]$ maps $\mathcal{D}(\mathbb{Q}_p)$ onto $\mathcal{D}(\mathbb{Q}_p)$. Its extension by continuity onto $L_2(\mathbb{Q}_p)$ determines an unitary operator in $L_2(\mathbb{Q}_p)$.

The Fourier transform $F[f]$ of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ is defined by the standard relation $\langle F[f], \phi \rangle = \langle f, F[\phi] \rangle, \forall \phi \in \mathcal{D}(\mathbb{Q}_p)$.

2.2. The operator D^α

The operator of differentiation is not defined in $L_2(\mathbb{Q}_p)$. Its role is played by the operator of fractional differentiation D^α (the Vladimirov pseudo-differential operator) which is defined as

$$D^\alpha f = \int_{\mathbb{Q}_p} |\xi|_p^\alpha F[f](\xi) \chi_p(-\xi x) d_p \xi, \quad \alpha > 0. \tag{2.2}$$

It is easy to see that $D^\alpha f$ is well defined for all $f \in \mathcal{D}(\mathbb{Q}_p)$. The element $D^\alpha f$ need not belong necessarily to $\mathcal{D}(\mathbb{Q}_p)$ (since the function $|\xi|_p^\alpha$ is not locally constant) however $D^\alpha f \in L_2(\mathbb{Q}_p)$ [18].

Since $\mathcal{D}(\mathbb{Q}_p)$ is not invariant with respect to D^α we cannot define D^α on the whole space $\mathcal{D}'(\mathbb{Q}_p)$. For a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ the operator D^α is well defined only if the right-hand side of (2.2) exists.¹

In what follows we will consider $D^\alpha, \alpha > 0$, as an unbounded operator in $L_2(\mathbb{Q}_p)$. In this case, the domain of definition $\mathcal{D}(D^\alpha)$ consists of those $f \in L_2(\mathbb{Q}_p)$ for which $|\xi|_p^\alpha F[f](\xi) \in L_2(\mathbb{Q}_p)$. Since D^α is unitarily equivalent to the operator of multiplication by $|\xi|_p^\alpha$, this operator is positive self-adjoint in $L_2(\mathbb{Q}_p)$ and its spectrum consists of eigenvalues $\lambda_m = p^{\alpha m} (m \in \mathbb{Z})$ of infinite multiplicity and their accumulation point $\lambda = 0$.

It was recently shown [19] that the set of eigenfunctions of D^α

$$\psi_{Nj\epsilon}(x) = p^{-\frac{N}{2}} \chi(p^{N-1} jx) \Omega(|p^N x - \epsilon|_p), \quad N \in \mathbb{Z}, \epsilon \in \mathbb{Q}_p / \mathbb{Z}_p, j = 1, \dots, p - 1, \tag{2.3}$$

¹ To overcome such an inconvenience, a p -adic analog of the Lizorkin spaces can be used instead of $\mathcal{D}(\mathbb{Q}_p)$ [2,3].

forms an orthonormal basis in $L_2(\mathbb{Q}_p)$ (p -adic wavelet basis) such that

$$D^\alpha \psi_{Nj\epsilon} = p^{\alpha(1-N)} \psi_{Nj\epsilon}. \tag{2.4}$$

Here the indexes N, j, ϵ serve as parameters of the basis. In particular, elements $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ can be described as $\epsilon = \sum_{i=1}^m \epsilon_i p^{-i}$ ($m \in \mathbb{N}, \epsilon_i = 0, \dots, p-1$).

The p -adic wavelet basis (2.3) does not depend on the choice of α and it provides a convenient framework for the investigation of D^α . In particular, analyzing the expansion of any element $u \in \mathcal{D}(D^\alpha)$ with respect to (2.3), it is not hard to establish the uniform convergence of the corresponding series for $\alpha > 1/2$. This fact and the property of eigenfunctions $\psi_{Nj\epsilon}$ to be continuous on \mathbb{Q}_p imply the next statement.

Proposition 2.1. (See [20].) *The domain $\mathcal{D}(D^\alpha)$ consists of functions continuous on \mathbb{Q}_p if and only if $\alpha > 1/2$.*

Let us consider an equation

$$(D^\alpha - \lambda I)h = \delta_{x_k}, \quad \lambda \in \mathbb{C}, \quad x_k \in \mathbb{Q}_p, \quad \alpha > 0, \tag{2.5}$$

where $D^\alpha : L_2(\mathbb{Q}_p) \rightarrow \mathcal{D}'(\mathbb{Q}_p)$ is understood in the distribution sense.

It follows from [18, Lemma 3.7] that Eq. (2.5) has no solutions belonging to $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1/2$.

Theorem 2.1. *The following statements are valid:*

1. *If $\alpha > 1/2$, then Eq. (2.5) has a unique solution $h = h_{k,\lambda} \in L_2(\mathbb{Q}_p)$ if and only if $\lambda \neq p^{\alpha m}$, where m runs $\mathbb{Z} \cup \{-\infty\}$.*
2. *If $\alpha > 1$ and $\lambda \neq p^{\alpha m}$ ($\forall m \in \mathbb{Z} \cup \{-\infty\}$), then $h_{k,\lambda} \in \mathcal{D}(D^{\alpha/2})$.*

Proof. First of all we remark that any function $u \in \mathcal{D}(D^\alpha)$ can be expanded in an uniformly convergent series with respect to the complex-conjugated p -adic wavelet basis $\{\overline{\psi_{Nj\epsilon}}\}$. This means (since $\{\overline{\psi_{Nj\epsilon}}\}$ are continuous functions on \mathbb{Q}_p) that $u(x_k) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} \langle u, \overline{\psi_{Nj\epsilon}} \rangle \overline{\psi_{Nj\epsilon}}(x_k)$ for $x = x_k$.

Obviously, $\overline{\psi_{Nj\epsilon}}(x_k) \neq 0 \Leftrightarrow |p^N x_k - \epsilon|_p \leq 1$. Here $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ and hence, $|\epsilon|_p > 1$ for $\epsilon \neq 0$. It follows from the strong triangle inequality and the condition $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ that $|p^N x_k - \epsilon|_p \leq 1 \Leftrightarrow \epsilon = \{p^N x_k\}_p$. But then, recalling (2.3), we obtain

$$\overline{\psi_{Nj\epsilon}}(x_k) = \begin{cases} 0, & \epsilon \neq \{p^N x_k\}_p, \\ p^{-N/2} \chi(-p^{N-1} j x_k), & \epsilon = \{p^N x_k\}_p. \end{cases} \tag{2.6}$$

Therefore,

$$\begin{aligned} \langle \delta_{x_k}, u \rangle = u(x_k) &= \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \langle u, \overline{\psi_{Nj\{p^N x_k\}_p}} \rangle \\ &= \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \langle \psi_{Nj\{p^N x_k\}_p}, u \rangle. \end{aligned} \tag{2.7}$$

Since $\mathcal{D}(\mathbb{Q}_p) \subset \mathcal{D}(D^\alpha)$ the equality (2.7) yields that

$$\delta_{x_k} = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \psi_{Nj\{p^N x_k\}_p}, \tag{2.8}$$

where the series converges in $\mathcal{D}'(\mathbb{Q}_p)$.

Suppose that a function $h \in L_2(\mathbb{Q}_p)$ is represented as a convergent series in $L_2(\mathbb{Q}_p)$:

$$h(x) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} c_{Nj\epsilon} \psi_{Nj\epsilon}(x).$$

Applying the operator $D^\alpha - \lambda I$ termwise, we get a series

$$(D^\alpha - \lambda I)h = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} c_{Nj\epsilon} (p^{\alpha(1-N)} - \lambda) \psi_{Nj\epsilon}, \tag{2.9}$$

converging in \mathcal{D}' (since $D^\alpha \mathcal{D}(\mathbb{Q}_p) \subset L_2(\mathbb{Q}_p)$). The comparison of (2.8) and (2.9) gives

$$c_{Nj\epsilon} = \begin{cases} 0, & \epsilon \neq \{p^N x_k\}_p, \\ p^{-N/2} \chi(-p^{N-1} jx_k) [p^{\alpha(1-N)} - \lambda]^{-1}, & \epsilon = \{p^N x_k\}_p. \end{cases}$$

Thus

$$h_{k,\lambda}(x) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} jx_k) [p^{\alpha(1-N)} - \lambda]^{-1} \psi_{Nj\{p^N x_k\}_p}(x) \tag{2.10}$$

is a unique solution of (2.5).

Since the functions $\psi_{Nj\{p^N x_k\}_p}(x)$ in (2.10) are elements of the orthonormal basis (2.3) in $L_2(\mathbb{Q}_p)$, the function $h_{k,\lambda}(x)$ belongs to $L_2(\mathbb{Q}_p)$ if and only if

$$(p-1) \sum_{N=-\infty}^{\infty} p^{-N} [p^{\alpha(1-N)} - \lambda]^{-2} < \infty.$$

This inequality holds $\Leftrightarrow \lambda \neq p^{\alpha m} (\forall m \in \mathbb{Z} \cup \{-\infty\})$. Assertion 1 is proved.

Let $\alpha > 1$. Taking (2.3) and (2.10) into account, it is easy to see that $h_{k,\lambda} \in \mathcal{D}(D^{\alpha/2})$ if and only if the following series converge in $L_2(\mathbb{Q}_p)$:

$$\begin{aligned} & \sum_{N=1}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} jx_k) [p^{\alpha(1-N)} - \lambda]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{pj\{p^N x_k\}_p} \\ & + \sum_{N=-\infty}^0 \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} jx_k) [p^{\alpha(1-N)} - \lambda]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{pj\{p^N x_k\}_p} \end{aligned}$$

(if the limit exists then it coincides with $D^{\alpha/2} h_k$). For the general term of the first series we have

$$|p^{-N/2} p^{\frac{\alpha}{2}(1-N)} \chi(-p^{N-1} jx_k) [p^{\alpha(1-N)} - \lambda]^{-1}|^2 \leq Cp^{-N(\alpha+1)}, \quad N \geq 1$$

(since $\lambda \neq p^{\alpha m}, \forall m \in \mathbb{Z} \cup \{-\infty\}$) that implies its convergence in $L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$.

Similarly, the general term of the second series can be estimated from above by $Cp^{(\alpha-1)N}$ ($N \leq 0$), which implies its convergence in $L_2(\mathbb{Q}_p)$ for $\alpha > 1$. Theorem 2.1 is proved. \square

Let us study the solutions $h_{k,\lambda}(x)$ of (2.5) in more detail for $\alpha > 1$. To do this we consider the family of functions $M_{p^\gamma}(\lambda)$ ($\gamma \in \mathbb{Z} \cup \{-\infty\}$) represented by the series

$$M_{p^\gamma}(\lambda) = \frac{p-1}{p} \sum_{N=-\infty}^{-\gamma} \frac{p^N}{p^{\alpha N} - \lambda} - \frac{p^{-\gamma}}{p^{\alpha(1-\gamma)} - \lambda}, \quad \gamma \in \mathbb{Z}, \tag{2.11}$$

$$M_{p^{-\infty}}(\lambda) := M_0(\lambda) = \frac{p-1}{p} \sum_{N=-\infty}^{\infty} \frac{p^N}{p^{\alpha N} - \lambda}. \tag{2.12}$$

Obviously, $M_0(\lambda)$ is differentiable for $\lambda \in \mathbb{C} \setminus \{p^{\alpha N} \mid \forall N \in \mathbb{Z} \cup \{-\infty\}\}$ and $M'_0(\lambda) = \frac{p-1}{p} \sum_{N=-\infty}^{\infty} \frac{p^N}{(p^{\alpha N} - \lambda)^2}$.

Proposition 2.2. *Let $\alpha > 1$ and $\lambda \neq p^{\alpha N}$ ($\forall N \in \mathbb{Z} \cup \{-\infty\}$). Then*

$$h_{k,\lambda}(x) = \begin{cases} M_0(\lambda) & \text{if } x = x_k, \\ M_{p^\gamma}(\lambda) & \text{if } |x - x_k|_p = p^\gamma, \end{cases} \quad \|h_{k,\lambda}\|^2 = M'_0(\lambda).$$

Proof. If $\alpha > 1$ and $\lambda \neq p^{\alpha N}$ ($\forall N \in \mathbb{Z} \cup \{-\infty\}$), then $h_{k,\lambda} \in \mathcal{D}(D^{\alpha/2})$, where $\alpha/2 > 1/2$ and hence, the series (2.10) point-wise converges to $h_{k,\lambda}(x)$.

Employing (2.3) and (2.12), we immediately deduce from (2.10) that $h_{k,\lambda}(x_k) = M_0(\lambda)$, $\|h_{k,\lambda}\|^2 = M'_0(\lambda)$, and

$$h_{k,\lambda}(x) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \frac{p^{-N} \chi(p^{N-1} j(x-x_k))}{p^{\alpha(1-N)} - \lambda} \cdot \Omega(|p^N x - \{p^N x_k\}_p|_p) \tag{2.13}$$

for $x \neq x_k$.

The expression (2.13) can be simplified with the use of the following arguments:

1. It follows from the strong triangle inequality and the definitions of $\{\cdot\}_p$ and $\Omega(\cdot)$ that $\Omega(|p^N x - \{p^N x_k\}_p|_p) = \Omega(|p^N x - p^N x_k|_p)$ and

$$\Omega(|p^N x - \{p^N x_k\}_p|_p) = 0 \iff |p^N(x-x_k)|_p > 1 \iff |x-x_k|_p > p^{-N}.$$

If $x \neq x_k$, then $|x-x_k|_p = p^\gamma$ for some $\gamma \in \mathbb{Z}$. Therefore, the terms of (2.13) with indexes $N < \gamma$ are equal to zero.

2. Since $|p^{N-1} j(x-x_k)|_p = |p^{N-1}|_p |j|_p |x-x_k|_p = p^{\gamma+1-N}$ the fractional part $\{p^{N-1} j(x-x_k)\}_p$ is equal to zero for $N \geq \gamma + 1$. Hence, $\chi(p^{N-1} j(x-x_k)) \equiv 1$ when $N \geq \gamma + 1$.
3. Denote for brevity $y = p^{N-1}(x-x_k)$ and consider the case when $N = \gamma$. Then $|y|_p = p$ and hence $\{y\}_p = p^{-1}y_0$, where $y_0 \in \{1, \dots, p-1\}$ is a first term in the canonical presentation of y (see (2.1)). Since p is a prime number, it is easy to verify that the set of numbers $\{jy\}_p$ ($j = 1 \dots p-1$) coincides with the set $p^{-1}j$ ($j = 1 \dots p-1$) by modulo p . This means that

$$\sum_{j=1}^{p-1} \chi(p^{\gamma-1} j(x-x_k)) = \sum_{j=1}^{p-1} \chi(jy) = \sum_{j=1}^{p-1} \exp\left(j \frac{2\pi i}{p}\right) = -1$$

(the last equality holds because $\sum_{j=1}^p \exp ji\omega = 0$ for $\omega = \frac{2\pi}{p}$).

Statements 1–3 allow one to rewrite (2.13) as follows

$$h_{k,\lambda}(x) = (p-1) \sum_{N=\gamma+1}^{\infty} \frac{p^{-N}}{p^{\alpha(1-N)} - \lambda} - \frac{p^{-\gamma}}{p^{\alpha(1-\gamma)} - \lambda} = M_{p^\gamma}(\lambda).$$

Proposition 2.2 is proved. \square

By Proposition 2.2, $h_{k,\lambda}(x)$ is a ‘radial’ function which takes exactly one value $M_{p^\gamma}(\lambda)$ for all points x of the sphere $S_\gamma(x_k) = \{x \in \mathbb{Q}_p \mid |x-x_k|_p = p^\gamma\}$. Such a property of the solution $h_{k,\lambda}(x)$ of Eq. (2.5) is related to the property of δ to be homogeneous of degree $|x|_p^{-1}$ [27].

In conclusion, we single out properties of the functions $M_{p^\gamma}(\lambda)$ and $M_0(\lambda)$ which will be useful for the spectral analysis in the next section.

Lemma 2.1. *Let $\alpha > 1$ and let $M_{p^\gamma}(\lambda)$ and $M_0(\lambda)$ be defined by (2.11) and (2.12). Then*

1. *The function $M_0(\lambda)$ is continuous and monotonically increasing on each interval $(-\infty, 0)$, $(p^{\alpha N}, p^{\alpha(N+1)})$ ($\forall N \in \mathbb{Z}$). Furthermore, $M_0(\lambda)$ maps $(-\infty, 0)$ onto $(0, \infty)$ and maps $(p^{\alpha N}, p^{\alpha(N+1)})$ onto $(-\infty, \infty)$.*
2. *The function $M_{p^\gamma}(\lambda)$ is continuous and monotonically increasing (decreasing) on $(-\infty, 0)$ (on $(p^{\alpha(1-\gamma)}, \infty)$). Furthermore, $M_{p^\gamma}(\lambda)$ maps $(-\infty, 0)$ onto $(0, \infty)$ and maps $(p^{\alpha(1-\gamma)}, \infty)$ onto $(0, \infty)$.*

The proof of Lemma 2.1 is quite elementary and it is based on a simple analysis of the series (2.11) and (2.12). In particular, rewriting the definition of $M_{p^\gamma}(\lambda)$ as

$$M_{p^\gamma}(\lambda) = \sum_{N=-\infty}^{-\gamma} \frac{p^N}{p^{\alpha N} - \lambda} - \sum_{N=-\infty}^{-\gamma+1} \frac{p^{N-1}}{p^{\alpha N} - \lambda}$$

$$= \sum_{N=-\infty}^{-\gamma} \frac{p^N}{p^{\alpha N} - \lambda} - \sum_{N=-\infty}^{-\gamma} \frac{p^N}{p^{\alpha(N+1)} - \lambda} = \sum_{N=-\infty}^{-\gamma} \frac{p^N (p^{\alpha(N+1)} - p^{\alpha N})}{(p^{\alpha N} - \lambda)(p^{\alpha(N+1)} - \lambda)}$$

we easily establish the assertion 2.

3. *p*-Adic Schrödinger-type operator with point interactions

In this section, we are going to study finite rank point perturbations of D^α determined by the expression

$$D^\alpha + V_Y, \quad V_Y = \sum_{i,j=1}^n b_{ij} \langle \delta_{x_j}, \cdot \rangle \delta_{x_i}, \quad b_{ij} \in \mathbb{C}, \quad Y = \{x_1, \dots, x_n\}. \tag{3.1}$$

Since $\delta_{x_j} \notin L_2(\mathbb{Q}_p)$ the expression (3.1) does not determine an operator in $L_2(\mathbb{Q}_p)$. Moreover, in contrast to the standard theory of point interactions [1], the potential V_Y is not defined on the domain of definition $\mathcal{D}(D^\alpha)$ of the unperturbed operator D^α for $\alpha \leq 1/2$ (Proposition 2.1). For this reason we will assume $\alpha > 1/2$.

3.1. Definition of operator realizations of $D^\alpha + V$ in $L_2(\mathbb{Q}_p)$

Let $\mathfrak{H}_2 \subset \mathfrak{H}_1 \subset L_2(\mathbb{Q}_p) \subset \mathfrak{H}_{-1} \subset \mathfrak{H}_{-2}$ be the standard scale of Hilbert spaces (*A*-scale) associated with the positive self-adjoint operator $A = D^\alpha$ in $L_2(\mathbb{Q}_p)$. Here $\mathfrak{H}_s = \mathcal{D}(A^{s/2})$, $s = 1, 2$ with the norm $\|u\|_s = \|(D^\alpha + I)^{s/2}u\|$ and \mathfrak{H}_{-s} is the completion of $L_2(\mathbb{Q}_p)$ with respect to the norm $\|u\|_{-s}$. In a natural way \mathfrak{H}_s and \mathfrak{H}_{-s} are dual and the inner product in $L_2(\mathbb{Q}_p)$ is extended to a pairing $\langle \phi, u \rangle = ((D^\alpha + I)^{s/2}u, (D^\alpha + I)^{-s/2}\phi)$, $u \in \mathfrak{H}_s, \phi \in \mathfrak{H}_{-s}$ (see [4] for details).

By virtue of Proposition 2.2, the solutions $h_{k,\lambda}$ of (2.5) satisfy the relation $\overline{h_{k,\lambda}} = h_{k,\bar{\lambda}}$. Taking this into account and using (2.7) and (2.10) we get

$$\langle \delta_{x_k}, u \rangle = u(x_k) = ((D^\alpha - \bar{\lambda}I)u, h_{k,\lambda})_{L_2(\mathbb{Q}_p)} \quad (u \in \mathcal{D}(D^\alpha), x_k \in \mathbb{Q}_p) \tag{3.2}$$

for any complex $\lambda \neq p^{\alpha m}$ ($\forall m \in \mathbb{Z} \cup \{-\infty\}$). Hence, $\delta_{x_k} \in \mathfrak{H}_{-2}$.

In order to give a meaning to (3.1) as an operator acting in $L_2(\mathbb{Q}_p)$, we consider the positive symmetric operator A_{sym} defined by

$$A_{\text{sym}} = D^\alpha \upharpoonright_{\mathcal{D}}, \quad \mathcal{D} = \{u \in \mathcal{D}(D^\alpha) \mid u(x_1) = \dots = u(x_n) = 0\}, \quad \alpha > 1/2. \tag{3.3}$$

It follows from (3.2) that A_{sym} is a closed densely defined symmetric operator in $L_2(\mathbb{Q}_p)$ and the linear span of $\{h_{k,\lambda}\}_{k=1}^n$ coincides with $\ker(A_{\text{sym}}^* - \lambda I)$. It is convenient to present the domain of the adjoint $\mathcal{D}(A_{\text{sym}}^*)$ as $\mathcal{D}(A_{\text{sym}}^*) = \mathcal{D}(D^\alpha) \dot{+} \mathcal{H}$, where $\mathcal{H} = \ker(A_{\text{sym}}^* + I)$. Then

$$A_{\text{sym}}^* f = A_{\text{sym}}^*(u + h) = D^\alpha u - h, \quad \forall f = u + h \in \mathcal{D}(A_{\text{sym}}^*) \tag{3.4}$$

($u \in \mathcal{D}(D^\alpha), h \in \mathcal{H}$).

In the additive singular perturbation theory, the algorithm of the determination of operator realizations of $D^\alpha + V_Y$ is well known [4] and it is based on the construction of some extension (regularization) $A_{\text{reg}} := D^\alpha + V_{Y\text{reg}}$ of (3.1) onto the domain $\mathcal{D}(A_{\text{sym}}^*) = \mathcal{D}(D^\alpha) \dot{+} \mathcal{H}$.

The $L_2(\mathbb{Q}_p)$ -part

$$\tilde{A} = A_{\text{reg}} \upharpoonright_{\mathcal{D}(\tilde{A})}, \quad \mathcal{D}(\tilde{A}) = \{f \in \mathcal{D}(A_{\text{sym}}^*) \mid A_{\text{reg}} f \in L_2(\mathbb{Q}_p)\} \tag{3.5}$$

of the regularization A_{reg} is called the *operator realization* of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$.

Since the action of D^α on elements of \mathcal{H} is defined by (2.5) the regularization A_{reg} depends on the definition of $V_{Y\text{reg}}$.

If $\alpha > 1$, Theorem 2.1 gives that $\delta_{x_k} \in \mathfrak{H}_{-1}$. Hence, the singular potential $V_Y = \sum_{i,j=1}^n b_{ij} \langle \delta_{x_j}, \cdot \rangle \delta_{x_i}$ is form bounded [1]. In this case, the set $\mathcal{D}(A_{\text{sym}}^*) \subset \mathfrak{H}_1$ consists of continuous functions on \mathbb{Q}_p (in view of Proposition 2.1 and Theorem 2.1) and δ_{x_k} are uniquely determined on elements $f \in \mathcal{D}(A_{\text{sym}}^*)$ by the formula (cf. (3.2))

$$\langle \delta_{x_k}, f \rangle = ((D^\alpha + I)^{1/2} f, (D^\alpha + I)^{1/2} h_{k,-1})_{L_2(\mathbb{Q}_p)} = f(x_k). \tag{3.6}$$

Thus the regularization $A_{Y\text{reg}}$ is uniquely defined for $\alpha > 1$ and formula (3.5) provides a unique operator realization of (3.1) in $L_2(\mathbb{Q}_p)$ corresponding to a fixed singular potential V_Y .

If $1/2 < \alpha \leq 1$, then the delta functions δ_{x_k} form a \mathfrak{H}_{-1} -independent system (since the linear span of $\{\delta_{x_k}\}_1^n$ does not intersect with \mathfrak{H}_{-1}) and $V_{Y\text{reg}}$ is not uniquely defined on $\mathcal{D}(A_{\text{sym}}^*)$ (see [20] for a detailed discussion of this part).

3.2. Description of operator realizations

Let η be an invertible bounded self-adjoint operator in $L_2(\mathbb{Q}_p)$.

An operator A is called η -self-adjoint in $L_2(\mathbb{Q}_p)$ if $A^* = \eta A \eta^{-1}$, where A^* denotes the adjoint of A [9]. Obviously, self-adjoint operators are η -self-adjoint ones for $\eta = I$. In that case we will use the simpler terminology ‘self-adjoint’ instead of ‘ I -self-adjoint.’

Our goal is to describe η -self-adjoint operator realizations of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$ for $\alpha > 1$.

Since the solutions $h_k := h_{k,-1}$ ($1 \leq k \leq n$) of (2.5) form a basis of \mathcal{H} any function $f \in \mathcal{D}(A_{\text{sym}}^*) = \mathcal{D}(D^\alpha) \dot{+} \mathcal{H}$ admits a decomposition $f = u + \sum_{k=1}^n c_k h_k$ ($u \in \mathcal{D}(D^\alpha)$, $c_k \in \mathbb{C}$). Using such a presentation we define the linear mappings $\Gamma_i : \mathcal{D}(A_{\text{sym}}^*) \rightarrow \mathbb{C}^n$ ($i = 0, 1$),

$$\Gamma_0 f = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}, \quad \Gamma_1 f = - \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \forall f = u + \sum_{k=1}^n c_k h_k \in \mathcal{D}(A_{\text{sym}}^*). \tag{3.7}$$

In what follows we assume that

$$D^\alpha \eta = \eta D^\alpha \quad \text{and} \quad \eta : \mathcal{H} \rightarrow \mathcal{H}. \tag{3.8}$$

By the second relation in (3.8), the action of η on elements of \mathcal{H} can be described by the matrix $\mathcal{Y} = \|y_{ij}\|_{i,j=1}^n$ where entries y_{ij} are determined by the relations $\eta h_j = \sum_{i=1}^n y_{ij} h_i$ ($1 \leq j \leq n$). In general, the basis $\{h_i\}_{i=1}^n$ of \mathcal{H} is not orthogonal and the matrix \mathcal{Y} is not Hermitian ($\mathcal{Y} \neq \overline{\mathcal{Y}^t}$).

Theorem 3.1. (See [20].) *Let $\alpha > 1$ and let \tilde{A} be the operator realization of $D^\alpha + V_Y$ defined by (3.5). Then \tilde{A} coincides with the operator*

$$A_{\mathcal{B}} = A_{\text{sym}}^* \upharpoonright_{\mathcal{D}(A_{\mathcal{B}})}, \quad \mathcal{D}(A_{\mathcal{B}}) = \{f \in \mathcal{D}(A_{\text{sym}}^*) \mid \mathcal{B} \Gamma_0 f = \Gamma_1 f\}, \tag{3.9}$$

where $\mathcal{B} = \|b_{ij}\|_{i,j=1}^n$ is the coefficient matrix of the potential V_Y .

The operator $A_{\mathcal{B}}$ is self-adjoint if and only if the matrix \mathcal{B} is Hermitian.

If η satisfies (3.8), then $A_{\mathcal{B}}$ is η -self-adjoint if and only if the matrix $\mathcal{Y}\mathcal{B}$ is Hermitian.

Example 1 (*\mathcal{P} -self-adjoint realizations*). Let $Y = \{x_1, x_2\}$, where $x_2 = -x_1$ and let $\eta = \mathcal{P}$ be the space parity operator $\mathcal{P}f(x) = f(-x)$ in $L_2(\mathbb{Q}_p)$. It follows from Proposition 2.2 that $\mathcal{P}h_1 = h_2$ and $\mathcal{P}h_2 = h_1$. Hence, the corresponding matrix \mathcal{Y} has the form $\mathcal{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and \mathcal{P} satisfies (3.8).

By Theorem 3.1 the formula (3.9) determines \mathcal{P} -self-adjoint realizations $A_{\mathcal{B}}$ of $D^\alpha + V_Y$ if and only if the entries b_{ij} of the matrix $\mathcal{B} = \|b_{ij}\|_{i,j=1}^2$ satisfy the relations $b_{12}, b_{21} \in \mathbb{R}$, $b_{11} = \bar{b}_{22}$.

Under such conditions imposed on b_{ij} the corresponding singular potential V_Y is not symmetric in the standard sense (except the case $b_{ij} \in \mathbb{R}$, $b_{11} = b_{22}$, $b_{12} = b_{21}$) but satisfies the condition of \mathcal{P} -symmetry $\mathcal{P}V_Y^* = V_Y\mathcal{P}$, where the adjoint V_Y^* is determined by the relation $\langle V_Y u, v \rangle = \langle u, V_Y^* v \rangle$ ($u, v \in \mathcal{D}(D^\alpha)$). Assuming formally that $\mathcal{T}V_Y = V_Y^*\mathcal{T}$, where \mathcal{T} is the complex conjugation operator $\mathcal{T}f(x) = \overline{f(x)}$, we can rewrite the condition of \mathcal{P} -symmetry as follows $\mathcal{P}\mathcal{T}V_Y = V_Y\mathcal{P}\mathcal{T}$. This means that the expression $D^\alpha + V_Y$ is $\mathcal{P}\mathcal{T}$ -symmetric (since $\mathcal{P}\mathcal{T}D^\alpha = D^\alpha\mathcal{P}\mathcal{T}$). Thus the \mathcal{P} -self-adjoint operators $A_{\mathcal{B}}$ described above are operator realizations of the $\mathcal{P}\mathcal{T}$ -symmetric expression $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$.

3.3. Spectral properties

As a rule, spectral properties of finite rank perturbations are described in terms of a Nevanlinna function (Krein–Langer Q -function) appearing as a parameter in a Krein’s type resolvent formula relating the resolvents of perturbed

and unperturbed operators [4,12,23]. The choice of a resolvent formula has to be motivated by simple links with the parameters of the perturbations.

Denote by \mathcal{L} and \mathcal{L}_Y the closed subspaces of $L_2(\mathbb{Q}_p)$ spanned by the p -adic wavelets $\psi_{Nj\epsilon}(x)$ ($N \in \mathbb{Z}$, $j = 1, \dots, p - 1$) with $\epsilon \neq \{p^N x_i\}_p$ ($\forall x_i \in Y$) and $\epsilon = \{p^N x_i\}_p$ ($\exists x_i \in Y$), respectively. Obviously, $\mathcal{L} \oplus \mathcal{L}_Y = L_2(\mathbb{Q}_p)$. Relations (2.4), (2.6), and (3.3) imply that the subspaces \mathcal{L} and \mathcal{L}_Y reduce the operators D^α and A_{sym} . Furthermore $A_{\text{sym}} = D^\alpha \upharpoonright_{\mathcal{L}} \oplus A_{\text{sym}} \upharpoonright_{\mathcal{L}_Y}$.

Let A_B be the operator realization of $D^\alpha + V$ defined by (3.9). Then $A_B = D^\alpha \upharpoonright_{\mathcal{L}} \oplus A_B \upharpoonright_{\mathcal{L}_Y}$. Therefore, the spectrum of A_B consists of eigenvalues $\lambda = p^{\alpha N}$ ($\forall N \in \mathbb{Z}$) of infinite multiplicity and their accumulation point $\lambda = 0$.

To describe eigenvalues of finite multiplicity we consider the matrix

$$M(\lambda) = \|M_{|x_i - x_j|_p}(\lambda)\|_{i,j=1}^n, \quad \forall \lambda \in \mathbb{C} \setminus \{p^{\alpha N} \mid \forall N \in \mathbb{Z} \cup \{-\infty\}\}, \tag{3.10}$$

where the functions $M_{|x_i - x_j|_p}(\lambda)$ ($|x_i - x_j|_p = p^{\nu(x_i, x_j)}$) are defined by (2.11) and (2.12).

Theorem 3.2. *Let the matrix B in (3.9) be invertible. Then a point $\lambda \in \mathbb{C} \setminus \{p^{\alpha N} \mid \forall N \in \mathbb{Z} \cup \{-\infty\}\}$ is an eigenvalue of A_B if and only if $\det[M(\lambda) + B^{-1}] = 0$. In this case, the (geometric) multiplicity of λ is $n - r$, where r is the rank of $M(\lambda) + B^{-1}$.*

If $\det[M(\lambda) + B^{-1}] \neq 0$, then $\lambda \in \rho(A_B)$ and the corresponding Krein’s resolvent formula has the form

$$(A_B - \lambda I)^{-1} = (D^\alpha - \lambda I)^{-1} - (h_{1,\lambda}, \dots, h_{n,\lambda}) [M(\lambda) + B^{-1}]^{-1} \begin{pmatrix} (\cdot, h_{1,\bar{\lambda}}) \\ \vdots \\ (\cdot, h_{n,\bar{\lambda}}) \end{pmatrix}. \tag{3.11}$$

Proof. It is easy to see from (2.10) that $h_{k,\lambda} = u + h_{k,-1}$, where $u \in \mathcal{D}(D^\alpha)$. This relation and (3.7) give

$$\Gamma_1 h_{k,\lambda} = (0, \dots, \underbrace{-1}_{k\text{th}}, \dots, 0)^t. \tag{3.12}$$

On the other hand, in view of Proposition 2.2 and (3.7),

$$\Gamma_0 h_{k,\lambda} = (M_{|x_k - x_1|_p}(\lambda), \dots, \underbrace{M_0(\lambda)}_{k\text{th}}, \dots, M_{|x_k - x_n|_p}(\lambda))^t. \tag{3.13}$$

It is clear that $\lambda \in \mathbb{C} \setminus \{p^{\alpha N} \mid \forall N \in \mathbb{Z} \cup \{-\infty\}\}$ is an eigenvalue of A_B if and only if there exists a non-trivial element $f_\lambda \in \ker(A_{\text{sym}}^* - \lambda I) \cap \mathcal{D}(A_B)$. Representing f_λ as $f_\lambda = \sum_{k=1}^n c_k h_{k,\lambda}$, using (3.10), (3.12), (3.13), and keeping in mind that $\mathcal{D}(A_B) = \ker(\Gamma_0 - B^{-1} \Gamma_1)$, we rewrite the latter condition as follows: $[M(\lambda) + B^{-1}](c_1, \dots, c_n)^t = 0$. Therefore, λ is an eigenvalue if and only if this matrix equation has a non-trivial solution. Obviously, the (geometric) multiplicity of λ is $n - r$, where r is the rank of $M(\lambda) + B^{-1}$.

The resolvent formula (3.11) can be established by a direct verification with the help of (3.2), (3.12), and (3.13). Theorem 3.2 is proved. \square

Remark. It is easy to see that the triple $(\mathbb{C}^n, -\Gamma_1, \Gamma_0)$, where Γ_i are defined by (3.7) is a boundary value space (BVS) of A_{sym} and the matrix $M(\lambda)$ is the corresponding Weyl–Titchmarsh function of A_{sym} [13]. From this point of view, Theorem 3.2 is a direct consequence of the general BVS theory. However, we prefer not to employ the general constructions in the cases where the required results can be established in a more direct way.

3.4. The case of η -self-adjoint operator realizations

One of the principal motivations for the study of η -self-adjoint operators in framework of the quantum mechanics is the observation that some of them have real spectrum (like self-adjoint operators) and, therefore, they can be used as alternates to standard Hamiltonians to explain experimental data [21].

Since an arbitrary η -self-adjoint operator A is self-adjoint with respect to the indefinite metric $[f, g] := (\eta f, g)$ ($f, g \in L_2(\mathbb{Q}_p)$), one can attempt to develop a consistent quantum theory for η -self-adjoint Hamiltonians with real spectrum. However, in this case, we encounter the difficulty of dealing with a Hilbert space $L_2(\mathbb{Q}_p)$ equipped by

the indefinite metric $[\cdot, \cdot]$. One of the natural ways to overcome this problem consists in the construction of a certain previously unnoticed physical symmetry \mathcal{C} for A (see, e.g., [10,11,22]).

By analogy with [10], we will say that an η -self-adjoint operator A acting in $L_2(\mathbb{Q}_p)$ possesses the property of \mathcal{C} -symmetry if there exists a bounded linear operator \mathcal{C} in $L_2(\mathbb{Q}_p)$ such that the following conditions are satisfied:

- (i) $AC = CA$;
- (ii) $\mathcal{C}^2 = I$;
- (iii) the sesquilinear form $(f, g)_{\mathcal{C}} := [\mathcal{C}f, g]$ ($\forall f, g \in L_2(\mathbb{Q}_p)$) determines an inner product in $L_2(\mathbb{Q}_p)$ that is equivalent to initial one.

The existence of a \mathcal{C} -symmetry for an η -self-adjoint operator A ensures unitarity of the dynamics generated by A in the norm $\|\cdot\|_{\mathcal{C}}^2 = (\cdot, \cdot)_{\mathcal{C}}$.

In ordinary quantum theory, it is crucial that any state vector can be expressed as a linear combination of the eigenstates of the Hamiltonian. For this reason, it is natural to assume that every physically acceptable η -self-adjoint operator must admit an unconditional basis composed of its eigenvectors, or at least, of its root vectors (see [25] for a detailed discussion of this point).

Theorem 3.3. *Let $A_{\mathcal{B}}$ be the η -self-adjoint operator realization of $D^\alpha + V$ defined by (3.9). Then the following statements are equivalent:*

- (i) $A_{\mathcal{B}}$ possesses the property of \mathcal{C} -symmetry;
- (ii) the spectrum $\sigma(A_{\mathcal{B}})$ is real and there exists a Riesz basis of $L_2(\mathbb{Q}_p)$ composed of the eigenfunctions of $A_{\mathcal{B}}$.

Proof. It is known that the property of \mathcal{C} -symmetry for η -self-adjoint operators is equivalent to their similarity to self-adjoint ones [6,22]. Hence, if $A_{\mathcal{B}}$ possesses \mathcal{C} -symmetry, then there exists an invertible bounded operator Z such that

$$A_{\mathcal{B}} = ZHZ^{-1}, \tag{3.14}$$

where H is a self-adjoint operator in $L_2(\mathbb{Q}_p)$. So, the spectrum of $A_{\mathcal{B}}$ lies on the real axis. Furthermore, it follows from Theorem 3.2 that $\sigma(A_{\mathcal{B}})$ has no more than a countable set of points of condensations. Obviously, this property holds for the spectrum of the self-adjoint operator H . Applying now Lemma 4.2.7 in [9], we immediately derive the existence of an orthonormal basis of $L_2(\mathbb{Q}_p)$ composed of the eigenfunctions of H . To complete the proof of the implication (i) \Rightarrow (ii) it is sufficient to use (3.14).

Let us verify that (ii) \Rightarrow (i). Indeed, if $\{f_i\}_1^\infty$ is a Riesz basis composed of the eigenfunctions of $A_{\mathcal{B}}$ (i.e., $A_{\mathcal{B}}f_i = \lambda_i f_i$, $\lambda_i \in \mathbb{R}$), then $f_i = Ze_i$, where $\{e_i\}_1^\infty$ is an orthonormal basis of $L_2(\mathbb{Q}_p)$ and Z is an invertible bounded operator. This means that (3.14) holds for a self-adjoint operator H defined by the relations $He_i = \lambda_i e_i$. Theorem 3.3 is proved. \square

The next statement is a direct consequence of Theorem 3.3.

Corollary 1. *An arbitrary self-adjoint operator realization $A_{\mathcal{B}}$ of $D^\alpha + V$ possesses a complete set of eigenfunctions in $L_2(\mathbb{Q}_p)$.*

In conclusion we note that the spectral properties of η -self-adjoint operators can have rather unexpected features. In particular, the standard one-dimensional Schrödinger operator with a certain kind \mathcal{PT} -symmetric zero-range potentials gives examples of \mathcal{P} -self-adjoint operators in $L_2(\mathbb{R})$ whose spectra coincide with \mathbb{C} [7].

3.5. The Friedrichs extension

Let A_F be the Friedrichs extension of the symmetric operator A_{sym} defined by (3.3). The standard arguments of the extension theory lead to the conclusion (see [20] for details) that $A_F = D^\alpha$ when $1/2 < \alpha \leq 1$ and

$$A_F = A_{\text{sym}}^* \upharpoonright_{\mathcal{D}(A_F)}, \quad \mathcal{D}(A_F) = \{f(x) \in \mathcal{D}(A_{\text{sym}}^*) \mid f(x_1) = \dots = f(x_n) = 0\}$$

when $\alpha > 1$. In the latter case, $\mathcal{D}(A_F) = \ker \Gamma_0$ and the operator A_F can formally be described by (3.9) with $\mathcal{B} = \infty$.

Obviously, the essential spectrum of A_F consists of the eigenvalues $\lambda = p^{\alpha N}$ ($N \in \mathbb{Z}$) of infinite multiplicity, and their accumulation point $\lambda = 0$.

Let $\alpha > 1$. Repeating step by step the proof of Theorem 3.2 and taking the relation $\mathcal{D}(A_F) = \ker \Gamma_0$ into account, we conclude that the discrete spectrum $\sigma_{\text{dis}}(A_F)$ coincides with the set of solutions λ of the equation $\det M(\lambda) = 0$. The obtained relation allows one to establish some connections between $\sigma_{\text{dis}}(A_F)$ and the geometrical characteristics of the set Y . To illustrate this fact we consider the two points case $Y = \{x_1, x_2\}$.

Indeed,

$$\lambda \in \sigma_{\text{dis}}(A_F) \iff 0 = \det \|M_{|x_i - x_j|_p}(\lambda)\|_{i,j=1}^2 = (M_0(\lambda) - M_{p^\gamma}(\lambda))(M_0(\lambda) + M_{p^\gamma}(\lambda)),$$

where $p^\gamma = |x_1 - x_2|_p$. Therefore, the discrete spectrum is determined by the equations $M_0(\lambda) - M_{p^\gamma}(\lambda) = 0$ and $M_0(\lambda) + M_{p^\gamma}(\lambda) = 0$.

In view of (2.11) and (2.12),

$$M_0(\lambda) - M_{p^\gamma}(\lambda) = \frac{p-1}{p} \sum_{N=-\gamma+2}^{\infty} \frac{p^N}{p^{\alpha N} - \lambda} + \frac{p^{1-\gamma}}{p^{\alpha(1-\gamma)} - \lambda}. \tag{3.15}$$

A simple analysis of (3.15) shows that the function $M_0(\lambda) - M_{p^\gamma}(\lambda)$ is monotonically increasing on the intervals $(-\infty, p^{\alpha(1-\gamma)})$ and $(p^{\alpha N}, p^{\alpha(N+1)})$, $\forall N \geq -\gamma + 1$ and it maps $(-\infty, p^{\alpha(1-\gamma)})$ onto $(0, \infty)$ and maps $(p^{\alpha N}, p^{\alpha(N+1)})$ onto $(-\infty, \infty)$. This means that the set of solutions of $M_0(\lambda) - M_{p^\gamma}(\lambda) = 0$ coincides with the infinite series of numbers $\lambda = \lambda_N^-$, $N \geq -\gamma + 1$ each of which is situated in the interval $(p^{\alpha N}, p^{\alpha(N+1)})$. We will call the series of numbers $\{\lambda_N^-\}_{N=-\gamma+1}^{\infty}$ the type-1 part of the discrete spectrum of A_F . So, the type-1 part σ_{dis}^- of $\sigma_{\text{dis}}(A_F)$ consists of solutions of the equation $M_0(\lambda) - M_{p^\gamma}(\lambda) = 0$.

By virtue of (2.11) and (2.12),

$$M_0(\lambda) + M_{p^\gamma}(\lambda) = 2 \frac{p-1}{p} \sum_{N=-\infty}^{-\gamma} \frac{p^N}{p^{\alpha N} - \lambda} + \frac{p-2}{p} \frac{p^{1-\gamma}}{p^{\alpha(1-\gamma)} - \lambda} + \frac{p-1}{p} \sum_{N=-\gamma+2}^{\infty} \frac{p^N}{p^{\alpha N} - \lambda}.$$

Analyzing this relation, it is easy to see that there exists exactly one solution $\lambda = \lambda_N^+$ of $M_0(\lambda) + M_{p^\gamma}(\lambda) = 0$ lying inside an interval $(p^{\alpha N}, p^{\alpha(N+1)})$, $\forall N \in \mathbb{Z}$. We will call the infinite series of numbers $\{\lambda_N^+\}_{N=-\infty}^{\infty}$ the type-2 part σ_{dis}^+ of the discrete spectrum $\sigma_{\text{dis}}(A_F)$.

Obviously, $\sigma_{\text{dis}}^- \cup \sigma_{\text{dis}}^+ = \sigma_{\text{dis}}(A_F)$. Let $N \geq -\gamma + 1$ and let $\lambda_N^\pm \in \sigma_{\text{dis}}^\pm$ be the corresponding discrete spectrum points in $(p^{\alpha N}, p^{\alpha(N+1)})$. It follows from Lemma 2.1 that $\lambda_N^+ < \lambda_N^-$. Therefore, $\sigma_{\text{dis}}^- \cap \sigma_{\text{dis}}^+ = \emptyset$.

Thus the discrete spectrum $\sigma_{\text{dis}}(A_F)$ consists of infinite series of eigenvalues of multiplicity one, which are disposed as follows: an interval $(p^{\alpha N}, p^{\alpha(N+1)})$ contains exactly one eigenvalue λ_N^+ if $N < -\gamma$ (type-2 only) and exactly two eigenvalues $\lambda_N^+ < \lambda_N^-$ if $N \geq -\gamma + 1$ (type-1 and type-2).

The obtained description shows that the type-1 part σ_{dis}^- of $\sigma_{\text{dis}}(A_F)$ uniquely determines the distance $|x_1 - x_2|_p$.

In the general case $Y = \{x_1, \dots, x_n\}$ the discrete spectrum $\sigma_{\text{dis}}(A_F)$ also contains the type-1 part. Indeed, denote by $p^{\gamma_{\text{min}}}$ the minimal distance between the points of Y . Without loss of generality we may assume that $|x_1 - x_2|_p = p^{\gamma_{\text{min}}}$. Then, by the strong triangle inequality, $|x_j - x_1|_p = |x_j - x_2|_p = p^{\gamma_j} \geq p^{\gamma_{\text{min}}}$ for any point $x_j \in Y$ ($j \neq 1, 2$). This means that the first two rows (columns) of the matrix $M(\lambda)$ (see (3.10)) differ from each other by the first two terms only. Subtracting the second row from the first one we get

$$\det M(\lambda) = (M_0(\lambda) - M_{p^{\gamma_{\text{min}}}}(\lambda)) \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ M_{p^{\gamma_{\text{min}}}}(\lambda) & M_0(\lambda) & M_{p^{\gamma_3}}(\lambda) & \dots & M_{p^{\gamma_n}}(\lambda) \\ M_{p^{\gamma_3}}(\lambda) & M_{p^{\gamma_3}}(\lambda) & \ddots & & \\ \vdots & \vdots & & \ddots & \\ M_{p^{\gamma_n}}(\lambda) & M_{p^{\gamma_n}}(\lambda) & & & \ddots \end{vmatrix}.$$

Thus the type-1 part σ_{dis}^- of the discrete spectrum always exists and it characterizes the minimal distance $p^{\gamma_{\text{min}}}$ between elements of Y .

3.6. Two points interaction

3.6.1. Invariance with respect to the change of points of interaction

Let $Y = \{x_1, x_2\}$ and let the symmetric potential $V_Y = \sum_{i,j=1}^2 b_{ij}(\delta_{x_j}, \cdot)\delta_{x_i}$ be invariant under the change $x_1 \leftrightarrow x_2$. This means that $b_{ij} \in \mathbb{R}$ and $b_{11} = b_{22}, b_{12} = b_{21}$. In this case, the inverse \mathcal{B}^{-1} of the coefficient matrix \mathcal{B} has the form $\mathcal{B}^{-1} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a = b_{11}/\Delta, b = -b_{12}/\Delta$, and $\Delta = b_{11}^2 - b_{12}^2 \neq 0$. (We omit the case $b_{11} = b_{12} = b_{21} = b_{22}$.)

The operator $A_{\mathcal{B}}$ is self-adjoint in $L_2(\mathbb{Q}_p)$ and (by Theorem 3.2)

$$\lambda \in \sigma_{\text{dis}}(A_{\mathcal{B}}) \iff (M_0(\lambda) - M_{p^\gamma}(\lambda) + a - b)(M_0(\lambda) + M_{p^\gamma}(\lambda) + a + b) = 0,$$

where $p^\gamma = |x_1 - x_2|_p$. Thus, the description of $\sigma_{\text{dis}}(A_{\mathcal{B}})$ is similar to the description of $\sigma_{\text{dis}}(A_F)$ and we can define some analogs of the type-1

$$\sigma_{\text{dis}}^-(A_{\mathcal{B}}) := \{\lambda \in \mathbb{R} \setminus \sigma(D^\alpha) \mid M_0(\lambda) - M_{p^\gamma}(\lambda) + a - b = 0\}$$

and the type-2 $\sigma_{\text{dis}}^+(A_{\mathcal{B}}) := \{\lambda \in \mathbb{R} \setminus \sigma(D^\alpha) \mid M_0(\lambda) + M_{p^\gamma}(\lambda) + a + b = 0\}$ parts of the discrete spectrum $\sigma_{\text{dis}}(A_{\mathcal{B}})$.

By analogy with the Friedrichs extension case (see (3.15)), $\sigma_{\text{dis}}^-(A_{\mathcal{B}})$ contains an infinite series of eigenvalues λ_N^- lying in the intervals $(p^{\alpha N}, p^{\alpha(N+1)})$, $\forall N \geq -\gamma + 1$. However, in contrast to the Friedrichs case, the interval $(-\infty, p^{\alpha(-\gamma+1)})$ contains an additional (unique) point $\lambda^- \in \sigma_{\text{dis}}^-(A_{\mathcal{B}})$ if and only if

$$0 < b - a \quad \text{and} \quad b - a \neq [M_0(\lambda) - M_{p^\gamma}(\lambda)]|_{\lambda=p^{\alpha m}}, \quad -\infty \leq m \leq -\gamma,$$

where the difference $[M_0(\lambda) - M_{p^\gamma}(\lambda)]|_{\lambda=p^{\alpha m}}$ is determined by (3.15). In particular, $\lambda^- < 0 \iff 0 < b - a < M_0(0) - M_{p^\gamma}(0) = p^{(1-\alpha)(-\gamma+1)}$.

The type-2 part $\sigma_{\text{dis}}^+(A_{\mathcal{B}})$ contains an infinite series of eigenvalues λ_N^+ lying in the intervals $(p^{\alpha N}, p^{\alpha(N+1)})$, $\forall N \in \mathbb{Z}$, covering positive semi-axis. An additional (unique) negative point $\lambda^+ \in \sigma_{\text{dis}}^+(A_{\mathcal{B}})$ arises $\iff b + a < 0$.

Obviously $\sigma_{\text{dis}}^-(A_{\mathcal{B}}) \cup \sigma_{\text{dis}}^+(A_{\mathcal{B}}) = \sigma_{\text{dis}}(A_{\mathcal{B}})$ but $\sigma_{\text{dis}}^-(A_{\mathcal{B}})$ and $\sigma_{\text{dis}}^+(A_{\mathcal{B}})$ need not be disjoint.

3.6.2. Examples of \mathcal{P} -self-adjoint realizations

Let $Y = \{x_1, x_2\}$, where $x_2 = -x_1$ and let $A_{\mathcal{B}}$ be \mathcal{P} -self-adjoint realizations of $D^\alpha + V_Y$ described in Example 1. We restrict ourselves to the case where the inverse \mathcal{B}^{-1} of the coefficient matrix \mathcal{B} has the form $\mathcal{B}^{-1} = \begin{pmatrix} -ia & b \\ -b & ia \end{pmatrix}$ ($a, b \in \mathbb{R}$).

The operator $A_{\mathcal{B}}$ is \mathcal{P} -self-adjoint in $L_2(\mathbb{Q}_p)$ and λ is an eigenvalue of $A_{\mathcal{B}}$ if and only if

$$(M_0(\lambda) - M_{p^\gamma}(\lambda))(M_0(\lambda) + M_{p^\gamma}(\lambda)) + a^2 + b^2 = 0 \quad (p^\gamma = |2x_1|_p).$$

Using properties of $M_0(\lambda) - M_{p^\gamma}(\lambda)$ and $M_0(\lambda) + M_{p^\gamma}(\lambda)$ presented in Section 3.5, it is easy to describe real eigenvalues of $A_{\mathcal{B}}$. Precisely:

- (i) The negative semi-axis $\mathbb{R}_- = (-\infty, 0)$ belongs to $\rho(A_{\mathcal{B}})$.
- (ii) If $N < -\gamma$, then the interval $(p^{\alpha N}, p^{\alpha(N+1)})$ contains an eigenvalue λ_N of $A_{\mathcal{B}}$ such that $p^{\alpha N} < \lambda_N < \lambda_N^+$, where λ_N^+ is the corresponding type-2 discrete spectrum point of the Friedrichs extension A_F .
- (iii) If $N \geq -\gamma + 1$, then eigenvalues of $A_{\mathcal{B}}$ may appear only in the subinterval $(\lambda_N^+, \lambda_N^-) \subset (p^{\alpha N}, p^{\alpha(N+1)})$, where λ_N^- is the type-1 point of $\sigma_{\text{dis}}(A_F)$. The existence of such eigenvalues in $(\lambda_N^+, \lambda_N^-)$ can be guaranteed by the decreasing of a and b (for fixed $N \geq -\gamma + 1$).

3.7. One point interaction

Without loss of generality we will assume $x_1 = 0$. Then the general expression (3.1) takes the form $D^\alpha + b\langle \delta_0, \cdot \rangle \delta_0$ ($b \in \mathbb{R} \cup \infty$) and the corresponding self-adjoint operator realizations A_b in $L_2(\mathbb{Q}_p)$ are defined by the formula

$$A_b f = A_b(u + \beta h_{1,-1}) = D^\alpha u - \beta h_{1,-1}, \tag{3.16}$$

where the parameter $\beta = \beta(u, b) \in \mathbb{C}$ is uniquely determined by the relation $\beta u(0) = -\beta[1 + bM_0(-1)]$. The operators A_b are self-adjoint extensions of the symmetric operator $A_{\text{sym}} = D^\alpha \upharpoonright_{\mathcal{D}}$, $\mathcal{D} = \{u \in \mathcal{D}(D^\alpha) \mid u(0) = 0\}$.

In our case, the subspace \mathcal{L}_Y is the closed linear span of $\psi_{Nj0}(x)$ ($N \in \mathbb{Z}$, $j = 1, \dots, p - 1$) and $A_b = D^\alpha \upharpoonright_{\mathcal{L}} \oplus A_b \upharpoonright_{\mathcal{L}_Y}$. The operator $A_b \upharpoonright_{\mathcal{L}_Y}$ is a self-adjoint extension of $A_{\text{sym}} \upharpoonright_{\mathcal{L}_Y}$ and the points $p^{\alpha(1-N)}$ are eigenvalues of multiplicity $p - 2$ of the symmetric operator $A_{\text{sym}} \upharpoonright_{\mathcal{L}_Y}$. The orthonormal basis $\{\tilde{\psi}_{Nj0}(x)\}_{j=1}^{p-2}$ of the corresponding subspace $\ker(A_{\text{sym}} \upharpoonright_{\mathcal{L}_Y} - p^{\alpha(1-N)}I)$ can be chosen as follows:

$$\tilde{\psi}_{Nj0}(x) = \left(\frac{j}{j+1}\right)^{1/2} \left[\psi_{N(j+1)0}(x) - \frac{1}{j} \sum_{i=1}^j \psi_{Ni0}(x) \right]. \tag{3.17}$$

The decomposition $A_b = D^\alpha \upharpoonright_{\mathcal{L}} \oplus A_b \upharpoonright_{\mathcal{L}_Y}$, Lemma 2.1, and Theorem 3.2 allow one to describe in detail the spectral properties of A_b ($b \neq 0$). Precisely:

- (i) The operator A_b is positive $\Leftrightarrow b > 0$. Otherwise ($b < 0$), the unique solution of the equation $M_0(\lambda) = -1/b$ on the semi-axis $(-\infty, 0)$ gives a negative eigenvalue λ_b^- of multiplicity one. The corresponding normalized eigenfunction has the form

$$\phi_b^-(x) = \frac{h_{1,\lambda_b^-}(x)}{\sqrt{M'_0(\lambda_b^-)}} = \frac{1}{\sqrt{M'_0(\lambda_b^-)}} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{p-1} \frac{p^{-m/2}}{p^{\alpha(1-m)} - \lambda_b^-} \psi_{mj0}(x). \tag{3.18}$$

- (ii) The positive part of the discrete spectrum of A_b consists of an infinite series of points λ_{Nb} of multiplicity one, each of which is the unique solution of $M_0(\lambda) = -1/b$ in the interval $(p^{\alpha N}, p^{\alpha(N+1)})$ ($N \in \mathbb{Z}$). The corresponding normalized eigenfunction is (cf. (3.18))

$$\phi_{Nb}(x) = \frac{1}{\sqrt{M'_0(\lambda_{Nb})}} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{p-1} \frac{p^{-m/2}}{p^{\alpha(1-m)} - \lambda_{Nb}} \psi_{mj0}(x). \tag{3.19}$$

- (iii) The points $p^{\alpha(1-N)}$ are eigenvalues of infinite multiplicity of A_b . The orthonormal basis of the corresponding subspace $\ker(A_b - p^{\alpha(1-N)}I)$ can be chosen as follows:

$$\psi_{Nj\epsilon}(x) \quad (1 \leq j \leq p - 1, \epsilon \neq 0), \quad \tilde{\psi}_{Nj0}(x) \quad (1 \leq j \leq p - 2),$$

where $\psi_{Nj\epsilon}(x)$ and $\tilde{\psi}_{Nj0}(x)$ are defined by (2.3) and (3.17), respectively.

- (iv) The coefficient b of the singular perturbation $b\langle \delta_0, \cdot \rangle \delta_0$ is uniquely recovered by any point of the discrete spectrum and

$$\sigma_{\text{dis}}(A_{b_1}) \cap \sigma_{\text{dis}}(A_{b_2}) = \emptyset \quad (b_1 \neq b_2); \quad \bigcup_{b \in \mathbb{R} \cup \{\infty\}} \sigma_{\text{dis}}(A_b) = \mathbb{R} \setminus \sigma(D^\alpha).$$

Combining properties (i)–(iii) with Corollary 1 we immediately establish the following statement.

Proposition 3.1. *The set of eigenfunctions of A_b*

$$\begin{aligned} &\psi_{Nj\epsilon}(x) \quad (N \in \mathbb{Z}, 1 \leq j \leq p - 1, \epsilon \neq 0), \\ &\tilde{\psi}_{Nj0}(x) \quad (N \in \mathbb{Z}, 1 \leq j \leq p - 2), \\ &\phi_{Nb}(x) \quad (N \in \mathbb{Z}), \\ &\phi_b^-(x) \quad (\text{for the case } b < 0 \text{ only}) \end{aligned} \tag{3.20}$$

forms an orthonormal basis of $L_2(\mathbb{Q}_p)$.

The Krein spectral shift $\xi_b(\lambda) = \frac{1}{\pi} \arg[1 + bM_0(\lambda + i0)]$ is easily calculated

$$\xi_b(\lambda) = \begin{cases} 0 & \text{if } \lambda \in (-\infty, \lambda_-) \cup [\bigcup_{-\infty}^{\infty} (\lambda_{N,b}, p^{\alpha(N+1)})], \\ 1 & \text{if } \lambda \in (\lambda_-, 0) \cup [\bigcup_{-\infty}^{\infty} (p^{\alpha N}, \lambda_{N,b})] \end{cases}$$

(the interval $(\lambda_-, 0)$ is omitted for $b > 0$). Therefore [24], the difference of the spectral projectors $P_\lambda(A_b) - P_\lambda(D^\alpha)$ ($P_\lambda := P_{(-\infty, \lambda)}$) is trace class and $\text{Tr}[P_\lambda(A_b) - P_\lambda(D^\alpha)] = 0$ for all $\lambda \in \ker \xi_b(\lambda)$.

Let us consider the transformation of dilation $Uf(x) = p^{-1/2}f(px)$. Obviously, U is an unitary operator in $L_2(\mathbb{Q}_p)$ and the p -adic wavelet basis $\{\psi_{Nj\epsilon}(x)\}$ is invariant with respect to the dilation

$$U\psi_{Nj\epsilon}(x) = \psi_{(N+1)j\epsilon}(x). \tag{3.21}$$

Furthermore, in view of (2.4)

$$U^m D^\alpha = p^{\alpha m} D^\alpha U^m, \quad m \in \mathbb{Z}. \tag{3.22}$$

In this sense the operator D^α is $p^{\alpha m}$ -homogeneous with respect to the one parameter family $\mathfrak{U} = \{U^m\}_{m \in \mathbb{Z}}$ of unitary operators [4,15].

Proposition 3.2. *Among self-adjoint operators A_b described by (3.16) there are only two $p^{\alpha m}$ -homogeneous operators with respect to the family \mathfrak{U} . One of them $A_0 = D^\alpha$ is the Krein–von Neumann extension of A_{sym} , another one coincides with the Friedrichs extension $A_\infty = A_F$.*

An orthonormal basis of $L_2(\mathbb{Q}_p)$ composed of the eigenfunctions of A_b and invariant with respect to the dilation U exists if and only if $b = 0$ or $b = \infty$.

Proof. The first part of the proposition is a direct consequence of [15, Subsection 4.4].

The p -adic wavelet basis $\{\psi_{Nj\epsilon}(x)\}$ is an example of an orthonormal basis composed of the eigenfunctions of A_0 and invariant with respect to U .

Let us show that the orthonormal basis of eigenfunctions of A_∞ defined by (3.20) also is invariant with respect to U . Indeed, relations (3.17) and (3.21) yield $U\tilde{\psi}_{Nj0} = \tilde{\psi}_{(N+1)j0}$.

It follows from (2.12) that

$$p^{\alpha-1}M_0(p^\alpha\lambda) = M_0(\lambda). \tag{3.23}$$

Using (3.23) and recalling that $\lambda_{N\infty}$ is the solution of $M_0(\lambda) = 0$ in the interval $(p^{\alpha N}, p^{\alpha(N+1)})$, we derive the recurrent relation $\lambda_{(N+1)\infty} = p^\alpha\lambda_{N\infty}$. The obtained relation and (3.19), (3.21) imply $U\phi_{N\infty}(x) = \phi_{(N-1)\infty}(x)$. Hence, the basis (3.20) is invariant with respect to U for $b = \infty$.

Let \mathcal{M} be an arbitrary orthonormal basis composed of the eigenfunctions of A_b ($b \in \mathbb{R} \setminus \{0\}$). Since $\lambda_{Nb} \in (p^{\alpha N}, p^{\alpha(N+1)})$ is an eigenvalue of A_b of multiplicity one the corresponding eigenfunction $\phi_{Nb}(x)$ belongs to \mathcal{M} . Assuming that \mathcal{M} is invariant with respect to U we get $A_b U\phi_{Nb} = \mu U\phi_{Nb}$, where $\mu \in \sigma(A_b)$. To find μ we note that the $p^{\alpha m}$ -homogeneity of A_0 and A_∞ with respect to \mathfrak{U} implies that A_{sym} and A_{sym}^* also are $p^{\alpha m}$ -homogeneous with respect to U . Therefore,

$$\lambda_{Nb} U\phi_{Nb} = U A_b \phi_{Nb} = U A_{\text{sym}}^* \phi_{Nb} = p^\alpha A_{\text{sym}}^* U\phi_{Nb} = p^\alpha A_b U\phi_{Nb} = p^\alpha \mu U\phi_{Nb}.$$

Thus $\mu = p^{-\alpha}\lambda_{Nb}$. Obviously $\mu \in (p^{\alpha(N-1)}, p^{\alpha N})$ and μ is the solution of $M_0(\lambda) = -1/b$ (since μ is an eigenvalue of A_b). Employing (3.23) for $\lambda = \mu$, we arrive at the following contradiction $-1/b = M_0(\mu) = p^{\alpha-1}M_0(p^\alpha\mu) = p^{\alpha-1}M_0(\lambda_{Nb}) = -p^{\alpha-1}/b$ that completes the proof of Proposition 3.2. \square

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